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AN APPLICATION OF PAPPUS' INVOLUTION THEOREM TO CAYLEY–KLEIN PROJECTIVE MODELS

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Abstract. Pappus' Involution Theorem is useful for proving incidence relations in the hyperbolic and elliptic planes. This fact is exemplified with the proof of a theorem about a family of 4-gons in the hyperbolic and elliptic planes. This non-Euclidean theorem is also re-interpreted in multiple ways, providing some other theorems for different figures in the hyperbolic plane.

1. Introduction

Involutions are a quite useful tool in theorem proving. Many geometric problems admit easier or shorter proofs using involutions (non-geometric problems too, of course—see [16] for an astonishing example!). In the projective context, the most ancient result involving involutions is the following theorem, due to Pappus:

Theorem 1.1 (Pappus' Involution Theorem). *The three pairs of opposite sides of a complete quadrangle meet any line not through a vertex in three reciprocal pairs of points of a projective involution.*

See [5, p. 49] for a proof. This is a partial version of Desargues' Involution Theorem (see [5, p. 81]; see also [7, pp. 292–293]). A *complete quadrangle* is the figure in the projective plane that is produced by a set of four points (the *vertices* of the complete quadrangle), no three of which are collinear, when all the lines joining any two of them (the *sides* of the complete quadrangle, see Fig. 2) are drawn. Two sides of the complete quadrangle are *opposite* if they don't share a vertex, and in this case they intersect in a *diagonal point* of the complete quadrangle. Throughout the paper the word "quadrangle" is always used with this projective meaning, and we identify a complete quadrangle with its set of vertices. With the notation of Fig. 2, the six sides of the complete quadrangle $\mathcal{Q} = \{A, B, C, D\}$ intersect the line r in the six points E, F, G, H, I, J. Pappus' Involution Theorem implies that

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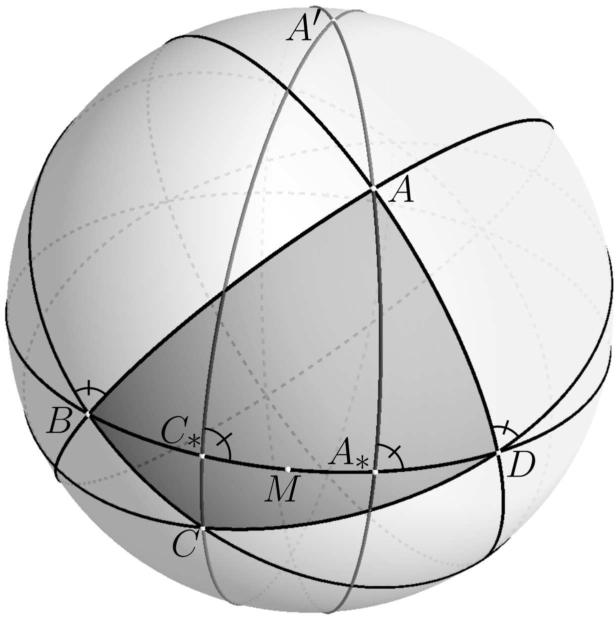

FIGURE 1. Elliptic construction (spherical view).

there is a unique projective involution $\tau_{\mathscr{Q}} : r \to r$ such that $\tau_{\mathscr{Q}}(E) = F$, $\tau_{\mathscr{Q}}(G) = H$ and $\tau_{\mathscr{Q}}(I) = J$. The involution $\tau_{\mathscr{Q}}$ is the *quadrangular involution* induced by $\mathscr{Q}$ in r.

During the author's research [15], Pappus' Involution Theorem was found as an extremely powerful tool for proving incidence relations in the hyperbolic and elliptic planes in the framework of Cayley–Klein models, and the first purpose of this article is to exhibit an example of this fact. This is done in Section 3, where a quadrangular involution is used to prove Theorem 1.2 below in a quite simple manner. A 4-sided polygon in the Euclidean, elliptic or hyperbolic plane is called a 4-*gon*. A 4-gon is *diametral*[1] if it has two right angles located at opposite vertices.

Theorem 1.2. *Let $ABCD$ be a diametral 4-gon in the hyperbolic or elliptic planes with right angles at B and D. Let A_*, C_* be the orthogonal projections of the points A, C into the diagonal BD, respectively. A midpoint*[2] *of the segment $\overline{BD}$ is also a midpoint of the segment $\overline{A_*C_*}$ (see Fig. 1).*

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[1]In the Euclidean case it is a cyclic 4-gon with a diagonal which is a diameter of the circumcycle.

[2]Note that we have written "a midpoint" instead of "the midpoint". Euclidean or hyperbolic segments have undoubtedly a unique midpoint (*the* midpoint). Since elliptic lines are closed, in the elliptic case this concept is subtler. For convenience, we consider that an elliptic segment with endpoints A and B has two midpoints: the two points collinear with A and B and equidistant from A and B.

This theorem also holds in the Euclidean plane and therefore it is a theorem of Bolyai's absolute geometry[3]. In the Euclidean context, it would be a reformulation of [10, problem II.214].

Since projective geometry is a general theory that includes many others, such as Euclidean, affine, hyperbolic and elliptic geometries, for example, it is well-known that any projective theorem is a general version of many other theorems in all those other settings. In the planar case, in the same way that a projective theorem involving conics can be seen in the Euclidean plane as a set of different theorems for the different conic sections, it can be seen also as a set of theorems for different figures in the hyperbolic and elliptic planes. This idea was used by Thurston [11, Ch. 2] for deducing, at one stroke, the trigonometry of many different figures in the hyperbolic plane: the *generalized triangles* (see also [4]), which are obtained by modifying the relative position of a projective triangle with respect to the absolute conic of the model. Using Thurston's trick, a unique projective proof of a non-Euclidean theorem about triangles provides a bunch of different non-Euclidean theorems, slightly different from the original one, which we call *projective variations* of the original one. Another purpose of the paper is to illustrate this remarkable property of Cayley–Klein models, that seems to be not very well known, by exploring the projective variations of Theorem 1.2. This is done in Sec. 4, where five such projective variations (Theorems 4.2 to 4.6) are presented.

Before that, in Sec. 2 the basic tools from projective geometry and Cayley–Klein projective models to be used in the subsequent sections are introduced. We finish in Sec. 5 with some questions that naturally arise from the previous theorems. In particular, we present an Euclidean higher-dimensional generalization of Theorem 1.2. In all figures right angles are denoted with the symbol ⦝.

2. Cayley–Klein models

It is assumed that the reader is familiar with the basic concepts of real and complex planar projective geometry: the projective plane and its fundamental subsets (points, lines, pencils of lines, conics), and their projectivities. Nevertheless, we give a brief review of some concepts and results needed for a better understanding of Sections 3 and 4. For the rigurous definitions and proofs we refer to [5], [14] or [8], for example. It is also assumed that the reader has some elementary background in non-Euclidean planar geometry (see [6], [11], or the elementary and delightful memoir [9] for example).

Although we work mostly with real elements, the real projective plane $\mathbb{RP}^2$ is considered standardly embedded in the complex projective plane $\mathbb{CP}^2$.

If A, B are two different points in the projective plane, AB denotes the line joining them. If a, b are two different lines, or a line and a conic, in the projective plane, $a \cdot b$ is their intersection set.

[3]In fact, since Theorem 1.2 only involves the notions of "perpendicularity" and of "midpoint", it makes sense in a much more general framework like Bachmann's theory of metric planes [2], or even in a weaker axiom system than that for metric planes, as that of Hjelmslev groups [3], for example. This was pointed out to the author by a referee.

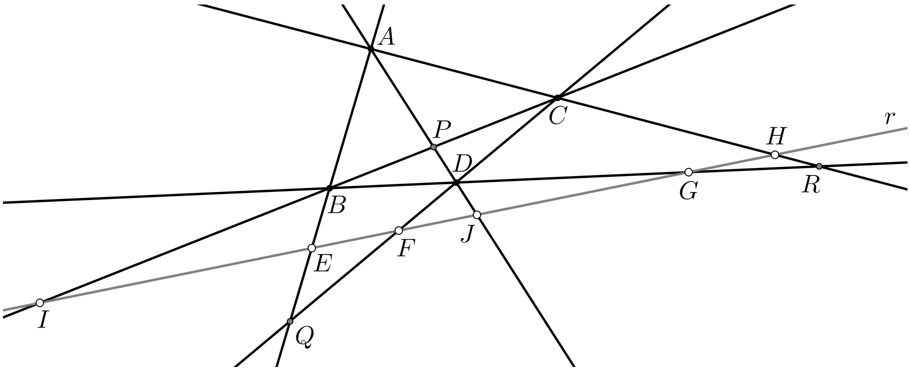

FIGURE 2. Quadrangles. Pappus' Involution Theorem.

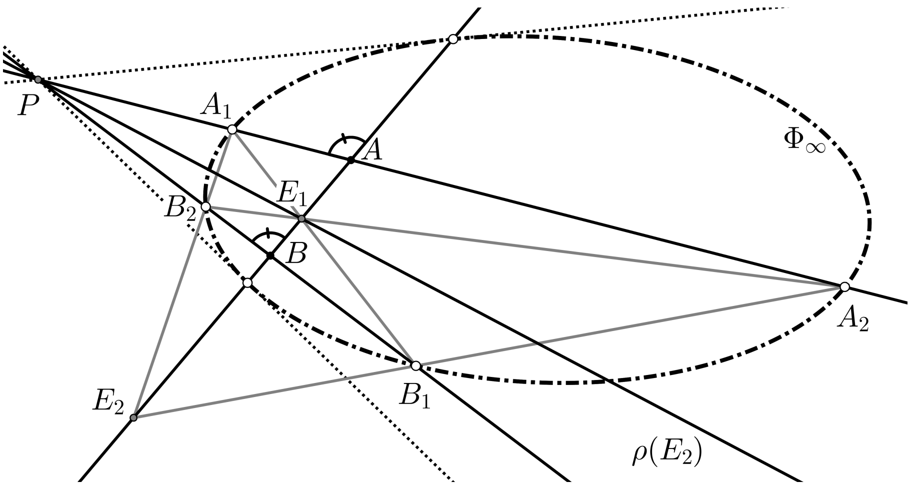

FIGURE 3. Midpoint and orthogonal bisector of a hyperbolic segment.

Let us review the construction of the projective models of the hyperbolic and elliptic planes. We will not give a full construction of these models: we will just show the projective interpretation of the basic geometric concepts that will be needed later. Detailed constructions of the models can be found in [6, 8, 9], for example.

For constructing the non-Euclidean planar models, a non-degenerate conic Φ_∞ is fixed (the *absolute conic*) such that the polar of each real point with respect to Φ_∞ is a real line. An equivalent formulation of this property is to require, working with homogeneous coordinates in $\mathbb{CP}^2$, that Φ_∞ can be expressed by an equation with real coefficients. Such a conic can be of two kinds: a *real conic*, if it has real points; or an *imaginary conic*, if it has no real points (see [14, vol. II, p. 186]).

When Φ_∞ is a real conic, the interior points of Φ_∞ constitute the hyperbolic plane, and when Φ_∞ is an imaginary conic the whole $\mathbb{RP}^2$ constitutes the elliptic plane. We use the common term *the non-Euclidean plane* either for the hyperbolic plane or for the elliptic plane. Geodesics in these models are given by the intersection with the non-Euclidean plane of real projective lines. If A, B are two different points in the non-Euclidean plane, $\overline{AB}$ denotes a segment having A and B as endpoints. Although in the hyperbolic case points on $\mathbb{RP}^2$ not interior to Φ_∞ are not points of the hyperbolic plane, they are of geometric interest for the model (see [11]). For instance, via the *absolute polarity* (the polarity with respect to Φ_∞), each point exterior to Φ_∞ parametrizes a line of the hyperbolic plane (its polar line). Therefore, we will talk about points or lines in a purely projective sense, even if the referred elements are imaginary or, in the hyperbolic case, not interior to Φ_∞.

Let us denote by ρ the absolute polarity. Given a point P and a line p, $\rho(P)$ and $\rho(p)$ are the polar of P and the pole of p with respect to Φ_∞, respectively.

The polarity ρ induces a natural involution on any line p not tangent to Φ_∞ (resp. on any pencil of lines passing through a point P not in Φ_∞): the *conjugacy involution*. This involution sends each point $P \in p$ (resp. each line p through P) to its *conjugate*, which is defined as $p \cdot \rho(P)$ (resp. the line $P\rho(p)$). The double points of the conjugacy involution on p are the two points on $p \cdot \Phi_\infty$. Equivalently, the double lines of the conjugacy involution on the pencil of lines through P are the two lines tangent to Φ_∞ through P. Two lines whose intersection point lies in the non-Euclidean plane are *perpendicular* if they are conjugate with respect to Φ_∞, that is, if each one contains the pole of the other with respect to Φ_∞. If A is a point of the non-Euclidean plane, a is a line through A and A' is the conjugate point of A in a with respect to Φ_∞, then $\rho(A')$ is the line perpendicular to a through A and conversely.

Let A, B be two points on the non-Euclidean plane and take $P = \rho(AB)$. Let a and b be the lines joining P with A and B, respectively. Each of the lines a, b has two (perhaps imaginary) different intersection points with Φ_∞. Let $\{A_1, A_2\}$ and $\{B_1, B_2\}$ be the intersection sets of a and b with Φ_∞ respectively. Since the complete quadrangle $\mathscr{Q} = \{A_1, A_2, B_1, B_2\}$ is inscribed in Φ_∞ the points

$$E_1 = A_1B_1 \cdot A_2B_2 \quad \text{and} \quad E_2 = A_1B_2 \cdot A_2B_1$$

lie on AB. We say that E_1 and E_2 are the *midpoints* of the segment $\overline{AB}$. Note that this definition of the midpoints of a segment is projective. Moreover, we could talk about *midpoints of a projective segment with respect to a conic* in a strictly projective context, as it will be done in the statement of Theorem 4.1[4]. This definition can be interpreted in multiple ways (see [6]) in the non-Euclidean context. For example:

[4]While the concept of "segment" in the Euclidean or non-Euclidean context is rather intuitive, to define "segment" in a complex projective setting is not a trivial task. For our purposes, it suffices to identify the projective segment $\overline{AB}$ with the set $\{A, B\}$.

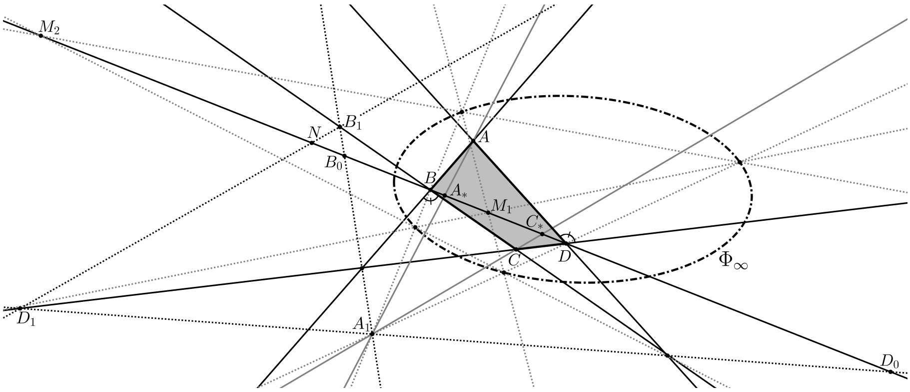

FIGURE 4. Hyperbolic construction (projective view).

- If Φ_∞ is imaginary, the points A_1, A_2, B_1, B_2 are also imaginary and E_1 and E_2 are the two points of AB which are equidistant from A and B in the elliptic plane.
- If Φ_∞ is a real conic and A and B are interior to Φ_∞, exactly one of the two points E_1, E_2, say E_1, is interior to Φ_∞: it is the midpoint of the hyperbolic segment $\overline{AB}$ (Fig. 3). The other point E_2 is the pole of PE_1, which is the orthogonal bisector of $\overline{AB}$.
- If Φ_∞ is a real conic and AB is exterior to Φ_∞, its pole P is interior to Φ_∞, and the lines PE_1, PE_2 are the two bisectors of the hyperbolic angle $\widehat{ab}$.

By construction, since E_1 and E_2 are diagonal points of the complete quadrangle $\mathscr{Q}$, the points A, B, E_1, E_2 form a harmonic set, which in terms of cross-ratios is equivalent to $(ABE_1E_2) = -1$. Since $\mathscr{Q}$ is inscribed in Φ_∞, the projective triangle PE_1E_2 is self-polar with respect to Φ_∞, and this implies that E_1 and E_2 are conjugate to each other in AB. Therefore, if U and V are the two intersection points of AB with Φ_∞, then $(UVE_1E_2) = -1$. A useful characterization of midpoints is the following (see [15]):

Lemma 2.1. *If C, D are two points of AB such that*

$$(ABCD) = (UVCD) = -1,$$

then $\{C, D\} = \{E_1, E_2\}$.

Proof. Let C, D be two points of AB such that $(ABCD) = (UVCD) = -1$. Take the harmonic involutions $\tau_{E_1E_2}$ and τ_{CD}, of AB with fixed points E_1, E_2 and C, D respectively. The composition $\tau_{E_1E_2} \circ \tau_{CD}$ fixes the points U, V, A, B and so it must be the identity on AB. This implies that $\{E_1, E_2\} = \{C, D\}$. □

3. Proof of Theorem 1.2

Let $ABCD$ be a 4-gon in the non-Euclidean plane such that the lines AB and AD are conjugate to BC and DC respectively with respect to Φ_∞ (Fig. 4). This means that $B_1 = \rho(AB)$ and $D_1 = \rho(AD)$ belong to BC and DC, respectively.

Let a be the line BD, and take $A_1 = \rho(a)$. Thus, $A_* = a \cdot AA_1$ and $C_* = a \cdot CA_1$.

Let B_0 and D_0 be the intersection points with a of the lines A_1B_1 and A_1D_1, respectively. The triangle $A_1B_1D_1$ is the polar triangle of ABD. Hence, B_0 and D_0 are the conjugate points of B and D in a with respect to Φ_∞, respectively. The point $N = a \cdot B_1D_1$ is the pole of AA_1, and so it is the conjugate point of A_* in a with respect to Φ_∞.

Let M_1, M_2 be the midpoints of $\overline{BD}$, and consider the complete quadrangle $\mathscr{Q} = \{C, A_1, B_1, D_1\}$.

Consider the following three involutions in a:

- the conjugacy involution ρ_a induced by ρ in a;
- the quadrangular involution τ induced by $\mathscr{Q}$ in a; and
- the harmonic involution σ in a with respect to M_1 and M_2.

The quadrangular involution τ sends B, D and C_* into D_0, B_0 and N, respectively. This implies that the composition $\rho_a\tau$ sends B, D, B_0 and D_0 into D, B, D_0 and B_0, respectively. Since

$$(B_0D_0M_1M_2) = (\rho_a(B)\rho_a(D)\rho_a(M_2)\rho_a(M_1)) = (BDM_2M_1) = -1, \tag{3.1}$$

then $\sigma(B_0) = D_0$. The projectivities σ and $\rho_a\tau$ of a agree over at least three different points of a, and so they coincide. Hence, $\sigma(C_*) = \rho_a\tau(C_*) = \rho_a(N) = A_*$. If the points A_* and C_* coincide, they coincide with a double point M_1 or M_2 of σ and the result trivially holds. If $A_* \neq C_*$, they are harmonic conjugates with respect to M_1 and M_2, and therefore

$$(A_*C_*M_1M_2) = -1\,.$$

If U and V are the intersection points of a with Φ_∞,

$$(UVM_1M_2) = -1$$

since M_1 and M_2 are the midpoints of $\overline{BD}$. By Lemma 2.1 the points M_1 and M_2 are the midpoints of $\overline{A_*C_*}$. □

4. Projective variations of Theorem 1.2

Disguised in the proof of Theorem 1.2 there is, in fact, a proof of the following projective theorem:

Theorem 4.1. *Let $\mathscr{Q}$ be a complete quadrangle in the projective plane with vertices A, B, C, D in* general position *with respect to Φ_∞ (vertices and diagonal points not in Φ_∞, sides and diagonal lines[5] not tangent to Φ_∞) such that the lines AB and AD are conjugate to BC and DC respectively with respect to Φ_∞. Let A_1 be the pole of BD, and let A_* and C_* be the intersection points of AA_1 and CA_1, respectively,*

[5]The *diagonal lines* of a complete quadrangle are the lines joining diagonal points.

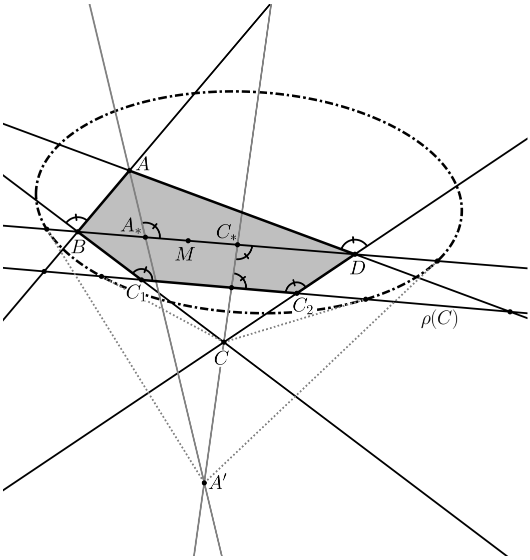

Figure 5. 4-right pentagon I.

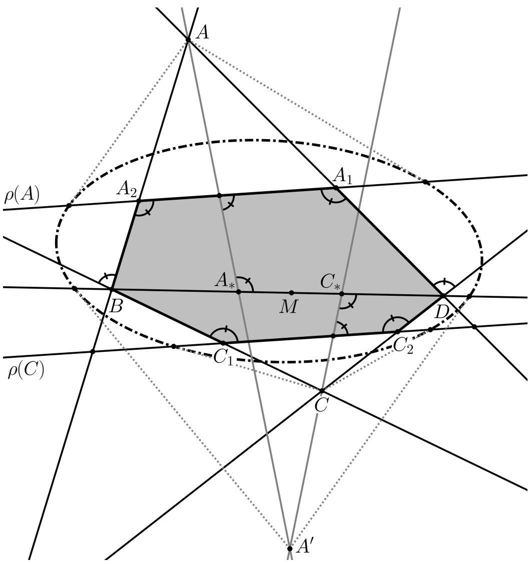

Figure 6. Right-angled hexagon I.

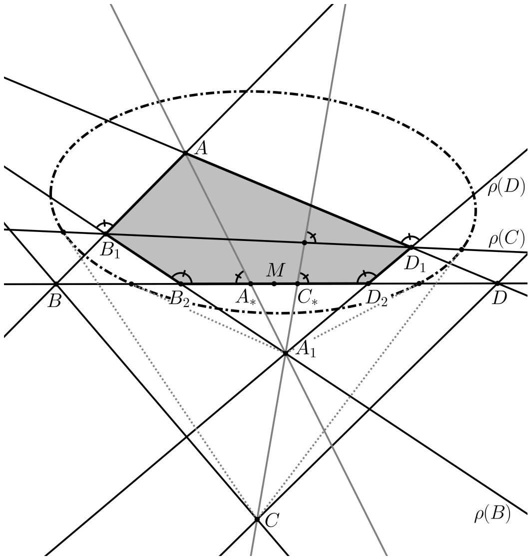

FIGURE 7. 4-right pentagon II.

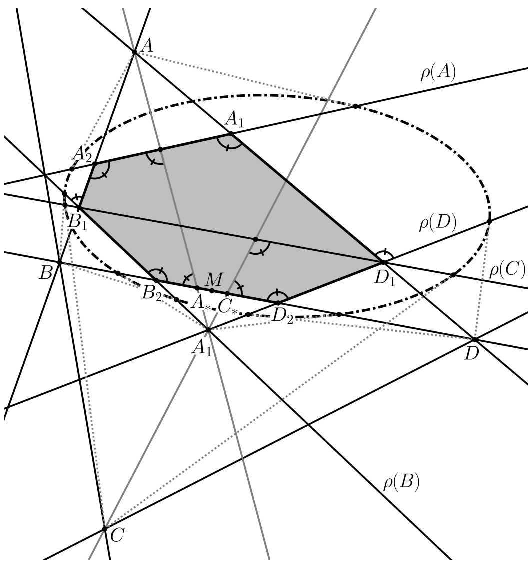

FIGURE 8. Right-angled hexagon II.

with BD. The midpoints of $\overline{A_*C_*}$ with respect to Φ_∞ (as defined projectively in Section 2) are also the midpoints of $\overline{BD}$ with respect to Φ_∞. □

Assume that $ABCD$ is a diametral 4-gon in the hyperbolic plane, as the one depicted in Fig. 4. After changing the relative position of A, B, C, D with respect to Φ_∞ and applying Theorem 4.1 to the resulting figure, some projective variations of Theorem 1.2 are obtained.

4.1. **4-right pentagon.** If in the 4-gon $ABCD$ it is assumed that the vertex C lies outside the absolute conic while the rest of vertices remain inside Φ_∞, the polar of C appears in the figure as the common perpendicular to the lines CB and CD. The hyperbolic polygon that appears is a *4-right pentagon*: a hyperbolic pentagon with, at least, four right angles at the vertices different from A (Fig. 5). In this case, Theorem 4.1 implies:

Theorem 4.2. *In the hyperbolic 4-right pentagon ABC_1C_2D, with non-right angle at most at A, let A_* be the orthogonal projection of A into BD, and let C_* be the intersection of BD with the common perpendicular to BD and C_1C_2. The midpoint of $\overline{BD}$ is also the midpoint of $\overline{A_*C_*}$.* □

4.2. **Right-angled hexagon.** If in the previous figure the vertex A is pushed out of Φ_∞, while B and D remain interior to Φ_∞, the polar of A becomes part of the figure as the common perpendicular to the lines AB and AD. The figure that appears is a *right-angled hexagon*: an hexagon in the hyperbolic plane with six right angles as the one depicted in Fig. 6. With the notation of this figure, the traslation of Theorem 4.1 for this configuration is:

Theorem 4.3. *Let $A_1A_2BC_1C_2D$ be a hyperbolic right-angled hexagon. Let A_* be the intersection point with BD of the common perpendicular to BD and A_1A_2, and let C_* be the intersection point with BD of the common perpendicular to BD and C_1C_2. The midpoint of $\overline{BD}$ is also the midpoint of $\overline{A_*C_*}$.* □

4.3. **4-right pentagon II.** If, after pushing C out of Φ_∞ for obtaining the 4-right pentagon of Fig. 5, the points B and D are also pushed out of Φ_∞ while the line BD remains secant to Φ_∞ and A remains interior to Φ_∞, the polars of B and D appear in the figure, drawing with the lines AB, AD and BD another 4-right pentagon $AB_1B_2D_2D_1$ as the one of Fig. 7. Because the pole of AB is collinear with B and C, the polars of B and C intersect at the point B_1 lying in AB, which is also the conjugate point of $C_1 = BC \cdot \rho(C)$ in $\rho(C)$ with respect to Φ_∞. In the same way, the polars of C and D intersect at the point D_1 lying in AD which is the conjugate point of $C_2 = CD \cdot \rho(C)$ in $\rho(C)$ with respect to Φ_∞. On the other hand, the polars of B and D intersect BD at B_2 and D_2, respectively, which are the conjugate points of B, D respectively in BD with respect to Φ_∞. The pentagon $AB_1B_2D_2D_1$ is a 4-right pentagon with right angles at all its vertices with the unique possible exception of A. By Lemma 2.1 and (3.1), the midpoints of $\overline{BD}$ are also the midpoints of $\overline{B_2D_2}$. With the notation of Fig. 7:

Theorem 4.4. *Let $AB_1B_2D_2D_1$ be a hyperbolic 4-right pentagon with right angles at the vertices B_1, B_2, D_1, D_2. Let A_* be the orthogonal projection of A into B_2D_2,*

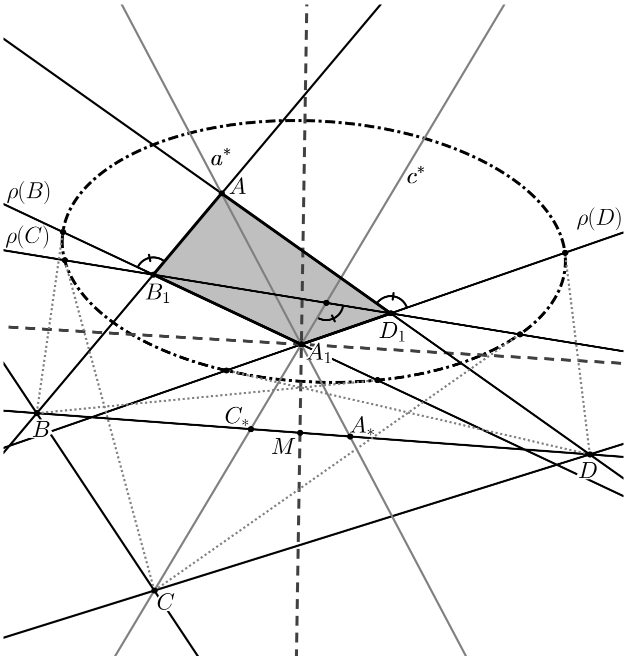

FIGURE 9. Hyperbolic quadrangle revisited.

and let C_* be the intersection with B_2D_2 of the common perpendicular to B_1D_1 and B_2D_2. The midpoint of $\overline{B_2D_2}$ is also the midpoint of $\overline{A_*C_*}$. □

4.4. **Right-angled hexagon II.** If in the previous figure the point A is also pushed out of the absolute conic, a theorem similar to Theorem 4.4 for right-angled hexagons is obtained. With the notation of Fig. 8:

Theorem 4.5. *Let $A_1A_2B_1B_2D_2D_1$ be a hyperbolic right-angled hexagon. Let a_* and c_* be the common perpendiculars to B_2D_2 and A_1A_2 and to B_2D_2 and B_1D_1, respectively, and let A_* and C_* be the intersection points of a_* and c_*, respectively, with B_2D_2. The midpoint of $\overline{A_*C_*}$ is the midpoint of $\overline{B_2D_2}$.* □

4.5. **Quadrangle II.** If in the configuration "4 right-pentagon II" (Fig. 7) the points B, D are moved until the line BD is exterior to Φ_∞, the point A_1 becomes interior to Φ_∞ and $AB_1A_1D_1$ is a hyperbolic diametral 4-gon with right angles at the opposite vertices B_1, D_1. After a reinterpretation of the points A_*, C_*, M_1, M_2 for this figure, the following theorem is obtained (see Fig. 9):

Theorem 4.6. *Let $AB_1A_1D_1$ be a hyperbolic diametral 4-gon with right angles at the opposite vertices B_1, D_1. Consider the line $a_* = AA_1$ and the line c_* perpendicular to B_1D_1 through A_1. The angle-bisectors of $\widehat{B_1A_1D_1}$ are also the angle-bisectors of $\widehat{a_*c_*}$.* □

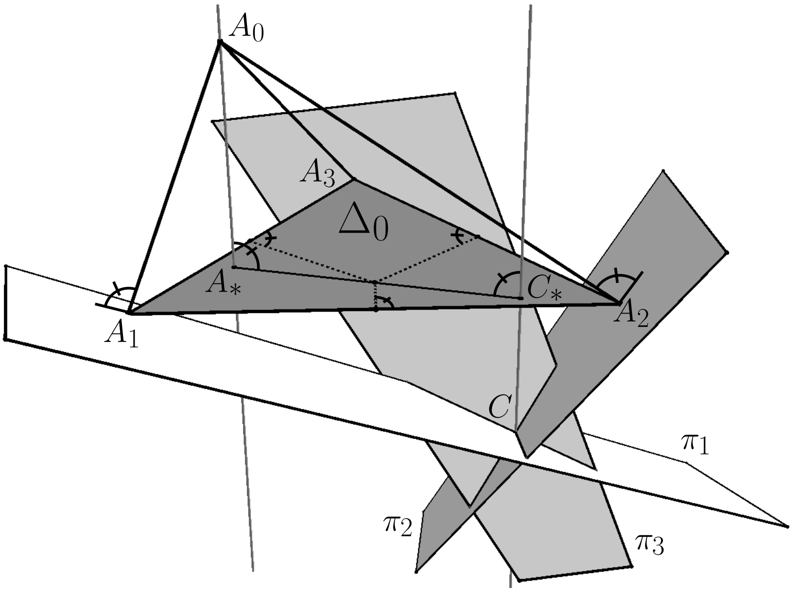

FIGURE 10. Three-dimensional version of Theorem 5.1.

The previous construction is valid also for the elliptic case, and it can be checked that this theorem also holds in the Euclidean plane.

5. Higher-dimensional generalizations

It should be interesting to generalize the above techniques to higher dimensions: to find higher-dimensional generalizations of Theorems 1.2, 4.2, 4.3, 4.4, 4.5 or 4.6, perhaps in terms of *generalized tetrahedra* [12, 13].

A higher-dimensional generalization of the Euclidean version of Theorem 1.2 is:

Theorem 5.1. *Let Δ be an n-simplex in Euclidean n-dimensional space with vertices $A_0, A_1, \dots, A_n$. Consider the opposite face Δ_0 to A_0 in Δ, and take the hyperplane α_0 containing Δ_0. Let $\pi_1, \pi_2, \dots, \pi_n$ be the hyperplanes orthogonal to $A_0A_1, A_0A_2, \dots, A_0A_n$ through $A_1, A_2, \dots, A_n$ respectively, and let C be the intersection point of $\pi_1, \pi_2, \dots, \pi_n$. If A_* and C_* are the orthogonal projections of A_0 and C respectively into α_0, then the midpoint of $\overline{A_*C_*}$ is the circumcenter of Δ_0.*

Proof. We proceed by induction on n.

For $n = 2$, the result is given by the Euclidean version of Theorem 1.2. We omit the proof.

Assume that the result is true for simplices in Euclidean $(n-1)$-dimensional space.

Let Δ be the simplex in Euclidean n-dimensional space of the statement. Let M be the midpoint of $\overline{A_*C_*}$. For $i = 1, 2, \dots, n$ let Δ'_i be the opposite face of A_i in

Δ_0, and let us denote $\pi_i' = \pi_i \cap \alpha_0$. Since π_i is orthogonal to A_0A_i, the line A_*A_i is orthogonal to π_i' for $i = 1, 2, \ldots, n$.

Let α_1 be the hyperplane containing $A_0, A_2, \ldots, A_n$, and consider $\alpha_1' = \alpha_0 \cap \alpha_1$. Since $\pi_1, \pi_2, \ldots, \pi_n$ intersect in a point in Euclidean n-dimensional space, the intersection of the $(n-2)$-dimensional subspaces $\pi_2' \cap \ldots \cap \pi_n'$ is a point C_1 of the $(n-1)$-dimensional space α_0. The line CC_1 coincides with $\pi_2 \cap \ldots \cap \pi_n$. Hence, it is orthogonal to α_1. This implies that C_*C_1, the orthogonal projection of CC_1 into α_0, is orthogonal to α_1'.

Let A_*^1 and C_*^1 be the orthogonal projections of A_* and C_*, respectively, into α_1'. Note that C_*^1 is also the orthogonal projection of C_1 into α_1'. Applying the induction hypothesis to the simplex in α_0 with vertices $A_*, A_2, \ldots, A_n$, the midpoint of $\overline{A_*^1C_*^1}$ is the circumcenter of Δ_1'. Therefore, the orthogonal projection of M into α_1' is the circumcenter of Δ_1', and this implies that M is equidistant from $A_2, \ldots, A_n$. Proceeding in the same way, we conclude that M is equidistant from $A_1, A_2, \ldots, A_n$, and M is the circumcenter of Δ_0. □

The three-dimensional version of this theorem is illustrated in Fig. 10. We have checked that Theorem 5.1 is not valid in the 3-dimensional hyperbolic and elliptic cases. However, there are geometric constructions that are equivalent in Euclidean geometry but non-equivalent in the non-Euclidean case, and this implies that there are Euclidean concepts that have multiple non-Euclidean interpretations. For instance, it is well-known that the Euler line of a triangle, as it is usually defined in the Euclidean plane, does not exist in general for non-Euclidean triangles. Nevertheless, alternative definitions of the circumcenter and the barycenter of a triangle, different to the standard ones but equivalent to them in Euclidean geometry, can be given in such a way that the Euler line *does exist* for any triangle in the hyperbolic and elliptic planes [1, 15].

Question 5.2. Is there a different formulation of Theorem 5.1 which is valid also in the non-Euclidean cases?

It must be noted that in Euclidean n-space the set of points $A_0, A_1, \ldots, A_n, C$ of the statement of Theorem 5.1 is *diametrically cyclic*, in the sense that all these points lie in an $(n-1)$-dimensional sphere in $\mathbb{R}^n$ in which A_0 and C are antipodal points, while this is not the case in the non-Euclidean context.

On the other hand, we have also tried to find an n-dimensional generalization of Theorem 4.6 without success.

Question 5.3. Is there an n-dimensional (Euclidean or non-Euclidean) version of Theorem 4.6?

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