ON THE NUMBER OF IRREDUCIBLE COMPONENTS OF THE REPRESENTATION VARIETY OF A FAMILY OF ONE-RELATOR GROUPS

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Let us consider the group $G = \langle x, y \mid x^m = y^n \rangle$ with m and n nonzero integers. The set $R(G)$ of representations of G over $SL(2, \mathbb{C})$ is a four dimensional algebraic variety which is an invariant of G. In this paper the number of irreducible components of $R(G)$ together with their dimensions are computed. We also study the set of metabelian representations of this family of groups. Finally, the behavior of the projection $t : R(G) \longrightarrow X(G)$, where $X(G)$ is the character variety of the group, and some combinatorial aspects of $R(G)$ are investigated.

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1. Introduction

Given a finitely generated group G, the set $R(G)$ of its representations over $SL(2, \mathbb{C})$ can be endowed with the structure of an affine algebraic variety (see [15]), the same holds for the set $X(G)$ of characters of representations over $SL(2, \mathbb{C})$ (see [3]). Since different presentations of a group G give rise to isomorphic representation and character varieties; the study of geometric invariants of $R(G)$ and $X(G)$, like the dimension or the number of irreducible components is of interest in combinatorial group theory (see [1,11,12,14] for instance). The varieties of representations and characters have also many applications in 3-dimensional geometry and topology as can be seen in [2,6,7,9,20] for instance.

Let us consider the group $G_{m,n} = \langle x, y \mid x^m = y^n \rangle$ with m and n nonzero integers. In [12] it was shown that $\dim R(G_{m,n}) = 4$ and in [13] the number of four dimensional irreducible components of $R(G_{m,n})$ was explicitly computed in terms of m and n. In previous work by the authors, see [18], an explicit description of $X(G_{m,2})$ was given and in [16,17] the character variety $X(G_{m,n})$ was studied giving an explicit decomposition into irreducible components, computing their number and dimension and completely describing their combinatorial structure. In this paper we use the results in [16,17], together with some new techniques, to compute the total number of irreducible components of $R(G_{m,n})$ and their dimensions; thus extending the work in [13]. In passing we also study the set of metabelian representations of $G_{m,n}$, the behavior of the projection $t : R(G_{m,n}) \longrightarrow X(G_{m,n})$ and some combinatorial aspects of the representation variety.

The paper is organized as follows. In Section 2, we recall the main definitions and results that will be used throughout the paper. In Section 3, we give a decomposition of $R(G_{m,n})$ into irreducible components, computing their number and dimension and studying the behavior of the projection $t : R(G_{m,n}) \longrightarrow X(G_{m,n})$. Finally, the set of metabelian representations is studied in detail in Section 4.

2. Preliminaries

2.1. *Representation and character varieties of finitely presented groups*

Let G be a group, a *representation* $\rho : G \longrightarrow SL(2,\mathbb{C})$ is just a group homomorphism. Two representations ρ and ρ' are said to be equivalent if there exists $P \in SL(2,\mathbb{C})$ such that $\rho'(g) = P^{-1}\rho(g)P$ for every $g \in G$. A representation ρ is *reducible* if the elements of $\rho(G)$ all share a common eigenvector, otherwise we say ρ is *irreducible*. The following proposition presents some useful characterizations of reducibility.

Proposition 1. *(see [3, Lemma 1.2.1. and Prop. 1.5.5.])*

(1) Let $\rho : G \longrightarrow SL(2,\mathbb{C})$ be a representation. The following conditions are equivalent:
   (a) ρ is reducible.
   (b) $\rho(G)$ is, up to conjugation, a subgroup of upper triangular matrices.
   (c) $\operatorname{tr}\rho(g) = 2$ for all g in the commutator $G' = [G,G]$.

(2) If G is generated by two elements g and h, then $\rho : G \longrightarrow SL(2,\mathbb{C})$ is reducible if and only if $\operatorname{tr}\rho([g,h]) = 2$.

Now, let us consider a finitely presented group $G = \langle x_1,\dots,x_k \mid r_1,\dots,r_s\rangle$ and let $\rho : G \longrightarrow SL(2,\mathbb{C})$ be a representation. It is clear that ρ is completely determined by the k-tuple $(\rho(x_1),\dots,\rho(x_k))$ and thus we can identify

$$R(G) = \{(\rho(x_1),\dots,\rho(x_k)) \mid \rho \text{ is a representation of } G\} \subseteq \mathbb{C}^{4k}$$

with the set of all representations of G into $SL(2,\mathbb{C})$, which is therefore (see [3]) a well-defined affine algebraic set, up to canonical isomorphism.

Recall that given a representation $\rho : G \longrightarrow SL(2,\mathbb{C})$ its character $\chi_\rho : G \longrightarrow \mathbb{C}$ is defined by $\chi_\rho(g) = \operatorname{tr}\rho(g)$. Note that two equivalent representations ρ and ρ' have the same character, and the converse is also true if ρ or ρ' is irreducible [3, Prop. 1.5.2.]. Now choose any $g \in G$ and define $t_g : R(G) \longrightarrow \mathbb{C}$ by $t_g(\rho) = \chi_\rho(g)$.

It can be seen that the ring T generated by $\{t_g \mid g \in G\}$ is a finitely generated ring ([3, Prop. 1.4.1.]) and, moreover, it can be shown using the well-known identities

$$\operatorname{tr} A = \operatorname{tr} A^{-1}\,, \qquad \operatorname{tr} AB = \operatorname{tr} BA\,, \qquad \operatorname{tr} AB = \operatorname{tr} A \operatorname{tr} B - \operatorname{tr} AB^{-1}\,, \tag{1}$$

which hold in $SL(2,\mathbb{C})$ (see [4, Cor. 4.1.2.]) that T is generated by the set:

$$\{t_{x_i}, t_{x_i x_j}, t_{x_i x_j x_h} \mid 1 \leq i < j < h \leq k\}.$$

Now choose $\gamma_1, \ldots, \gamma_\nu \in G$ such that $T = \langle t_{\gamma_i} \mid 1 \leq i \leq \nu \rangle$ and define the map $t : R(G) \longrightarrow \mathbb{C}^\nu$ by $t(\rho) = (t_{\gamma_1}(\rho), \ldots, t_{\gamma_\nu}(\rho))$. Observe that $\nu \leq \frac{k(k^2+5)}{6}$. Put $X(G) = t(R(G))$, then $X(G)$ is an algebraic variety which is well defined up to canonical isomorphism [3, Cor. 1.4.5.] and is called the *character variety* of the group G in $SL(2,\mathbb{C})$. Note that $X(G)$ can be identified with the set of all characters χ_ρ of representations $\rho \in R(G)$.

2.2. *A family of polynomials*

In forthcoming sections, we will need a particular family of polynomials whose definition is given below.

$$h_k(T) : \begin{cases} h_0(T) = 0, \\ h_1(T) = 1, \\ h_k(T) = T h_{k-1}(T) - h_{k-2}(T), \; k \in \mathbb{Z} \setminus \{0,1\}. \end{cases}$$

Note that this family is closely related to the Chebyshev polynomials (see [19]), in fact it can be shown that $h_k(2X) = U_{k-1}(X)$ for all k, where U_k is the Chebyshev polynomial of the second kind. Although this family has many interesting properties, we would like to emphasize one which will be used later, namely that $h_k(\pm 2) \neq 0$.

2.3. *Combinatorial Structure of $X(G_{m,n})$*

The decomposition of $X(G_{m,n})$ into irreducible components as well as its combinatorial structure were previously studied in [16,17]. It was proved that $X(G_{m,n}) = \mathcal{L} \cup \mathcal{C}$ where $\mathcal{L}$ consists of disjoint straight lines (after an adequate change of coordinates) and $\mathcal{C}$ consists of disjoint smooth curves. Moreover, the number of irreducible components is given by the following result.

Theorem 2. *Let d be the greatest common divisor of m and n. The number of irreducible components of $X(G_{m,n})$ is:*

$$\begin{cases} \dfrac{(|m|-1)(|n|-1)}{2} + \dfrac{d+1}{2} & \text{if } d \text{ is odd,} \\[2ex] \dfrac{(|m|-1)(|n|-1)+1}{2} + \dfrac{d+2}{2} & \text{if } d \text{ is even,} \end{cases}$$

where the first summand corresponds to the number of straight lines in $\mathcal{L}$ and the second one corresponds to the number of irreducible components of $\mathcal{C}$.

Example 3. The following figure shows the combinatorial structure of $X(G_{42,30})$.

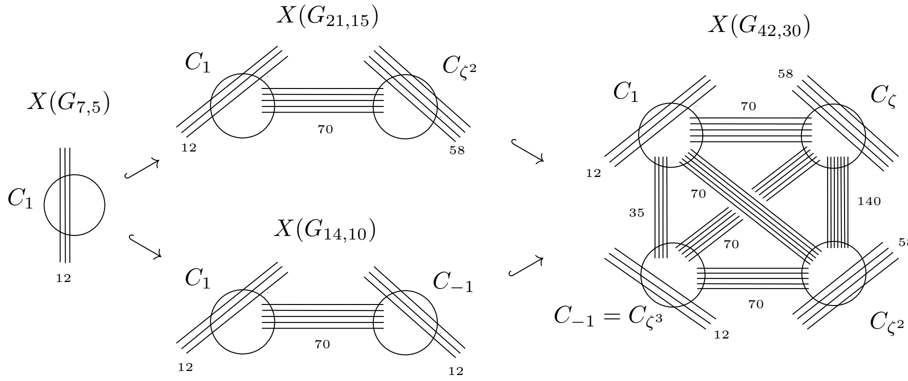

Fig. 1. Combinatorial structure of $X(G_{42,30})$.

3. The Decomposition of $R(G_{m,n})$ into Irreducible Components

The main goal of this section will be to compute the number of irreducible components of $R(G_{m,n})$ and to study the behavior of the projection $t : R(G_{m,n}) \longrightarrow X(G_{m,n})$. In particular we will see that like in [5, Rem. 3.17.] (where the variety of characters in $PSL(2,\mathbb{C})$ is considered), the projection t always induces a bijection between irreducible components.

Given the set $R(G_{m,n})$ we define the following subsets:

$$\begin{aligned} Irr &= \{\rho \in R(G_{m,n}) \mid \rho \text{ is irreducible}\}, \\ Red &= \{\rho \in R(G_{m,n}) \mid \rho \text{ is reducible}\}, \\ Ab &= \{\rho \in R(G_{m,n}) \mid \rho(G_{m,n}) \text{ is abelian}\}, \\ M_R &= \{\rho \in Red \mid \rho(G_{m,n}) \text{ is metabelian}\}. \end{aligned} \tag{2}$$

Clearly we have the partitions $R(G_{m,n}) = Irr \cup Red = Irr \cup Ab \cup M_R$. We know that Ab is an algebraic set, while Irr and M_R are not. We can obtain a decomposition of $R(G_{m,n})$ into closed subsets just by taking closures, but the unions will no longer be disjoint. Namely, we have $R(G_{m,n}) = \overline{Irr} \cup Ab \cup \overline{M_R}$. Moreover, since it can be seen that $\overline{M_R} \subseteq \overline{Irr}$ we have that $R(G_{m,n}) = \overline{Irr} \cup Ab$ and we will study these two subsets separatedly in order to count the number of irreducible components of $R(G_{m,n})$.

In the light of [16,17], it can be seen that $t(\overline{Irr}) = \mathcal{L}$, $t(Ab) = \mathcal{C} = t(Red)$ and $t(\overline{M_R}) = \mathcal{L} \cap \mathcal{C} = t(\overline{Irr}) \cap t(Ab)$. In particular this implies (t being continuous) that the previous decomposition is not redundant.

3.1. *Irreducible components of* $\overline{Irr}$

The following result can be shown by elementary facts of general topology.

Lemma 4. *Let* $\varphi : X \longrightarrow Y$ *be a continuous surjective map between two topological spaces. Then the number of irreducible components of* Y *is less than or equal to the number of irreducible components of* X*. Moreover, if these numbers are different then there exist* X_1 *and* X_2 *two different components of* X *such that* $\overline{\varphi(X_1)} \subseteq \overline{\varphi(X_2)}$*.*

Note that if X_1 and X_2 are two different components of $R(G_{m,n})$ which contain irreducible representations, then $t(X_1) \cap t(X_2) = \emptyset$ and thus $t|_{\overline{Irr}} : \overline{Irr} \longrightarrow t(\overline{Irr})$ preserves the number of irreducible components.

Remark 5. Let G be an arbitrary finitely presented group. From Propositions 1.5.2 and 1.1.1 in [3], $t(X_1) \cap t(X_2) \subseteq t(Red)$ and hence $t(X_1) \nsubseteq t(X_2)$. Therefore $t|_{\overline{Irr}} : \overline{Irr} \longrightarrow t(\overline{Irr})$ always preserves the number of irreducible components.

As was proved in [16,17], $t(\overline{Irr})$ has either $\frac{(|m|-1)(|n|-1)}{2}$ irreducible components if d is odd or $\frac{(|m|-1)(|n|-1)+1}{2}$ if d is even. Thus we have found the number of irreducible components of $\overline{Irr}$. Note that that due to [3, Cor. 1.5.3.] all these irreducible components are of dimension 4.

Since it is known that $\dim R(G_{m,n}) = 4$ and we will see that $\dim Ab = \dim \overline{M_R} = 3$, we have; in particular, the following result:

Theorem 6. *The number or irreducible 4-dimensional components of* $R(G_{m,n})$ *is:*

- $\dfrac{(|m|-1)(|n|-1)}{2}$ *if* d *is odd.*
- $\dfrac{(|m|-1)(|n|-1)+1}{2}$ *if* d *is even.*

This result can be found in [13, Theorem A], where a more direct approach is used. Note that we obtained it as a consequence of our study of $X(G_{m,n})$ in [16,17].

3.2. *Irreducible components of Ab*

This section is devoted to count the number of irreducible components of Ab. First we note that $Ab = R(G^{\mathrm{ab}}_{m,n})$ where $G^{\mathrm{ab}} = G/G'$ denotes the abelianization of G. In our case $G^{\mathrm{ab}}_{m,n} = \langle x, y \mid x^m = y^n,\ [x, y] = 1\rangle$. In the following lemma we give another presentation of $G^{\mathrm{ab}}_{m,n}$ which will be easier to work with.

Lemma 7. $G^{ab}_{m,n} \cong H_d = \langle a, b \mid a^d = 1 = [a, b]\rangle$*, where* $d = \gcd(m, n)$*.*

Proof. Put $m' = \frac{m}{d}$ and $n' = \frac{n}{d}$ and consider Bezout's identity $\alpha m - \beta n = d$. The claimed isomorphism is then given by $\phi : G^{\mathrm{ab}} \longrightarrow H_d$ with $\phi(x) = b^{n'}a^{\alpha}$, $\phi(y) = b^{m'}a^{\beta}$ and $\psi : H_d \longrightarrow G^{\mathrm{ab}}$ with $\psi(a) = x^{m'}y^{-n'}$, $\psi(b) = x^{-\beta}y^{\alpha}$. □

Thus, $Ab \cong R(H_d)$ and we will count the irreducible components of the latter. To do so we introduce some notation. If $\xi \in \{1, \ldots, \zeta^{d-1}\}$ is a d-th root of unity we put $A_\xi = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}$ and we define the set $V_\xi = \{P^{-1} A_\xi P \mid P \in SL(2,\mathbb{C})\}$. Note that $V_1 = \{I_2\}$, $V_{-1} = \{-I_2\}$ and if $\xi \neq \pm 1$ then V_ξ is an irreducible affine algebraic variety of dimension 2 (see [12, Cor. 1.5.]).

Lemma 8. *If $X \in SL(2,\mathbb{C})$ is such that $X^d = I_2$, then $X \in V_\xi$ for some ξ d-th root of unity.*

Proof. Since $\mathbb{C}$ is algebraically closed there exists $P \in SL(2,\mathbb{C})$ such that $PXP^{-1} = \begin{pmatrix} a & \alpha \\ 0 & a^{-1} \end{pmatrix} = Y$. Clearly $I_2 = Y^d = \begin{pmatrix} a^d & \alpha h_d(a+a^{-1}) \\ 0 & a^{-d} \end{pmatrix}$ so $a^d = 1$ and $\alpha h_d(a + a^{-1}) = 0$ and two cases arise.

1. If $a = \pm 1$, since $h_d(\pm 2) \neq 0$ it must be $\alpha = 0$ and $X = Y = \pm I_2 \in V_{\pm 1}$.
2. If $a = \xi \neq \pm 1$ then $a \neq a^{-1}$ and X is diagonalizable so there exists $P \in SL(2,\mathbb{C})$ such that $X = P^{-1} A_\xi P \in V_\xi$. □

If we now define $M_\xi = \{(A,B) \in SL(2,\mathbb{C}) \mid A \in V_\xi,\ [A,B] = I_2\}$, then the previous lemma shows that $R(H_d) = \bigcup_{i=0}^{d-1} M_{\zeta^i}$. Clearly $M_1 = \{I_2\} \times SL(2,\mathbb{C})$ and $M_{-1} = \{-I_2\} \times SL(2,\mathbb{C})$ are irreducible affine algebraic varieties of dimension 3. Now, we want to study the case M_ξ with $\xi \neq \pm 1$.

Proposition 9. *M_ξ is an affine irreducible algebraic variety of dimension 3 for all ξ d-th root of unity.*

Proof. We can assume $\xi \neq \pm 1$. In this case we define $\Psi : M_\xi \longrightarrow V_\xi \times \mathbb{C}^*$ as follows: given $(A,B) \in M_\xi$ there exists $P \in SL(2,\mathbb{C})$ such that $PAP^{-1} = A_\xi$, now since A and B commute, PBP^{-1} and A_ξ must also commute and it follows that $PBP^{-1} = \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix}$ must be diagonal. We define $\Psi(A,B) = (A,b)$.

Let us see that Ψ is well defined: if $PAP^{-1} = A_\xi = QAQ^{-1}$, then QP^{-1} commute with A_ξ and it must be diagonal. Consequently $QBQ^{-1} = QP^{-1} \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix} (QP^{-1})^{-1} = \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix}$ and $\Psi(A,B)$ does not depend on the choice of P.

Now, we claim that Ψ is bijective. Let us suppose that $\Psi(A_1,B_1) = (A_1,b_1) = (A_2,b_2) = \Psi(A_2,B_2)$, then $A_1 = A_2$ and $b_1 = b_2$. Since $A_1 = A_2$ and Ψ is well defined there must exist $P \in SL(2,\mathbb{C})$ such that $PB_1P^{-1} = \begin{pmatrix} b_1 & 0 \\ 0 & b_1^{-1} \end{pmatrix} = \begin{pmatrix} b_2 & 0 \\ 0 & b_2^{-1} \end{pmatrix} = PB_2P^{-1}$ so we have that $(A_1,B_1) = (A_2,B_2)$ and Ψ is injective. Since surjectivity of Ψ is obvious the claim follows.

To finish the proof it is enough to observe that Ψ induces a birational equivalence between M_ξ and $V_\xi \times \mathbb{C}$. □

Now, in order to be able to count the number of irreducible components, we must remove the redundant components in the decomposition $R(H_d) = \bigcup_{i=0}^{d-1} M_{\zeta^i}$. This is done as follows.

Proposition 10. *$M_\xi = M_\eta$ if and only if $\xi = \eta^{\pm 1}$. Moreover, if $M_\xi \neq M_\eta$, then they are disjoint.*

Proof. $M_\xi = M_\eta \Leftrightarrow V_\xi = V_\eta \Leftrightarrow A_\xi \in V_\eta \overset{[11,\, 1.4.]}{\Leftrightarrow} \operatorname{tr} A_\xi = \operatorname{tr} A_\eta \Leftrightarrow \xi = \eta^{\pm 1}$. □

As a consequence of this proposition we have that Ab has $\frac{d+1}{2}$ irreducible components if d is odd and $\frac{d+2}{2}$ if d is even.

3.3. *Counting the irreducible components of $R(G_{m,n})$*

We can now combine the results obtained in the previous sections to explicitly compute the number of irreducible components of $R(G_{m,n})$. Namely we have the following.

Theorem 11. *The number of irreducible components of $R(G_{m,n})$ is*

$$\begin{cases} \dfrac{(|m|-1)(|n|-1)}{2} + \dfrac{d+1}{2} & \text{if } d \text{ is odd,} \\[2ex] \dfrac{(|m|-1)(|n|-1)+1}{2} + \dfrac{d+2}{2} & \text{if } d \text{ is even,} \end{cases}$$

where the first summand corresponds to the number of irreducible components in $\overline{Irr}$ and the second one to the irreducible components of Ab.

Note that, in the light of Theorem 11 and recalling Theorem 2, the projection $t : R(G_{m,n}) \longrightarrow X(G_{m,n})$ preserves the number of irreducible components. In particular, if we consider the restrictions $t|_{\overline{Irr}} : \overline{Irr} \longrightarrow \mathcal{L}$ and $t|_{Ab} : Ab \longrightarrow \mathcal{C}$, we have that both of them induce bijections between irreducible components. As an easy consequence we also obtain that the irreducible components of $\overline{Irr}$ are disjoint and the same holds for those of Ab.

Also note that, due to [3, Cor. 1.5.3.], if Irr_0 is an irreducible component of $\overline{Irr}$, then $\dim Irr_0 = \dim t(Irr_0) + 3$. In our case $\dim Irr_0 = 4$ and $\dim t(Irr_0) = 1$. Nevertheless, it is known that the result is not true in general and, for instance, we have that $\dim Ab_0 = 3$ while $\dim t(Ab_0) = 1$ for every irreducible component of Ab.

4. Metabelian Representations

Although we have just computed the number of irreducible components of $R(G_{m,n})$ it can be interesting to compute the number of components of $\overline{M_R}$ and to see if $t|_{\overline{M_R}}$ preserves such number. Recall that M_R is the set of reducible metabelian representations of $G_{m,n}$ and $t(\overline{M_R}) = t(\overline{Irr}) \cap t(Ab)$. In the following lemma we give some properties of such representations which will be useful in the sequel.

Lemma 12. *Let $\rho \in M_R$. Then $\rho(x)^m = \rho(y)^n = \pm I_2$ and $\operatorname{tr} \rho(x), \operatorname{tr} \rho(y) \neq 2$.*

Proof. We can assume that $\rho(x) = X = \left(\begin{smallmatrix} a & \alpha \\ 0 & a^{-1} \end{smallmatrix}\right)$ and $\rho(y) = Y = \left(\begin{smallmatrix} b & \beta \\ 0 & b^{-1} \end{smallmatrix}\right)$. Let us suppose that $\operatorname{tr} X = 2$. Then $a = 1$ and $b^n = 1$ and two cases arise:

1. If $b = \pm 1$ then $\rho(G)$ is abelian, which contradicts the hypothesis $p \in M_R$.
2. If $b \neq \pm 1$ then $I_2 = Y^n = X^m = \left(\begin{smallmatrix} 1 & m\alpha \\ 0 & 1 \end{smallmatrix}\right)$ and $m\alpha = 0$, thus $\alpha = 0$ and $X = I_2$ which implies again that $\rho(G)$ is abelian.

Analogously it can be proved that $\operatorname{tr} X \neq -2$ and $\operatorname{tr} Y \neq \pm 2$.

Now, suppose that $X^m = Y^n \neq \pm I_2$, in which case $a^m, b^n \neq \pm 1$ and $h_m(a + a^{-1}) \neq 0 \neq h_n(b+b^{-1})$. From $A^m = B^n$ it follows that $\alpha h_m(a+a^{-1}) = \beta h_n(b+b^{-1})$, so $\alpha = 0$ if and only if $\beta = 0$ and, since $\alpha = \beta = 0$ implies that $\rho(G)$ is abelian we deduce that $\alpha\beta \neq 0$. Finally $a^m + a^{-m} = b^n + b^{-n} \Leftrightarrow (a - a^{-1})h_m(a + a^{-1}) = (b + b^{-1})h_n(b + b^{-1}) \Leftrightarrow \beta(a - a^{-1}) = \alpha(b - b^{-1}) \Leftrightarrow \rho(G)$ is abelian. This, again, is a contradiction and the lemma follows. □

We will now introduce some notation. Let us denote by Θ the set

$$\begin{aligned} \Theta &= \{\xi \mid \xi^m = \pm 1,\ \xi \neq \pm 1\} = \\ &= \{\xi \mid \xi^m = 1,\ \xi \neq \pm 1\} \cup \{\xi \mid \xi^m = -1,\ \xi \neq -1\} = \Theta^+ \cup \Theta^-. \end{aligned} \tag{3}$$

Analogously, $\Upsilon = \{\eta \mid \eta^n = \pm 1,\ \eta \neq \pm 1\} = \Upsilon^+ \cup \Upsilon^-$. Now, given $\xi \in \Theta$ and $\eta \in \Upsilon$ we put $A_\xi = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}$ and $B_\eta = \begin{pmatrix} \eta & 1 \\ 0 & \eta^{-1} \end{pmatrix}$. Finally, let us define $V_{\xi,\eta} = \{(P^{-1}A_\xi P, P^{-1}B_\eta P) \mid P \in SL(2,\mathbb{C})\}$.

Lemma 13. *If $\rho \in M_R$, there exist $\xi \in \Theta$ and $\eta \in \Upsilon$ such that $(\rho(x), \rho(y)) \in V_{\xi,\eta}$.*

Proof. Put $X = \rho(x)$ and $Y = \rho(y)$. By Lemma 12 we have that there exists $P \in SL(2,\mathbb{C})$ such that $PXP^{-1} = \begin{pmatrix} \xi & \alpha \\ 0 & \xi^{-1} \end{pmatrix} = X'$ and $PYP^{-1} = \begin{pmatrix} \eta & \beta \\ 0 & \eta^{-1} \end{pmatrix} = Y'$ for some $\xi \in \Theta$, $\eta \in \Upsilon$ and $\alpha, \beta \in \mathbb{C}$. By straightforward computations it is easy to see that there exists an upper triangular matrix $Q \in SL(2,\mathbb{C})$ such that $Q^{-1}X'Q = A_\xi$ and $Q^{-1}Y'Q = B_\eta$. This completes the proof. □

As a consequence of this lemma we obtain a decomposition

$$\overline{M_R} = \bigcup_{\substack{\xi \in \Theta^+ \\ \eta \in \Upsilon^+}} \overline{V_{\xi,\eta}} \ \cup \bigcup_{\substack{\xi \in \Theta^- \\ \eta \in \Upsilon^-}} \overline{V_{\xi,\eta}}. \tag{4}$$

Therefore, in order to find the dimension and the number of irreducible components of $\overline{M_R}$ we will study each $\overline{V_{\xi,\eta}}$ separatedly.

Proposition 14. *Given $\xi \in \Theta^\pm$ and $\eta \in \Upsilon^\pm$, the set $\overline{V_{\xi,\eta}}$ is an affine irreducible algebraic variety of dimension 3.*

Proof. Let us define $\Phi : V_{\xi,\eta} \longrightarrow PSL(2,\mathbb{C})$ as follows: given $(A, B) \in V_{\xi,\eta}$, there exists $P \in SL(2,\mathbb{C})$ such that $PAP^{-1} = A_\xi$ and $PBP^{-1} = B_\eta$; we define $\Phi(A, B) = [P]$.

Now we will see that Φ is well defined. If $PAP^{-1} = A_\zeta = QAQ^{-1}$, and $PBP^{-1} = B_\eta = QBQ^{-1}$ it follows that PQ^{-1} commutes with A_ξ, so it must be diagonal. Moreover as PQ^{-1} commutes with B_η and it is diagonal it must be $PQ^{-1} = \pm I_2$ so $P = \pm Q$ and $[P] = [Q]$ in $PSL(2,\mathbb{C})$.

Let us see the injectivity, since Φ is trivially surjective. If $(A_1, B_1), (A_2, B_2) \in V_{\xi,\eta}$ are such that $\Phi(A_1, B_1) = \Phi(A_2, B_2) = [P]$, then either $PA_1P^{-1} = A_\xi = PA_2P^{-1}$ or $PA_1P^{-1} = (-P)A_2(-P)^{-1}$. But in any case $A_1 = A_2$ and analogously $B_1 = B_2$.

Now, since Φ clearly induces a birational equivalence between $V_{\xi,\eta}$ and $PSL(2,\mathbb{C}) \cong SO_3(\mathbb{C})$, the proof is complete. $\square$

Note that $V_{\xi_i,\eta_j} = V_{\xi_k,\eta_l}$ if and only if $\xi_i = \xi_k$ and $\eta_j = \eta_l$. Moreover if $V_{\xi_i,\eta_j} \neq V_{\xi_k,\eta_l}$ then they are disjoint. Finally, as in the previous sections, we have to remove the redundant components in our decomposition of $\overline{M_R}$.

Proposition 15. *$\overline{V_{\xi_i,\eta_j}} = \overline{V_{\xi_k,\eta_l}}$ if and only if $\xi_i = \xi_k$ and $\eta_j = \eta_l$.*

Proof. Let us suppose that $\overline{V_{\xi_i,\eta_j}} = \overline{V_{\xi_k,\eta_l}}$, then $(\xi_i + \xi_i^{-1}, \eta_j + \eta_j^{-1}, \xi_i\eta_j + (\xi_i\eta_j)^{-1}) = t\left(\overline{V_{\xi_i,\eta_j}}\right) = t\left(\overline{V_{\xi_k,\eta_l}}\right) = (\xi_k + \xi_k^{-1}, \eta_l + \eta_l^{-1}, \xi_k\eta_l + (\xi_k\eta_l)^{-1})$. This implies that either $\xi_i = \xi_k$ and $\eta_j = \eta_l$ or $\xi_i = \xi_k^{-1}$ and $\eta_j = \eta_l^{-1}$.

Now, if $\xi_i = \xi_k^{-1}$ and $\eta_j = \eta_l^{-1}$ we know that $V_{\xi_i,\eta_j} \cap V_{\xi_k,\eta_l} = \emptyset$. Consequently $V_{\xi_i,\eta_j} \subseteq \overline{V_{\xi_k,\eta_l}} - V_{\xi_k,\eta_l}$. This is a contradiction since $\dim(\overline{V_{\xi_k,\eta_l}} \backslash V_{\xi_k,\eta_l}) < \dim V_{\xi_k,\eta_l} = \dim V_{\xi_i,\eta_j}$ (see [10, §8.3.] for instance).

The converse is obvious. $\square$

Corollary 16. *The number of irreducible components of $\overline{M_R}$ is equal to*

$$\begin{cases} 2(|m|-1)(|n|-1) & \text{if } d \text{ is odd,} \\ \\ 2[(|m|-1)(|n|-1)+1] & \text{if } d \text{ is even.} \end{cases}$$

It can be seen that $t|_{\overline{M_R}}$ maps every irreducible component of $\overline{M_R}$ to a single point which lies in $\mathcal{L} \cap \mathcal{C}$. Moreover, there are 2 irreducible components of $\overline{M_R}$ mapping to the same point in $\mathcal{L} \cap \mathcal{C}$, namely:

$$\{\overline{V_{\xi,\eta}}, \overline{V_{\xi^{-1},\eta^{-1}}}\} \xrightarrow{t} \left(\xi + \xi^{-1}, \eta + \eta^{-1}, \xi\eta + (\xi\eta)^{-1}\right), \tag{5}$$

for all $(\xi, \eta) \in (\Theta^+ \times \Upsilon^+) \cup (\Theta^- \times \Upsilon^-)$.

This implies that $\overline{Irr} \cap Ab$ consists at least of $2(|m|-1)(|n|-1)$ (resp. $2[(|m|-1)(|n|-1)+1]$) if d is odd (resp. even) irreducible components of dimension less than or equal to 2.

Also observe that although [3, Cor. 1.5.3.] cannot be applied in this situation, if M_{R_0} is an irreducible component of $\overline{M_R}$, then we still have that $3 = \dim M_{R_0} = \dim t(M_{R_0}) + 3 = 0 + 3$.

Conclusion and Future Work

As far as the authors know, the techniques provided in this paper have never been used before to study the representation variety of a group. On the other hand they seem to be useful only for one-relator groups. The fundamental group of the complement of two-bridge knots in the sphere S^3 satisfy the previous condition. In [8] an irreducible component of the character variety, called *excellent curve*, of these kind of knots has been obtained. Therefore it would be very interesting to compute an easy description of $R(G)$ and see how much information is codified in the representation variety in such a case. We think that metabelian representations could play an important role.

Since the combinatorial structure of $X(G_{m,n})$ does not define a complete invariant of the group (but nearly), another interesting question would be to improve our understanding of the combinatorial structure of $R(G_{m,n})$, which seems to contain more information, in order to see if it defines a complete invariant of the family of groups $G_{m,n}$.

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