THE CORRECTION TERM FOR THE RIEMANN-ROCH FORMULA OF CYCLIC QUOTIENT SINGULARITIES AND ASSOCIATED INVARIANTS

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Abstract. This paper deals with the invariant R_X called the RR-correction term, which appears in the Riemann Roch and Numerical Adjunction Formulas for normal surface singularities. Typically, $R_X = \delta_X^{\text{top}} - \delta_X^{\text{an}}$ decomposes as difference of topological and analytical local invariants of its singularities. The invariant δ_X^{top} is well understood and depends only on the dual graph of a good resolution. The purpose of this paper is to give a new interpretation for δ_X^{an}, which in the case of cyclic quotient singularities can be explicitly computed via generic divisors.

We also include two types of applications: one is related to the McKay decomposition of reflexive modules in terms of special reflexive modules in the context of the McKay correspondence. The other application answers two questions posed by Blache [5] on the asymptotic behavior of the invariant R_X of the pluricanonical divisor.

Introduction

For a projective normal surface S the following generalized Riemann-Roch formula can be proved (see e.g. [29, 7, 5])

$$\chi(\mathcal{O}_S(D)) = \chi(\mathcal{O}_S) + \frac{1}{2}D\cdot(D-K_S) + R_S(D), \tag{1}$$

where $R_S : \operatorname{Weil}(S)/\operatorname{Cart}(S) \to \mathbb{Q}$ is a map defined on the group of Weil divisors up to Cartier. This invariant descends to a map $R_{S,x} : \operatorname{Weil}(S,x)/\operatorname{Cart}(S,x) \to \mathbb{Q}$ at each singular point $x \in S$. Moreover, R_S can be recovered from the local information as follows $R_S(D) = \sum_{x\in \operatorname{Sing}(S)} R_{S,x}(D_x)$, where D_x is the class of D in the local surface (S,x).

One major breakthrough accomplished in [5, Thm. 2.1] is to provide an interpretation of a closely related object. Consider $\sigma : (\tilde{S}, E) \to (S,x)$ a resolution of S at x, where E is the exceptional part of the resolution, $\sigma^*D = E_D + \hat{D}$, $\hat{D}$ is the strict transform of D, E_D is its exceptional part, and $K_{\tilde{S}}$ (resp. K_S) is the canonical divisor of $\tilde{S}$ (resp. S). Then

$$A_{S,x}(D) := -R_{S,x}(D+K_S) = \frac{1}{2}E_D(\hat{D}+K_{\tilde{S}}) - h^0(\sigma_*\mathcal{O}_{\hat{D}}/\mathcal{O}_D). \tag{2}$$

Note that both summands depend on σ. Our purpose is to study this invariant for good resolutions of (S,D). In that case

$$\delta^{\text{top}}(D) := \frac{1}{2}E_D(\hat{D}+K_{\tilde{S}}) = -\frac{1}{2}E_D(E_D+Z_{\tilde{S}}), \tag{3}$$

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($Z_{\tilde{S}}$ is a numerical canonical cycle) and

$$\delta^{\text{an}}(D) := h^0(\sigma_* \mathcal{O}_{\hat{D}}/\mathcal{O}_D) \tag{4}$$

do not depend on σ. Denote by $K_{\tilde{S}/S} := K_{\tilde{S}} - \sigma^* K_S$ the relative canonical divisor, then the canonical cycle $Z_{\tilde{S}}$ is numerically equivalent to $-K_{\tilde{S}/S}$.

The invariant $\delta^{\text{top}}(D)$ is in general a rational number which appears in the literature in different disguises, such as $\chi(-E_D)$ –see [24, Section 3]– and $\delta(D)$ –see [12, Section 4.2]. We use this notation to indicate its topological nature since it contributes to the genus of the normalization of D and it can be obtained directly from the dual graph of the resolution. The invariant $\delta^{\text{an}}(D)$ however is a non-negative integer which only depends on the abstract curve D. Our purpose in this paper is to introduce a closely related invariant $\kappa(D)$ which admits a more geometrical interpretation, can be easily calculated in generic situations, and it coincides numerically with $\delta^{\text{an}}(D)$ for quotient singularities. In a separate paper we intend to analyze exactly in which extend both invariants coincide.

Roughly speaking, the invariant $\kappa(D)$ measures the number of conditions on meromorphic forms φ on S with poles along D so that their pull-back is logarithmic and extends holomorphically to E_D. This invariant was introduced in [11] for smooth surfaces and in [27] to study the cohomology ring of complements of curves in weighted projective planes. A similar invariant appeared before in the context of quasi-adjunction ideals –see [19].

The main results of this paper can be summarized as follows. Let $X = (S, x)$ a cyclic quotient surface and consider D a divisor in X. Throughout this paper, and unless otherwise stated divisor means Weil divisor. Define

$$\Delta_X(D) := \delta^{\text{top}}(D) - \kappa(D). \tag{5}$$

In [11] it was shown that $\Delta_X(\cdot)$ is a well-defined rational-valued function $\Delta_X : \operatorname{Weil}(X)/\operatorname{Cart}(X) \to \mathbb{Q}$.

Theorem 0.1. *If D is a divisor on X, then $A_X(D) = \Delta_X(D)$ and hence $\delta^{an}(D) = \kappa(D)$.*

This result allows one to compute the correction term $A_X(D)$ for any divisor D via calculating $\Delta_X(D') = \Delta_X(D)$ for D' a generic divisor such that $D - D'$ is Cartier. In order to give this description consider the Hirzebruch-Jung resolution of X and its associated numerical data $\mathbf{q} = [q_1 = q, \ldots, q_n = 1]$ which is the list of remainders in the Hirzebruch-Jung continued fraction expansion of $\frac{d}{q}$ as in Definition 1.4. Any non-Cartier divisor D in a cyclic quotient singularity of order d has an associated index , say $0 < k < d$. We associate a list of integers $[k] = [k_1, \ldots, k_n]$ to k and $\mathbf{q}$ as the *greedy* X-decomposition of k defined as the quotients of the divisions of

$$\begin{aligned}
k &= k_1 q_1 + r_1 \\
r_1 &= k_2 q_2 + r_2 \\
&\ldots \\
r_{n-1} &= k_n q_n + 0
\end{aligned}$$

–see section 2.

Theorem 0.2. *If D is generic, then $\kappa(D) = r_D - 1$ where r_D is the number of branches of D. Moreover, if D has index $k \neq 0$ in X and $[k] = [k_1, ..., k_n]$ is the X-greedy decomposition of k. Then $r_D = \sum_i k_i$.*

Remark 0.3. Note that the local invariants $\Delta_X(D), \kappa(D), \delta^{\text{top}}(D)$ and $\delta^{\text{an}}(D)$ can be extended for any quotient singularity. However $\kappa(D) = r_D - 1$ is not true in general for non-cyclic singularities. It would be very interesting to obtain such simple formulas for general quotient surface singularities.

Corollary 0.4. *Let D be a divisor of index k and D' a generic divisor in its class, then*

$$\Delta_X(D) = \delta^{top}(D') - \sum_i k_i + 1.$$

To describe the next main result we introduce some notation and briefly recall some concepts on cyclic quotient surface singularities. Let $X := \mathbb{C}^2/G$, $G \subset \mathrm{GL}(2,\mathbb{C})$ be a cyclic quotient surface singularity. By McKay's correspondence any (indecomposable) reflexive module over $\mathcal{O}_X$ is associated with an (irreducible) representation of G −see [16, 17] and [32] for a general view of this geometric correspondence. Since G is abelian, its irreducible representations coincide with $\mathrm{Hom}(G,\mathbb{C}^*) = \boldsymbol{C}_d$, the multiplicative group of d-th roots of unity. After fixing ζ_d a generator of $\boldsymbol{C}_d$, a local system $\rho_k : G \to \mathbb{C}^*$ is given as $\rho_k(g) = \zeta_d^k$ for $g \in G$ a generator of G and its McKay corresponding module is denoted by $\mathcal{O}_X(\zeta_d^k)$, where $k = 0, \ldots, d-1$ is called its *index*.

The elements of $\mathcal{O}_X(\zeta_d^k)$ can be interpreted as the germs $f \in \mathbb{C}\{x,y\}$ such that $f(g\cdot(x,y)) = \rho_k(g)f(x,y) = \zeta_d^k f(x,y)$. The set of zeroes $\{f = 0\} \subset X$ of such a germ defines a divisor. These divisors are Cartier only if f is invariant under the action of G, $\mathcal{O}_X = \mathcal{O}_X(\zeta_d^0) = \mathbb{C}\{x,y\}^{\mathrm{inv}\, G}$. Alternatively,

$$\mathcal{O}_X(\zeta_d^k) := \mu_*(\mathcal{O}_{(\mathbb{C}^2,0)} \otimes V_{\rho_k^*})^G,$$

where $\mu : \mathbb{C}^2 \to X$ is the canonical projection and $V_{\rho_k^*}$ denotes the eigenspace associated with the dual character ρ_k^* −see [31].

Recall that a reflexive $\mathcal{O}_X$-module M is *special* if $M \otimes \omega_X$ is reflexive (c.f. [31, Theorem 5]). The special McKay correspondence establishes the bijection between the *special indecomposable reflexive* $\mathcal{O}_X$-modules and the special irreducible representations.

Note that the modules $\mathcal{O}_X(\zeta_d^{q_i})$ where q_i appears in the Hirzebruch-Jung resolution of X are exactly the special indecomposable reflexive modules by the special McKay correspondence.

Our next goal will be to explicitly give the *McKay decomposition* of the reflexive $\mathcal{O}_X$-modules $\mathcal{O}_X(\zeta_d^k)$ into special indecomposable reflexive $\mathcal{O}_X$-modules.

The McKay decomposition of $\mathcal{O}_X(\zeta_d^k)$ is given as follows.

Theorem 0.5. *If $k \in \mathbb{Z}$ and $[k] = [k_1, \ldots, k_n]$ is the greedy X-decomposition of k, then*

$$\mathcal{O}_X(\zeta_d^k) = \bigotimes_{i=1}^n \mathcal{O}_X(\zeta_d^{q_i})^{\otimes k_i}.$$

Geometrically, the generic divisors in a special indecomposable reflexive $\mathcal{O}_X$-module are the push forward of smooth irreducible curves intersecting the exceptional part of the resolution transversally, which will be called *curvettes* −see section 1.6. The previous result can be understood as a decomposition of a generic germ of index $k \neq 0$ as a product of curvettes.

Finally we present two applications of these main results.

In the first one, we give the McKay decomposition of the module of LR-logarithmic forms M_D^{nul} associated with a generic divisor D. This module is the main object studied in this paper and appears in the definition of the κ-invariant. To define it, consider $\pi : Y \to X$ a minimal resolution of X and $\hat{D}$ the strict transform of D by π, then $M_D^{\mathrm{nul}} = \pi_*(\omega_Y(\hat{D}))$. This module can be defined as the submodule of 2-forms whose pull-back after resolution can be extended holomorphically over the exceptional divisors. The following theorem states that if D is a generic Weil divisor of index k, then M_D^{nul} coincides with $\omega_X(k) = \omega_X \otimes \mathcal{O}_X(\zeta_d^k)$.

Theorem 0.6. *If $[k]=[k_1,\ldots,k_n]$ is the greedy X-decomposition of k and $\mathbf{c}$ is the Hirzebruch-Jung continued fraction of $\frac{d}{q}$, then*

$$M_D^{nul}=\omega_X(k)=\bigotimes \mathcal{O}_X(\zeta_d^{q_i})^{\otimes(k_i+c_i-2)}.$$

As a consequence of this in Corollary 4.5 we give necessary conditions for M_D^{nul} to be generated by monomials and characterize when it is a reflexive module, that is, when M_D^{nul} coincides with the *Auslander-Reiten translate* $\tau(\mathcal{O}_X(\zeta_d^k))=(\mathcal{O}_X(\zeta_d^k)\otimes\omega_X)^{**}$ of $\mathcal{O}_X(\zeta_d^k)$, where k is the index of D –see [31].

As a second application, we solve two questions posed in [5, pg. 337] in the affirmative. Both questions have to do with the correction term $R_X(kK_X)=-\Delta_X((k-1)K_X)$ of the pluricanonical divisor. Let $X=\frac{1}{d}(1,q)$ with $\gcd(d,q)=1$ and denote by K_X its canonical divisor. Let $I:=\min\{k\in\mathbb{N}\mid kK_X \text{ is Cartier}\}$ –it can arithmetically be expressed as $I=\frac{d}{\gcd(d,q+1)}$. Finally, consider the polynomial function $f_I(x)=\frac{(x-1)(I-x)}{I-1}$. Blache asks the following two questions regarding the map $\mathbb{Z}\to\mathbb{Q}$ defined by $k\mapsto R_X(kK_X)$. Note that this map has period I and vanishes at 1 and I.

(1) Does the inequality $|R_X(kK_X)|<f_I(k)$ hold for all $k=2,\ldots,I-1$?
(2) Is $|R_X((k+1)K_X)-R_X(kK_X)|<1$ for all $k\in\mathbb{Z}$?

In particular, question (2) can be improved by showing

Proposition 0.7.

$$|R_X((k+1)K_X)-R_X(kK_X)|<1-\operatorname{lct}(X,\mathfrak{m}),$$

where $\operatorname{lct}(X,\mathfrak{m})$ *is the log-canonical threshold of X with respect to the maximal ideal* $\mathfrak{m}$.

The paper is organized as follows: in section 1 we give the necessary definitions and notation to set the concepts to be dealt with in this paper such as the space of germs, curvettes, generic curves, δ^{top}-invariant, κ-invariant, Δ_X, Milnor number, and Newton polygons. In section 2 we describe the invariants κ_X and Δ_X and the geometric interpretation of the McKay decomposition of eigenmodules via the use of curvettes. In section 3 we prove the main Theorems 0.1,0.2, and 0.5. A final section 4 is devoted to some interesting applications and examples of the main results of this paper such as the McKay decomposition of M_D^{nul} for generic divisors given in Theorem 0.6 and in section 4.4 answers to Blache's conjectures on the R_X invariant of the pluricanonical divisor mentioned above.

1. Settings and Definitions

Let us recall some definitions and properties on quotient surfaces, embedded $\mathbb{Q}$-resolutions, and weighted blow-ups –see [3, 15, 18] for a more detailed exposition.

1.1. **Quotient surface singularities.** Let $\boldsymbol{C}_d$ be the cyclic group of d-th roots of unity generated by a root of unity ζ_d. Consider a vector of weights $(a,b)\in\mathbb{Z}^2$ and the action

$$\begin{array}{ccc}\boldsymbol{C}_d\times\mathbb{C}^2 & \longrightarrow & \mathbb{C}^2,\\ (\zeta,(x,y)) & \mapsto & (\zeta^a\,x,\zeta^b\,y).\end{array}\tag{6}$$

The set of all orbits $\mathbb{C}^2/\boldsymbol{C}_d$ is called a *cyclic quotient space of type* $\frac{1}{d}(a,b)$. Unless otherwise stated, $\gcd(d,a)=\gcd(d,b)=1$.

1.2. **Spaces of germs.** Consider $X = \frac{1}{d}(a,b)$ and $\rho : (\mathbb{C}^2, 0) \to X$ the universal abelian cover whose fibers are given by the cyclic action of order d on $(\mathbb{C}^2, 0)$ as defined in (6). The local ring $\mathcal{O}$ of functions on $(\mathbb{C}^2, 0)$, admits a $\boldsymbol{C}_d$-graduation given by the characters of $\boldsymbol{C}_d$

$$\mathcal{O} = \bigoplus_{\zeta \in \boldsymbol{C}_d} \mathcal{O}_X(\zeta) = \bigoplus_{k=0}^{d-1} \mathcal{O}_X(\zeta_d^k), \tag{7}$$

where

$$\mathcal{O}_X(\zeta_d^k) = \{f \in \mathcal{O} \mid f(\zeta \cdot (x,y)) = \zeta^k f(x,y), \forall \zeta \in \boldsymbol{C}_d\},$$

for any $k \in \mathbb{Z}$. The notation $\mathcal{O}_X(\zeta_d^k)$ is justified since its elements, even though they do not define functions on X, determine a well-defined set of zeroes in X. In other words, $\{(x,y) \in \mathbb{C}^2 \mid f(x,y) = 0\}$ is invariant under the cyclic action given in (6) and hence it defines a germ in X. This explains why $\mathcal{O}_X(\zeta_d^k)$ is also called the *eigenmodule* associated with k (c.f. [30]). More precisely, it is the space of eigenfunctions of the morphism $\zeta\cdot : \mathcal{O} \to \mathcal{O}$ defined by $f(x,y) \mapsto f(\zeta \cdot (x,y))$ with eigenvalue ζ^k. Each element in $\mathcal{O}_X(\zeta_d^k)$ defines a Weil divisor. Note that:

Properties 1.1.

(1) $\mathcal{O}_X(\zeta_d^0) = \mathcal{O}_X$ *is the ring of functions on* X*,*
(2) $x \in \mathcal{O}_X(\zeta_d^a)$ *and* $y \in \mathcal{O}_X(\zeta_d^b)$*,*
(3) $\mathcal{O}_X(\zeta_d^{k_1}) = \mathcal{O}_X(\zeta_d^{k_2})$ *whenever* $k_1 \equiv k_2 \mod d$*,*
(4) $\mathcal{O}_X(\zeta_d^k)$ *is a finitely generated monomial* $\mathcal{O}_X$*-module, that is,* $\mathcal{O}_X(\zeta_d^k) \subset \mathbb{C}\{x,y\}$ *can be generated over* $\mathcal{O}_X$ *by a finite set of monomials.*
(5) $\operatorname{Hom}(\mathcal{O}_X(\zeta_d^{k_1}), \mathcal{O}_X(\zeta_d^{k_2})) = \mathcal{O}_X(\zeta_d^{k_2-k_1})$*,*
(6) $(\mathcal{O}_X(\zeta_d^k))^* = \mathcal{O}_X(\zeta_d^{-k})$*,*
(7) $\mathcal{O}_X(\zeta_d^{k_1}) \otimes \mathcal{O}_X(\zeta_d^{k_2}) \subset (\mathcal{O}_X(\zeta_d^{k_1}) \otimes \mathcal{O}_X(\zeta_d^{k_2}))^{**} = \mathcal{O}_X(\zeta_d^{k_1+k_2})$*.*

Proof. Properties (1)-(3) are immediate. To prove (4) note that any germ in $\mathcal{O}_X(\zeta_d^k)$, say $f(x,y) = \sum_{r,s} a_{r,s} x^r y^s$ satisfies that $f(\zeta^a x, \zeta^b y) = \zeta^k f(x,y)$, where ζ is a primitive d-th root of unity. Hence $\zeta^{ar+bs} = \zeta^k$ for all i,j such that $a_{r,s} \neq 0$, that is, $ar + bs \equiv k \mod d$ and hence, each non-trivial monomial of $f(x,y)$ is in $\mathcal{O}_X(\zeta_d^k)$. Moreover, this module is generated by the finite set of monomials $\{x^r y^s \mid r,s \in \{0,...,d-1\},\ ar+bs \equiv k \mod d\}$.

In order to prove (5) note that multiplication by an element in $\mathcal{O}_X(\zeta_d^{k_2-k_1})$ induces an element of $\operatorname{Hom}(\mathcal{O}_X(\zeta_d^{k_1}), \mathcal{O}_X(\zeta_d^{k_2}))$. It remains to show that all morphisms are of this type. Consider $\varphi \in \operatorname{Hom}(\mathcal{O}_X(\zeta_d^{k_1}), \mathcal{O}_X(\zeta_d^{k_2}))$ and $x^r, y^s \in \mathcal{O}_X(\zeta_d^{k_1})$ monomials such that $ar \equiv bs \equiv k_1 \mod d$ –they exist since $\gcd(a,d) = \gcd(b,d) = 1$. Then,

$$\varphi(x^r) = h_1(x,y) \in \mathcal{O}_X(\zeta_d^{k_2}), \quad \varphi(y^s) = h_2(x,y) \in \mathcal{O}_X(\zeta_d^{k_2}).$$

Since both x^d and $x^{d-r}y^s$ are invariant functions in $\in \mathcal{O}_X$, one has $\varphi(x^d y^s) = x^d h_2$ and $\varphi(x^d y^s) = (x^{d-r}y^s)h_1$. Therefore $y^s h_1 = x^r h_2$, which implies $h_1 = x^r h$ and $h_2 = y^s h$ for a certain $h \in \mathcal{O}_X(\zeta_d^{k_2-k_1})$. This can done for all monomials in $\mathcal{O}_X(\zeta_d^{k_1})$. Using (3) this shows that φ is given as multiplication by an element in $\mathcal{O}_X(\zeta_d^{k_1})$.

Property (6) is an immediate consequence of (5) and (1). Finally the equality in (7) follows from (5) and (6) using that the functors $\operatorname{Hom}(M, -)$ and $- \otimes M$ are adjoint. ☐

In light of Property (7) one has the following.

Definition 1.2. Given two reflexive $\mathcal{O}_X$-modules M and N we call

$$\kappa_X(M,N) = \dim_{\mathbb{C}} \frac{(M \otimes N)^{**}}{M \otimes N}$$

the *reflexiveness index* of M and N.

This definition can be given for general modules and general quotient surface singularities after factoring out the possible torsion in $M \otimes N$. Note that in that case $\kappa_X(M, \mathcal{O}_X) = 0$ if and only if M is reflexive and in addition $\kappa_X(M, \omega_X) = 0$ if and only if M is special – see [31, Theorem 5].

Following [15] and [34, Definition 1.16] we introduce two useful concepts in the space of germs.

Definition 1.3. A germ $f \in \mathcal{O}_X(\zeta_d^k)$ is called *quasi-smooth* if there is another germ $g \in \mathcal{O}_X(\zeta_d^{\bar{k}})$ such that $\mathbb{C}\{f, g\} = \mathcal{O} = \mathbb{C}\{x, y\}$. Moreover, a germ that admits an equation of the type $f^n g^m \in \mathcal{O}_X(\zeta_d^{nk+m\bar{k}})$ for two quasi-smooth germs generating $\mathcal{O}$ is said to be in $\mathbb{Q}$-*normal crossing*.

Note that, if $f \in \mathcal{O}_X(\zeta_d^k)$ is a quasi-smooth germ, then k can only be either a or b as in Property 2.

1.3. **Embedded $\mathbb{Q}$-resolutions and weighted blow-ups.** An *embedded $\mathbb{Q}$-resolution* of a germ $\{f = 0\} \subset \frac{1}{d}(a, b)$ is a proper analytic map $\pi : Y \to \frac{1}{d}(a, b)$ such that:

(1) Y is an orbifold having abelian (cyclic) quotient singularities,
(2) π is an isomorphism over $X \setminus \{0\}$ when restricted to $Y \setminus \pi^{-1}(0)$,
(3) $\pi^{-1}(f)$ is a $\mathbb{Q}$-normal crossing divisor on Y.

As a key tool to construct embedded $\mathbb{Q}$-resolutions we will recall toric transformations or weighted blow-ups in this context (see [26] as a general reference), which can be interpreted as blow-ups of $\mathfrak{m}$-primary ideals.

The (p, q)-weighted blow-up is a birational morphism $\pi : \widehat{\frac{1}{d}(a, b)} \to \frac{1}{d}(a, b)$ that can be described by covering $\widehat{\frac{1}{d}(a, b)}$ with two charts $\widehat{U}_1$ and $\widehat{U}_2$. More precisely, $\widehat{U}_1$ (and analogously for $\widehat{U}_2$) is of type $\frac{e}{pd}(1, \frac{-q+a'pb}{e})$ with $a'a = b'b \equiv 1 \mod (d)$ and $e = \gcd(d, pb - qa)$, and the equations are given by

$$
\begin{array}{ccc}
\frac{e}{pd}(1, \frac{-q+a'pb}{e}) & \longrightarrow & \widehat{U}_1, \\
\big[(x^e, y)\big] & \mapsto & ((x^p, x^q y), [1 : y]_\omega)
\end{array}
\tag{8}
$$

In particular, if the determinant $\left|\begin{smallmatrix} a & b \\ p & q \end{smallmatrix}\right| = 0$, then $e = d$ and $\widehat{U}_1 = \frac{1}{p}(-d, q)$. The discussion for the second chart is analogous.

The exceptional divisor $E = \pi^{-1}(0)$ is identified with $\mathbb{P}^1_{(p,q)}/\boldsymbol{C}_d$. The singular points are cyclic quotients and correspond to the origins of the charts.

1.4. **Hirzebruch-Jung resolution.** This is a classical well-known resolution of a cyclic surface singularity X related to a Hirzebruch-Jung continued fraction. In order to describe it, consider a cyclic quotient surface singularity X described as $X := \frac{1}{d}(1, q)$. Note that this notation is not canonical since $\frac{1}{d}(1, q) = \frac{1}{d}(1, q')$, where $q'q = 1 \mod d$.

Definition 1.4. Let $q_0 := d$, $q_1 := q$, and define $c_i := \left\lceil \frac{q_{i-1}}{q_i} \right\rceil$ as the round-up of the fraction $\frac{q_{i-1}}{q_i}$, where $q_{i-1} = c_i q_i - q_{i+1}$, $1 \leq i \leq n$ such that $q_n = 1$ and $c_n = q_{n-1}$. The list $\mathbf{c} := [c_1, \ldots, c_n = q_{n-1}]$, (resp. $\mathbf{q} := [q_1 = q, ..., q_n = 1]$) is known as the *Hirzebruch-Jung continued fraction* (resp. *remainder*) of $\frac{d}{q}$.

Also note that

$$
\frac{d}{q} = c_1 - \cfrac{1}{c_2 - \frac{1}{c_3 - \ldots}}
$$

These numerical data encode all the necessary information of the resolution of X as follows.

Consider the $(1,q)$-weighted blow-up at the origin of X. One obtains an exceptional divisor $\tilde{E}_1$ with self-intersection number $-\frac{d}{q}=-\frac{q_0}{q_1}$. If $q=1$ the new ambient space is smooth, $\tilde{E}_1=E_1$ isomorphic to $\mathbb{P}^1$, and the resolution process is over. If $q>1$, then $\tilde{E}_1$ contains a singular point of type $(q;1,-d)$ which is equal to $(q;1,q_2)$ since $-d\equiv q_2 \mod q$. After a $(1,q_2)$-weighted blow-up at this point one obtains a new exceptional divisor $\tilde{E}_2$ whose self-intersection number is $-\frac{q_1}{q_2}$. Note that the strict transform of $\tilde{E}_1$ becomes E_1 – a divisor isomorphic to $\mathbb{P}^1$ in the smooth part of the surface. Note that $E_1^2=-c_1$. Repeating the same procedure one obtains exceptional divisors $\tilde{E}_i$ with $\tilde{E}_i^2=-\frac{q_{i-1}}{q_i}$. Once the surface is smooth the final exceptional divisors $E_1,\ldots,E_n$ form a bamboo-shaped graph where $E_i^2=-c_i$.

1.5. **LR-Logarithmic eigenmodules.** Let $\{f=0\}$ be a germ in $X=\frac{1}{d}(a,b)$ with $f\in\mathcal{O}_X(\zeta_d^k)$ and consider $D=(f)$ its associated Weil divisor. Following Steenbrink [35], if $\tilde{X}=\operatorname{Reg}X$ is the regular part of X one can define $\tilde{\Omega}^p_X=i_*\Omega^p_{\tilde{X}}$, where $i:\tilde{X}\hookrightarrow X$. This has a natural $\mathcal{O}_X$-module structure. Equivalently $\tilde{\Omega}^p_X$ can be seen as the push-forward of Ω^p_Y via a resolution of X. Finally, this is also isomorphic to the module of invariant p-forms on $\mathbb{C}^2$ by the cyclic action (6).

Let us now fix a $\mathbb{Q}$-resolution $\pi:Y\to X$ of D.

Definition 1.5. ([28, 13]) The *log-resolution logarithmic eigenmodule* (LR for short) of X w.r.t. π is

$$\Omega^p_X(\mathrm{LR}\langle D\rangle)=\pi_*\tilde{\Omega}^p_Y(\log\langle\pi^*D\rangle).$$

The *null*-submodule of X w.r.t. π, is defined as

$$M^{\mathrm{nul}}_{f,\pi}=\{h\in\mathcal{O}_X(\zeta_d^{k+w})\mid \pi^*((h)-(f)-\omega_X)\geq 0\},$$

where $w=-a-b \mod d$ is the index of ω_X, the canonical divisor on X.

Remark 1.6. Alternatively, $M^{\mathrm{nul}}_{f,\pi}$ can be seen as the $\mathcal{O}_X$-eigenmodule resulting as a pull-back of $\Omega^2_X(\mathrm{LR}\langle D\rangle)$ as follows. Multiplication by $\frac{dx\wedge dy}{f}$ induces a morphism

$$\begin{array}{cccc}\varphi: & \mathcal{O}_X(\zeta_d^{k+w}) & \longrightarrow & \tilde{\Omega}^2_X[D] \\ & h & \mapsto & h\frac{dx\wedge dy}{f}.\end{array} \tag{9}$$

where $\tilde{\Omega}^2_X[D]:=(\tilde{\Omega}^2_X)_{(f)}$. Hence $h\in M^{\mathrm{nul}}_{f,\pi}$ if

$$h\frac{dx\wedge dy}{f}\in\Omega^p_X(\mathrm{LR}\langle D\rangle)$$

admits a holomorphic extension outside $\hat{D}$ the strict transform of D under π. Note that the quotient $\mathcal{O}_X(\zeta_d^{k+w})/M^{\mathrm{nul}}_{f,\pi}$ has the structure of a finite dimensional complex vector space as long as f defines an isolated singularity.

The following integer number does not depend on the chosen resolution:

$$\kappa_X(f):=\dim_{\mathbb{C}}\frac{\mathcal{O}_X(\zeta_d^{k+w})}{M^{\mathrm{nul}}_f}. \tag{10}$$

For instance, it is known (see [11, Chapter 2]) that if $X=(\mathbb{C}^2,0)$ and f is a holomorphic germ, then $\kappa_X(f)=\delta^{\mathrm{top}}_X(f)$.

1.6. **Curvettes, valuations, and generic germs.** In this section $X = \frac{1}{d}(1,q)$ and the Hirzebruch-Jung resolution π described above will be fixed. Recall that π is a composition of n weighted blow-ups centered at singular points. Consider E_i the exceptional component obtained at the i-th blow-up according to π. Following C.T.C. Wall [38], an E_i*-curvette* on X is the image of a smooth curve transversal to E_i at a smooth point. The terminology *curvette* is due to P. Deligne [14], however the definition followed here is slightly different.

If $\Gamma = (V_\Gamma, E_\Gamma)$ denotes the dual resolution graph of X (a bamboo in this case), Σ denotes the link of X, $L := H_2(\tilde{X};\mathbb{Z}) \cong \oplus_{i\in V_\Gamma}\mathbb{Z}E_i$, $L' := H_2(\tilde{X},\Sigma;\mathbb{Z})$, and then L' is generated by the E_i-curvettes $\{E_i^*\}_{i\in V_\Gamma}$, where $(E_i^*, E_j) = \delta_{i,j}$. The vanishing of $H_2(\Sigma;\mathbb{Z})$ makes L sit inside L' in the relative homology long exact sequence as follows:

$$\begin{array}{ccccccccc} 0 & \to & L = H_2(\tilde{X};\mathbb{Z}) & \to & L' = H_2(\tilde{X},\Sigma;\mathbb{Z}) & \to & H_1(\Sigma;\mathbb{Z}) & \to & 0 \\ & & E_i & \mapsto & \sum_j (E_j^*, E_i)E_j^* & & & & \end{array}$$

The first homology group of the link $L'/L \cong H_1(\Sigma;\mathbb{Z})$ is the group of characters on $\pi_1(X)$, which in the cyclic quotient case is $\boldsymbol{C}_d$.

The Hirzebruch-Jung resolution introduced in section 1.1 defines valuations $v_i : \mathcal{O}_X \to \mathbb{Z}\cup\{\infty\}$ associated with each exceptional divisor E_i, $i = 1,...,n$ by computing the order of vanishing of a germ $f \in \mathcal{O}_X$ along each exceptional divisor E_i in the resolution process. Note that this definition can be naturally extended to $\mathcal{O}_X(\zeta_d^k)$ as follows: $v_i(f) := \frac{1}{d}v_i(f^d)$, where $f^d \in \mathcal{O}_X$. This results into a finite family of maps: $v_i : \mathcal{O}_X(\zeta_d^k) \to \frac{1}{d}\mathbb{Z}\cup\{\infty\}$ satisfying

$$v_i(h\cdot f) = v_i(h) + v_i(f), \qquad \forall\ h \in \mathcal{O}_X, f \in \mathcal{O}_X(\zeta_d^k),$$

and

$$v_i(f+g) \geq \min\{v_i(f), v_i(g)\}, \qquad \forall\ f,g \in \mathcal{O}_X(\zeta_d^k).$$

We will denote by $v = \sum_i v_i : \cup_k(\mathcal{O}_X(\zeta_d^k))^* \to (\frac{1}{d}\mathbb{Z})^n$ the morphism $v(f) = (v_i(f))_i$.

The group $(\frac{1}{d}\mathbb{Z})^n$ has a partial order induced coordinatewise, namely

$$\frac{1}{d}(i_1,\dots,i_n) \leq \frac{1}{d}(j_1,\dots,j_n)$$

if and only if $i_m \leq j_m$ for all $m = 1,\dots,n$.

Definition 1.7. A germ $D = \{f = 0\}$, $f \in \mathcal{O}_X(\zeta_d^k)$ is called *generic* if $v(f)$ is a minimal element in $v(\mathcal{O}_X(\zeta_d^k)) \subset (\frac{1}{d}\mathbb{Z})^n$ with its induced partial order, that is, f is minimal if $v(g) \leq v(f)$ implies $v(g) = v(f)$ for any $g \in \mathcal{O}_X(\zeta_d^k)$.

Remark 1.8. As a side note, the word generic in the previous definition is consistent with being generic with respect to a linear system in the following sense. Recall that in the regular ring context $X = \mathbb{C}^2$, a set of germs S is called a *linear system* if there exists an ideal $I \subset \mathcal{O}_X$ such that $D = \{f = 0\} \in S$ if and only if $f \in I\setminus\{0\}$ –see [10, section 2.7]. One can extend this definition to the quotient singularity case $X = \mathbb{C}^2/G$ by asking that $I \subset \mathcal{O}_X(\rho)$, for some $\rho \in \check{G} \cong \operatorname{Weil}(X)/\operatorname{Cart}(X)$ be an $\mathcal{O}_X$-module. Given a generic germ $D = \{f = 0\}$ of index k as above define $\mathcal{O}_f$ as the preimage $I_f = v^{-1}(v_+(f)) \cap \mathcal{O}_X(\zeta_d^k)$, where $v_+(f) = \{\bar{\imath} \in (\frac{1}{d}\mathbb{Z})^n \mid \bar{\imath} \geq v(f)\}$. Note that $I_f \subset \mathcal{O}_X(\zeta_d^k)$ is a $\mathcal{O}_X$-module.

An equivalent notion called *minimal* element was introduced by Némethi in [25, section 10.3].

1.7. **Newton polygon, Milnor number, and δ-invariant.** A very useful tool for the study of germs in $(\mathbb{C}^2, 0)$ is the information encoded in the Newton polygon of an equation. In the cyclic case, a natural extension of Newton polygons can be defined. We will only give a brief description of it. A more detailed description of invariants such as Newton numbers will be given in a follow-up paper by the authors.

Let $D = \{f = 0\}$ be a germ with $f \in \mathcal{O}_X(\zeta_d^k)$. Define the *Newton cloud* (or *Newton support*) of $f = \sum a_{r,s} x^r y^s$ as

$$S_{\mathbb{L}}(f) = \{(r,s) \in \mathbb{L} \mid a_{r,s} \neq 0\} \subset \mathbb{L}(k),$$

where $\mathbb{L}(k) := \{(r,s) \in \mathbb{N}^2 \mid r + qs \equiv k \mod d\}$ and $\mathbb{L} := \mathbb{L}(0)$ is the lattice of exponents of monomials in $\mathcal{O}_X$. The collection of the compact faces of the boundary of the convex hull of $\bar{S}_{\mathbb{L}}(f) := S_{\mathbb{L}}(f) + \mathbb{L}$ is called the $\mathbb{L}$-*Newton polygon* of f and denoted by $N_{\mathbb{L}}(f)$. This extends the notion of Newton polygon given in [30] for functions on X. The following properties are an immediate consequence of the definitions:

Proposition 1.9.

(1) $N_{\mathbb{L}}(f_1 f_2) = N_{\mathbb{L}}(f_1) \oplus N_{\mathbb{L}}(f_2)$, where $\oplus$ denotes the Minkowski sum,
(2) The number of faces in $N_{\mathbb{L}}(f)$ is an upper bound of the number of irreducible branches of f,
(3) $\bigcup_{f \in \mathcal{O}_X(\zeta_d^k)} \bar{S}_{\mathbb{L}}(f) = \mathbb{L}(k)$,
(4) $\bar{S}_{\mathbb{L}}(f) \subset \bar{S}_{\mathbb{L}}(g) \Rightarrow v(g) \leq v(f)$.

As a consequence one obtains the following interpretation of generic germs.

Proposition 1.10. *If $f \in \mathcal{O}_X(\zeta_d^k)$ is a generic germ, then $\bar{S}_{\mathbb{L}}(f) = \mathbb{L}(k)$.*

In [12] we extended the concept of Milnor fiber and Milnor number of a curve singularity allowing the ambient space to be a quotient surface singularity. Alternative generalizations of Milnor numbers can be found, for instance, in [6, 33, 36]. The Milnor number proposed here (cf. [12]) seems natural for surfaces and allows for a generalization of the local δ-invariant and can be described in terms of a $\mathbb{Q}$-resolution of the curve singularity.

Our purpose is to define a Milnor number $\mu_X(D)$ associated with a Weil divisor D on a quotient surface singularity X as a quotient of Milnor fibers.

Definition 1.11. Let $D = (f)$ be a Cartier divisor in $X = \mathbb{C}^2/G$ defined by a function $f : X \to \mathbb{C}$. Then $M_X(D) := \{f = t\} \subset X$ is a well-defined hypersurface called the *Milnor fiber* of f. In this case, the *Milnor number* of f can be defined as

$$\mu_X(D) := 1 - \chi(M_X(D)). \tag{11}$$

In a more general situation, let $D = (f)$ be a Weil divisor and

$$f^G := \prod_{g \in G} f(g \cdot x)$$

its associated Cartier divisor. Then the *Milnor number* of f is defined as

$$\mu_X(D) := 1 - \frac{1}{|G|}\chi(M_X(f^G)). \tag{12}$$

Also define the *topological delta invariant* δ_X of f as the rational number verifying

$$\mu_X(D) = 2\delta_X(f) - r_X(f) + 1,$$

where $r_X(f)$ is the number of local branches of D at 0.

Also note that this definition of Milnor fiber does not coincide with [8] or [2] since $M_X(D)$, μ_X, and hence δ_X depend on the embedding of D in X.

This definition matches the one presented in [12] since $M_X(D)$ does not contain any orbifold point.

Example 1.12. Note that, with this definition, $\delta_X(u) = 0$ for $u \in \mathcal{O}_X^*$ a unit in the ring of functions on X. For instance, any quasi-smooth curve in $\frac{1}{d}(1,q)$ satisfies that $\mu_X(D) = \frac{d-1}{d}$ –see [12]. However, according to the definition in [8], any quasi-smooth curve coincides with their Milnor fiber and hence the Milnor number is 0.

Remark 1.13. An equivalent definition of the Milnor number can be given as the *rational* degree of the characteristic polynomial $Z(t^{\frac{1}{d}})$ associated to the local monodromy of $\operatorname{Supp}(D) \subset X$ –see [21, p. 962], cf. [1]. An alternative definition can be given via the $Z(t)$ series relative to a curve germ as defined by Campillo, Delgado, Gusein-Zade [9] and Nemethi in [24] for rational surface singularities.

The following formula for the δ-invariant of the product will be useful in the future.

Lemma 1.14 ([12, Corollary 4.8])**.** *For any f_1, f_2 defining reduced Weil divisors on X without common components, the following holds*

$$\delta_X(f_1 f_2) = \delta_X(f_1) + \delta_X(f_2) + (f_1, f_2)_X,$$

where $(f_1, f_2)_X$ denotes the intersection multiplicity of f_1 and f_2 in X.

Recall that in our context $(f_1, f_2)_X$ can be defined as

$$(f_1, f_2)_X := \frac{1}{|G|^2} \dim_{\mathbb{C}} \frac{\mathbb{C}\{x,y\}}{\langle f_1^G, f_2^G \rangle}, \tag{13}$$

where f_i^G is the function germ associated with the Weil divisor $(f_i) = D_i$, $i = 1, 2$ as in Definition 1.11. For a general definition see [23] or [18].

To end this section we will note that the δ-invariant defined here coincides with the invariant δ^{top} as defined in (3) for quotient surface singularities.

Proposition 1.15. *If X is a quotient singularity and $f \in \mathcal{O}_X(\rho)$ a ρ-invariant germ, then $\delta_X(f) = \delta_X^{top}(f)$.*

Proof. This is an immediate consequence of the formula

$$\delta_X(f) = \frac{1}{|G|}\delta_{\mathbb{C}^2}(f^G) + \frac{1}{2}\left(r_X - \frac{r}{|G|}\right),$$

(see [12, section 4.2]) where r_X (resp. r) is the number of local branches of f (resp. f^G) in X (resp. $\mathbb{C}^2$). ∎

2. The invariant Δ_X and combinatorics of curvettes

In this section we will define the basic arithmetic data associated with the cyclic quotient singularity X and we will describe a generic germ for a given divisor class $k \in \operatorname{Weil}(X)/\operatorname{Cart}(X)$. The main problem can be described as follows. Given a resolution $\pi : \tilde{X} \to X$, a generic germ $f \in \mathcal{O}_X(\zeta_d^k)$, its associated divisor $D = (f)$, and the relative homology class of its strict transform $[\hat{D}] \in H_2(\tilde{X}, \Sigma; \mathbb{Z})$ as in section 1.6. One is interested in finding the non-negative coefficients $k_i \in \mathbb{Z}_{\geq 0}$ such that

$$[\hat{D}] = \sum_{i \in V_\Gamma} k_i E_i^* = \sum_{i \in V_\Gamma} ([\hat{D}], E_i^*) E_i^* \in L'.$$

This will be useful in order to describe the invariant $\Delta_X(k)$.

2.1. **The invariant Δ_X.** The invariants κ_X described in (10) and δ_X^{top} can be combined to define an invariant of the Weil divisor class as follows.

Proposition-Definition 2.1 ([27, 13]). *Let $X = \frac{1}{d}(1,q)$ and $f \in \mathcal{O}_X(\zeta_d^k)$ reduced. Then $\Delta_X(f) := \delta_X^{top}(f) - \kappa_X(f)$ defines a map*

$$\Delta_X : \operatorname{Weil}(X)/\operatorname{Cart}(X) \cong \mathbb{Z}_d \to \mathbb{Q},$$

that is, $\Delta_X(f) = \Delta_X(g)$ for any $f, g \in \mathcal{O}_X(\zeta_d^k)$ reduced.

Moreover, $\Delta_X(0) = 0$ and $\Delta_X(f) = \frac{d-1}{2d}$ if f defines a quasi-smooth germ.

Assuming $D = \{f = 0\}$, $f \in \mathcal{O}_X(\zeta_d^k)$, the isomorphism $\operatorname{Weil}(X)/\operatorname{Cart}(X) \cong \mathbb{Z}_d$ will allow us to write $\Delta_X(f)$, $\Delta_X(D)$ or simply $\Delta_X(k)$ depending on the context. Alternatively, the choice of the isomorphism $\operatorname{Weil}(X)/\operatorname{Cart}(X) \cong \mathbb{Z}_d$ is equivalent to the choice of a quasi-smooth germ, which determines the index of a Weil divisor.

A remarkable property of the map $\Delta_X : \operatorname{Weil}(X)/\operatorname{Cart}(X) \to \mathbb{Q}$ is that it characterizes a cyclic quotient surface singularity. Its proof requires an independent result (Proposition 3.5) that will be shown later, but for exposition purposes we present it here.

Proposition 2.2. *Let $D \in \operatorname{Weil}(X)/\operatorname{Cart}(X)$ be a quasi-smooth germ in a cyclic quotient surface singularity X and denote by $d_k := \Delta_X(kD)$. Then, $d_0 = 0$ and*

$$X \cong \frac{1}{d}(1,q),$$

where $d := \frac{1}{1-2d_1}$ and $q := dd_2+1$. In particular, $\Delta_X(D)$ and $\Delta_X(2D)$ characterize the quotient singularity X.

Proof. Since $\Delta_X(0)$ and $\Delta_X(D)$ have already been discussed in Proposition-Definition 2.1, let us calculate $\Delta_X(2D)$ in $X = \frac{1}{d}(1,q)$. By the definition of quasi-smooth (see Definition 1.3) we can assume $D \in \mathcal{O}_X(\zeta_d^q)$ and $\Delta_X(2D) = \delta_X^{\text{top}}(f) - \kappa_X(f)$, where f is any element in $\mathcal{O}_X(\zeta_d^{2q})$. Note that $f_\lambda = y + \lambda x^q$ is quasi-invariant of index ζ_d^q. Hence $f = (y+x^q)(y-x^q) \in \mathcal{O}_X(\zeta_d^{2q})$. Then Lemma 1.14 and Remark 1.12 implies

$$\delta_X^{\text{top}}(f) = \frac{d-1}{2d} + \frac{d-1}{2d} + \frac{q}{d},$$

since $(f_1, f_{-1})_X = \frac{q}{d}$. In order to compute $\kappa_X(f)$, let us blow-up the origin of $\frac{1}{d}(1,q)$ with weights $(1,q)$. In this case the blow-up is a $\mathbb{Q}$-resolution and hence Proposition 3.5 tells us that

$$\kappa_X(f) = \#\{(i,j) \mid i,j \geq 1,\ 2q \geq qi + j \equiv 2q \mod d\}.$$

There is only one point in this set, namely $(1,q')$. Therefore $\Delta_X(2D) = \frac{q-1}{d}$. Summarizing, one obtains

$$\Delta_X(D) = d_1 = \frac{d-1}{2d}, \qquad \Delta_X(2D) = d_2 = \frac{q-1}{d},$$

which proves the claim. ◻

2.2. **Further numerical properties of cyclic quotient surface singularities.** Consider $X = \frac{1}{d}(1,q)$ a cyclic quotient surface singularity and recall from section 1.4 $q_0 := d$, $\mathbf{q} = [q_1 = q, q_2, ..., q_{n-1}, q_n = 1]$, $\mathbf{c} = [c_1, c_2, ..., c_{n-1}, c_n = q_{n-1}]$ as described in section 1.3 such that $q_{i-1} = c_i q_i - q_{i+1}$ for $i = 1, \ldots, n$ Note that $c_i \geq 2$ for all $1 \leq i \leq n$. Define $\bar{q}_i$ as the smallest positive integer such that $q_1 \bar{q}_i \equiv q_i \mod d$. This way one can define a new sequence $\bar{\mathbf{q}} = [\bar{q}_1 = 1, \bar{q}_2, ..., \bar{q}_{n-1}, \bar{q}_n]$.

We will describe some useful properties relating the sequences $\mathbf{q}$, $\bar{\mathbf{q}}$, and $\mathbf{c}$, which will be used in the upcoming sections.

Lemma 2.3. *Let $X_i = \frac{1}{q_i}(1, q_{i+1})$ and $(\bar{q}_{X_i})_j$ denote the numbers $\bar{q}_j$ associated with the space X_i. Under the conditions above one has the following:*

(1) $\bar{q}_i = dq_i\left(\frac{1}{q_0q_1} + \cdots + \frac{1}{q_{i-1}q_i}\right)$,
(2) $\bar{q}_i = c_{i-1}\bar{q}_{i-1} - \bar{q}_{i-2}$,
(3) $\bar{q}_jq_i - q_j\bar{q}_i = d\cdot(\bar{q}_{X_i})_{j-i}$, $\forall j \geq i$,
(4) $q_i = q\bar{q}_i - d\cdot(\bar{q}_{X_1})_{i-1}$,
(5) $\bar{q}_{i+1}q_i - q_{i+1}\bar{q}_i = d$,
(6) $(\bar{q}_{X_i})_j(\bar{q}_{X_{i+1}})_j - (\bar{q}_{X_i})_{j+1}(\bar{q}_{X_{i+1}})_{j-1} = 1$.

Proof. Let $\overline{Q}_i := dq_i\left(\frac{1}{q_0q_1} + \cdots + \frac{1}{q_{i-1}q_i}\right)$. Formulas (2)–(6), replacing $\bar{q}_i$ by $\overline{Q}_i$, are easily checked after some simple calculations. Hence $q_1\overline{Q}_i \equiv q_i \mod d$. To end the proof it is enough to show $\overline{Q}_i < d$ for $i = 1, \ldots, n$. Let us fix $1 \leq i \leq n$, since $q_j > q_{j+1}$, one obtains $q_j \geq q_i + (i-j)$ for $j = 0, 1, \ldots, i$. Therefore,

$$\begin{aligned}\overline{Q}_i &\leq dq_i\left(\frac{1}{(q_i+i)(q_i+i-1)} + \frac{1}{(q_i+i-1)(q_i+i-2)} + \cdots + \frac{1}{(q_i+1)q_i}\right)\\ &= dq_i\left(\frac{2}{(q_i+i)(q_i+i-2)} + \cdots + \frac{1}{(q_i+1)q_i}\right) = \cdots = dq_i\frac{i}{(q_i+i)q_i} < d.\end{aligned}$$

□

2.3. **The X-decomposition of k.** Consider now a vector with n non-negative coordinates $\alpha = [\alpha_1, \ldots, \alpha_n] \in (\mathbb{Z}_{\geq 0})^n$. Then we define $\|\alpha\|_X := \alpha\cdot\mathbf{q}$. In this context, if $k \equiv \|\alpha\|_X \mod d$, we will say α is an X-*decomposition* of k. We will also define $\|\alpha\|_1 := \sum \alpha_i$.

Remark 2.4. Consider $D = \{f = 0\}$, with $f \in \mathcal{O}(\zeta_d^k)$, a germ in X and $v : \cup_k(\mathcal{O}(\zeta_d^k))^* \to \frac{1}{d}(\mathbb{Z}^n)$ the morphism as defined in section 1.6. The vector $\alpha(f) := [0, dv_1(f), \ldots, dv_n(f), 0]$ has integral coordinates. Note that $\|\alpha(f)\|_X \equiv k \mod d$.

From yet another point of view, an X-*decomposition* of k is a solution to the following *coin change-making scenario*: given an integer k and a sequence of coin values $\mathbf{q} = [q_1, \ldots, q_n]$, one is interested in the amount of coins of each type $[k_1, \ldots, k_n]$ that add up to k, that is, such that $\sum_i k_iq_i = k$.

Among all possible solutions to the coin change-making scenario, there is an effective one following the *greedy algorithm* resulting from picking the largest value coin which is not greater than the remaining amount. In our case, this results in the following.

Definition 2.5. Let $X = \frac{1}{d}(1, q)$ be a surface and $\mathbf{q}$ defined as above, then the *greedy* X-decomposition of k is the following list of integers $[k_1, ..., k_n]$, resulting from the quotients of the division of $0 \leq k' < d$, $k \equiv k' \mod d$, by $\mathbf{q}$, that is, $k' = k_1q_1 + k_1'$, and $k_i' = k_{i+1}q_{i+1} + k_{i+1}'$ for $i \geq 1$.

The greedy X-decomposition of k will be denoted by $[k]$. Note $\|[k]\|_1 := \sum k_i = [k]\cdot\mathbf{1}$ which will be simply denoted by $\|k\|_1$.

The following result will be useful for answering one of Blache's questions and it is related to the log canonical threshold of X with respect to the maximal ideal, see section 4.4. Although its proof requires Theorem 0.5, it is presented here for completeness.

Definition-Lemma 2.1. *Define $\bar{k}$ as the smallest positive integer such that $\bar{k}q \equiv k \mod d$, cf. section 2.2. If $[k] = [k_1, \ldots, k_n]$, then $\bar{k} = \sum_{i=1} k_i\bar{q}_i$.*

Proof. Since $\sum_{i=1}^n k_i\bar{q}_i$ satisfies the condition $(\sum_{i=1}^n k_i\bar{q}_i)q \equiv k \mod d$ and by definition $\bar{k}$ is the smallest one verifying such a condition, $\bar{k} \leq \sum_{i=1}^n k_i\bar{q}_i$. Assume by contradiction that $\bar{k} < \sum_{i=1}^{n} k_i \bar{q}_i$. Then $y^{\bar{k}}$ is in $\mathcal{O}_X(\zeta_d^k)$ but not in $\otimes_{i=1}^{n} \mathcal{O}_X(\zeta_d^{q_i})^{\otimes k_i}$ which is a contradiction by Theorem 0.5. □

Also, among all possible solutions to the coin change-making scenario, one can state the following *knapsack type problem* called the *coin change-making problem.*

Problem 2.6 (Coin Change-Making Problem)**.** *Given k and $\mathbf{q}$, find a solution α to the coin change-making scenario which minimizes the number of coins, that is, such that $\|\alpha\|_1$ is minimal.*

The greedy algorithm does not provide in general a solution to the coin change-making problem, for instance, for $k = 6$ and $\mathbf{q} = [4, 3, 1]$ note that the greedy algorithm provides the following solution to the change-making scenario $[6] = [1, 0, 2]$ which is not a solution to the problem since $[0, 2, 0]$ uses fewer coins.

In our case however the answer is positive.

Lemma 2.7. *Given a quotient surface $X = \frac{1}{d}(1, q)$ and $\mathbf{q}$ as above, the greedy X-decomposition of $k \in \mathbb{Z}$ is a solution to the Coin Change-Making Problem 2.6.*

Proof. The result is a direct consequence of the proof of the main result in [20] (see also [22, Theorem p.4]) which we summarize here for convenience. Denote by $\mathrm{Opt}(\mathbf{q}, k)$ (resp. $\mathrm{G}(\mathbf{q}, k)$) the number of coins in a solution (resp. greedy candidate) to the coin change-making problem for $\mathbf{q}$ and k. Denote by $c_i := \lceil \frac{q_{i-1}}{q_i} \rceil$. Then $\mathrm{Opt}(\mathbf{q}, k) = \mathrm{G}(\mathbf{q}, k)$ for all k if $\mathrm{G}(\mathbf{q}, c_i q_i - q_{i-1}) \leq c_i - 1$ for all $i = 1, \ldots, n$. In our situation, $c_i q_i - q_{i-1} = q_{i+1}$ and $c_i \geq 2$, hence $\mathrm{G}(\mathbf{q}, c_i q_i - q_{i-1}) = 1$ and the result follows. □

3. Proof of Main Theorems

In this section we plan to prove the main results of the paper: Theorems 0.1, 0.2, and 0.5. Corollary 0.4 follows immediately from Theorems 0.1 and 0.2.

3.1. **Proof of Theorem 0.1.** In the introduction we have already discussed that $A_X(D) = \delta_X^{\mathrm{top}}(D) - \delta_X^{\mathrm{an}}(D)$ and $\Delta_X(D) = \delta_X^{\mathrm{top}}(D) - \kappa_X(D)$, see (2), (3), (4), (5). Therefore $A_X(D) = \Delta_X(D)$ implies $\delta_X^{\mathrm{an}}(D) = \kappa_X(D)$. Let us prove the former equality. Consider $k \geq 1 + d + q$ and choose a divisor D in the weighted projective plane $\mathbb{P}^2_w$ with $w = (1, d, q)$ of degree k. On the one hand, according to [13, Theorem 1.1],

$$L_w(k - 1 - d - q) = g_{w,k} - \sum_{P \in \mathrm{Sing}(\mathbb{P}^2_w)} \Delta_{\mathbb{P}^2_w, P}(k), \tag{14}$$

where $L_w(\ell)$ is the dimension of the vector space of the w-quasi-homogeneous polynomials of degree ℓ and $g_{w,k} = 1 + \frac{k(k-1-d-q)}{2dq}$. On the other hand, due to (1), one has

(15)
$$\chi(\mathbb{P}^2_w, \mathcal{O}_{\mathbb{P}^2_w}(D + K_{\mathbb{P}^2_w})) = \chi(\mathbb{P}^2_w, \mathcal{O}_{\mathbb{P}^2_w}) + \frac{D(D + K_{\mathbb{P}^2_w})}{2} + \sum_{P \in \mathrm{Sing}(\mathbb{P}^2_w)} R_{\mathbb{P}^2_w, P}(D + K_{\mathbb{P}^2_w}).$$

Note that $\chi(\mathbb{P}^2_w, \mathcal{O}_{\mathbb{P}^2_w}(D + K_{\mathbb{P}^2_w})) = L_w(k - 1 - d - q)$, $\chi(\mathbb{P}^2_w, \mathcal{O}_{\mathbb{P}^2_w}) = 1$, $\frac{D(D+K_{\mathbb{P}^2_w})}{2} = \frac{k(k-1-d-q)}{2dq}$, and $R_{\mathbb{P}^2_w, P}(D + K_{\mathbb{P}^2_w}) = -A_{\mathbb{P}^2_w, P}(D)$. Then, subtracting (14) and (15), one obtains

$$\sum_{P \in \mathrm{Sing}(\mathbb{P}^2_w)} \left(\Delta_{\mathbb{P}^2_w, P}(k) - A_{\mathbb{P}^2_w, P}(k) \right) = 0.$$

Since the weighted projective plane $\mathbb{P}^2_w$ has two singular points of type $\frac{1}{d}(1,q)$ and $\frac{1}{q}(1,d)$, the previous equation reduces to

$$\Delta_{\frac{1}{d}(1,q)}(k) - A_{\frac{1}{d}(1,q)}(k) = -\Delta_{\frac{1}{q}(1,d)}(k) + A_{\frac{1}{q}(1,d)}(k).$$

If $q=1$, then both $\Delta_{\frac{1}{q}(1,d)}(k)$ and $A_{\frac{1}{q}(1,d)}(k)$ are zero and $\Delta_{\frac{1}{d}(1,q)}(k) = A_{\frac{1}{d}(1,q)}(k)$. Otherwise, $\Delta_{\frac{1}{d}(1,q)}(k) = A_{\frac{1}{d}(1,q)}(k)$ if and only if $\Delta_{\frac{1}{q}(1,d)}(k) = A_{\frac{1}{q}(1,d)}(k)$. Since q is strictly less than d and $\frac{1}{q}(1,d) = \frac{1}{q}(1, d \bmod q)$ the claim follows by induction. $\square$

3.2. **Generic germs as a product of curvettes.** As a preparation for Theorem 0.2 our goal here is to describe the irreducible curvettes in X. We will provide equations as quasi-invariant binomials. This description will be useful in order to calculate their κ-invariant.

Lemma 3.1. *The quasi-invariant germ $f = x^{q_i} - y^{\bar{q}_i}$ is a curvette in $\mathcal{O}_X(\zeta_d^{q_i})$.*

Proof. In order to show this result, we will use a recursive argument on the length of the canonical resolution of X (see section 1.3). Let us perform the $(1,q)$-blow-up of X. The strict transform of f is $\widehat{f} = x^{q_i} + y^{\frac{q\bar{q}_i - q_i}{d}} \in \mathcal{O}_{\widehat{X}}(\zeta_d^{q_i})$, where $\widehat{X} = \frac{1}{q}(1,q_2)$. By Lemma 2.3(4), $\widehat{f} = x^{q'_j} - y^{\bar{q}'_j}$ where $q' = q_{\widehat{X}}$ and $j = i-1$.

Hence it is enough to check the result for q_1, equivalently that the Newton polygon $N_{\mathbb{L}}(q_1)$ has only one compact face. This is immediate since $N_{\mathbb{L}}(q)$ is the $\mathbb{L}(k)$-convex hull of $\{(0,1)\} + \mathbb{L}$ and $\{(q,0)\} + \mathbb{L}$ as in Figure 1 which has only one compact face. $\square$

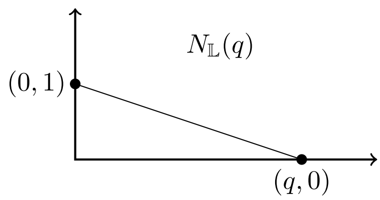

FIGURE 1.

Remark 3.2. Note that f as above might not be irreducible as a germ in $\mathcal{O}_{\mathbb{C}^2}$. For instance, in $X = \frac{1}{6}(1,5)$ one has $\mathbf{q} = [5,4,3,2,1]$, $\bar{\mathbf{q}} = [1,2,3,4,5]$ and thus by Lemma 3.1, $x^4 - y^2 \in \mathcal{O}_X(\zeta_d^4)$ is irreducible in $\mathcal{O}_X(\zeta_d^4)$. Note that neither $x^2 - y$ nor $x^2 + y$ are quasi-invariant.

Proposition 3.3. *Let $X = \frac{1}{d}(1,q)$ be a cyclic surface singularity and k an integer $0 \le k < d$, then the germ*

$$f = \prod_{i=1}^{n} \prod_{j=1}^{k_i} (x^{q_i} - \lambda_j^i y^{\bar{q}_i}) \in \mathcal{O}_X(\zeta_d^k) \subset \mathbb{C}\{x,y\}, \tag{16}$$

with $\lambda_j^i \in \mathbb{C}^*$ and $\lambda_{j_1}^i \neq \lambda_{j_2}^i$, $j_1 \neq j_2$ is generic in $\mathcal{O}_X(\zeta_d^k)$, where $[k] = [k_0, \dots, k_{n+1}]$ is the greedy X-decomposition of k and $\bar{q}_i$ is the inverse class of q_i modulo d. Moreover, any generic germ $g \in \mathcal{O}_X(\zeta_d^k)$ is such that $N_{\mathbb{L}}(g) = N_{\mathbb{L}}(f)$ for an f as above.

Proof. Following the notation introduced in Remark 2.4 note that $\|\alpha(f)\|_X = k$. Assume $g \in \mathcal{O}_X(\zeta_d^k)$ is a germ such that $v(g) \le v(f)$. To show the minimality of f it is enough to prove that $v(g) = v(f)$. On the one hand $k \equiv \|\alpha(g)\|_X \le \|\alpha(f)\|_X = k$ implies $\|\alpha(g)\|_X = k$. On the other hand, by Lemma 2.7 $\alpha(f)$ is a solution to the coin change-making problem associated with $\|\alpha\|_X = k$, that is, $\|\alpha(f)\|_1$ is minimal. Since $\|\alpha(g)\|_1 \leq \|\alpha(f)\|_1$, one has $\|\alpha(g)\|_1 = \|\alpha(f)\|_1$, which together with $v(g) \leq v(f)$, implies $v(g) = v(f)$.

The *moreover* part is equivalent to proving $N_{\mathbb{L}}(g) = N_{\mathbb{L}}(f)$ for any generic germ $g \in \mathcal{O}_X(\zeta_d^k)$. This is a consequence of Proposition 1.10 since $\bar{S}_{\mathbb{L}}(f) = \mathbb{L}(k) = \mathbb{L}(g)$. □

3.3. **Calculation of $\kappa(D)$ for generic germs.** For better exposition the proof is presented divided into several results. One of the key points is to write a generic germs in terms of curvettes. Let us start with a technical result.

Let $\pi : \widehat{X} \to X$ be the (p,q)-weighted blow-up defined in section 1.3. Let us consider $f \in \mathcal{O}_X(\zeta_d^k)$ and denote by $\nu_{p,q}(f)$ the multiplicity of $f(x^p, y^q)$.

Lemma 3.4. *Under the conditions above*

$$\begin{gathered}\{(i,j) \mid pi + qj \leq \nu_{p,q}(f),\ ai + bj \equiv k \mod d\} = \\ \{(i,j) \mid \nu_{p,q}(f) \geq pi + qj \equiv \nu_{p,q}(f) \mod e,\ ai + bj \equiv k \mod d\}.\end{gathered}$$

Proof. Consider the system $\{pi + qj + \ell = \nu_{p,q}(f), ai + bj \equiv k \mod d\}$. It will be shown that e divides ℓ. Since $0 \neq f \in \mathcal{O}_X(\zeta_d^k)$, there exist $0 \neq (i_0, j_0) \in \mathbb{N}^2$ such that $\nu_{p,q}(f) = pi_0 + qj_0$ with $k = ai_0 + bj_0$. Then,

$$\begin{cases} p(i - i_0) + q(j - j_0) \equiv -\ell, \\ a(i - i_0) + b(j - j_0) \equiv 0. \end{cases}$$

Multiplying the first equation by a and the second one by p, one obtains $(aq - pb)(j - j_0) \equiv -a\ell$, thus $e := \gcd(d, aq - pb)$ divides $a\ell$. Analogously, e divides $b\ell$ too and hence $e|\ell$ because $\gcd(d, a, b) = 1$. □

Proposition 3.5. *Let $\kappa_\pi = \#\{(i,j) \in \mathbb{Z}^2 \mid i,j \geq 1,\ pi + qj \leq \nu_{p,q}(f),\ ai + bj \equiv k \mod d\}$. Then,*

$$\kappa_X(f) = \kappa_\pi + \sum_{P \in E \cap V(\widehat{f})} \kappa_P(\widehat{f}). \tag{17}$$

Proof. Consider $h \in \mathcal{O}_X(\zeta_d^{k+w})$ and the 2-form $\psi = h\dfrac{dx \wedge dy}{f}$ and let us calculate the pull-back of ψ after the blowing-up π,

$$\psi \xleftarrow{\pi} \frac{p}{e} x^N\, \widehat{h}\, \frac{dx \wedge dy}{\widehat{f}}, \tag{18}$$

where $N = (\nu_{p,q}(h) - \nu_{p,q}(f) + p + q - e)/e$ and $e = \gcd(d, pb - qa)$, see section 1.3. Thus $h \in M_f^{\text{nul}}(k + w)$ if and only if $N \geq 0$ and $\widehat{h} \in M_{\widehat{f},P}^{\text{nul}}$ for all $P \in E \cap V(\widehat{f})$. This proves

$$\kappa_X(f) = \tilde{\kappa}_\pi + \sum_{P \in E \cap V(\widehat{f})} \kappa_P(\widehat{f}),$$

where $\tilde{\kappa}_\pi = \dim_{\mathbb{C}} \left(\frac{\mathcal{O}_X(\zeta_d^{k+w})}{\{h \mid \nu_{p,q}(h) \geq \nu_{p,q}(f) - p - q + e\}} \right)$.

It remains to show that $\tilde{\kappa}_\pi = \kappa_\pi$. Since both $\mathcal{O}_X$-modules considered above are monomial, the dimension of the quotient can be computed simply by counting the monomials in $\mathcal{O}_X(\zeta_d^{k+w})$ not in $\{h \in \mathcal{O}_X(\zeta_d^{k+w}) \mid \nu_{p,q}(h) \geq \nu_{p,q}(f) - p - q + e\}$. Identifying each monomial $x^i y^j$ with the integral point (i,j) in $\mathbb{Z}^2_{\geq 0}$, one obtains

$$\begin{aligned}\tilde{\kappa}_\pi &= \#\{(i,j) \mid i,j \geq 0,\ pi + qj < \nu_{p,q}(f) - p - q + e,\ ai + bj \equiv k - a - b \mod d\} \\ &= \#\{(i,j) \mid i,j \geq 1,\ pi + qj < \nu_{p,q}(f) + e,\ ai + bj \equiv k \mod d\} \\ &= \#\{(i,j) \mid i,j \geq 1,\ pi + qj \leq \nu_{p,q}(f),\ ai + bj \equiv k \mod d\}.\end{aligned}$$

The last equality is a direct consequence of Lemma 3.4. □

3.4. **Proof of Theorem 0.2.** By the second part of Proposition 3.3 we can assume that f has the form given in (16). Consider the $(1,q)$-weighted blow-up at $0 \in X$ and denote by $\widehat{D}$ the strict transform of $D := \{f=0\} \subset X$. By Proposition 3.5,

$$\kappa_X(D) = \kappa_\pi + \sum_{P \in E \cap \widehat{D}} \kappa_P(\widehat{D}), \tag{19}$$

where $\kappa_\pi := \#\{(i,j) \in \mathbb{Z}^2 \mid i,j \geq 1,\ k \geq i+qj \equiv k \mod d\}$. The unique pairs (i,j) satisfying both the inequality and the congruence are $(k-q,1), (k-2q,2), \ldots, (k-jq,j)$ as long as $k-jq>0$. Then,

$$\kappa_\pi = \begin{cases} k_1 - 1 & \text{if } k = k_1 q, \\ k_1 & \text{otherwise.} \end{cases}$$

On the other hand, by construction, $\kappa_P(\widehat{D}) \neq 0$ only when P is the singular point of type $(q;1,q_2)$.

Equation (19) allows us to repeat the same arguments until X is smooth, see section 1.3. The conclusion is that $\kappa_X(f) = \sum_{i=1}^n k_i - 1 = \|k\|_1 - 1$, that is, the number of local branches of f minus 1. □

3.5. **Proof of Theorem 0.5.** Since $(\mathcal{O}_X(\zeta_d^{q_0}))^{k_0} = \mathcal{O}_X$, without loss of generality, we can assume $0 \leq k < d$. Three cases can be distinguished.

Case 1. Note that if $k = q_i$ there is nothing to prove.

Case 2. If $[k] = [0,\ldots,0,k_i,0,\ldots,0]$ it is enough to show that

$$I := \dim \frac{\mathcal{O}_X(\zeta_d^k)}{\otimes \mathcal{O}_X(\zeta_d^{q_i})^{\otimes k_i}} = 0,$$

that is, there are no $\mathbb{L}(k)$-points under the polygon $\oplus_i N_{\mathbb{L}}(q_i)^{k_i}$. In order to do this consider $f = (x^{q_i} - \lambda_1 y^{\bar{q}_i}) \cdots (x^{q_i} - \lambda_{k_i} y^{\bar{q}_i})$.

The $(\bar{q}_i, q_i)$-blow-up π of X is a $\mathbb{Q}$-resolution of f and thus, by Proposition 3.5,

$$\kappa_X(f) = \kappa_\pi + \sum_{P \in E \cap V(\widehat{f})} \kappa_P(\widehat{f}) = \kappa_\pi, \tag{20}$$

where $\kappa_\pi = \#\{(r,s) \in \mathbb{Z}^2 \mid r,s \geq 1,\ k\bar{q}_i \geq \bar{q}_i r + q_i s \equiv k\bar{q}_i \mod d\}$.

Note that both I and κ_π describe a certain number of $\mathbb{L}(k)$-points as follows: let s denote the number of points on the compact segment L of $\oplus_{k_i} N_{\mathbb{L}}(q_i)$, I_1 (resp. I_2) the number of points on the x-axis (resp. y-axis) under or on L. Then I and κ_π are related by

$$I + s = \kappa_\pi + I_1 + I_2. \tag{21}$$

On the other hand it is clear that $s = k_i + 1$. Also, $I_1 = 1$ (and analogously by symmetry $I_2 = 1$). Therefore formula (21) becomes $\kappa_\pi = k_i - 1 + I$.

Finally, by Theorem 0.2 one has $\kappa_X(f) = k_i - 1$. Since $\kappa_X(f) = \kappa_\pi$ by (20), one obtains $I = 0$.

Case 3. In general, since $0 \leq k < d$ one has the greedy X-decomposition of k, namely $[k] = [k_1, \ldots, k_n]$. Again, we will show that $I := \dim \frac{\mathcal{O}_X(\zeta_d^k)}{\otimes \mathcal{O}_X(\zeta_d^{q_i})^{\otimes k_i}} = 0$. Assume $k_1 \neq 0$, otherwise the result will be proved by induction. Consider $x^{r_0} y^{s_0}$ a monomial in $\mathcal{O}_X(\zeta_d^k)$, that is, $0 \leq r_0 + q s_0 \equiv k \mod d$. Note that the slope of the compact segment in $N_{\mathbb{L}}(q_i)$ is $-\frac{\bar{q}_i}{q_i}$, hence the biggest slope among the compact segments in $\bigoplus_{i,k_i} N_{\mathbb{L}}(q_i)$ is $-\frac{\bar{q}_1}{q_1}$, and its corresponding line has equation $i + qj = k$. Note that $x^{r_0} y^{s_0}$ must be such that $k \leq r_0 + q s_0 \equiv k \mod d$. Therefore, after substituting $x \mapsto x$, $y \mapsto x^q y$ one can construct the following monomial $x^k (x^{r_1} y^{s_1})$, where $r_1 = (r_0 + q_1 s_0 - k)/d$, $s_1 = s_0$, $r_1 + q_2 s_1 \equiv k^{(1)} \mod q$, $k = k_1 q_1 + k^{(1)}$, and

$x^{r_1}y^{s_1} \in \mathcal{O}_{X_1}(\zeta_d^{k^{(1)}})$, $X_1 = \frac{1}{q_1}(1, q_2)$. Note that, given $x^{r_1}y^{s_1} \in \mathcal{O}_{X_1}(\zeta_d^{k^{(1)}})$, one can recover the original monomial in $\mathcal{O}_X(\zeta_d^k)$ by writing $s_0 := s_1$ and $r_0 := r_1 d + k - q s_1$. Recursively, at the last step, one can use Case 2 to prove that such a monomial must belong to $\mathcal{O}_{X_{n'}}(\zeta_d^{k^{(n')}})$, where n' is the last non-zero entry of $[k]$, namely, $[k_0, k_1, ..., k_{n'}, 0, ..., 0]$. Hence, this implies that $x^{r_0}y^{s_0} \in \otimes\mathcal{O}_X(\zeta_d^{q_i})^{\otimes k_i}$. □

4. Applications and examples

We begin this section describing the canonical bundle of a cyclic quotient surface singularity X. This description will allow us to give an alternative view on the discrepancy divisor of X. Finally, the $\mathcal{O}_X$-eigenmodule of LR-logarithmic forms is calculated for generic germs.

4.1. **The canonical bundle of X.** Recall that $\mathbf{c} = [c_1, ..., c_n]$ denotes the vector of coefficients of the Hirzebruch-Jung continued fraction of $\frac{d}{q}$ as defined in section 2. Also recall that the canonical bundle is given by $\omega_X = \mathcal{O}_X(K_X) = \mathcal{O}_X(\zeta_d^w)$, where $w := d - 1 - q$.

Proposition 4.1. *Under the previous notation,*

$$\omega_X = \bigotimes_{i=1}^{n} \mathcal{O}_X(\zeta_d^{q_i})^{\otimes(c_i-2)}.$$

Proof. The result will follow from Theorem 0.5 after calculating the greedy X-decomposition of $w = d - 1 - q$. Let us denote by $[w] = [w_1, \dots, w_n]$ such decomposition. Since $d = c_1 q_1 - q_2$ and $c_1 \geq 2$, one has

$$w_1 = d - 1 - q_1 = (c_1 - 2)q_1 + (q_1 - q_2 - 1)$$

$w_1 = c_1 - 2$. Analogously, by induction assuming $w_i = c_i - 2$,

$$w_{i+1} = q_i - q_{i+1} - 1 = (c_{i+1} - 2)q_{i+1} + (q_{i+1} - q_{i+2} - 1)$$

where $q_i = c_{i+1}q_{i+1} - q_{i+2}$ (note that $c_i \geq 2$), which implies $w_{i+1} = c_{i+1} - 2$. □

4.2. **Another view on the discrepancy of cyclic quotient surfaces.** As a consequence of Proposition 4.1 one can give an alternative proof of the formula for the discrepancy divisor of a cyclic quotient singularity, cf. [4, 37]. Recall that, given $\pi : \tilde{X} \to X$ a resolution of the singularity X and $E_1, \dots, E_n$ the exceptional divisors of the resolution, the *discrepancy divisor* (or *relative canonical divisor*) $K_{\tilde{X}/X} = \sum_i \varepsilon_i E_i$ associated with $\tilde{X}$ and π is given by the formula

$$K_{\tilde{X}} = \pi^*(K_X) + K_{\tilde{X}/X}, \tag{22}$$

where $K_{\tilde{X}}$ is the canonical divisor on $\tilde{X}$ and $K_X = \pi_*(K_{\tilde{X}})$.

In case $X = \frac{1}{d}(1, q)$ and π is the Hirzebruch-Jung resolution described in section 1.1, the canonical divisor K_X is given by a Weil divisor $f \in \mathcal{O}_X(\zeta_d^w) = \mathcal{O}_X(\zeta_d^{\mathbf{c}-2})$ (Proposition 4.1), therefore multiplying formula (22) by each E_j one obtains the following vectorial equation

$$(0, \dots, 0) = (c_1 - 2, \dots, c_n - 2) + (\varepsilon_1, \dots, \varepsilon_n)(E_i \cdot E_j),$$

where $c_j - 2 = (f) \cdot E_j$ and $\sum_i \varepsilon_i (E_i \cdot E_j) = K_{\tilde{X}/X} \cdot E_j$. Since

$$E_i \cdot E_j = \begin{cases} -c_i & \text{if } i = j \\ 1 & \text{if } |i - j| = 1 \\ 0 & |i - j| > 1 \end{cases}$$

one obtains $\varepsilon_i = \frac{q_i + \bar{q}_i}{d} - 1$.

Example 4.2. To continue Example 4.4 note that the discrepancy divisor associated with $X = \frac{1}{14}(1, 11)$ is

$$-\frac{1}{7}\left(E_1 + 2E_2 + 3E_3 + 4E_4 + 2E_5\right).$$

4.3. **Proof of Theorem 0.6.** The purpose of this section is to give a description of M_f^{nul} for a generic f. Before we give the proof, the following technical result is needed.

Lemma 4.3. *Consider* $0 \le k < d$ *and* $[k] = [0, \ldots, 0, k_r, \ldots, k_s, 0 \ldots, 0]$ *its greedy* X*-decomposition, where* $k_r, k_s \neq 0$*. Then the greedy* X*-decomposition of* $k + w$ *is*

$$[k + w] = [c_* - 2, c_{r-1} - 1, k_r - 1, k_*, k_s - 1, c_{s+1} - 1, c_* - 2].$$

Proof. We will consider two cases:

- Case 1. If $k \ge q + 1$, then $k + w \equiv k - q - 1 \mod d$, with $0 \le k - q - 1 < d$.
- Case 2. If $k < q + 1$, in which case $k + w \equiv k + d - q - 1 \mod d$, with $0 \le k + d - q - 1 < d$.

Before we prove the result for each case, let us note the following properties:

(1) $\alpha'_j := [0, \ldots, 0, c_j - 1, c_{j+1} - 2, \ldots, c_n - 2]$ is a greedy X-decomposition of $\|\alpha'_j\|_X = q_{j-1} - 1$. The proof follows that of Proposition 4.1.
(2) $\alpha_i := [c_1 - 2, \ldots, c_{i-1} - 2, c_i - 1, 0, \ldots, 0]$ is a greedy X-decomposition of $\|\alpha'_i\|_X = d - q + q_{i+1}$.
(3) If α_1 and α_2 are X-decompositions, α_1 is greedy, and $\alpha_2 \le \alpha_1$, then α_2 is greedy.
(4) If $\alpha = [a_1, \ldots, a_i, 0, \ldots, 0]$ (resp. $\beta = [0, \ldots, 0, b_{i+1}, \ldots, b_n]$) is a greedy X-decomposition of a (resp. b) with $b < q_i$, then $\alpha + \beta$ is a greedy X-decomposition of $a + b$.

In order to prove case 1, first note that $k > q$ implies $k_1 > 0$, that is $r = 1$ in the statement. Let us define $\beta := [k_1 - 1, k_2, \ldots, k_s - 1, 0, \ldots, 0]$ which is the greedy X-decomposition of $\|\beta\|_X = k - q_1 - q_s = k - q - q_s$ by (3). Using (4) above, one can see that $\beta + \alpha_{s+1} = [k_1 - 1, k_2, \ldots, k_{s-1}, k_s - 1, c_{s+1} - 1, c_{s+2} - 2, \ldots, c_n - 2]$ is the required greedy X-decomposition of $k - q - 1 = k + w$.

In order to prove case 2, similarly as before, $\beta := [0, \ldots, 0, k_r - 1, k_{r+1}, \ldots, k_{s-1}, k_s - 1, 0, \ldots, 0]$ is the greedy X-decomposition of $\|\beta\|_X = k - q_r - q_s$ by (3). Then using (4) $\alpha_{r-1} + \beta + \alpha'_{s+1}$ is the required greedy X-decomposition. □

Proof of Theorem 0.6. The inclusion $\mathcal{O}_X(\zeta_d^k) \otimes \mathcal{O}_X(\zeta_d^w) \subseteq M_f^{\text{nul}}$ is a consequence of f being generic as follows. Since $\mathcal{O}_X(\zeta_d^k) \otimes \mathcal{O}_X(\zeta_d^w)$ is generated by monomials it is enough to show the result for its monomial generators. Consider $h = x^{i_1+i_2} y^{j_1+j_2}$, where $i_1 + qj_1 = k + m_1 d$ and $i_2 + qj_2 = w + m_2 d$ for $m_i \ge 0$. Let us check that the pull-back of the $(1, q)$-blow-up of $h\frac{dx\wedge dy}{f}$ can be extended holomorphically to the exceptional divisor. Using formula (18) is it enough to check that $N = \frac{\nu_{1,q}(h) - \nu_{1,q}(f) + 1 + q - d}{d}$ is non-negative, that is, $(i_1 + i_2) + q(j_1 + j_2) - k + 1 + q - d = (i_1 + qj_1 - k) + (i_2 + qj_2 - (d - q - 1)) = md$, where $m = m_1 + m_2 \ge 0$. An induction argument on the number of required blow-ups similar to the one used in Lemma 3.1 gives the result.

It remains to show that $\kappa(f)$ is the reflexiveness index of $\mathcal{O}_X(\zeta_d^k)$ (see Definition 1.2), that is,

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_X(\zeta_d^{k+w})}{M_f^{\text{nul}}} = \dim_{\mathbb{C}} \frac{\mathcal{O}_X(\zeta_d^{k+w})}{\mathcal{O}_X(\zeta_d^k) \otimes \mathcal{O}_X(\zeta_d^w)} =: \kappa_X(\mathcal{O}_X(\zeta_d^k), \omega_X).$$

The left-hand side dimension is $\kappa_X(f)$ which equals $\sum_{i=1}^n k_i - 1$ by Theorem 0.2. In other words, one needs to check

$$\kappa_X(\mathcal{O}_X(\zeta_d^k), \omega_X) = \sum_{i=1}^n k_i - 1. \tag{23}$$

Let us denote by Γ_1 (resp. Γ_2) the Newton polygon associated with $\mathcal{O}_X(\zeta_d^{k+w})$ (resp. $\mathcal{O}_X(\zeta_d^k) \otimes \mathcal{O}_X(\zeta_d^w)$). If $S_i \subset \mathbb{L}(k+w)$ denotes the set of $\mathbb{L}(k+w)$-points in the first quadrant under Γ_i, $i = 1, 2$, then note that S_i are both finite, $S_2 \subset S_1$, and $\#(S_2 \setminus S_1) = \kappa_X(\mathcal{O}_X(\zeta_d^k), \omega_X)$. The result will follow from counting $\#(S_2 \setminus S_1) = \sum_{i=1}^n k_i - 1$. Let us write $[k] = [0, \ldots, 0, k_r, \ldots, k_s, 0, \ldots, 0]$, where $k_r, k_s \neq 0$. By Lemma 4.3 we know

$$\ell := [k+w] = [c_* - 2, c_{r-1} - 1, k_r - 1, k_*, k_s - 1, c_{s+1} - 1, c_* - 2].$$

By Proposition 4.1, $m := [k] + [w] = [k_* + c_* - 2]$. Given an X-decomposition $\alpha = [\alpha_1, \ldots, \alpha_n]$, we will use the following notation:

$$\|\alpha\|_j := \sum_{i=j}^n \alpha_i q_i, \qquad \text{and} \qquad \|\alpha\|^j := \sum_{i=1}^j \alpha_i \bar{q}_i.$$

Note that $(\|m\|_j, \|m\|^{j-1})$ denotes the coordinates of a vertex in Γ_2 joining the group of segments of slope $-\frac{\bar{q}_{j-1}}{q_{j-1}}$ and the group of segments of slope $-\frac{\bar{q}_j}{q_j}$ as shown in Figure 2.

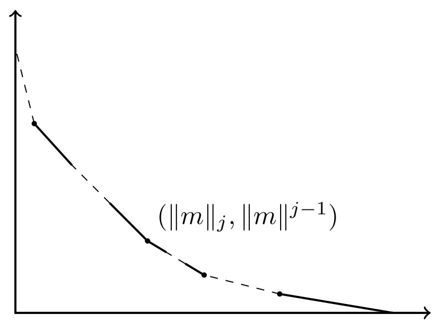

FIGURE 2.

Let us show following:

1. $\|m\|^j = \|\ell\|^j$, $j = 1, \ldots, r-2$,
2. $\|m\|^{r-1} - \|\ell\|^{r-1} = -\bar{q}_{r-1}$,
3. $\|m\|_j = \|\ell\|_j$, $j = s+2, \ldots, n$,
4. $\|m\|_r - \|\ell\|_r = q_{r-1}$,
5. $\|m\|_j = \|\ell\|_j$, $j = 1, \ldots, r-1$,
6. $\|m\|^j = \|\ell\|^j$, $j = s+2, \ldots, n$,

Equalities (1) and (3) are immediate since the first $r-1$ (resp. last $(n-s-1)$) coordinates of ℓ and m coincide. In order to obtain (4) note that:

$$\|m\|_r - \|\ell\|_r = (c_r - 1)q_r + \sum_{i=r+1}^{s-1} (c_i - 2)q_i + (c_s - 1)q_s - q_{s+1}.$$

Using $c_i q_i = q_{i-1} + q_{i+1}$ one obtains the required formula. Also, in order to obtain (5),

$$\|m\|_1 - \|\ell\|_1 = \|m\|_j - \|\ell\|_j = -q_{r-1} + \|m\|_r - \|\ell\|_r.$$

Analogous calculations prove (2) and (6). Equalities (1)-(6) show that Γ_1 and Γ_2 share the first $r-1$ groups of segments of slopes $-\frac{\bar{q}_j}{q_j}$, $j=1,\dots,r-2$, and all but one of the segments of slope $-\frac{\bar{q}_{r-1}}{q_{r-1}}$ as shown in Figure 3.

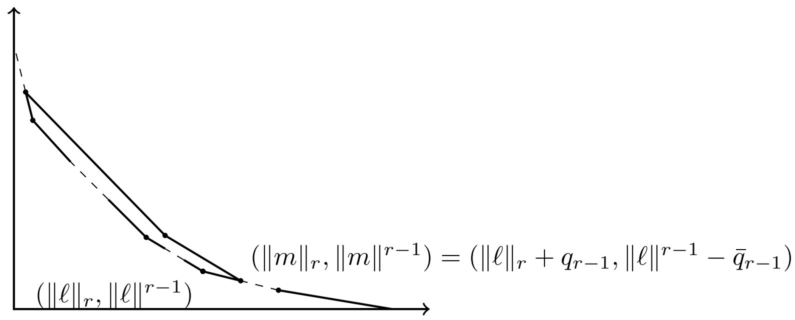

FIGURE 3.

In particular, the difference $\#(S_2 \setminus S_1)$ is invariant if we assume

$$\ell := [0,\dots,0,1,k_r-1,k_*,k_s-1,1,0,\dots,0] \tag{24}$$

and

$$m := [0,\dots,0,0,k_r+c_r-1,k_*+c_*-1,k_s+c_s-1,0,0,\dots,0].$$

Consider now the quadrilateral H_r given by the vertices $P_r = (\|\ell\|_r, \|\ell\|^{r-1})$, $P_{r+1} = (\|\ell\|_{r+1}, \|\ell\|^r)$, $Q_r = (\|m\|_r, \|m\|^{r-1})$, and $Q_{r+1} = (\|m\|_{r+1}, \|m\|^r)$. Note that $\overrightarrow{P_rQ_r} = (q_r - q_{r+1}, \bar{q}_{r+1} - \bar{q}_r)$, $\overrightarrow{P_rP_{r+1}} = (q_{r+1}, -\bar{q}_{r+1})$ (see Figure 4). A simple

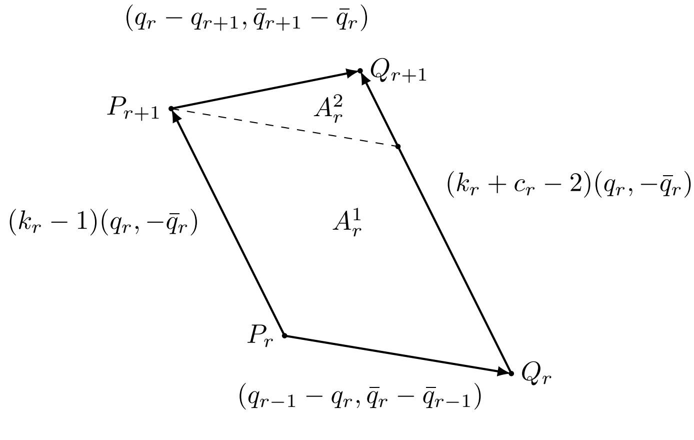

FIGURE 4.

calculation gives the area of this polygon H_r after decomposing it as a parallelogram and a triangle (see Figure 4) as $A_r = A_r^1 + A_r^2$, where

$$A_r^1 = (k_r-1)(\bar{q}_r, q_r)\cdot(q_{r-1}-q_r, \bar{q}_r - \bar{q}_{r-1}) = (k_r-1)d$$

and

$$A_r^2 = \tfrac{1}{2}(q_r - q_{r+1}, \bar{q}_{r+1} - \bar{q}_r)\cdot(c_r-1)(\bar{q}_r, q_r) =$$

$$\tfrac{1}{2}(c_r-1)(\bar{q}_r q_r - \bar{q}_r q_{r+1} + q_r\bar{q}_{r+1} - \bar{q}_r q_r) = \tfrac{1}{2}(c_r-1)d,$$

where Lemma 2.3 is used for these equalities. Hence $A_r = \frac{1}{2}d(2k_r + c_r - 3)$. Using Pick's Theorem for the lattice $\mathbb{L}(k+w)$ one obtains

$$\frac{A_r}{d} = \frac{1}{2}B_r + I_r - 1 = \frac{1}{2}(k_r + k_r + c_r - 1) + I_r - 1,$$

where B_r is the number of $\mathbb{L}(k+w)$-boundary points on the polygon H_r, namely $\Big((k_r-1)+1\Big)+\Big((k_r+c_r-2)+1\Big)$ and I_r is the number of $\mathbb{L}(k+w)$-interior points on H_r. Therefore

$$\frac{1}{2}(2k_r + c_r - 3) = \frac{1}{2}(2k_r + c_r - 1) + I_r - 1$$

which implies $I_r = 0$. Analogously, one can prove that $I_i = 0$, where I_i is the number of $\mathbb{L}(k+w)$-interior points on H_i, the polygon determined by P_i, P_{i+1}, Q_i, and Q_{i+1}, $i = r+1, \dots, s$.

Finally, this implies that $\#(S_2 \setminus S_1)$ can be calculated as the number minus two of boundary $\mathbb{L}(k+w)$-points on the Newton polygon given by ℓ in (24), that is,

$$\Gamma(\ell) = \Gamma(q_{r-1}) \oplus \Gamma(q_r)^{k_r-1} \oplus (\oplus_{i=r+1}^{s-1}\Gamma(q_i)^{k_i}) \oplus \Gamma(q_s)^{k_s-1} \oplus \Gamma(q_{s+1}),$$

which coincides with $(\|\ell\|_1 + 1) - 2 = \sum_i k_i - 1 = \|k\|_1 - 1$ as required. $\square$

Example 4.4. Consider $X = \frac{1}{14}(1,11)$, then

$$\begin{array}{rcl} \mathbf{q} & = & [11, \;\; 8, \;\; 5, \;\; 2, \;\; 1] \\ \mathbf{c} & = & [2, \;\;\; 2, \;\; 2, \;\; 3, \;\; 2] \end{array}$$

We will calculate M_h^{nul} for a generic $h \in \mathcal{O}_X(\zeta_d^{10})$. Note that $[k+w] = [10+2] = [1,0,0,0,1]$ and $[k]+[w] = [0,1,0,2,0]$. Therefore, by Theorems 0.5 and 0.6, $M_h^{\text{nul}} = \mathcal{O}_X(\zeta_d^8) \otimes \mathcal{O}_X(\zeta_d^2)^{\otimes 2}$ and $\mathcal{O}_X(\zeta_d^{12}) = \mathcal{O}_X(\zeta_d^{11}) \otimes \mathcal{O}_X(\zeta_d^1)$. Finally,

$$\kappa_X(\mathcal{O}_X(\zeta_d^{10}), \omega_X) = \dim_{\mathbb{C}} \frac{(\mathcal{O}_X(\zeta_d^{10}) \otimes \omega_X)^{**}}{\mathcal{O}_X(\zeta_d^{10}) \otimes \omega_X} = \dim_{\mathbb{C}} \frac{\mathcal{O}_X(\zeta_d^{11}) \otimes \mathcal{O}_X(\zeta_d^1)}{\mathcal{O}_X(\zeta_d^8) \otimes \mathcal{O}_X(\zeta_d^2)^{\otimes 2}} = 1,$$

which agrees with Theorem 0.2.

Corollary 4.5. *For a given $f \in \mathcal{O}_X(\zeta_d^k)$ the module M_f^{nul} is monomial if f is generic. Moreover M_f^{nul} is only reflexive if $\mathcal{O}_X(\zeta_d^k)$ is a special reflexive module, that is, $k = q_i$.*

Proof. By Theorem 0.6 if $f \in \mathcal{O}_X(\zeta_d^k)$ is generic, then $M_f = \omega_X \otimes \mathcal{O}_X(\zeta_d^k)$ which is generated by monomials. For the *moreover* part, note that $\kappa_X(\mathcal{O}_X(\zeta_d^k), \omega_X) = 0$ if and only if $\omega_X \otimes \mathcal{O}_X(\zeta_d^k)$ is reflexive, which, by (23) is equivalent to $k = q_i$, that is, $\mathcal{O}_X(\zeta_d^k)$ is a special reflexive $\mathcal{O}_X$-module –see [31, Theorem 5]. $\square$

Remark 4.6. In general, M_f^{nul} is not monomial if f is not generic, even if it is a product of curvettes on X, as the following examples shows.

Example 4.7. Let $f = (x+y^4)^2 - y^{18} \in \mathcal{O}_X(\zeta_d^2)$ be a nongeneric germ in $X = \frac{1}{5}(1,4)$. Two consecutive blow-ups of weight $(4,1)$ and $(1,1)$ respectively serve as a $\mathbb{Q}$-resolution of $(f,0)$. This resolution allows one to use the recursive formula [12, Theorem 4.5], which results in $\delta_X(f) = \frac{13}{5}$. Let us calculate M_f^{nul}. One can easily check that $x^2 - y^8 \in M_f^{\text{nul}}$, since

$$(x^2-y^8)\ \frac{dx\wedge dy}{(x+y^4)^2-y^{18}} \quad \xleftarrow{\substack{x=\bar u_1\bar v_1^4,\ u_1=\bar u_1+1\\ y=\bar v_1,\ v_1=\bar v_1^5}} \quad \bar v_1^8\ (\bar u_1^2-1)\ \frac{d\bar u_1\wedge d\bar v_1^5}{\bar v_1^8((\bar u_1+1)^2-\bar v_1^{10})} =$$

$$u_1(u_1-2)\ \frac{du_1\wedge dv_1}{(u_1^2-v_1^2)} \quad \xleftarrow{\substack{u_1=u_2v_2\\ v_1=v_2}} \quad u_2\ (u_2v_2-2)\frac{du_2\wedge dv_2}{(u_2-1)(u_2+1)}.$$

Analogously one can also check that $y^{13},\ x^2+xy^4\in M_f^{\mathrm{nul}}$. Using the curvette $h=x^2-y^3\in\mathcal{O}_X(\zeta_d^2)$, one obtains $\kappa_X(2)=\|2\|_1-1=0$ (Theorem 0.2). Also, it is easy to check that $\delta_X(2)=\frac{3}{5}$ (via a $(3,2)$-blow-up as a resolution as mentioned above), hence $\Delta_X(2)=\delta_X(2)-\kappa_X(2)=\frac{3}{5}$. Since $\Delta_X(2)=\frac{3}{5}=\delta_X(f)-\kappa_X(f)=\frac{13}{5}-\kappa_X(f)$, one obtains

$$\kappa_X(f)=2=\dim_{\mathbb{C}}\frac{\mathcal{O}_X(\zeta_d^2)}{M_f^{\mathrm{nul}}}\le\dim_{\mathbb{C}}\frac{\mathcal{O}_X(\zeta_d^2)}{\mathbb{C}\{x^2-y^8,y^{13},x^2+xy^4\}}=2.$$

Hence, $M_f^{\mathrm{nul}}=\mathbb{C}\{x^2-y^8,y^{13},x^2+xy^4\}$, which is not a monomial module.

4.4. **Answers to Blache's questions.** Let $X=\frac{1}{d}(1,q)$ with $\gcd(d,q)=1$ and denote by K_X its canonical divisor. Let $I:=\min\{n\in\mathbb{N}\mid nK_X \text{ is Cartier}\}$ – it can arithmetically be expressed as $I=\frac{d}{\gcd(d,q+1)}$. Finally, consider the polynomial function $f_I(x)=\frac{(x-1)(I-x)}{I-1}$. In one of his works, Blache asks the following two questions concerning the map $\mathbb{Z}\to\mathbb{Q}$ defined by $k\mapsto R_X(kK_X)$, see [5, p. 337]. Note that this map has period I and vanishes at 1 and I.

(Bl-1) Does the inequality $|R_X(kK_X)|<f_I(k)$ hold for all $k=2,\dots,I-1$?
(Bl-2) Is $|R_X((k+1)K_X)-R_X(kK_X)|<1$ for all $k\in\mathbb{Z}$?

As pointed out by Blache, after resolving the singularity X, Proposition 5.2 in [5] allows one to calculate the explicit value of the previous function. However, the meaning of his formula remains a little mysterious. Our geometric and topological interpretation of the correction term provides a recursive formula giving a positive answer to both questions. Our approach is as follows: recall that $R_X(kK_X)=-\Delta_X(k(1+q))$, then we show recursive formulas for $\Delta_X(k)$ using generic elements. This is summarized in the following result, where $\{x\}$ denotes the fractional part of x and q' is such that $qq'\equiv 1 \mod d$.

Proposition 4.8. *Using the previous notation:*

(1) $\Delta_X(k+1)=\Delta_X(k)+\Delta_X(1)+\left\{\frac{kq'}{d}\right\}-1$, $\forall k\not\equiv 0 \mod d$.
(2) $\Delta_X(k+q)=\Delta_X(k)+\Delta_X(q)+\left\{\frac{k}{d}\right\}-1$, $\forall k\not\equiv 0 \mod d$.
(3) $\Delta_X(k+1+q)=\Delta_X(k)+\left\{\frac{k}{d}\right\}+\left\{\frac{kq'}{d}\right\}-1$, $\forall k\not\equiv 0 \mod d$.
(4) $\Delta_X(k(1+q))=\sum_{i=1}^{k-1}\left(\left\{\frac{i(1+q)}{d}\right\}+\left\{\frac{i(1+q')}{d}\right\}\right)-(k-1)$, $k=1,\dots,d$.
(5) $\Delta_X((k+1)(1+q))=\Delta_X(k(1+q))+\left\{\frac{k(1+q)}{d}\right\}+\left\{\frac{k(1+q')}{d}\right\}-1$, $\forall k\not\equiv 0 \mod d$.

Proof. The proof of (3)–(5) follows from (1) and (2). The proof of (1) and (2) are similar. Hence we will only show (2).

Let $f\in\mathcal{O}_X(\zeta_d^k)$ be generic. Due to Proposition 3.3, $f=\prod_{i=1}^n\prod_{j=1}^{k_i}(x^{q_i}-\lambda_j^i y^{\bar q_i})$ where $\sum_{i=1}^n k_iq_i=(k \mod d)$. By equation (5), since $yf\in\mathcal{O}_X(\zeta_d^{k+q})$, the difference $\Delta_X(k+q)-\Delta_X(k)$ can be rewritten as follows

$$\Delta_X(yf)-\Delta_X(f)=\delta_X(yf)-\delta_X(f)-(\kappa_X(yf)-\kappa_X(f)).$$

By Lemma 1.14, $\delta_X(yf) - \delta_X(f) = \delta_X(y) + (y,f)_X$. The κ_X-invariant of a quasi-smooth curve is zero, see section 1.5, thus $\delta_X(y) = \Delta_X(y) = \Delta_X(q)$. From equation (13), the intersection multiplicity $(y,f)_X$ is $\frac{(y,f)_{\mathbb{C}^2}}{d} = \left\{\frac{k}{d}\right\}$.

It remains to show that $\kappa_X(yf) - \kappa_X(f) = 1$. Consider $\pi : Y \to X$ the $(1,q)$-blow-up at the origin of X. After the blowing-up the curve $y = 0$ remains smooth and it does not intersect the strict transform of $f = 0$ in Y, so $\kappa_Y(\widehat{yf}) = \kappa_Y(\widehat{f})$. By Proposition 3.5, $\kappa_X(yf) = \kappa_\pi(yf) + \kappa_Y(\widehat{f})$ and $\kappa_X(f) = \kappa_\pi(f) + \kappa_Y(\widehat{f})$. Therefore $\kappa_X(yf) - \kappa_X(f) = \kappa_\pi(yf) - \kappa_\pi(f)$. By definition,

$$\kappa_\pi(yf) = \#\{(i,j) \in \mathbb{Z}^2 \mid i,j \geq 1,\ i + qj \geq \nu_{1,q}(f) + q,\ i + qj \equiv k + q \mod d\},$$

which can be rewritten as $\kappa_\pi(yf) = \#A$, where A is the subset of $\mathbb{Z}^2$ defined by

$$A := \{(i,j) \mid i \geq 1,\ j \geq 0,\ i + qj \geq \nu_{1,q}(f),\ i + qj \equiv k \mod d\}.$$

Finally, consider the decomposition $A = (A \cap \{j \geq 1\}) \sqcup (A \cap \{j = 0\})$. Note that $\kappa_\pi(f) = \#(A \cap \{j \geq 1\})$ and $A \cap \{j = 0\} = \{(k \mod d, 0)\}$. This concludes the proof of (2). □

4.5. **Question** (Bl-1)**.** Assume $k = 2, \ldots, I-1$. Let us denote by $e := \gcd(d, 1+q)$ so that $I = d/e$. Consider the subgroup H of $\mathbb{Z}/\mathbb{Z}d$ generated by $1+q$. It has order I and hence it can also be generated by e, that is, $H = \{0, e, 2e, \ldots, (I-1)e\} \subset \mathbb{Z}/\mathbb{Z}d$. To calculate $\sum_{i=1}^{k-1}(i(1+q) \mod d)$, one has to take $k-1$ non-zero pairwise different elements in H and add them up. The minimum (resp. maximum) of all possible choices is $e + 2e + \cdots + (k-1)e = e\frac{k(k-1)}{2}$ (resp. $d - e + \cdots + d - (k-1)e = d(k-1) - e\frac{k(k-1)}{2}$). Since the order of $1 + q^{-1}$ as an element of H is also e, the same bound applies to $\sum_{i=1}^{k-1}(i(1+q^{-1}) \mod d)$. Therefore, by Proposition 4.8(4),

$$\frac{2}{d}e\frac{k(k-1)}{2} - (k-1) \leq \Delta_X(k(1+q)) \leq \frac{2}{d}\left(d(k-1) - e\frac{k(k-1)}{2}\right) - (k-1).$$

This shows that

$$|\Delta_X(k(1+q))| \leq \frac{(k-1)(I-k)}{I} = \frac{I-1}{I}f_I(k)$$

for $k = 1, \ldots, I$ and this implies the required bound.

In Figure 5 one sees how the value of $\Delta_X(k(1+q))$ fits the bound depending on the singular type of X. In particular, the bound is sharp for $X = \frac{1}{d}(1,1)$ as in the left-hand side of the figure.

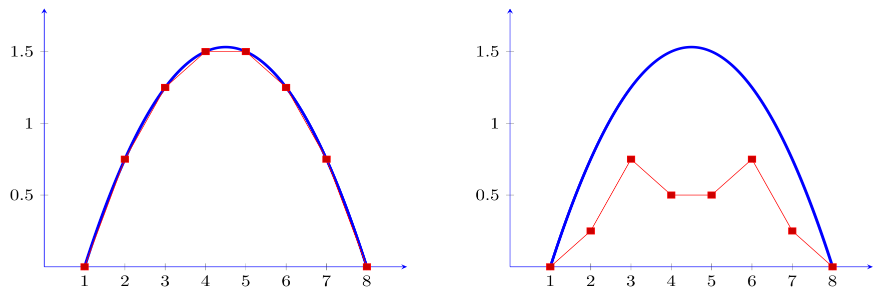

FIGURE 5. Comparison between $\Delta_X(k(1+q))$ and $\frac{(k-1)(I-k)}{I}$ for $X = \frac{1}{16}(1,1)$ and $X = \frac{1}{16}(1,5)$ respectively.

4.6. **Question** (Bl-2)**.** Let $k \in \mathbb{Z}$. If $k(1+q) \equiv 0 \mod d$, then $\Delta_X((k+1)(1+q)) - \Delta_X(k(1+q)) = 0$. Otherwise, according to Proposition 4.8(5), the previous difference can be expressed as $\left\{\frac{k(2+q+q')}{d}\right\} - 1$. Working out as above one obtains that

$$|\Delta_X((k+1)(1+q)) - \Delta_X(k(1+q))| \leq \frac{I-1}{I} f_I(2) = 1 - \frac{2}{I},$$

as long as $k(1+q) \not\equiv 0 \mod d$. This already gives a positive answer to question 2. Although this bound is sharp by $X = \frac{1}{d}(1,1)$, we were still able to find another interesting bound using the log canonical threshold of X with respect to the maximal ideal $\mathfrak{m}$, denoted by $\operatorname{lct}(X, \mathfrak{m})$, see below.

Proof of Proposition 0.7. Let $m := \min\{q_i + \bar{q}_i\}_{i=1}^n$ and denote by $i_0 \in \{1, \ldots, n\}$ the index where this minimum is reached. Then $\operatorname{lct}(X, \mathfrak{m}) = \frac{m}{d} = \frac{q_{i_0} + \bar{q}_{i_0}}{d} \geq \frac{2}{d}$, see e.g. [37]. Let us write $k(1+q) \mod d = \sum_{i=1}^n k_i q_i$, then, by Definition-Lemma 2.1, $k(1+q') \mod d = \sum_{i=1}^n k_i \bar{q}_i$ and hence

$$\Delta_X((k+1)(1+q)) - \Delta_X(k(1+q)) = \sum_{i=1}^n k_i \frac{q_i + \bar{q}_i}{d} - 1 \geq \operatorname{lct}(X, \mathfrak{m}) - 1. \tag{25}$$

The letter inequality holds for all $k \in \mathbb{Z}$. Now let us use Serre's duality from [5] to obtain an upper bound. Denote by $\ell = -(k+2)$, then by Serre one has

$$\Delta_X((k+1)(1+q)) - \Delta_X(k(1+q)) = -\Big(\Delta_X((l+1)(1+q)) - \Delta_X(l(1+q))\Big)$$

which is less than or equal to $1 - \operatorname{lct}(X, \mathfrak{m})$ by equation (25).

This shows that $|\Delta_X((k+1)(1+q)) - \Delta_X(k(1+q))| \leq 1 - \operatorname{lct}(X) < 1$. Moreover, the inequality is sharp if $k(1+q) \equiv q_{i_0} \mod d$ has a solution, that is, whenever $\gcd(d, 1+q)$ divides q_{i_0}. □

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