Results in Mathematics manuscript No.
(will be inserted by the editor)

Explicit expressions for higher order binomial convolutions of numerical sequences

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Received: date / Accepted: date

Abstract We give explicit expressions for higher order binomial convolutions of sequences of numbers having a finite exponential generating function. Illustrations involving Cauchy, Bernoulli, and Apostol-Euler numbers are presented. In these cases, we obtain formulas easy to compute in terms of Stirling numbers.

Keywords Higher order convolutions · exponential generating function · Cauchy numbers · Bernoulli numbers · Apostol-Euler numbers · Stirling numbers.

Mathematics Subject Classification (2000) 05A19 · 60E05

1 Introduction

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The Cauchy numbers of the first kind $\boldsymbol{c} = (c_n)_{n \geq 0}$ are defined via their exponential generating function as

$$G(\boldsymbol{c}, z) = \frac{z}{\log(1+z)} = \sum_{n=0}^{\infty} c_n \frac{z^n}{n!}, \quad z \in \mathbb{C}, \quad |z| < 1, \tag{1}$$

(see, for instance, Comtet [5, Ch. VII] or Merlini *et al.* [13]), or in integral form as

$$c_n = \int_0^1 (\theta)_n \, d\theta, \quad n \in \mathbb{N}_0, \tag{2}$$

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where $(\theta)_n$ is the descending factorial, i.e., $(\theta)_n = \theta(\theta-1)\cdots(\theta-n+1)$, $n \in \mathbb{N}$, $(\theta)_0 = 1$. Different generalizations of these numbers can be found in Komatsu and Yuan [12], Pyo *et al.* [15], and the references therein.

The starting point of this note is a recent paper by Komatsu and Simsek [11], in which these authors pose the problem of finding rational numbers $a_0, \ldots, a_{m-1}$, such that

$$\sum_{l_1+\cdots+l_m=\mu} \binom{\mu}{l_1,\ldots,l_m} \sum_{k_1+\cdots+k_m=n} \binom{n}{k_1,\ldots,k_m} c_{k_1+l_1}\cdots c_{k_m+l_m} = \sum_{j=0}^{m-1} a_j c_{n+\mu-j}, \tag{3}$$

where

$$\binom{n}{k_1,\ldots,k_m} = \frac{n!}{k_1!\cdots k_m!}, \quad k_1,\ldots,k_m \in \mathbb{N}_0, \quad k_1+\cdots+k_m = n,$$

is the multinomial coefficient. Actually, Komatsu and Simsek [11] find explicit formulae for $m=3$ and $m=4$ using umbral calculus. Observe that the right-hand side in (3) is a linear combination of Cauchy numbers.

The aim of this note is to provide explicit expressions for the left-hand side in (3), where the Cauchy numbers are replaced by arbitrary sequences of numbers having a finite exponential generating function. This is done in Theorem 1 in the following section. When applied to various sequences of numbers, such as the Cauchy, the Bernoulli, and the Apostol-Euler numbers, Theorem 1 gives us formulas easy to compute, since they depend on the Stirling numbers of the first and second kinds (see Section 3).

2 The main result

Unless otherwise specified, we assume from now on that $n, \mu \in \mathbb{N}_0$, $m \in \mathbb{N}$, and $z \in \mathbb{C}$ with $|z| < r$, where $r > 0$ may change from line to line. Let $\mathcal{H}$ be the set of real sequences $\boldsymbol{u} = (u_n)_{n\geq 0}$ such that

$$\sum_{n=0}^{\infty} |u_n| \frac{r^n}{n!} < \infty,$$

for some radius $r > 0$. If $\boldsymbol{u} \in \mathcal{H}$, we denote its exponential generating function by

$$G(\boldsymbol{u}, z) = \sum_{n=0}^{\infty} u_n \frac{z^n}{n!}. \tag{4}$$

If $\boldsymbol{u}$ and $\boldsymbol{v}$ are in $\mathcal{H}$, the binomial convolution of $\boldsymbol{u}$ and $\boldsymbol{v}$, denoted by $\boldsymbol{u} \times \boldsymbol{v} = ((u \times v)_n)_{n\geq 0}$, is defined as

$$(u \times v)_n = \sum_{k=0}^{n} \binom{n}{k} u_k v_{n-k}.$$

It was shown in [2] that if $\boldsymbol{u}^{(k)} = (u_n^{(k)})_{n\geq 0} \in \mathcal{H}$, $k = 1, \ldots, m$, then $\boldsymbol{u}^{(1)} \times \cdots \times \boldsymbol{u}^{(m)} \in \mathcal{H}$ and

$$(u^{(1)} \times \cdots \times u^{(m)})_n = \sum_{j_1+\cdots+j_m=n} \binom{n}{j_1, \ldots, j_m} u_{j_1}^{(1)} \cdots u_{j_m}^{(m)}. \tag{5}$$

In addition, $\boldsymbol{u}^{(1)} \times \cdots \times \boldsymbol{u}^{(m)}$ is characterized by its exponential generating function

$$G(\boldsymbol{u}^{(1)} \times \cdots \times \boldsymbol{u}^{(m)}, z) = G(\boldsymbol{u}^{(1)}, z) \cdots G(\boldsymbol{u}^{(m)}, z). \tag{6}$$

Finally, denote by

$$\begin{aligned} &J(\boldsymbol{u}^{(1)} \times \cdots \times \boldsymbol{u}^{(m)}; \mu, n) \\ &= \sum_{l_1+\cdots+l_m=\mu} \binom{\mu}{l_1, \ldots, l_m} \sum_{k_1+\cdots+k_m=n} \binom{n}{k_1, \ldots, k_m} u_{k_1+l_1}^{(1)} \cdots u_{k_m+l_m}^{(m)}. \end{aligned} \tag{7}$$

With these ingredients, we state the following.

Theorem 1 *Let* $\boldsymbol{u}^{(k)} \in \mathcal{H}$, $k = 1, \ldots, m$. *Then,*

$$J(\boldsymbol{u}^{(1)} \times \cdots \times \boldsymbol{u}^{(m)}; \mu, n) = (u^{(1)} \times \cdots \times u^{(m)})_{\mu+n}.$$

Proof If $\boldsymbol{u} \in \mathcal{H}$ and $l \in \mathbb{N}_0$, we consider the shifted sequence $\boldsymbol{u}(l) \in \mathcal{H}$ defined as

$$\boldsymbol{u}(l) = (u_{l+n})_{n\geq 0}. \tag{8}$$

Differentiating term by term in (4), we see that

$$G(\boldsymbol{u}(l), z) = G^{(l)}(\boldsymbol{u}, z).$$

Hence, using Leibniz's rule for differentiation in (6), we get

$$\begin{aligned} &G((\boldsymbol{u}^{(1)} \times \cdots \times \boldsymbol{u}^{(m)})(\mu), z) = G^{(\mu)}(\boldsymbol{u}^{(1)} \times \cdots \times \boldsymbol{u}^{(m)}, z) \\ &= \sum_{l_1+\cdots+l_m=\mu} \binom{\mu}{l_1, \ldots, l_m} G^{(l_1)}(\boldsymbol{u}^{(1)}, z) \cdots G^{(l_m)}(\boldsymbol{u}^{(m)}, z) \\ &= \sum_{l_1+\cdots+l_m=\mu} \binom{\mu}{l_1, \ldots, l_m} G(\boldsymbol{u}^{(1)}(l_1) \times \cdots \times \boldsymbol{u}^{(m)}(l_m), z), \end{aligned} \tag{9}$$

By (8), we have

$$G((\boldsymbol{u}^{(1)} \times \cdots \times \boldsymbol{u}^{(m)})(\mu), z) = \sum_{n=0}^{\infty} (u^{(1)} \times \cdots \times u^{(m)})_{\mu+n} \frac{z^n}{n!}. \tag{10}$$

By (5) and (8), we see that

$$\begin{aligned} &G(\boldsymbol{u}^{(1)}(l_1) \times \cdots \times \boldsymbol{u}^{(m)}(l_m), z) = \sum_{n=0}^{\infty} (u^{(1)}(l_1) \times \cdots \times u^{(m)}(l_m))_n \frac{z^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{k_1+\cdots+k_m=n} \binom{n}{k_1, \ldots, k_m} u_{l_1+k_1}^{(1)} \cdots u_{l_m+k_m}^{(m)}. \end{aligned}$$

This, in conjunction with (9) and (10), shows the result.

According to Theorem 1, in order to compute expression (7), we only need to look at the $\mu+n$ term of the binomial convolution $\boldsymbol{u}^{(1)} \times \cdots \times \boldsymbol{u}^{(m)}$, which, in turn, can be obtained by differentiating the exponential generating function in (6). In many occasions, such a term can be described in probabilistic terms, as it happens in the case of the Cauchy numbers considered in the following section.

3 Examples

Let $(U_j)_{j\geq 1}$ be a sequence of independent identically distributed random variables having the uniform distribution on $[0,1]$ and denote by

$$S_m = U_1 + \cdots + U_m \qquad (S_0 = 0). \tag{11}$$

It is well known (cf. Feller [8, p. 27] or Adell and Sangüesa [4, Proposition 2.1]) that the probability density of S_m is given by the spline funcion

$$\rho_m(\theta) = \frac{1}{(m-1)!}\sum_{k=0}^{m-1}\binom{m}{k}(-1)^k(\theta-k)_+^{m-1}, \quad \theta\in[0,m], \tag{12}$$

where $x_+ = \max(x,0)$. On the other hand, recall that the Stirling numbers of the first and second kind, respectively denoted by $s(n,k)$ and $S(n,k)$, $k = 0,1\ldots,n$, are defined by

$$(x)_n = \sum_{k=0}^{n} s(n,k)x^k, \qquad x^n = \sum_{k=0}^{n} S(n,k)(x)_k, \quad x\in\mathbb{R}, \tag{13}$$

(see, for instance, Abramowitz and Stegun [1] or Olver *et al.* [14]). Finally, Sun [16] (see also [2]) showed the following probabilistic representation for the Stirling numbers of the second kind

$$S(n,m) = \binom{n}{m}\mathbb{E}S_m^{n-m}, \quad m = 0,1\ldots,n, \tag{14}$$

where $\mathbb{E}$ stands for mathematical expectation.

With the preceding notations, we state the following.

Corollary 1 *Let $\boldsymbol{c} = (c_n)_{n\geq 0}$ be the Cauchy numbers. Then,*

$$J(\boldsymbol{c}\times \overset{m}{\smile}{\cdots} \times \boldsymbol{c};\mu,n) = \int_0^m (\theta)_{\mu+n}\rho_m(\theta)\,d\theta = \sum_{k=0}^{\mu+n}\frac{s(\mu+n,k)S(m+k,m)}{\binom{m+k}{m}}.$$

Proof From (1) and (12), we see that

$$G(\boldsymbol{c},z) = \frac{z}{\log(1+z)} = \int_0^1 (1+z)^\theta\,d\theta = \mathbb{E}(1+z)^{U_1}.$$

By (6), (11), and the independence between the random variables involved, we have

$$G(\boldsymbol{c} \times \overset{m}{\smile}{\cdots} \times \boldsymbol{c}, z) = \mathbb{E}(1+z)^{U_1} \cdots \mathbb{E}(1+z)^{U_m} = \mathbb{E}(1+z)^{S_m} = \sum_{n=0}^{\infty} \mathbb{E}(S_m)_n \frac{z^n}{n!}, \tag{15}$$

where in the last equality we have used the binomial expansion. This means that

$$(c \times \overset{m}{\smile}{\cdots} \times c)_{\mu+n} = \mathbb{E}(S_m)_{\mu+n} = \int_0^m (\theta)_{\mu+n} \rho_m(\theta)\, d\theta, \tag{16}$$

as follows from (12). Finally, we get from (13) and (14)

$$\mathbb{E}(S_m)_{\mu+n} = \sum_{k=0}^{\mu+n} s(\mu+n,k) \mathbb{E} S_m^k = \sum_{k=0}^{\mu+n} \frac{s(\mu+n,k) S(m+k,m)}{\binom{m+k}{m}}.$$

This, together with (16) and Theorem 1, completes the proof.

The Bernoulli numbers $\boldsymbol{B} = (B_n)_{n\geq 0}$ are defined by their exponential generating function as

$$G(\boldsymbol{B}, z) = \frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}.$$

The Cauchy numbers of the first kind defined in (1) are also named Bernoulli numbers of the second kind, denoted in many papers by $\boldsymbol{b} = (b_n)_{n\geq 0}$. This is the reason why we use capital letters to denote the Bernoulli numbers. Recently, we have shown in [3, Theorem 6] that

$$(B \times \overset{m}{\smile}{\cdots} \times B)_n = \sum_{k=0}^{n} \frac{S(n,k) s(m+k,m)}{\binom{m+k}{m}}.$$

This formula and Theorem 1 immediately yield the following result.

Corollary 2 *Let $\boldsymbol{B} = (B_n)_{n\geq 0}$ be the Bernoulli numbers. Then,*

$$J(\boldsymbol{B} \times \overset{m}{\smile}{\cdots} \times \boldsymbol{B}; \mu, n) = \sum_{k=0}^{\mu+n} \frac{S(\mu+n,k) s(m+k,m)}{\binom{m+k}{m}}.$$

Comparing Corollaries 1 and 2, it is interesting to note that the roles of the Stirling numbers of the first and second kinds are interchanged.

As a last example, consider the Apostol-Euler numbers $\boldsymbol{e}^{(\beta)} = (e_n^{(\beta)})_{n\geq 0}$ defined by means of their exponential generating function as

$$G(\boldsymbol{e}^{(\beta)}, z) = \frac{1}{1+\beta(e^z - 1)} = \sum_{n=0}^{\infty} e_n^{(\beta)} \frac{z^n}{n!}, \quad 0 < \beta \leq 1. \tag{17}$$

For $\beta = 1/2$, $\boldsymbol{e}^{(\beta)}$ are the classical Euler numbers. Apostol-type numbers and polynomials extending formula (17) can be found in Hernández-Llanos *et al.* [10]. The following formula has been proved in [3, Theorem 8]

$$(e^{(\beta)} \times \overset{m}{\smile\cdots} \times e^{(\beta)})_n = \sum_{k=0}^{n} (-m)_k \beta^k S(n,k). \tag{18}$$

Similar identities to (18) were obtained by Dilcher [6], He and Araci [9], and Dilcher and Vignat [7]. Formula (18), in conjunction with Theorem 1, gives us the following corollary.

Corollary 3 *Let $\boldsymbol{e}^{(\beta)} = (e_n^{(\beta)})_{n\geq 0}$ be the Apostol-Euler numbers. Then,*

$$J(\boldsymbol{e}^{(\beta)} \times \overset{m}{\smile\cdots} \times \boldsymbol{e}^{(\beta)}; \mu, n) = \sum_{k=0}^{\mu+n} (-m)_k \beta^k S(\mu+n,k).$$

Acknowledgements We would like to thank the referee, whose comments and suggestions greatly improved the final outcome.

The authors are partially supported by Research Projects DGA (E-64) and MTM2015-67006-P.

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