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Computation of Families of Periodic Orbits and Bifurcations around a Massive Annulus

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Abstract:	This paper studies the main features of the dynamics around a planar annular disk. It is addressed an appropriated closed expression of the gravitational potential of a massive disk, which overcomes the difficulties found in previous works in this matter concerning its numerical treatment. This allows us to define the differential equations of motion that describes the motion of a massless particle orbiting the annulus. We describe the computation methods proposed for the continuation of uni-parametric families of periodic orbits, these algorithms have been applied to analyze the dynamics around a massive annulus by means of a description of the main families of periodic orbits found, their bifurcations and linear stability.
Response to Reviewers:	All the reviewer's comments have been taken into consideration. The response to specific comments follows the order listed by the reviewer. 1.-Related to the motivation of our work we have added references to other astrophysical rings systems, from disks around supermassive black holes to protoplanetary disks; and we have clarified that we follow the procedure for the computation of the potential of a circular wire proposed in literature (Scheeres 1992; Breiter, Dybczynski, Elipe 1996; Kalvouridis 1999; Arribas & Elipe 2005; Arribas, Elipe & Kalvouridis 2007; Alberti & Vidal 2007; Elipe, Arribas & Kalvouridis 2007). We have also added references to Breiter (1996) & Arribas (2007). Page 1, Section: Introduction. 2.- We have removed the sentence: like Asteroid Belts or flight formation. Page 1, Section: Introduction. 3.- It has been added some explanation related to the motion geometry. Page 3-4, Section 1: The annular disk and its potential function. a) Explanation about the integrals of motion “Prior to computing families of periodic […] “. End of Page 3 and first paragraph of Page 4. b) Explanation about equilibria computation . We have added the motion equations

	that govern the movement when the particle is restricted to the equatorial plane or the polar axis. Concerning the equilibria in the polar axis, it is explained that the only equilibrium point is the origin of the system and, as it is analyzed in (Tresaco, Elipe & Riaguas, 2011) it is a stable critical point. Page 4, first column. c) We have clarified the meaning of Polar plane, which refers to any plane perpendicular to the Equatorial plane and containing the origin, thanks to the symmetry of the force field. Page 4, First paragraph of second column. 4.- A discussion about the equivalence of relative equilibria and Equatorial circular orbits has been added in Page 4 (first column): "Concerning the movement on the Equatorial plane […] and therefore, stationary points are circular solutions of the complete problem (r,\lambda,z) i.e. critical points of the effective potential, named r_0, for values of the angular momentum $\Lambda\neq 0$, will correspond to circular orbits on the plane Oxy: (r_0\cos(\Lambda t/{r_0}^2),r_0\sin(\Lambda t/{r_0}^2),0) of period T=2\pi{r_0}^2/\Lambda. Page 4, Section 1: The annular disk and its potential function. 5.- It was incorrectly written "Family 1 orbits are stated to start with an infinitesimal radius…", it has been changed to Family 1 orbits are stated to start with an infinite radius…". Page 7, Section 4: Dynamics on the polar plane. 7.- English mistakes have been corrected.

Computation of Families of Periodic Orbits and Bifurcations around a Massive Annulus

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Abstract This paper studies the main features of the dynamics around a planar annular disk. It is addressed an appropriated closed expression of the gravitational potential of a massive disk, which overcomes the difficulties found in previous works in this matter concerning its numerical treatment. This allows us to define the differential equations of motion that describes the motion of a massless particle orbiting the annulus. We describe the computation methods proposed for the continuation of uni-parametric families of periodic orbits, these algorithms have been applied to analyze the dynamics around a massive annulus by means of a description of the main families of periodic orbits found, their bifurcations and linear stability.

Keywords periodic orbits; bifurcation of families; solid annulus disk potential

Introduction

Outer planets of the Solar System and probably many of the Extrasolar ones have rings. Scientific exploration missions aimed to study the boundaries of the solar System, such as the space probes Pioner11 (1979), Voyager1 (1980), Voyager2 (1981) or the most recent and relevant program of planetary exploration Cassini-Huygens (2004), have provided an in-depth knowledge of the planetary ring systems, determining the structure, composition and dynamical behavior of the planet, rings and moons. Planetary rings are made of millions of rocky and icy particles, each maintaining their own orbit around the planet inside its Roche limit; these small orbiting particles can be considered, from a distance, as a continuous solid annular ring. Ring systems can be also found from disks around supermassive black holes to protoplanetary disks that give rise to planets, the Kuiper belt for example is the remnant of the disk that rotated around the Sun.

Many authors studied the dynamics of planetary ring systems. One of the pioneers, Maxwell (1859) proposed a model for the motion of the particles surrounding Saturn considering a polygonal configuration for the planar $(n+1)$ body problem, in such a way that n bodies of equal mass are located at the vertices of a regular n-gon centered at the remaining body. This model attracted the interest of researchers (Scheeres 1992; Breiter, Dybczynski, Elipe 1996; Kalvouridis 1999; Arribas & Elipe 2005; Arribas, Elipe & Kalvouridis 2007; Alberti & Vidal 2007; Elipe, Arribas & Kalvouridis 2007) in the last years because of the possibility of considering this type of configuration for different dynamical systems. The dynamical models proposed consider a gravitation potential created by a ring, where the ring is described as a finite number of particles placed in a ring configuration, or a solid circular wire.

On the other hand we find authors, such as Stone (1996) or Tiscareno (2007) working on the dynamics of disk formation, angular momentum transport, density waves or disk instability and mass transfer; or such as Longaretti (1989), Sicardy (1991) or Benet & Merlo (2009) that carried out different studies based on the data observed by spacecraft missions to planetary rings, aimed to analyze the physical properties of the rings: composition, distribution of the masses or the relation between planet resonance relationship and the stability of the rings and incomplete arcs.

These are some examples of problems that motivated previous works about the dynamics around a circular ring. We focus our research on the models proposed by Scheeres (1992); Kalvouridis (1999); Arribas & Elipe (2005); Alberti

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& Vidal (2007), with the aim of extend the dynamical systems proposed in their works to a planetary ring. Work is currently underway to analyze the dynamics when a central elliptical body is introduced into the model.

The aim of the present study is to consider a massive bidimensional annular disk as a first approach, which provides a more precise approximation of the rings that can be found in the planets of our Solar System like Saturn for example, that is surrounded by a thin, flat ring that extends over hundreds of kilometers around it; or also considered a dynamical model of astrophysical disks that exists during the early stage of stellar system formation.
The extension to this new model, while fairly simple in its approach, entails many difficulties concerning the analytic treatment of the potential function due to the elliptic integrals involved. Some works in literature deal with the computation of the gravitational potential of a massive disk, like Krough, Ng & Snyder (1982) or Lass & Blitzer (1983). They provide expressions that are mathematically correct, but not appropriated for numeric evaluation and do not cover the whole space. Works by Alberti & Vidal (2007) or Fukushima (2010) deal with the dynamics of the problem but makes use of integral form for the potential and no numerical computation of orbits are made. This paper investigates the dynamics of a particle orbiting the annulus through the search of periodic orbits. Periodic orbits have been widely studied over the last century and are still a topic of great interest for understanding the dynamics of non-integrable Hamiltonian systems. For this purpose a first analysis of the equilibrium points of the system is performed, followed by the computation of initial periodic orbits needed for the continuation of their families.

Some software packages have been derived in order to perform these numeric computations. For the calculation of the initial conditions we have used the representation of Poincaré Sections, together with program *Zeros* (Abad & Elipe 2011) based on evolution strategies to detect periodic orbits in dynamical problems. The continuation of the families of periodic orbits has been carried out through two methods derived following lines of the algorithm of Deprit & Henrard (1967) and the algorithm based on the computation of Poincaré maps (Scheeres 1999). The continuation methods proposed are not restricted to symmetric problems and, since the procedure involves the computation of the variational equations, a side effect is the trivial computation of the linear stability of the periodic orbits. Thus, the evolution of a wide number of periodic orbits is described, allowing to illustrate relevant structures of the phase space and their implications.

The paper is organized as follows. In Section 1 we formulate the problem and derive a proper mathematical expression of the potential function. Next, Section 2 presents the methods we used for both detecting periodic orbits and continuing the families. The analysis of periodic orbits on the plane which contains the annulus is given in Section 3 and the polar orbits are studied in Section 4.

1 The annular disk and its potential function

We address the study of the dynamics of an infinitesimal particle moving under the gravitational field of a massive bi-dimensional annular disk. There are several possibilities in obtaining the potential and the force of such planar body. Broucke & Elipe (2005) obtained both for a solid circular ring in closed form in terms of a complete elliptic integral of the first kind, thus the gravitational potential of the disk simply is the definite integral with the limits of the integral the radii of the annulus.

Let us now consider a homogeneous annulus of radii $b < a$ placed on the Oxy-plane of a Cartesian coordinate system of total mass M and surface density σ,(see Fig. 1).

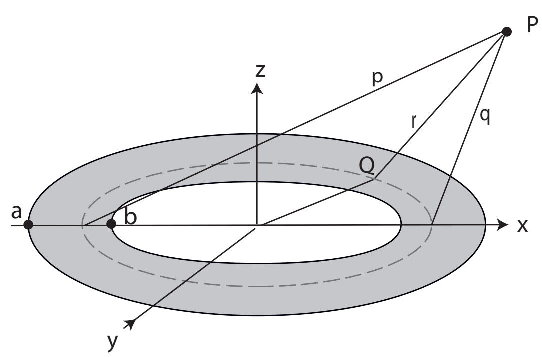

Fig. 1 Bidimensional annulus of radii $b < a$.

The potential created by an annulus of radii a and b is computed from the disk potential by subtracting two concentric disks of radius a and b, respectively. According to Kellogg (1929) this potential is a *single layer potential* with essential discontinuities at the boundary of the annulus but, otherwise, it is a continuous function. Its gradient is a continuous function everywhere except at points in the circular annulus. It is not defined for points at the boundary and it has a step discontinuity at points in the annulus but outside its boundary.

This potential has been already derived by Krough, Ng & Snyder (1982) and by Lass & Blitzer (1983). Nonetheless, the formula given there does not represent the potential function at every point in the space for which the potential function has a real finite value. It cannot be evaluated at significant regions of the space where the potential is a well defined function, or in such a way that produces wrong evaluations when numerically computed.

In order to circumvent the problems arisen in the numerical treatment of the potential expression, we use several formulas from the textbook Byrd & Friedman (1945), and the computational approach by Bulirsch (1971), Carlson (1979) and Fukushima (2009, 2010) to overcome difficulties in evaluating the Elliptic Integrals; details of it can be found in Tresaco, Elipe & Riaguas (2011).
The potential function is depicted in Eq. (1),

$$
\begin{aligned}
U = \frac{2\mu}{\pi(a^2-b^2)} \Big[& -p_a E(k_a) - \frac{a^2-r^2}{p_a} K(k_a) + \\
& |z| \left(\frac{\pi}{2} + \frac{\pi}{2}\operatorname{sign}(a-r) \right) - |z| \operatorname{sign}(a-r) \\
& [E(k_a)F(\phi_a, k'_a) + K(k_a)E(\phi_a, k'_a) - K(k_a)F(\phi_a, k'_a)] \\
& + p_b E(k_b) + \frac{b^2-r^2}{p_b} K(k_b) - |z| \left(\frac{\pi}{2} + \frac{\pi}{2}\operatorname{sign}(b-r) \right) \\
& + |z| \operatorname{sign}(b-r) \Big[E(k_b)F(\phi_b, k'_b) + K(k_b)E(\phi_b, k'_b) \\
& - K(k_b)F(\phi_b, k'_b) \Big] \Big],
\end{aligned}
\tag{1}
$$

where we introduced the following auxiliaries quantities

$$
\begin{aligned}
& r^2 = x^2 + y^2, \qquad R^2 = x^2 + y^2 + z^2, \\
& p_a^2 = (a+r)^2 + z^2, \quad q_a^2 = (a-r)^2 + z^2, \\
& k_a^2 = 4ar/p_a^2, \qquad k_a'^2 = 1 - k_a^2, \qquad \phi_a = \arcsin \frac{|z|}{q_a}, \\
& p_b^2 = (b+r)^2 + z^2, \quad q_b^2 = (b-r)^2 + z^2, \\
& k_b^2 = 4br/p_b^2, \qquad k_b'^2 = 1 - k_b^2, \qquad \phi_b = \arcsin \frac{|z|}{q_b}.
\end{aligned}
$$

The gradient of this function is given by

$$
\begin{aligned}
\frac{\partial U}{\partial x} = & \frac{2\mu}{\pi(a^2-b^2)} \frac{x}{r^2} \left(\sqrt{R^2+a^2+2ar} \Big[(1 - \frac{1}{2}k_a^2) K(k_a) \right. \\
& \left. - E(k_a) \Big] - \sqrt{R^2+b^2+2br} [(1 - \frac{1}{2}k_b^2) K(k_b) - E(k_b)] \right) \\
\frac{\partial U}{\partial y} = & \frac{2\mu}{\pi(a^2-b^2)} \frac{y}{r^2} \left(\sqrt{R^2+a^2+2ar} \Big[(1 - \frac{1}{2}k_a^2) K(k_a) \right. \\
& \left. - E(k_a) \Big] - \sqrt{R^2+b^2+2br} [(1 - \frac{1}{2}k_b^2) K(k_b) - E(k_b)] \right)
\end{aligned}
\tag{2}
$$

$$
\begin{aligned}
\frac{\partial U}{\partial z} = & -\frac{\mu}{\pi(a^2-b^2)} \left(\frac{2z}{\sqrt{R^2+a^2+2ar}} K(k_a) - 2\operatorname{sign}(z) \right. \\
& \left(\frac{\pi}{2} + \frac{\pi}{2}\operatorname{sign}(a-r) - \operatorname{sign}(a-r) \Big[(E(k_a) - K(k_a)) \right. \\
& \left. F(\phi, k'_a) + K(k_a)E(\phi, k'_a) \Big] \right) - \frac{2z}{\sqrt{R^2+b^2+2br}} K(k_b) \\
& + 2\operatorname{sign}(z) \left(\frac{\pi}{2} + \frac{\pi}{2}\operatorname{sign}(b-r) - \operatorname{sign}(b-r) \Big[\right. \\
& \left. \left. (E(k_b) - K(k_b)) F(\phi_b, k'_b) + K(k_b)E(\phi_b, k'_b) \Big] \right) \right).
\end{aligned}
$$

Under this form, the potential function and the force function derived from it can be properly evaluated at any point in the space where they are defined.

In next section we analyze the dynamics of an infinitesimal particle under the attraction of a planar annulus; in order to do this, it is necessary to fix values of the physical parameters of the problem. We can assume without loss of generality that the outer radius of the annulus a, and the gravitational constant μ are equal to one. We will take the value $b = 0.75$ for the inner radius of the annulus ($0 < b < a$). Different values of b do not modify qualitatively the nature of the results, only quantitative variations in the orbital elements of the periodic orbits and families are obtained. Therefore, all the following computations of periodic orbits have been done taking the parameters values $\mu = GM = 1$, $a = 1$ and $b = 0.75$.

Once we have a convenient expression of the potential, it is time to study the dynamics of an infinitesimal particle moving under the gravitational field of a massive bidimensional circular annulus.
The Lagrangian function is

$$
\mathscr{L} = T - U = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(x, y, z),
$$

where $U(x,y,z)$ is the potential function (1), and T the kinetic energy. We are in presence of an autonomous problem, and hence, the energy is an integral of this dynamical system

$$
E = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + U(x, y, z).
$$

The equations of motion are

$$
\ddot{x} = -U_x, \quad \ddot{y} = -U_y, \quad \ddot{z} = -U_z, \tag{3}
$$

where U_x, U_y, U_z denote the partial derivatives.

Equilibrium points are the simplest invariant objects along with periodic orbits; they are important not only for their existence, also because they structure the global dynamics of the system. Prior to computing families of periodic orbits, we focus on the calculation of some particular stationary solutions analyzing the critical points of the system (Tresaco, Elipe & Riaguas 2011). Since the annulus model has axial symmetry, it is natural to use cylindrical coordinates (r,λ,z) to have the Lagrangian

$$\mathscr{L} = \frac{1}{2}(\dot{r}^2 + r^2\dot{\lambda}^2 + \dot{z}^2) - U(r,z);$$

due to the fact that the angle λ is a cyclic variable, its conjugate moment $\Lambda = \partial\mathscr{L}/\partial\dot{\lambda} = r^2\dot{\lambda}$ is constant and the equations of motion in cylindric coordinates are

$$\begin{aligned} \ddot{r} &= -\partial U/\partial r + \Lambda^2/r^3, \\ \ddot{z} &= -\partial U/\partial z. \end{aligned} \tag{4}$$

In order to find the stationary solutions we had to analyze the equations of motion; due to the complexity of the expressions containing elliptic Integrals, analytic solutions are not feasible in general. This is the reason why we only studied the equilibria when the motion is reduced either to the *xy*-plane, or when it is confined to the *Oz* axis.
The movement on the *Oz*-axis is determined by the following differential equation

$$\ddot{z} = \frac{\mu z}{\pi(a^2-b^2)}\left(\frac{1}{\sqrt{z^2+a^2}} - \frac{1}{\sqrt{z^2+b^2}}\right),$$

the only equilibrium point is $z = 0$. Considering that the motion is confined to the 0*z*-axis, this point corresponds to the origin. Concerning the movement on the Equatorial plane, since the system is conservative, the energy is

$$E = T + U(r) = \frac{1}{2}\left(\dot{r}^2 + \frac{\Lambda^2}{r^2}\right) + U(r) = \frac{1}{2}(\dot{r}^2 + W(r)),$$

where $W(r) = \Lambda^2/r^2 + U(r)$ is the so-called effective potential, thus

$$\dot{r} = \sqrt{2(E - W(r))},$$

and therefore, stationary points are circular solutions of the complete problem (r,λ,z) i.e. critical points of the effective potential, named r_0, for values of the angular momentum $\Lambda \neq 0$, will correspond to circular orbits on the plane *Oxy*: $\left(r_0\cos(\Lambda t/{r_0}^2), r_0\sin(\Lambda t/{r_0}^2), 0\right)$ of period $T = 2\pi {r_0}^2/\Lambda$.

It has been proved that the there is a in-plane stable equilibrium inside the annulus $(b < r < a)$, and the origin is also a stationary point, it is linearly unstable for small displacements along the Equatorial plane of the annulus, while it is a stable position for perturbations along the polar axis *Oz*, and only for Energy values greater than the Energy at the origin we will obtain periodic orbits, see details in Tresaco, Elipe & Riaguas (2011). We also determined the existence of two critical points in the exterior of the annulus corresponding to one stable and one unstable circular equatorial orbits. We observed that as we increase the angular momentum, one of the equatorial orbits tends to the annulus while the other goes to orbits of increasing radius.
In order to study the evolution of the orbits we carry out the numerical computation of families of periodic orbits on the fundamental planes: we named the Equatorial plane and the Polar plane. The Equatorial plane is the one containing the disk, whereas the Polar plane refers to any plane perpendicular to the Equatorial plane and containing the origin, thanks to the symmetry of the force field. We address the analysis of the families of periodic orbits found joined with their stability computation.

2 Computation tools of Periodic Orbits

Some software tools have been developed in order to perform the numeric computation of families of periodic orbits. Usually, algorithms for continuing families of periodic orbits need as starter a periodic orbit. To find such initial orbit we use two different ways, on the one hand the classical way of the Poincaré Sections, and on the other, the program *Zeros* (Abad & Elipe 2011) based on evolution strategies to detect periodic orbits in dynamical systems.

Poincaré Surface of Section is a well-known tool to represent the intersection of an orbit in the phase space of a continuous dynamical system with a certain lower dimensional manifold, usually one of the coordinate planes. An autonomous dynamical system provides a reduction of the degrees of freedom of the system thanks to the Energy integral. The Poincaré section is a map obtained though the numeric integration of two conjugated variables, when intersect a fundamental plane for a given Energy level. These plots allow us to distinguish between the quasi-periodic regions from unstable chaotic motion. Concentric lines (islands) identify quasi-periodic orbits while unstable periodic orbits corresponds to hyperbolic points on these plots and are usually more difficult to spot.

The computation of Poincaré sections requires a high computational cost, and besides, highly unstable orbits are difficult to identify. In these cases the application of the program *Zeros* is very useful. *Zeros* is an evolution strategy algorithm belonging to the general class of Genetic Algorithms. It converts the problem of finding periodic orbits into a problem of finding minima of a certain function. To solve this problem an adapted evolution strategy algorithm is applied. Since the problem of finding periodic orbits has no unique solution, but is dense in the phase space, in order to avoid accumulation of solutions around a particular point while abandoning other regions with solutions, some modifications are applied to the algorithm to adapt it to particular functions which has lines of zeros. By using the dynamics of the problem, it is possible to reduce the number of variables to be used, which dramatically reduces the time of computation of a wide set of exact periodic orbits.

The numerical continuation of one-parameter families of periodic orbits has been carried out through two methods, developed following lines of the algorithm based on the computation of Poincaré Maps (Scheeres 1999) and the algorithm of Deprit & Henrard (1967). These continuation methods follow periodic orbits along paths in the parameter plane showing the evolution of the family and its bifurcations. Both methods consist of general algorithms of computing periodic orbits, but they take advantage of the simplifications due to the nature of the problems treated, namely autonomous Hamiltonian systems. We developed both continuation methods *ad-hoc* and applied to different problems, see (Abad, Elipe & Tresaco 2009; Lara, Deprit & Elipe 1995; Riaguas, Elipe & Lara 1999; Elipe & Lara 2003; Tresaco & Ferrer 2010), with the purpose of computing uniparametric families of periodic orbits.

The Deprit and Henrard continuation algorithm addresses a boundary value problem for the variational equations relative to a conservative dynamical system. It consists on separating the normal displacements along an orbit from the tangential ones. This decomposition is meant to separate purely periodic contribution from the secular effects, the latter in the tangential displacements. Thus, the formulation of the variational equations in the Frenet frame directly reduces the dimension of the state transition matrix to compute, and eliminates the trivial eigenvalues that exist for any closed trajectory in a time invariant system. Of course, this algorithm is not restricted to symmetric problems, and is valid for the computation of families of periodic orbits for variations of any parameter or integral for a conservative dynamical system with two or three degrees of freedom.

The second method we used is based on the computation of Poincaré Maps, with the surface of section chosen to be normal to a convenient surface in the phase space (see Scheeres (1999) for details). The Poincaré map is defined as the map from one transversal crossing of the surface to the next. This transversal condition together with the conservation of the Energy integral, make possible to remove the two variables from consideration, creating a four-dimensional map from the Poincaré surface to itself. The reduced monodromy matrix has its unity eigenvalues removed; this allows the reduced map to be used to iteratively solve for the fixed points of the map that correspond to the closed periodic orbits.

Continuing families of periodic orbits is reduced to find the displacements to the orbit, which are the solutions of the variational equations described. A side effect is then the computation of the linear stability with no additional effort. Linear stability of periodic orbits depends on the eigenvalues of the resolvent of the variational equations associated with the fundamental period of the periodic orbit, the monodromy matrix. When the trace of that matrix, Tr, satisfies $|Tr| < 2$, it applies for linearly stable.

We will continue periodic orbits through their family which depends on a certain parameter σ, so its eigenvalues also vary continuously with σ. It follows that a periodic orbit can lose its linear stability when a pair of nontrivial eigenvalues having modulus 1 (i.e., $Tr = 2$) that is a singularity of both algorithms, leading to possible bifurcations deriving to a possible bifurcation with another family, or when a pair of eigenvalues take the value -1 through a period-doubling phenomena ($Tr = -2$).

In the event of dealing with 3-D Hamiltonian systems, taking into account that the eigenvalues appear in reciprocal pairs $(\lambda_i, 1/\lambda_i)$, $(i = 1,2,3)$, and that one eigenvalue takes the value 1 with multiplicity 2, linear stability is determined by two stability indexes, $k_1 = \lambda_1 + 1/\lambda_1$ and $k_2 = \lambda_2 + 1/\lambda_2$, where λ_1 and λ_2 are the nontrivial eigenvalues. The condition $|k_i| < 2$, $(i = 1,2)$ implies linear stability while any other possibility means instability. Finally, in reference to planar solutions, these two stability indexes correspond to the in-plane stability and to the out-of-plane stability, and are denoted in the literature by k_n and k_b, respectively.

Hénon (1965), pointed out six important types of critical orbits according to the structure of the monodromy matrix:

1. There is an extremum of the Energy and no bifurcation with another family of symmetric periodic orbits.
2. There is a bifurcation with a symmetric family of the same period.
3. There is an extremum of the Energy and also a bifurcation with another family of symmetric periodic orbits.
4. There is a bifurcation with a non-symmetric family of the same period.
5. There is a bifurcation with another family of symmetric periodic orbits with double period.
6. There is a bifurcation with another family of non-symmetric periodic orbits with double period.

In the above mentioned cases, the corresponding monodromy matrix is of the type

$$\mathtt{T}_1 = \mathtt{T}_2 = \mathtt{T}_3 := \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad \mathtt{T}_4 := \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix},$$

$$\mathtt{T}_5 := \begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix}, \quad \mathtt{T}_6 := \begin{pmatrix} -1 & 0 \\ c & -1 \end{pmatrix}.$$

Therefore, the different types of bifurcations may be recognized by the structure of the Hénon matrix. In what follows we compute the Hénon matrix for all periodic orbits analyzed, and we also check that their determinant that is the unity.

Let us now show some of the most relevant families we found. For each family we compute the stability index along the variation of the parameter. The parameter selected for continuation has been the x-coordinate of the state vector, which allows the computation of the family for different energy levels and also provides an evolution of the size of the orbit to detect possible collision with the annulus.

3 Dynamics on the equatorial plane

In this case the motion is restricted to the xy-plane containing the annulus.

We make the analysis in the 3-D space, thus, we are able to detect two types of possible bifurcations, orbits on the equatorial plane (determined by the k_n index) and out of the equatorial plane (given by k_b). Figure 2 presents the evolution of the stability indexes in-plane k_n and out-of-plane k_b when the family of equatorial orbits in the exterior of the annulus is continued.

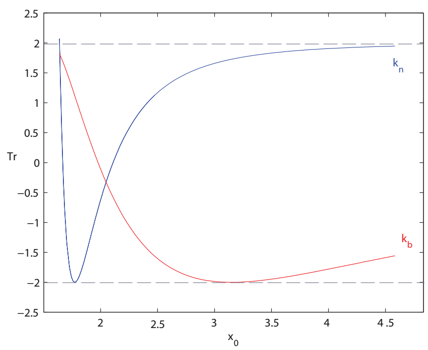

Fig. 2 Evolution of the stability indexes of the family of equatorial orbits exterior to the annulus $(r > a)$

It is immediately detected a family of trivial circular periodic orbits outside the annulus. This is a stable family that shows a transition to instability when the orbital radius approaches the annulus; this transition is consequence of the coexistence of the two critical points of the effective potential, one stable exterior to the annulus, and the other unstable but closer to it (Tresaco, Elipe & Riaguas 2011).

The evolution of the stability index k_n for this family shows that when the radius of the orbit goes towards infinity, the index tends asymptotically to 2, whereas the family ends into a collision with the annulus when its orbital energy decreases. This family of circular orbits also shows a critical value ($k_n = -2$) at $x_0 = 1.55$; this critical point indicates an in-plane bifurcation with a family of doubling period (Type 5). This new double-period family of orbits on the equatorial plane has been also continued leading again to another doubling bifurcation. Repeating the same procedure we find successive doubling period families whose trace patterns can be seen in Fig. 3.

It is worth to notice that the trace $Tr(m)$ of an orbit with multiple period m is related with the trace Tr of the single period orbit by the formula

$$Tr(m) = 2\cos(m \arccos(Tr/2)), \qquad |Tr| \leq 2,$$

where k denotes the trace.

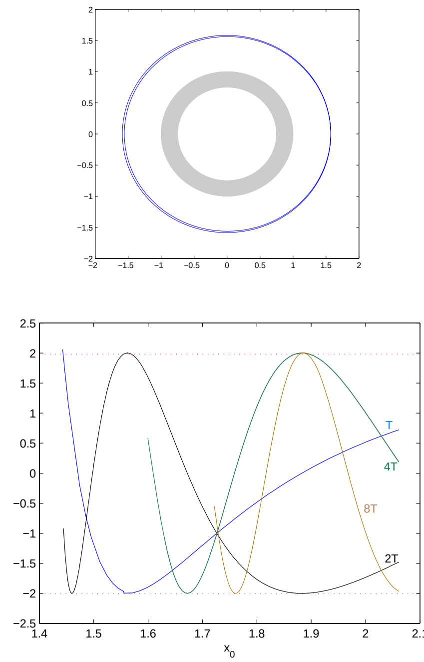

Fig. 3 Orbit resulting from the period-doubling bifurcation, and Trace evolution for the doubling period families

Further, we also analyzed the dynamics of equatorial orbits inside the annulus $(b < r < a)$. Although it may have no physical sense, it is mathematically possible since the potential presents essential discontinuities on the boundary of the annulus but, over it, it is well defined. The family is stable with big oscillations between the limit values, and when the radius of these orbits approaches the outer edge of the annulus, both stability indexes grow rapidly in magnitude preventing its continuation (See Fig. 4).

Finally, we show the behavior of the orbits in the interior of the annulus, $r < b$. It is observed that when the orbital radius approaches the annulus, the orbits become highly unstable, whereas when the radius of the orbits decreases, they remain stable but tends to $Tr = 2$ and when they get close to the origin they change to instability (see Fig. 5). This is due to the existence of an unstable equilibrium at the origin.

To conclude, we find that the annulus is surrounded by an in-plane instability regions when the orbits are close to the annulus from the interior and exterior of the annulus, this joined with the existence of a stable equilibrium inside the annulus may explain how it is organized the dynamics of particles of the annulus and around it.

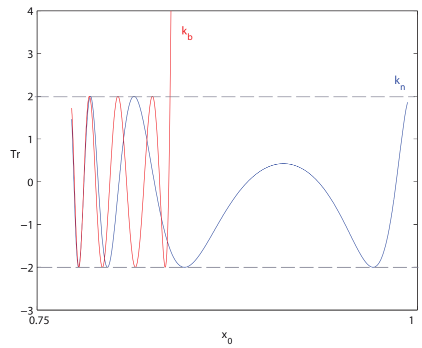

Fig. 4 Evolution of the stability indexes of the family of equatorial orbits inside the annulus $(b < r < a)$

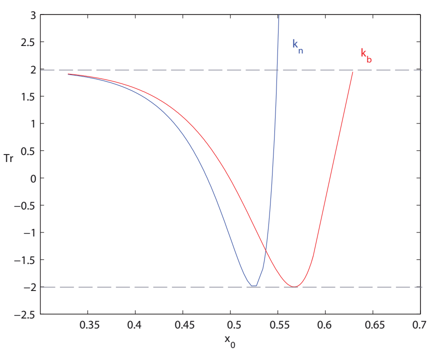

Fig. 5 Evolution of the stability indexes of the family of equatorial orbits in the interior the annulus $(0 < r < b)$

4 Dynamics on the polar plane

Hereafter, we address the search of periodic orbits perpendicular to the plane containing the annulus. This is a more difficult and interesting problem as it is a non-integrable system. Let us remind that the equatorial case is an integrable problem since it has two first integrals. We start by plotting some Poincaré sections in order to get a preliminary information of the dynamics of the system (Fig. 6).

From this plot we identify a set of approximated periodic orbits, whose family evolution is detailed next. In Fig. 7 we plot 20 of the most remarkable orbits we find. In each plot is represented the annulus (black segments) in polar projection. Their initial conditions, period T and trace Tr are listed in Table 1. Note that the family's number in that table runs from 1 to 20 and corresponds to Fig. 7 numbering.

We present now the complete evolution and stability computation of each of these families.

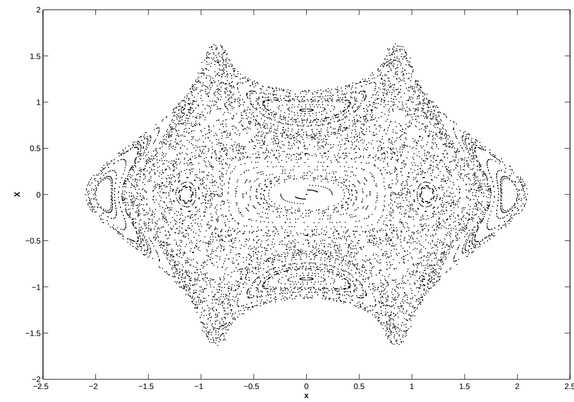

Fig. 6 Poincaré Surface of Section for $E = -0.5$

Family 1 is composed of stable circular orbits exterior to the annulus, which begins with orbits of infinite radius and ends with collision orbits on the annulus. The trace tends asymptotically to the critical value $Tr = 2$ as the radius of the orbit increases.

Families 2, 3, 4, 5 are made of 8-shape orbits that consist of 2-arc periodic orbits.
Let us consider an orbit of Family 2. Its stability evolution (see Fig. 8) shows that when the orbit radius increases the family becomes unstable, whereas when the radius decreases, the family stays within a stable region until it crosses the boundary value, leading to bifurcations with new families of periodic orbits.
The monodromy matrix at the bifurcation point ($Tr = 2$ at $x_0 \simeq 2.75$) is

$$M = \begin{pmatrix} 0.99837 & 0.00064 \\ -3.08100 & 0.99963 \end{pmatrix}, \qquad (5)$$

of Hénon's Type 4, leading to a bifurcation with a non-symmetric family of the same period, namely, Family 3.
The critical point closer to the annulus ($Tr = 2$), presents a Pitchfork bifurcation where the stable family moves to an unstable region while two new families of stable orbits appear, Families 4 and 5 , which evolve to collision orbits with the annulus (see Fig. 8).
The monodromy matrix at this bifurcation point is

$$M = \begin{pmatrix} 0.99999 & 0.11089 \\ -0.00013 & 0.99999 \end{pmatrix} \qquad (6)$$

which corresponds to a Hénon's Type 3.

Family 6 is made of *pretzel*-like orbits. Their evolution shows that for both big and small radius, the orbits collide with the annulus.

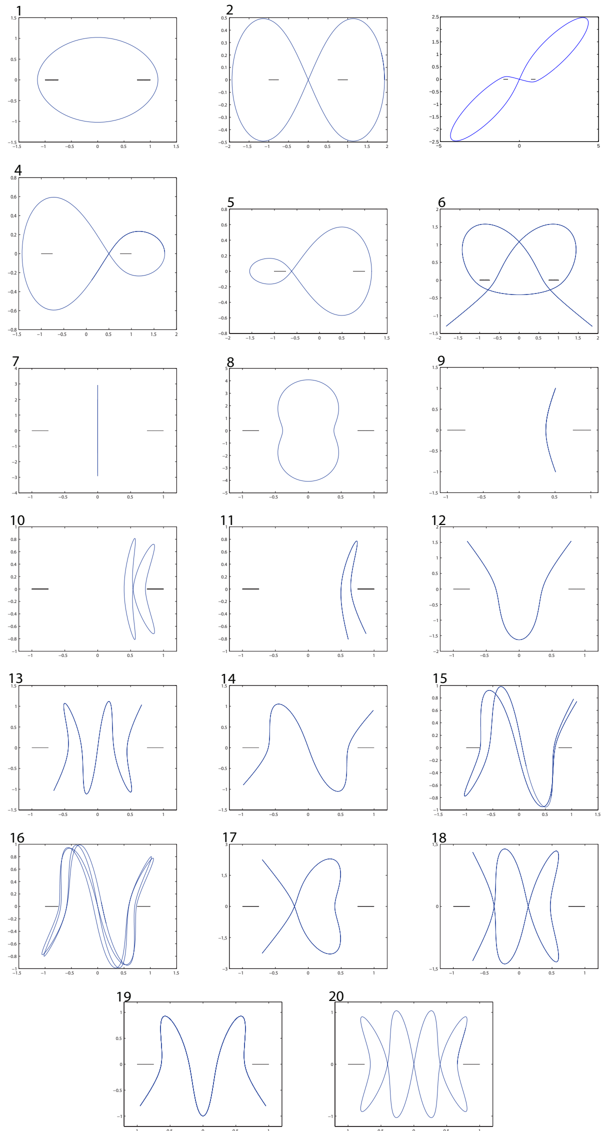

Fig. 7 Representation of the 20 polar orbits described along the paper

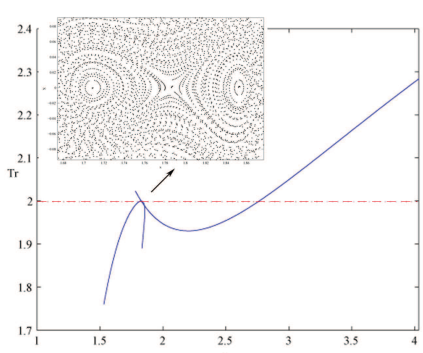

Fig. 8 Stability index evolution for Family 2, and the two bifurcated Families 4 and 5 which spring from a pitchfork bifurcation

The trace (see Fig. 9) passes through the value -2, which means a bifurcation with a non-symmetric family with double period, corresponding to Type 6 in Hénon classification.

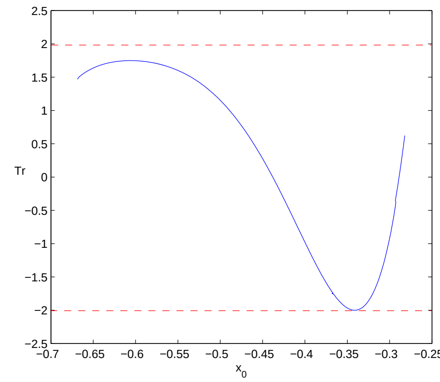

Fig. 9 Trace evolution of Family 6, made of *pretzel*-like orbits

Family 7 consists of one-dimensional vertical oscillations on the z-axis. The stability curve of the rectilinear orbits typically oscillates and crosses the critical values $Tr = \pm 2$ several times, leading to new families of periodic orbits. This is a well know phenomena observed in different dynamical systems such as the Sitnikov problem or the Hénon-Heiles problem Belbruno, Llibre & Olle (1994); Brack (2001); MaoDelos (1992).

Family 8 is an unstable family originated from a bifurcation of the previous family of vertical oscillations on the z-axis at the stability index value of 2. Its trace increases very fast as the orbit size grows until it terminates with a collision on the annulus.

Families 9, 10, 11 Family 9 is a stable family that originates out a pitchfork bifurcation with vertical oscillations, and ends up with a collision orbit with the annulus.
Its trace evolution shows a behavior similar to the stability graph of Family 6 (Fig. 9). As pointed out before, there is a bifurcation with a doubling-period family (let us recall that this doubling family at the bifurcation point have all unit eigenvalues), and thus its trace is $Tr = 2$, giving birth to new families. Plotting a Poincaré section at this critical value (see Fig. 10), the bifurcation can be easily identified. Indeed, we can see a central point that corresponds to the single-arc orbit, surrounded by four islands which belongs to a new bifurcated stable family, namely Family 11, whereas the other four hyperbolic points are related to a new unstable family, the Family 10.

The evolution of those bifurcated families and the doubling family is represented in Fig. 11.

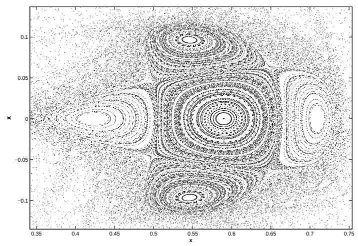

Fig. 10 Poincaré section at the bifurcation point of the doubling family 9

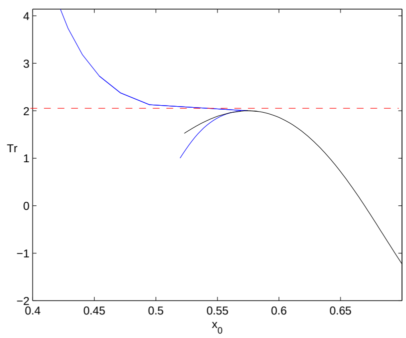

Fig. 11 Stability evolution of Family of 9 and its bifurcated Families 10 and 11

Families 12, 13 and their symmetric ones with respect to the *Ox*-axis come from a bifurcation of the vertical oscillations on the *z*-axis at the stability index $Tr = 2$; they end up with a collision orbit with the annulus. Their stability behavior is analogous to Family 6.

Families 14, 15, 16 Family 14 and its symmetric one with respect to *Ox*-axis is easily identified in a ring of islands surrounding the vertical orbit plotted in the corresponding Poincaré section. The evolution of the stability index intersects again the value $Tr = -2$, originating new families, through the doubling-period phenomena. Repeating the same procedure followed for Family 9, we continue doubling the family that bifurcates in two new families, one stable, Family 15, and another unstable, Family 16.

Families 17, 18 also originate from a bifurcation of the vertical orbit, and end up in a collision with the orbit as their radius grow. Their stability evolution begins at the trace value $Tr = 2$ and decreases its value until their end.

Families 19, 20 Family 19 shows the same behavior as Family 14, but it is made of highly unstable orbits that end up colliding with the annulus at a value $Tr \approx 37$. Finally, Family 20 corresponds to the double period bifurcated family from Family 19.

5 Conclusions

This paper analyzes the motion of an infinitesimal particle under the attraction of a planar annulus. We perform a systematic search of the most relevant solutions: periodic orbits. The evolution of a wide number of these periodic orbits globally describes how the dynamic around the annulus is organized.

This mathematical model is considered as a first approach for further studies of more complex dynamical systems. We should point out that the conclusions we can draw from the phase-space structure are generic and therefore of interest in the context of more realistic models, thus, an extension of the results obtained about the dynamics around a circular annulus is posed. Work is currently underway to analyze the dynamical changes when a central body is introduced into the model, studying how will affect a flatness coefficient of that planet and also a composition of annulus like the ring system of planet Saturn. Such a modification of our model should have applications to the study of the structure of planetary rings and, also, to determine locations that are more stable and suitable to place a spacecraft in scientific exploration mission around that body.

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Table 1 State vector, period and stability index of the 20 orbits presented in Fig. 7

	x	z	$\dot{x}$	$\dot{z}$	T	Tr
1	1.14498558	0.00000000	0.00000000	1.08570023	8.2283	-0.3716
2	1.93191413	0.00005241	-0.00005393	0.31172844	13.3033	1.9627
3	0.61431996	0.74536418	0.81006851	0.78959165	50.2524	-0.0072
4	1.73809965	-0.00000046	0.00000097	0.19712184	10.3537	1.5428
5	1.42672039	0.00005097	-0.00005961	0.61536913	10.3556	1.5436
6	-0.60617609	-0.00231883	0.62563965	1.19156741	46.8767	1.7487
7	0.00000000	0.00175956	0.00000000	1.27311072	25.0000	2.0000
8	0.38899022	0.00000771	0.00000213	1.39390805	39.9107	2.4758
9	0.45581169	-0.00030177	-0.00015502	0.98791160	7.6428	-1.4308
10	0.52211293	0.00759401	-0.11630520	0.95542967	12.8028	1.0986
11	0.47123829	0.00552701	-0.05452622	0.92828539	12.6087	2.3750
12	-0.00281186	-1.63333399	0.13129963	-0.00547436	25.5104	1.7511
13	0.44689964	-0.00069537	-0.04625679	1.03202274	44.0945	1.7951
14	-0.00001481	0.00006441	-0.21096655	0.91717358	24.7412	-0.9258
15	0.12994765	-0.29903286	-0.22850822	0.80424227	44.9477	3.2061
16	0.15868044	-0.26844813	-0.24850226	0.81956673	45.4211	1.2182
17	0.40325558	-0.00000000	0.00000185	1.27599360	57.1274	1.9978
18	0.47472898	-0.00000005	0.00000000	1.13845831	54.0839	1.9871
19	0.51239367	0.00820320	-0.07699219	1.06502319	34.5377	1.1231
20	0.23290426	-0.00000001	-0.00000004	1.01432950	40.1399	2.0005

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