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A Framework for Control System Design Subject to Average Data-Rate Constraints

Eduardo I. Silva, Milan S. Derpich and Jan Østergaard

Abstract

This paper studies discrete-time control systems subject to average data-rate limits. We focus on a situation where a noisy linear system has been designed assuming transparent feedback and, due to implementation constraints, a source-coding scheme (with unity signal transfer function) has to be deployed in the feedback path. For this situation, and by focusing on a class of source-coding schemes built around entropy coded dithered quantizers, we develop a framework to deal with average data-rate constraints in a tractable manner that combines ideas from both information and control theories. As an illustration of the uses of our framework, we apply it to study the interplay between stability and average data-rates in the considered architecture. It is shown that the proposed class of coding schemes can achieve mean square stability at average data-rates that are, at most, 1.254 bits per sample away from the absolute minimum rate for stability established by Nair and Evans. This rate penalty is compensated by the simplicity of our approach.

Index Terms

Networked control systems, average data-rate, signal-to-noise ratio, perfect reconstruction.

I. INTRODUCTION

PRACTICAL control systems often use non-transparent communication links and, thus, communication constraints arise [1]. Such constraints include random delays, data-loss and data-rate limits (quantization) [23], [36], [41]. This paper focuses on average data-rate constraints.

It might be argued that the communication capacity of modern networks is in general sufficiently large, so as to make quantization issues irrelevant (see, e.g., [23], [41]). However, there exist situations where the communication resources assigned to a particular relevant control signal are limited and, hence, quantization effects become important [36]. Quantization is a highly non-linear operation on signals and, accordingly, it is hard to analyze [21].

In control theory, quantization has usually been treated as an undesirable effect that should be compensated for (see, e.g., [32]). This stands in contrast to the perspective adopted by information theory, where quantization is considered as an integral part of systems (see, e.g., [8], [45]). This line of reasoning has recently been brought to the control arena (see, e.g., [34], [39], [40], [57], [62]). (An alternative view of quantization, closer to non-linear control theory than to information theory, has also been developed inside the control community; see, e.g., [7], [9], [13], [37].)

In a quantized discrete-time control framework, a key problem is being able to characterize the minimal average data-rate (in, e.g., bits per sample) that allows one to achieve a given control objective. In the context of noiseless digital channels, i.e., channels that transmit data without errors or delays at a constrained rate, this question is related to rate-distortion theory

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(i.e., lossy source-coding problems; see, e.g., [4], [8], [17], [46]). The associated design problem is how to quantize a signal, with the smallest average data-rate, whilst achieving a prescribed degree of fidelity or performance. A typical performance measure is the mean square error, but other measures are also possible. For instance, [54] suggests discrete measures to address *black-and-white* control problems such as, e.g., stabilizability and observability.

Standard results in information theory (and in particular in rate-distortion theory) rely upon coding arbitrarily long sequences which incur arbitrarily long time delays. In addition, most of the general results on rate-distortion theory do not take stability nor causality into account [4], [46]. It thus becomes clear that standard rate-distortion theory is not useful to deal with control problems. Some progress has been made in the information theory community towards a causal rate-distortion theory, but most results use coding schemes that, even though causal, allow for arbitrary delays [25], [38]. Only recently, [11] established upper bounds on the zero-delay causal rate distortion function for Gaussian stationary sources. However, the best bounds provided in [11] are of algorithmic nature, and derived for open loop systems. Thus, stability issues are not addressed in [11]. This is also the case of the results in [5], where sequential quantization of Markov sources is addressed.

The discussion in the previous paragraph makes the work documented in [34], [55], [60] specially relevant, even though the focus in those works lies only on stability. The first results that pointed out that there exists a data-rate under which (memoryless) quantized control cannot keep the state of a noiseless plant bounded were presented in [3], [60]. The results were later extended in [33], [55] using encoders and decoders with memory, and adaptive quantizer scaling policies (so-called *zooming* techniques [7], [60]). A landmark result was published in [34], where the authors focus on noisy plant models subject to mild conditions on the noise sources statistics. It was shown in [34] that it is possible to find causal coders, decoders and controllers such that the resulting closed loop system is mean square stable if and only if the average data-rate (in bits per sample), say $\mathcal{R}$, satisfies

$$\mathcal{R} > \sum_{i=1}^{n_p} \log_2 |p_i|, \tag{1}$$

where p_i denotes the ith unstable plant pole. The above result establishes a fundamental limitation in networked control systems (NCSs) closed over digital channels, when the problem of interest is mean square stabilization. Bounds similar to (1) arise as solutions to different problems (e.g., observability, deterministic stability, etc.) and under different assumptions on the channels and coding schemes (see, e.g., [14], [33], [36], [54], [55]). Indeed, the quantity on the right hand side of (1) is a fundamental measure of the difficulty of stabilizing a system, as discussed in [35], [40].

All constructions known to date that achieve stability at average data-rates arbitrarily close to (1) use complex non-linear time-varying coding schemes that, in principle, have infinite memory. The consideration of coding schemes with limited (or no memory) is much more involved [36] and no explicit solutions are currently available (see also Section VI in [55]). An alternative simpler approach has been presented in [61], although for the scalar plant case only.

Almost all the work referred to above focuses on stability questions only. A performance-oriented approach has been pursued in [36], [57]. In that work, conditions for separation and certainty equivalence have been investigated in the context of quadratic stochastic problems for fully observed plants with data-rate constraints in the feedback path. If the encoder has a specific recursive structure, then certainty equivalence and a *quasi*-separation principle hold [36]. This result is interesting, but [36] does not give a computable characterization of the optimal encoding policies. A similar drawback is shared by the results reported in [57]. In that work, performance related results are expressed in terms of the so-called sequential rate-distortion function (a rate-distortion function with causality constraints), which is difficult to compute in general. For fully observed Gaussian first order autoregressive systems, [57] provides an expression for the sequential rate-distortion function.

However, it is not clear from the results in [57] whether or not the sequential rate-distortion function is operationally tight (see Section IV-C in [57]). Related work can be found in [62], where estimation problems are addressed.

The main contribution of this paper is a novel, though restricted, bridge between information theory and control theory. The link is restricted in that it holds for a specific class of source-coding schemes based on entropy coded dithered quantizers (see, e.g., [63], [64], [66]). Nevertheless, the link is useful, enabling one to address control system design problems subject to average data-rate constraints in a systematic manner [47]. Our approach is constructive and based upon standard building blocks. As such, it yields bounds on average data-rates that are guaranteed to be achievable with conceptually simple source-coding schemes. An additional feature of our approach is that it does not rely on asymptotic approximations (e.g, high-rate or high vector dimensions assumptions).

As both a motivation for our approach, and also to illustrate a possible application, we consider a problem where a noisy linear system has been designed assuming transparent feedback and, due to implementation constraints, a source-coding scheme with unity signal transfer function has to be deployed in the feedback path. For this situation, we discuss how to obtain bounds on the minimal average data-rate that allows one to attain a certain performance level, and also provide a detailed characterization of the interplay between stability and average data-rates. It is shown that the proposed class of coding schemes can achieve mean square stability at average data-rates that are guaranteed to be at most 1.254 bits per sample away from the absolute minimum in (1). This rate penalty is compensated by the simplicity of our approach.

A key enabling result in the paper is that, when the proposed class of source-coding schemes is employed, average data-rate constraints can be enforced by imposing signal-to-noise ratio (SNR) constraints in a related analog additive noise channel (see also [6], [51]). Our results thus establish a formal relationship between SNR constraints and average data-rates constraints in noiseless digital channels. As such, our work goes beyond [6], [51] where no such relationship is presented. Early versions of the results reported in the paper can be found in [47], [50].

The remainder of the paper is organized as follows: Section II presents notation. Section III describes the setup considered in the paper. Section IV presents a lower bound on average data-rates that motivates the remainder of the paper. Section V introduces the class of source-coding schemes considered in the paper and relates average data-rate limits to SNR constraints. Section VI focuses on the interplay between stability and average data rate constraints. Section VII draws conclusions. For ease of reference, the Appendix presents basic information-theoretic facts.

II. NOTATION

$\mathbb{R}, \mathbb{R}_0^+, \mathbb{R}^+, \mathbb{N}_0$ stand for the reals, the non-negative reals, the strictly positive reals, and the non-negative integers, respectively. $|x|$ denotes the magnitude of the complex scalar x; X^H denotes the conjugate transpose of the matrix X. $\mathcal{RH}_2$ is the set of all strictly proper and stable real rational transfer functions, and $\mathcal{U}_\infty$ is the set of all stable, biproper and minimum phase real rational transfer functions. The usual norm in $\mathcal{L}_2$ is written $||\cdot||_2$ [58]. If x is an asymptotically wide sense stationary (wss) process [2], then $\sigma_x^2, S_x, \Omega_x$ denote its stationary variance, its stationary power spectral density (PSD), and the corresponding spectral factor, respectively. If x is a discrete time signal, then $x(k)$ is the kth sample, and x^k is shorthand for $x(0), \cdots, x(k)$. If $\{\mathbb{X}(k)\}_{k\in\mathbb{N}_0}$ is a family of sets, then $\mathbb{X}^k \triangleq \mathbb{X}(0) \times \cdots \times \mathbb{X}(k)$. If $\mathbb{X}$ is a set that does not depend on any index, then $\mathbb{X}^k \triangleq \mathbb{X} \times \cdots \times \mathbb{X}$ (k times), as usual. We write $x \perp\!\!\!\perp y$ if and only if (iff) x and y are independent. We write $x - y - z$ iff x, y and z form a Markov chain (see Appendix). $\mathcal{E}\{\cdot\}$ and $\mathcal{P}\{\cdot\}$ stand for the expectation and probability of $(\cdot)$, respectively. Definition of information-theoretic quantities and related notation is given in the Appendix.
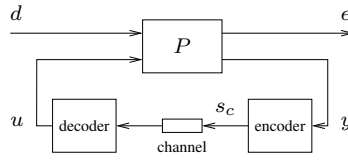
Fig. 1. NCS closed over a digital channel.

III. PROBLEM SETUP

This paper focuses on the NCS of Figure 1, where P is a given linear time-invariant (LTI) system such that

$$P \triangleq \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \quad \begin{bmatrix} e \\ y \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} d \\ u \end{bmatrix},$$

d models exogenous signals, e is a signal related to closed loop performance, y is a signal available for measurement, and u is a control input. We assume that P has been designed so as to achieve satisfactory performance when $u = y$. The feedback path, however, comprises an error free digital channel. Hence, the quantization of the signal y becomes mandatory. This task is performed by an encoder that outputs the sequence of binary symbols s_c. Once these symbols are available at the receiving end, a decoder generates the signal u that is fed back to P.

The situation described above arises naturally if, for example, P corresponds to the interconnection of an LTI plant and an LTI controller that has been designed without taking into account data-rate limits in the feedback path.

Throughout this paper we assume that the following holds:

Assumption 3.1:

(a) P is a proper real rational transfer function, P_{22} is single-input single-output and strictly proper, $P_{12} \neq 0$ and $P_{21} \neq 0$. If $u = y$, then the feedback system of Figure 1 is internally stable and well-posed.

(b) The initial state of P, say x_o, is a second order random variable, and d is a second order wss process with spectral factor $\Omega_d \in \mathcal{U}_\infty$.■

Assuming that the loop is stable when $u = y$ is consistent with our setup where P has been designed supposing transparent feedback. Assuming P_{22} to be strictly proper guarantees the well-posedness of the NCS of Figure 1 for all causal mappings between y and u. This assumption can be removed at the expense of additional care. However, removing the constraint of P_{22} being single-input single-output, requires additional effort for our approach to be useful.

The remainder of this paper aims at building a framework to study the interplay between the average rate at which the channel symbols s_c are transmitted, and the performance and stability of the NCS of Figure 1. To that end, we begin by first establishing a general lower bound on average data-rates in feedback systems.

IV. AVERAGE DATA-RATE LIMITS

A. Background

The use of digital communication systems requires the coding of analog signals [8]. Shannon's separation theorem states that this coding process can be separated into two problems [45]: source coding and channel coding (see also [17]). Source coding deals with the representation of continuous symbols using a countable alphabet and, as such, involves quantization [21]. On the other hand, channel coding focuses on the reliable and efficient communication of digital data over an underlying analog channel. We note that separation holds, and is useful, for point-to-point communications where causality and delays are not
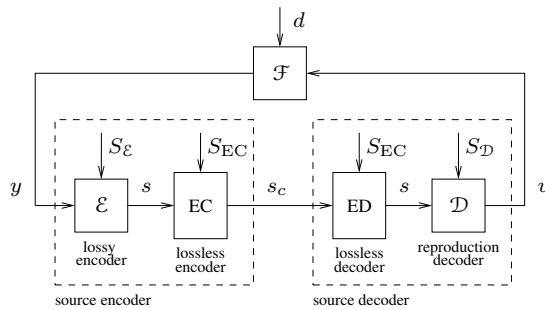
Fig. 2. General source-coding scheme used within a feedback loop.

an issue. If causality constraints are imposed, then separation does not hold in general (see also [57]). Nonetheless, the study of causal source-coding problems in isolation constitutes a key open problem in information theory [5], [11], [25].

The study of optimal source-coding (or quantization) problems is the subject of rate-distortion theory. Rate-distortion theory does not take channel coding into account, and assumes an idealized digital link between the sending and receiving ends [4], [21]. In this paper, we adopt a purely source-coding perspective[1] and consider source encoders whose output symbols s_c have a variable instantaneous length, but a bounded average length (see also [21]). We note, however, that guaranteeing bounded average data-rates does not guarantee bounded instantaneous data-rates [8], [21]. (Conditions for this to happen are explored in [22].)

Without loss of generality, we consider source-coding schemes with the structure depicted in Figure 2 [21]. In that figure, $\mathcal{E}$ is a *lossy encoder*, $\mathcal{D}$ a *reproduction decoder*, and the blocks EC and ED form a lossless encoder-decoder pair (also called *entropy coder* (EC) - *entropy decoder* (ED) pair; see, e.g., Chapter 5 in [8]). The lossy encoder maps continuously valued random variables into a countable set of symbols. These symbols are then mapped by the EC into a countable set of prefix-free binary words that, in general, changes at every time instant [8].[2] At the receiving end, the ED recovers the lossy encoder output symbols from the binary words generated by the EC, and the reproduction decoder maps the recovered symbols back into real numbers. A precise characterization of $\mathcal{E}, \mathcal{D}$, EC and ED is provided below.

B. General source-coding schemes

In this paper we focus on single-input single-output source-coding schemes within feedback loops, as depicted in Figure 2. Accordingly, we consider lossy encoders $\mathcal{E}$, reproduction decoders $\mathcal{D}$ and EC-ED pairs that are causal and, moreover, operate on a sample by sample basis, without delay. We also assume that side information is available at both the encoder and decoder sides. The new side information that becomes available at time instant k is denoted by $S_{\mathcal{E}}(k)$, for the source encoder, and by $S_{\mathcal{D}}(k)$ for the source decoder. Such side information is contained in suitably defined sets $\mathcal{S}_{\mathcal{D}}(k)$ and $\mathcal{S}_{\mathcal{E}}(k)$, where $S_{\mathcal{E}}(k) \in \mathcal{S}_{\mathcal{E}}(k)$ and $S_{\mathcal{D}}(k) \in \mathcal{S}_{\mathcal{D}}(k)$. We also define the set $\mathcal{S}_{\mathrm{EC}}(k) \triangleq \mathcal{S}_{\mathcal{E}}(k) \cap \mathcal{S}_{\mathcal{D}}(k)$, which contains the common side information that becomes available at both the encoder and decoder sides at instant k.

In Figure 2, the dynamic system $\mathcal{F}$ is assumed to be such that its (scalar) output y satisfies

$$y(k) = \mathcal{F}_k(u^{k-1}, d^k, x_o), \tag{2}$$

---

[1]Thus, the encoder and decoder in Figure 1 become source encoder and source decoder, respectively.

[2]The rationale behind an EC is to assign short binary words to frequent lossy encoder output symbols, and long words to infrequent ones, so as to reduce the average length of the symbols s_c sent through the channel.

where $\mathcal{F}_k : \mathbb{R}^k \times \mathbb{R}^{(k+1)n_d} \times \mathbb{R}^{n_o} \mapsto \mathbb{R}$ is a (possibly time varying) deterministic mapping, $x_o \in \mathbb{R}^{n_o}$ is the initial state of $\mathcal{F}$, u is the (scalar) input, and d, with $d(k) \in \mathbb{R}^{n_d}$, is an exogenous random process. We also characterize the output s of the lossy encoder via

$$s(k) = \mathcal{E}_k(y^k, S_{\mathcal{E}}^k), \tag{3}$$

where y is the input to the encoder, $S_{\mathcal{E}}(k)$ is as before, $\mathcal{E}_k : \mathbb{R}^{k+1} \times \mathcal{S}_{\mathcal{E}}^k \mapsto \mathcal{A}_s$ is a (possibly time varying) deterministic mapping, and $\mathcal{A}_s$ is a fixed countable set. The symbols s are then used by the EC to construct the binary words s_c via

$$s_c(k) = \mathcal{H}_k\left(s^k, S_{\mathrm{EC}}^k\right), \tag{4}$$

where $S_{\mathrm{EC}}(k) \in \mathcal{S}_{\mathrm{EC}}(k)$, $\mathcal{S}_{\mathrm{EC}}(k)$ is as before, $\mathcal{H}_k : \mathcal{A}_s^{k+1} \times \mathcal{S}_{\mathrm{EC}}^k \to \mathcal{A}(k)$ is a time varying deterministic mapping, and $\mathcal{A}(k)$ is a countable set of prefix-free binary words. The output s_c of the EC is transmitted to the receiving end assuming ideal digital communication (consistent with the source-coding point of view adopted in this paper). Once $s_c(k)$ becomes available at the receiving end, the ED recovers s via

$$s(k) = \mathcal{H}_k^{-1}\left(s_c^k, S_{\mathrm{EC}}^k\right), \tag{5}$$

where $\mathcal{H}_k^{-1} : \mathcal{A}^k \times \mathcal{S}_{\mathrm{EC}}^k \to \mathcal{A}_s$ is a time varying mapping that satisfies

$$\mathcal{H}_k^{-1}(\mathcal{H}_0(s(0), S_{\mathrm{EC}}(0)), \cdots, \mathcal{H}_k(s^k, S_{\mathrm{EC}}^k), S_{\mathrm{EC}}^k) = s(k) \tag{6}$$

for any $s^k \in \mathcal{A}_s^{k+1}$, any $S_{\mathrm{EC}}^k \in \mathcal{S}_{\mathrm{EC}}^k$, and any $k \in \mathbb{N}_0$. Condition (6) reinforces the fact that the EC-ED pairs considered here operate in real time, without delay.[3] Finally, the reproduction decoder constructs its output u via

$$u(k) = \mathcal{D}_k(s^k, S_{\mathcal{D}}^k), \tag{7}$$

where $S_{\mathcal{D}}(k)$ is as before, and $\mathcal{D}_k : \mathcal{A}_s^{k+1} \times \mathcal{S}_{\mathcal{D}}^k \mapsto \mathbb{R}$ is a (possibly time varying) deterministic mapping.

Before the reception of $s_c(k)$, both s^{k-1} and S_{EC}^k are available at the decoder side. It thus follows that the expected length $R(k)$, measured in nats,[4] of any binary description $s_c(k)$ of the lossy encoder output symbol $s(k)$ satisfies (see [8, Chapter 5], [45] and also [24])

$$H(s(k)|s^{k-1}, S_{\mathrm{EC}}^k) \le R(k), \tag{8}$$

where $H(\cdot|\cdot)$ denotes conditional entropy (see Appendix). The gap between both sides of (8) depends on how efficient the EC is at encoding s. It is known that there exist ECs such that [8]

$$0 \le R(k) - H(s(k)|s^{k-1}, S_{\mathrm{EC}}^k) < \ln 2. \tag{9}$$

That is, the gap in (8) is smaller than $\ln 2$ nats (1 bit) when suitable encoding policies are employed (e.g., Huffman[5] coding [8]).

In this paper we focus on the time average of $R(k)$:

Definition 4.1: The average data-rate of the source-coding scheme described above, measured in nats per sample, is defined via

$$\mathcal{R} \triangleq \lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} R(i), \tag{10}$$

---

[3]Equation (6) is a consequence of the fact that the EC-ED pair acts as a transparent link between the lossy encoder output alphabet $\mathcal{A}_s$ and the binary words in $\mathcal{A}(k)$ [8].

[4]1 nat equals $\ln 2$ bits.

[5]For Huffman coding the gap is actually upper bounded by $\min\{1, P_1 + 0.086\}$, where P_1 is the conditional probability of the most likely symbol of the alphabet of $s(k)$, given $(s^{k-1}, S_{\mathrm{EC}}^k)$ [16].

where $R(i)$ is the expected length of $s_c(i)$. ∎

C. Lower bounds on average data-rates

We will now study a lower bound on $\mathcal{R}$ that depends only on the joint statistics of the source encoder input y and its output u. This bound will play a key role in the remainder of this paper.

Our derivations require the following assumption:

Assumption 4.1: The systems $\mathcal{F}$, $\mathcal{E}$, EC, ED and $\mathcal{D}$ in Figure 2 are causal, described by (2)-(7), and such that $S_\mathcal{D} \perp\!\!\!\perp (d, x_o)$. ∎

Assumption 4.1 can be thought as being a "fairness" assumption. Indeed, it is consistent with the reasonable requirement that the source decoder uses only past and present symbols, and side information not related to the message being sent, to construct the current output value. In other words, we assume that the channel is the only link from y to u.

Definition 4.2: The source-coding scheme described by (2)-(7) is said to have an invertible reproduction decoder $\mathcal{D}$ (in short, an invertible decoder) iff, $\forall i \in \mathbb{N}_0$, there exists a deterministic mapping g_i such that $s^i = g_i(u^i, S_\mathcal{D}^i)$. ∎

If a source-coding scheme has an invertible decoder, then knowledge of $(u^i, S_\mathcal{D}^i)$ is equivalent to knowledge of $(s^i, S_\mathcal{D}^i)$. The generality of source-coding schemes with invertible decoder is examined next:

Lemma 4.1: Consider any source-coding scheme described by (2)-(7). Define $\check{R}(k) \triangleq H(s(k)|\ s^{k-1}, S_{\text{EC}}^k)$. If, $\forall k \in \mathbb{N}_0$, $u(k) = u_o(k)$, $\check{R}(k) = \check{R}_o(k)$, and the corresponding decoder is not invertible, then there exists another causal source-coding scheme, with an invertible decoder, such that $u(k) = u_o(k)$ and $\check{R}(k) \le \check{R}_o(k)$, $\forall k \in \mathbb{N}_0$.

Proof: Assume that the encoder-decoder pair $(\mathcal{E}, \mathcal{D})$ is such that it has been possible to use $S_\mathcal{D}^i$ to recover s^i from u^i, for all $i \le k - 1$ (such assumption is not needed at time instant $k = 0$). If at time instant k $\mathcal{D}$ cannot be inverted (i.e., if there exists no deterministic mapping g_k such that $s^k = g_k(u^k, S_\mathcal{D}^k)$), then there exist $s_1, s_2 \in \mathcal{A}_s$, $s_1 \ne s_2$, such that $u(k) = \mathcal{D}(s_1, s^{k-1}, S_\mathcal{D}^k) = \mathcal{D}(s_2, s^{k-1}, S_\mathcal{D}^k)$. Denote by p_i the conditional probability of the output of $\mathcal{E}$ being s_i at time instant k, given $(s^{k-1}, S_{\text{EC}}^k)$. Consider now another encoder-decoder pair $(\bar{\mathcal{E}}, \bar{\mathcal{D}})$ that behaves like $(\mathcal{E}, \mathcal{D})$, except for the fact that $\mathcal{E}$ outputs s_1 instead of either s_1 or s_2 at time instant k. At time instant k, $\bar{\mathcal{E}}$ outputs the value s_1 with conditional probability $p_1 + p_2$, given $(s^{k-1}, S_{\text{EC}}^k)$. Thus,

$$
\begin{aligned}
\check{R}(k)|_{(\mathcal{E}, \mathcal{D})} &= \check{R}_o(k) \\
&\overset{(a)}{=} -\sum_{i \notin \{1,2\}} p_i \ln p_i - p_1 \ln p_1 - p_2 \ln p_2 \\
&\overset{(b)}{\ge} -\sum_{i \notin \{1,2\}} p_i \ln p_i - (p_1 + p_2) \ln (p_1 + p_2) \\
&\overset{(c)}{=} \check{R}(k)|_{(\bar{\mathcal{E}}, \bar{\mathcal{D}})},
\end{aligned}
$$

where (a) follows from the definition of $\check{R}(k)$ and that of entropy, (b) follows from Jensen's inequality, and (c) follows from the definition of $\check{R}(k)$ and that of $(\bar{\mathcal{E}}, \bar{\mathcal{D}})$.

By repeating the above procedure until there are no two symbols of $\mathcal{A}_s$ that are mapped into the same value $u(k)$ at time instant k, one constructs a source-coding scheme where knowing $(u^i, S_\mathcal{D}^i)$ is equivalent to knowing $(w^i, S_\mathcal{D}^i)$, $u(i) = u_o(i)$, and $\check{R}(i) \le \check{R}_o(i)$, $\forall i \le k$. The result follows upon repeating the above for every $k \in \mathbb{N}_0$. ∎

Given (9), it follows from Lemma 4.1 that one can always focus on source-coding schemes with invertible decoders, without

loss of any generality.[6]

The next result will be used to prove the main result in this section.

Lemma 4.2: Consider a source-coding scheme inside a feedback loop, as depicted in Figure 2. If Assumption 4.1 holds and the decoder is invertible, then the Markov chain $S_\mathcal{D}^i - u^{i-1} - y^i$ holds and, conditioned upon $(s^{i-1}, S_\mathcal{D}^i)$, $y^i - s(i) - u(i)$ is also true.

Proof: Given u^{i-1}, it follows from (2) that there exists a deterministic mapping T_i such that $y^i = T_i(d^i, x_o)$. Since $S_\mathcal{D} \perp\!\!\!\perp (d, x_o)$, it immediately follows that y^i and $S_\mathcal{D}^i$ are independent upon knowledge of u^{i-1}, thus proving our first claim. The second claim is immediate upon noting that $u(i)$ depends deterministically upon $(s^i, S_\mathcal{D}^i)$. ∎

We are now in a position to state the main result of this section.

Definition 4.3: The directed mutual information rate across a system with random input r and random output s (or between two random processes r and s) is defined via [31]

$$I_\infty(r \to s) \triangleq \lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} I(s(i); r^i | s^{i-1}), \tag{11}$$

where $I(\cdot; \cdot | \cdot)$ denotes conditional mutual information (see Appendix). ∎

Theorem 4.1: Consider a source-coding scheme inside a feedback loop, as depicted in Figure 2. If Assumption 4.1 holds and the decoder is invertible, then $\mathcal{R} \geq I_\infty(y \to u)$.

Proof: Using (8) we have[7]

$$\begin{aligned}
R(i) &\overset{(a)}{\geq} H(s(i)|s^{i-1}, S_\mathcal{D}^i) - H(s(i)|s^{i-1}, S_\mathcal{D}^i, y^i) \\
&= I(s(i); y^i | s^{i-1}, S_\mathcal{D}^i) \\
&\overset{(b)}{=} I(s(i); y^i | u^{i-1}, S_\mathcal{D}^i) \\
&\overset{(c)}{\geq} I(u(i); y^i | u^{i-1}, S_\mathcal{D}^i) \\
&\overset{(I3)}{=} I(u^i, S_\mathcal{D}^i; y^i) - I(u^{i-1}, S_\mathcal{D}^i; y^i) \\
&\overset{(M2)}{\geq} I(u^i; y^i) - I(u^{i-1}, S_\mathcal{D}^i; y^i) \\
&\overset{(I3)}{=} I(u(i); y^i | u^{i-1}) - I(S_\mathcal{D}^i; y^i | u^{i-1}) \\
&\overset{(d)}{=} I(u(i); y^i | u^{i-1}), \tag{12}
\end{aligned}$$

where (a) follows from Property $(H1)$ in the Appendix and the fact that $\mathcal{S}_{\text{EC}}(k) \subseteq \mathcal{S}_\mathcal{D}(k)$, (b) follows from the fact that the decoder is invertible, (c) follows from $(M3)$, the fact that the decoder is invertible, and the second claim of Lemma 4.2, and (d) follows from $(I4)$ and the first claim of Lemma 4.2. The result now follows using (10), (11) and (12). ∎

Theorem 4.1 states that, when causality constraints are imposed, directed mutual information rate across a source-coding scheme serves as a lower bound on the associated average data-rate. The result relates a physical quantity (average data-rate) to an information-theoretic quantity (directed mutual information rate). Theorem 4.1 also suggests that the appropriate information-theoretic definition of average data-rates in causal source-coding schemes is the directed mutual information rate. However, showing that the *infimum*, over all joint input and output distributions that satisfy a causality constraint, of the directed mutual

---

[6]Lemma 4.1 also implies that, if the average data-rate across a causal source-coding scheme is to be minimized, then it is not suboptimal to focus on source-coding schemes with invertible decoders.

[7]In this paper, $x \overset{(Pn)}{\geq} y$, etc., means that Property (Pn) in the Appendix implies $x \geq y$. Uncommented steps follow from the definitions.

information rate across a source-coding scheme provides an operationally tight lower bound on the corresponding average data-rate, remains an open problem (see also [11]).

To our knowledge, Theorem 4.1 provides, for the first time, a characterization of the relationship between directed mutual information rate and the operational rate of source-coding schemes within feedback loops. The result in the literature that is closest to Theorem 4.1 is Theorem 2 in [66]. However, that result is derived for entropy coded dithered quantizers only (see Section V-B), as opposed to the general causal source-coding schemes considered here. Related results are Lemma 4.8.1 in [53] and Theorem B.1.1 in [27] where feedback data processing inequalities are presented. However, those results do not focus on operational data-rates, and assume no feedback between the signals at the physical ends of the processing chains.

Other relevant and related works are [28], [56]. In [28], the authors study fundamental inequalities involving directed mutual information rates across channels within feedback loops, and presents Bode-like fundamental limitations that arise due to finite capacity communication (see also [29]). On the other hand, [56] establishes a relationship between operational data-rates and directed mutual information rate from a *channel coding perspective*. In that work, the authors show that the *supremum*, over all joint channel input and output distributions that satisfy a causality constraint, of the directed mutual information rate across a channel equals Shannon's capacity with feedback for that channel. Despite all the work referred to above, no relationship (besides that of Theorem 4.1) between average operational data-rates and directed mutual information rate from a *source-coding perspective*, and valid in general settings, is currently available in the literature.

V. A CLASS OF SOURCE CODING SCHEMES

This paper aims at establishing a bridge between control and information theories, when a specific class of source-coding schemes is employed. This section presents such class.

A. Independent source-coding schemes

In order to obtain a simple (yet useful) framework for the study of the NCS of Figure 1, we will focus on the following class of source-coding schemes:

Definition 5.1: A source-coding scheme is said to be independent iff Assumption 4.1 holds, its reproduction decoder is invertible, and the (coding or quantization) noise sequence n, defined via

$$n \triangleq u - y,$$

obeys $n = \Omega q$, where q is a second order zero-mean i.i.d. sequence, $q(k)$ has finite differential entropy, $q \perp\!\!\!\perp (d, x_o)$, and the filter $\Omega \in \mathcal{U}_\infty$ has a deterministic initial state; see Figure 3(a).∎

The class of independent source-coding schemes is restrictive. However, it is a sensible choice when data-rate constraints arise in systems that have already been designed to perform satisfactorily in the absence of quantization. In such cases, which include the situation of interest in this paper (see Section III), it is desirable to introduce quantization effects in an additive fashion so as not to alter the nominal design relations (see also [20]). We will describe a practical independent source-coding scheme in Section V-B.

We begin our study of independent source-coding schemes by noting that the following holds:

Lemma 5.1: Any independent source-coding scheme can be written as shown in Figure 3(b), where v and w are auxiliary signals, q is as in Definition 5.1, and $A \in \mathcal{U}_\infty$ and $F \in \mathcal{RH}_2$ are auxiliary filters with deterministic initial states, such that $(1 - F) \in \mathcal{U}_\infty$. Moreover, in Figure 3(b), $I_\infty(y \to u) = I_\infty(v \to w)$.

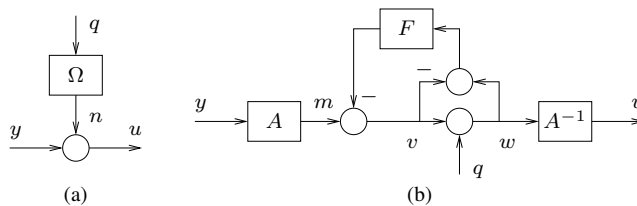Fig. 3. (a) Independent source-coding scheme and (b) equivalent rewriting.

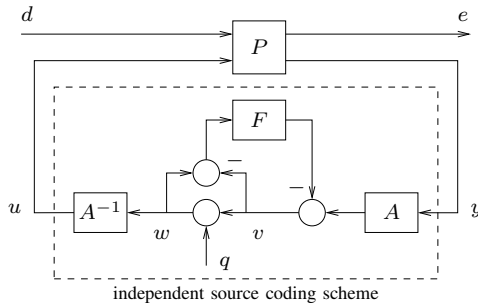independent source coding scheme

Fig. 4. Considered NCS closed over an independent source-coding scheme.

Proof: Our first claim follows upon defining $\Omega \triangleq A^{-1}(1 - F)$. To prove our second claim, we note that the assumptions on A and F imply that there exist deterministic mappings $g_1, \cdots, g_6$, with g_1 and g_2 invertible, such that (see Figure 3(b))

$u^i = g_1(w^i)$, $y^i = g_2(m^i)$, $y^i = g_3(v^i, u^{i-1})$, $v^i = g_4(y^i, u^{i-1})$, $u(i) = g_5(u^{i-1}, w(i))$ and $w(i) = g_6(u^i)$, $\forall i \in \mathbb{N}_0$. Hence,

$$
\begin{aligned}
I(w(i); v^i | w^{i-1}) &= I(w(i); v^i | u^{i-1}) \\
&\overset{(M3)}{=} I(w(i); y^i | u^{i-1}) \\
&\overset{(M3)}{=} I(u(i); y^i | u^{i-1}).
\end{aligned}
\tag{13}
$$

The proof is completed upon using (13) in (11). ∎

Since Lemma 5.1 holds, the system that arises when an independent source-coding scheme is employed in the feedback loop of Figure 1 can be written as shown in Figure 4. (Note that the error free digital channel of Figure 1 is embedded in the independent source-coding scheme of Figure 4. In Section V-B, we will make the channel explicit again.)

A key feature of independent source-coding schemes is that the directed mutual information rate across them can be bounded by the directed mutual information rate that would arise if all random sources were replaced by Gaussian ones. To be precise, we introduce the following definition:

Definition 5.2: Consider an LTI system with random inputs and random initial state. If x is a state, input, or output variable of the system, then x_G refers to the signal that would arise in the place of x, when all inputs and initial states are replaced by jointly Gaussian random variables (or processes) having the same first and second order (cross-)moments, and maintaining the same statistical dependence relationships, as in the original situation; x_G is called the Gaussian counterpart of x. ∎

Lemma 5.2: Consider the NCS of Figure 1, where the source-coding scheme is independent. If Assumption 3.1 holds, then

$$
I_\infty(v_G \to w_G) + h_\infty(w) - h_\infty(w_G) \le I_\infty(v \to w) \le I_\infty(v_G \to w_G) + D(q(k) || q_G(k)),
\tag{14}
$$

where v and w are the auxiliary variables introduced in Figure 3 (see Lemma 5.1), $h_\infty(\cdot)$ denotes entropy rate, and $D(\cdot || \cdot)$ denotes relative entropy (see Appendix). Equalities in (14) hold iff v and w are Gaussian.

Proof: The definition of q (resp. q_G) and the fact that there exists strictly causal feedback from y to u imply that $q(k)$ (resp. $q_G(k)$) is independent of v^k and w^{k-1} (resp. v_G^k and w_G^{k-1}). Exploiting the fact that $q(k)$ has finite differential entropy, one concludes that

$$
\begin{aligned}
I(w(k); & v^k|w^{k-1}) - I(w_G(k); v_G^k|w_G^{k-1}) \\
= \ & h(w(k)|w^{k-1}) - h(q(k) + v(k)|w^{k-1}, v^k) \\
& - h(w_G(k)|w_G^{k-1}) + h(q_G(k) + v_G(k)|w_G^{k-1}, v_G^k) \\
\overset{(h2,h1)}{=} \ & h(q_G(k)) - h(q(k)) + h(w(k)|w^{k-1}) \\
& - h(w_G(k)|w_G^{k-1}) \\
\overset{(D2)}{=} \ & D(q(k)||q_G(k)) \\
& - D(w(k)|w^{k-1}||w_G(k)|w_G^{k-1}).
\end{aligned}
\tag{15}
$$

The result now follows from Properties $(D1)$ and $(h2)$ in the Appendix, the definition of entropy rate, (15), (11), and the fact that both q and q_G are i.i.d.. ∎

Remark 5.1: If the conditions of Lemma 5.2 hold and, in addition, (d, x_o) is jointly Gaussian, then (see Chapter 5 in [47] and also [10])

$$
0 \leq I_\infty(v \to w) - I_\infty(v_G \to w_G) \leq D(q(k)||q_G(k)).
$$

Thus, if (d, x_o) is jointly Gaussian, then $I_\infty(v \to w)$ is minimized by choosing q to be Gaussian. ∎

The relevance of Lemma 5.2 lies in that the characterization of directed mutual information rate under Gaussianity assumptions is straightforward:

Theorem 5.1: Consider the NCS of Figure 1, where the source-coding scheme is independent. If Assumption 3.1 holds, then

$$
I_\infty(v \to w) \leq \frac{1}{4\pi} \int_{-\pi}^{\pi} \ln \frac{S_w}{\sigma_q^2} d\omega + D(q(k)||q_G(k))
\tag{16}
$$

$$
\leq \frac{1}{2} \ln(1 + \gamma) + D(q(k)||q_G(k)),
\tag{17}
$$

where

$$
\gamma \triangleq \frac{\sigma_v^2}{\sigma_q^2},
$$

S_w is the stationary PSD of w, σ_v^2 is the stationary variance of v, and σ_q^2 is the variance of q. Equality holds in (16) iff (d, x_o, q) is Gaussian, whereas equality holds in (17) iff S_w/σ_q^2 is constant for $\omega \in [-\pi, \pi]$. (The ratio γ defined in (17) will be referred to as the stationary signal-to-noise ratio (SNR) of the source-coding scheme.)

Proof: Proceeding as in the proof of Lemma 5.2 (see (15)) we conclude that

$$
\begin{aligned}
I_\infty(v_G & \to w_G) \\
& = \lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} \left(h(w_G(i)|w_G^{i-1}) - h(q_G(i)) \right) \\
& \overset{(h2,h3,R1)}{=} \frac{1}{4\pi} \int_{-\pi}^{\pi} \ln(2\pi e S_{w_G}) d\omega - \frac{1}{2} \ln(2\pi e \sigma_{q_G}^2),
\end{aligned}
\tag{18}
$$

where S_{w_G} and $\sigma_{q_G}^2$ are guaranteed to exists by Assumption 3.1 and the definition of q_G. Since $S_{w_G} = S_w$ and $\sigma_{q_G}^2 = \sigma_q^2$, our first claim follows from Lemma 5.2 and (18). Use of Jensen's inequality and the fact that $v(k) \perp\!\!\!\perp q(k)$ (see proof of Lemma 5.2) completes the proof. ∎

Theorem 5.1 provides explicit upper bounds on the directed mutual information rate across any independent source-coding scheme embedded in a stable and causal feedback loop. These bounds are, essentially, expressed in terms of the spectral characteristics of the auxiliary variables v and w in the scheme of Figure 3(b). Interestingly enough, there exists a one-to-one correspondence between the SNR γ of an independent source-coding scheme, and upper bounds on the directed mutual information rate across it. (Given Theorem 4.1, we can thus infer that there exists a link between the SNR of an independent source-coding scheme and the associated average data-rate. A precise characterization of such link will be given in Section V-B.)

We note that [12] also presents a relationship between directed mutual information rate and Bode-like integrals, when Gaussian distributions are assumed. Our result extends Theorem 4.6 in [12] to feedback loops with arbitrary disturbance and initial state distributions. Indeed, Theorem 4.6 in [12] can be recovered from the first inequality in (17) upon assuming (d, x_o, q) to be jointly Gaussian distributed.

We end this section by showing that, for any given independent source-coding scheme, there exists another independent source-coding scheme, with the same noise color Ω and the same directed mutual information rate across it, such that the gap between the right hand side of (16) and (17) can be made arbitrarily small:

Theorem 5.2: Consider the NCS of Figure 1, where the source-coding scheme is independent and has a fixed noise source q. Suppose that Assumption 3.1 holds and define $\phi \triangleq \frac{1}{4\pi} \int_{-\pi}^{\pi} \ln \frac{S_w}{\sigma_q^2} d\omega$. If the choice $(A, F) = (A_0, F_0) \in \mathcal{U}_\infty \times \mathcal{RH}_2$ is such that $\Omega = \Omega_0$, $I_\infty(v \to w) = I_0$, and $\phi = \phi_0$, then, for any arbitrarily small $\delta > 0$, there exist a choice of filters, $(A, F) = (A_1, F_1) \in \mathcal{U}_\infty \times \mathcal{RH}_2$, such that $\Omega = \Omega_0$, $I_\infty(v \to w) = I_0$, $\phi = \phi_0$ and, in addition, $\frac{1}{2} \ln(1 + \frac{\sigma_v^2}{\sigma_q^2}) = \phi_0 + \delta$. Moreover, if $S_w(e^{j\omega}) \neq 0 \ \forall \omega$ when $(A, F) = (A_0, F_0)$, then one can choose (A_1, F_1) so as to whiten w and, thus, $\delta = 0$ is achievable.

Proof: Denote by w_i and v_i the signals w and v that arise when $(A, F) = (A_i, F_i)$, $i \in \{0, 1\}$. Write $S_{w_0} = |\Omega_{w_0}|^2$, where Ω_{w_0} is stable, biproper, and has all its zeros in $\{z \in \mathbb{C} : |z| \leq 1\}$. Denote by $c_1, \cdots, c_{n_c}$ the zeros of Ω_{w_0} that lie on the unit circle. Define $\tilde{\Omega}_{w_0} \triangleq \Omega_{w_0} \prod_{i=1}^{n_c} z(z - c_i)^{-1} \in \mathcal{U}_\infty$ and, $\forall \epsilon \in (0, 1)$, $B_\epsilon \triangleq \tilde{\Omega}_{w_0}^{-1} \tilde{\Omega}_{w_0}(\infty) \prod_{i=1}^{n_c} z(z - \epsilon c_i)^{-1} \in \mathcal{U}_\infty$. Since $B_\epsilon \in \mathcal{U}_\infty$ and $B_\epsilon(\infty) = 1$, we have from the Bode integral Theorem [44] and the definition of B_ϵ that

$$
\begin{aligned}
\phi_0 &= \frac{1}{4\pi} \int_{-\pi}^{\pi} \ln \frac{|\Omega_{w_0}|^2}{\sigma_q^2} d\omega \\
&= \frac{1}{4\pi} \int_{-\pi}^{\pi} \ln \frac{|B_\epsilon \Omega_{w_0}|^2}{\sigma_q^2} d\omega = \frac{1}{2} \ln \frac{|\tilde{\Omega}_{w_0}(\infty)|^2}{\sigma_q^2}.
\end{aligned}
\tag{19}
$$

Define $(A_1, F_1) \triangleq (B_\epsilon A_0, 1 - B_\epsilon(1 - F_0)) \in \mathcal{U}_\infty \times \mathcal{RH}_2$. For this choice, it is immediate to see that $\Omega = \Omega_0$, that $I_\infty(v_1 \to w_1) = I_0$ (see proof of Lemma 5.1), and that

$$
S_{w_1} = |B_\epsilon|^2 S_{w_o}.
\tag{20}
$$

Hence, for $(A, F) = (A_1, F_1)$, (19) and (20) yield $\phi = \phi_0$ and

$$
\begin{aligned}
\frac{1}{2} \ln \left(1 + \frac{\sigma_{v_1}^2}{\sigma_q^2} \right) &\overset{(a)}{=} \frac{1}{2} \ln \frac{\sigma_{w_1}^2}{\sigma_q^2} \\
&\overset{(b)}{=} \frac{1}{2} \ln \frac{||B_\epsilon \Omega_{w_0}||_2^2}{\sigma_q^2} \\
&\overset{(c)}{=} \phi_0 + \frac{1}{2} \ln \left\| \prod_{i=1}^{n_c} \frac{z - c_i}{z - \epsilon c_i} \right\|_2^2,
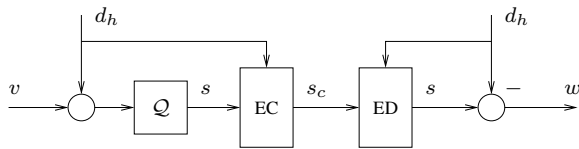\end{aligned}
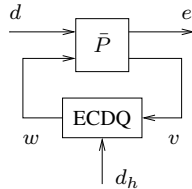\tag{21}
$$

Fig. 5. Entropy coded dithered quantizer.

Fig. 6. Entropy coded dithered quantizer inside a feedback loop.

where (a) follows from the fact that $v(k) \perp\!\!\!\perp q(k) \; \forall k$, (b) follows from (20), and (c) follows from the definition of B_ϵ and (19). Proceeding as in the proof of Lemma 10, p. 171, in [58], we conclude that the last term in (21) can be made arbitrarily small with $\epsilon \to 1$. Our last claim now follows from the last claim of Theorem 5.1. (If $S_{w_0} = 0$ for some ω, then $\epsilon = 1$ is not admissible since it implies $A_1 \notin \mathcal{U}_\infty$ and, thus, the representation of the independent source-coding scheme of Figure 3(b) would be internally unstable.) ■

B. Entropy coded dithered quantizers

Entropy coded dithered quantizers (ECDQs) are devices that have convenient properties that make them suitable for use as building blocks when dealing with average data-rate constraints [63], [64], [66]. In particular, we show below that ECDQs can be used to construct the noise source q that defines an independent source-coding scheme.

The structure of an ECDQ is shown in Figure 5. In that figure, d_h is a dither signal which is assumed available at both the sending and receiving ends, the EC-ED pair is as described in Section IV-B with $S_{\text{EC}}(k) = d_h(k)$, and $\mathcal{Q}$ corresponds to a uniform quantizer, i.e., $\forall x \in \mathbb{R}$,

$$\mathcal{Q}(x) \triangleq i\Delta, \quad \text{for } \left(i - \frac{1}{2}\right)\Delta \leq x < \left(i + \frac{1}{2}\right)\Delta, \quad i \in \mathbb{Z},$$

where $\Delta > 0$ is the quantization step (a designer's choice). The output s of the quantizer satisfies

$$s(k) = \mathcal{Q}(v(k) + d_h(k)) \in \mathcal{A}_s,$$

where $\mathcal{A}_s = \{x \in \mathbb{R} : x = i\Delta, i \in \mathbb{Z}\}$, and the output w of the ECDQ is given by

$$w(k) = s(k) - d_h(k).$$

Modelling quantization noise as an independent i.i.d. source is common in the signal processing literature [42]. In general, this model is not exact [26], [52] but becomes exact in ECDQs when the dither d_h is appropriately chosen (see, e.g., [43], [63]). The next theorem shows that this key property remains valid when ECDQs are embedded in strictly causal feedback loops:

Theorem 5.3: Consider the setup of Figure 6, where the ECDQ is as described above and has a finite quantization step Δ. Assume that $\bar{P}$ is a proper real rational transfer function, that the open-loop transfer function from w to v is single-input single-output and strictly proper, that the closed loop system is internally stable and well-posed when $w = v$, that the signal d
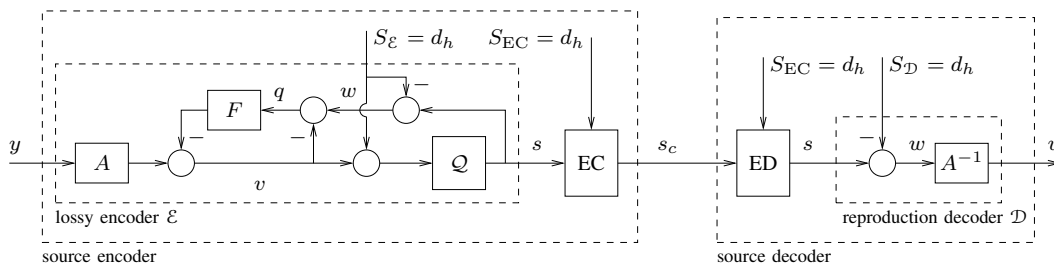
Fig. 7. Explicit rewriting of an independent source-coding scheme that employs an ECDQ as the link between v and w.

is a second order wss process, and that the initial state of $\bar{P}$, say $x_{\bar{P}}(0)$, is a second order random variable. If d_h is such that[8] $f(d_h(i)|x_{\bar{P}}(0), d, d_h^{i-1}) = f(d_h(i)) = U_\Delta(d_h(i))$, then the noise $q \triangleq w - v$ is such that $f(q(i)|x_{\bar{P}}(0), d, q^{i-1}) = f(q(i)) = U_\Delta(q(i))$.

Proof: Similar to the proof of Theorem 1 in [64] (see Chapter 5 in [47] for details). ■

Remark 5.2: The definition of ECs and EDs implies that Theorem 5.3 holds irrespective of how the EC-ED pair is chosen. (In particular, it holds if the EC-ED pair is omitted; see [43].) It is also worth noting that, if the dither is not subtracted at the decoder side, then only moment-independence is achieved [59]. ■

For the remainder of the paper, the following consequence of Theorem 5.3 is relevant:

Corollary 5.1: Consider the system of Figure 3(b) with A and F as in Lemma 5.1. If an ECDQ, with dither chosen as in Theorem 5.3 and finite quantization step Δ, is used as the link from v to w, then the system of Figure 3(b) becomes an independent source-coding scheme.

Proof: Given Lemma 5.1 and Theorem 5.3, it suffices to show that the resulting source-coding scheme satisfies Assumption 4.1 and has an invertible decoder. Since in the present situation $S_{\mathcal{E}}(k) = S_{\mathcal{D}}(k) = d_h(k)$, the assumptions on d_h imply that Assumption 4.1 is satisfied. Also, since $A \in \mathcal{U}_\infty$, its initial state is deterministic and, by definition of ECDQs, $s(k) = \mathcal{Q}(w(k))$ and $d_h(k) = \mathcal{Q}(w(k)) - w(k)$, we conclude that knowledge of u^k is equivalent to knowledge of (d_h^k, s^k). The invertibility of the decoder thus follows, and the proof is completed. ■

When an ECDQ is used as the link between v and w in the system of Figure 3(b), the resulting scheme can be rearranged as illustrated in Figure 7 (cf. Figure 2). From that figure, it is clear that feedback from w to the input of F in Figure 3(b) does not require explicit feedback around the digital channel. (Recall that we consider error-free digital channels.)

Remark 5.3: In source-coding schemes using ECDQs, the encoder and decoder share a common source of randomness (the dither). In principle, this implies that both the encoder and decoder must share information about the dither. This imposes an additional degree of implementation complexity, but to the best of our knowledge, there is no other simple way of satisfying Assumption 4.1. In practice, one can use synchronized pseudo-random number generators initialized with the same seeds. ■

1) Average data-rates in independent source-coding schemes that use ECDQs: We are now in a position to present an upper bound on the average data-rate in independent source-coding schemes that use ECDQs. We start with the following result:

Theorem 5.4: Consider the NCS of Figure 1, where the source-coding scheme is independent. Suppose that Assumption 3.1 holds, and that the link between the auxiliary signals v and w (see Figure 3 and Lemma 5.1) is an ECDQ with dither chosen as in Theorem 5.3 and finite quantization step Δ. Then, $H(s(k)|s^{k-1}, S_{EC}^k) = I(w(k); v^k|w^{k-1})$ and there exists an EC-ED pair such that $I_\infty(v \to w) \le \mathcal{R} < I_\infty(v \to w) + \ln 2$.

---

[8]Here, $f(x)$ (resp. $f(x|y)$) denotes the (resp. conditional) probability distribution of x (resp. of x, given y). U_Δ denotes the distribution of a uniform random variable with support $\left(-\frac{\Delta}{2}, \frac{\Delta}{2}\right)$.

Proof: By definition of ECDQs $H(s(k)|s^{k-1}, S_{\mathrm{EC}}^k) = H(s(k)|s^{k-1}, d_h^k)$. Also, from Theorem 5.3 and its proof we have that $(v^k, w^{k-1}) \perp\!\!\!\perp d_h(k)$ and $q(k) \perp\!\!\!\perp (w^{k-1}, v^k)$. Using the above, our first claim follows from the proof of Theorem 2 in [66]. The second claim follows from the first, (9) and (11). (A direct proof can be constructed by using the definition of ECDQs, and the fact that knowledge of u^k is equivalent to knowledge of (d_h^k, s^k) (see proof of Corollary 5.1), to show equality in all but the first inequality of (12).) ∎

Theorem 5.4 shows that using ECDQs inside independent source-coding schemes allows one to achieve average data-rates that are close to the absolute lower bounds established in Theorem 4.1. The worst case gap, which originates in the inefficiency of the EC and is smaller than $\ln 2$ nats, is intrinsic to any scalar lossless coder and cannot be removed, unless one assumes $\mathcal{R} \to \infty$ (high rate regime; [21]), uses block entropy coding (which may introduce unbounded delays; [8], [45]), or allows the coding scheme to operate in a non-stationary fashion by using time-varying policies [34], [55]. In practice the gap may be smaller than $\ln 2$ nats [21, p. 2333], [16].

A useful corollary of Theorems 5.1 and 5.4 is presented next:

Corollary 5.2: Consider the setup and assumptions of Theorem 5.4. There exists an ECDQ such that

$$\mathcal{R} < \frac{1}{4\pi} \int_{-\pi}^{\pi} \ln \frac{S_w}{\sigma_q^2} d\omega + \frac{1}{2} \ln \left(\frac{2\pi e}{12} \right) + \ln 2. \tag{22}$$

Proof: Immediate from Theorems 5.1 and 5.4, $(D3)$ in the Appendix, and the fact that the coding noise q in an ECDQ is uniformly distributed. ∎

Remark 5.4: Theorem 5.4 and Corollary 5.2 are only existence-type results. The implementation of EC-ED pairs inside ECDQs falls outside the scope of this paper, and we refer the reader to Remark 5.10 in [47] for related remarks. ∎

Corollary 5.2 provides a closed form upper bound on the average data-rate in an independent source-coding scheme that uses an ECDQ. The bound is given in terms of spectral properties of the ECDQ output w, and two additional constant terms. The second term in (22), i.e., $\frac{1}{2} \ln \left(\frac{2\pi e}{12} \right)$ nats per sample (i.e., ≈ 0.254 bits per sample), corresponds to the divergence of the ECDQ quantization noise distribution from Gaussianity and arises because ECDQs generate uniform quantization noise (not Gaussian noise; see also [63]–[65]). This term can also be given an alternative interpretation in terms of the *space filling loss* incurred when using a finite-dimensional quantizer instead of an infinite-dimensional one, with spherical quantization cells. We refer the interested reader to [18], [19], [65] for further details.[9] As mentioned before, the term $\ln 2$ (1 bit) arises due to the inefficiency of the EC inside the ECDQ. Interestingly, our result holds without Gaussianity assumptions on the external signal d nor on the initial state x_o.

Remark 5.5: If the conditions of Corollary 5.2 hold and, in addition, (d, x_o) is Gaussian, then a lower bound for $\mathcal{R}$ in (22) is given by the first term on the right hand side of (22). That is, (22) becomes tight up to $\frac{1}{2} \ln \left(\frac{2\pi e}{12} \right) + \ln 2$ nats per sample (see Remark 5.1 and Theorem 5.4). ∎

2) Independent source-coding schemes with memoryless ECDQs: So far, we have considered ECDQs where the EC-ED pair is allowed to exploit all past and present symbols s^k, binary words s_c^k, and side information $S_{\mathrm{EC}}^k = d_h^k$. Such ECDQs have *unrestricted* memory and its implementation requires the knowledge of the conditional distribution of $s(k)$, given (d_h^k, s^{k-1}). That distribution can be difficult to characterize. In order to simplify implementation, it is common to consider ECDQs without memory (see also [64]):

Definition 5.3: An ECDQ is said to be memoryless iff the associated EC-ED pair is such that $\mathcal{H}_k(s^k, S_{\mathrm{EC}}^k) = \mathcal{H}_k(s(k), d_h(k))$ and $\mathcal{H}_k^{-1}(s_c^k, S_{\mathrm{EC}}^k) = \mathcal{H}_k^{-1}(s_c(k), d_h(k))$, for all $k \in \mathbb{N}_0$. ∎

---

[9]It is worth noting that most results in [18], [19] are derived under high-rate assumptions.

When using a memoryless ECDQ, the EC can only exploit the knowledge of $d_h(k)$ to encode $s(k)$. Thus, (8) must be replaced by $R(k) \geq H(s(k)|d_h(k))$. Again, it is possible to design coding policies such that [8] (compare with (9))

$$0 \leq R(k) - H(s(k)|d_h(k)) < \ln 2.$$

We now present a definition and two results that allow one to state the counterpart of Corollary 5.2 for the case of independent source-coding schemes that use memoryless ECDQs:

Definition 5.4: Consider two random processes r and s. The scalar mutual information rate between r and s is defined via

$$I_\infty(r;s) \triangleq \lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} I(r(i);s(i)). \tag{23}$$

■

Theorem 5.5: Consider the setup and assumptions of Theorem 5.4. If the ECDQ is memoryless, then $H(s(k)|d_h(k)) = I(v(k);w(k))$ and there exists an EC-ED pair such that $I_\infty(v;w) \leq \mathcal{R} < I_\infty(v;w) + \ln 2$.

Proof: The result follows from the proof of Theorem 2 in [64], (9) and (23). ■

Theorem 5.6: Consider the NCS of Figure 1, where the source-coding scheme is independent. If Assumption 3.1 holds, then

$$I_\infty(v;w) \leq I_\infty(v_G;w_G) + D(q(k)\|q_G(k)) = \frac{1}{2}\ln(1+\gamma) + D(q(k)\|q_G(k)),$$

where all symbols are as in Lemma 5.2 and Theorem 5.1.

Proof: The result follows proceeding as in the proof of Theorem 5.1 (see details in [47]). ■

Corollary 5.3: Consider the setup and assumptions of Theorem 5.4. There exists a memoryless ECDQ such that

$$\mathcal{R} < \frac{1}{2}\ln(1+\gamma) + \frac{1}{2}\ln\left(\frac{2\pi e}{12}\right) + \ln 2. \tag{24}$$

Proof: Immediate from Theorems 5.5 and 5.6. ■

Remark 5.6: In analogy to Remark 5.5, we note that if the conditions of Corollary 5.3 hold, and (d, x_o) is Gaussian, then (24) becomes tight up to $\frac{1}{2}\ln\left(\frac{2\pi e}{12}\right) + \ln 2$ nats per sample. ■

The EC inside a memoryless ECDQ uses less information to encode s than the EC inside an ECDQ with unrestricted memory. As a consequence, average data-rates achievable with memoryless ECDQs will be, in general, larger than those achievable with ECDQs that have unrestricted memory. This conclusion is consistent with Corollaries 5.2 and 5.3, as Jensen's inequality reveals. However, the conclusion is pessimistic when ECDQs are used inside independent source-coding schemes. In this case, Theorem 5.2 guarantees that the rate penalty, as measured by the gap between the right hand sides of (22) and (24), incurred when using a memoryless ECDQ instead of an ECDQ with memory, can be made arbitrarily small if an appropriate choice for the auxiliary filters A and F is made. Hence, without being unduly conservative, it suffices to study the SNR γ to give upper bounds on the average data-rate of independent source-coding schemes, irrespective of whether they use ECDQs with unrestricted memory or not.

C. Discussion

Consider the NCS of Figure 1 when the source-coding scheme is independent and uses an ECDQ as the link between the auxiliary signals v and w (see Figure 7). For this setup, the results of Sections V-A and V-B allow one to restate control problems involving average data-rate constraints as control problems involving stationary SNR constraints. This enables one to design NCSs subject to average data-rate constraints in a systematic fashion that uses standard quadratic optimization methods. For instance, if performance is measured by means of the stationary variance σ_e^2 of the error signal e, then, irrespective of

whether the ECDQ has unrestricted memory or not, the minimal average data-rate needed to achieve a performance level D, say $\mathcal{R}_D$, satisfies (see Corollaries 5.2 and 5.3)

$$\mathcal{R}_D \triangleq \inf_{\sigma_e^2 < D} \mathcal{R} < \frac{1}{2} \ln\left(1 + \gamma_D\right) + \frac{1}{2} \ln\left(\frac{2\pi e}{12}\right) + \ln 2. \tag{25}$$

In (25), the optimization is carried out with respect to all A, F, Δ and EC-ED pairs that guarantee stability in an appropriate sense, and γ_D is the minimal stationary SNR of the source coder that allows one to achieve the performance level D, i.e.,

$$\gamma_D \triangleq \inf_{\sigma_e^2 < D} \gamma.$$

Once γ_D is characterized (which, in principle, is a standard quadratic control problem), one readily obtains a bound on $\mathcal{R}_D$ (see also Section VI below).

The framework provided here is conservative since it is based on inequalities that are not tight in general.[10] However, the framework provides bounds that are guaranteed, by construction, to be *achievable* with *conceptually simple* building blocks, and *do not rely on any asymptotic* approximation. As illustrated above, the framework also opens the door to use *standard synthesis methods* to deal with average data-rate limits in control system design, as explored in [47]–[49]. These are the main features that distinguish our approach from the literature surveyed in Section I.

VI. EXAMPLE: MEAN SQUARE STABILIZATION

We now illustrate the approach outlined in Section V-C by studying bounds on the minimal average data-rate for stability, i.e., we consider $D = \infty$ in (25). The case of finite D is omitted due to space constraints (see [48]).

A. Mean square stability subject to SNR constraints

We start by studying the interplay between constraints on the source coder SNR γ and stability. By virtue of Theorem 5.3, we focus on the feedback system of Figure 4, and use the following notion of stability:

Definition 6.1: Consider the linear system $x(k+1) = M_x x(k) + M_w w(k)$, where $k \in \mathbb{N}_0$, $x(k) \in \mathbb{R}^n$ is the system state at time instant k, $x(0) = x_o$, where x_o is a second order random variable, M_x and M_w are constant matrices of appropriate dimensions, and w is a second order wss process. The system is said to be mean square stable[11] (MSS) iff there exist finite $\mu \in \mathbb{R}^n$ and a finite and positive semi-definite $M \in \mathbb{R}^{n \times n}$, both not depending on x_o, such that $\lim_{k \to \infty} \mathcal{E}\{x(k)\} = \mu$, $\lim_{k \to \infty} \mathcal{E}\{x(k)x(k)^T\} = M$. ∎

For LTI systems, MSS is equivalent to internal stability [2]. Thus, if Assumption 3.1 holds and the noise q in Figure 4 satisfies Definition 5.1, then the feedback system of Figure 4 is MSS iff $A \in \mathcal{U}_\infty$ and $F \in \mathcal{RH}_2$.

For future reference define $S \triangleq (1 - P_{22})^{-1}$ and $T_{de} \triangleq P_{11} + P_{12}SP_{21}$.

Theorem 6.1: Consider the NCS of Figure 4, where q is as in Definition 5.1 and Assumption 3.1 holds. Then,

$$\gamma_{\mathrm{inf}} \triangleq \inf_{\substack{\sigma_q^2 \in \mathbb{R}_0^+ \\ A \in \mathcal{U}_\infty, F \in \mathcal{RH}_2}} \gamma = \left(\prod_{i=1}^{n_p} |p_i|^2\right) - 1,$$

where p_i is the ith unstable pole of P_{22}. The infimum γ_{inf} is not achievable in an MSS NCS. (Indeed, to achieve $\gamma \to \gamma_{\mathrm{inf}}$, it is necessary to have $\sigma_q^2 \to \infty$.) However, for any $A \in \mathcal{U}_\infty$, there exists a sequence of filters $\{F_i\}_{i \in \mathbb{N}_0}$, $F_i \in \mathcal{RH}_2$, such that $\gamma \to \gamma_{\mathrm{inf}}$ when $i \to \infty$ and $\sigma_q^2 \to \infty$.

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[10]Save for the Gaussian case where a characterization of the worst case gap was provided (see Remarks 5.5 and 5.6).

[11]We will make a slight abuse of notation and use MSS for both "mean square stability" and "mean square stable".

Proof: Our assumptions imply that

$$\sigma_v^2 = ||ASP_{21}\Omega_d||_2^2 + \sigma_q^2 ||1 - S + SF||_2^2. \tag{26}$$

Define $\xi_S \triangleq \prod_{i=1}^{n_p}(1 - z\bar{p}_i)(z - p_i)^{-1}$ (here, $\bar{p}_i$ stands for the complex conjugate of p_i). Using the proof techniques employed in [6], [51], we conclude from (26) that

$$\gamma = \frac{||ASP_{21}\Omega_d||_2^2}{\sigma_q^2} + ||1 - S + SF||_2^2$$

$$\overset{(a)}{\geq} ||1 - S + SF||_2^2$$

$$= ||\xi_S(\infty) - \xi_S S + \xi_S SF||_2^2$$

$$\qquad + ||\xi_S - \xi_S(0)||_2^2 + ||\xi_S(0) - \xi_S(\infty)||_2^2 \tag{27}$$

$$\overset{(b)}{\geq} \left(\prod_{i=1}^{n_p}|p_i|^2\right) - 1,$$

where (b) follows upon using the Residue Theorem [44] on the last two terms of (27). The gap in (a) can me made arbitrarily small with a sufficiently large σ_q^2 (and cannot be made equal to zero since, by assumption, $ASP_{21}\Omega_d \neq 0$). Lemma 10, p. 171, in [58] allows one to conclude that there exists $F \in \mathcal{RH}_2$ such that the gap in (b) is arbitrarily small. The result thus follows. (The existence of the family $\{F_i\}_{i \in \mathbb{N}_0}$ is also guaranteed by the results in [58].) ∎

Theorem 6.1 states that, for independent source-coding schemes, the minimal SNR that is compatible with MSS, i.e., $\gamma_{\inf}$, is only a function of the unstable poles of P_{22}. Hence, for any given $\Gamma \in \mathbb{R}^+$, the condition $\Gamma > \gamma_{\inf}$ is necessary and sufficient to be able to find A, F and a noise variance σ_q^2 (equivalently, a quantization step Δ) such that the resulting NCS is MSS and the SNR satisfies $\gamma \leq \Gamma$.

We conclude this section with a simple corollary of Theorem 6.1:

Corollary 6.1: Consider the setup and assumptions of Theorem 6.1. If $\gamma \to \gamma_{\inf}$, then the stationary variance σ_e^2 of the error signal e grows unbounded.

Proof: The result is immediate since, by Assumption 3.1, $P_{12}SA^{-1}(1 - F) \neq 0$, $\forall(A, F) \in \mathcal{U}_\infty \times \mathcal{RH}_2$ and, by Theorem 6.1, $\sigma_q^2 \to \infty$ as $\gamma \to \gamma_{\inf}$. ∎

We see from Corollary 6.1 that the study of conditions on γ for MSS are insufficient to give performance guarantees. This fact is unsurprising, and consistent with results in [15], [51].

B. Mean square stability subject to average data-rate constraints

We now return to the NCS of Figure 1 when the source-coding scheme is independent and uses an ECDQ. Given the definition of an independent source-coding scheme, it follows that the notion of MSS described in Definition 6.1 is still valid in this setting. It is also clear that, provided Assumption 3.1 holds, $\Delta < \infty$, and regardless of the EC-ED pair, the considered NCS is MSS iff $A \in \mathcal{U}_\infty$ and $F \in \mathcal{RH}_2$.

Corollary 6.2: Consider the setup and assumptions of Theorem 5.4. Then, irrespective of whether the ECDQ has memory or not, the following holds:

(a) The minimal average data-rate compatible with MSS, i.e., $\mathcal{R}_\infty$ as defined in (25), satisfies

$$\sum_{i=1}^{n_p}\ln|p_i| \leq \mathcal{R}_\infty < \bar{\mathcal{R}}_\infty, \tag{28}$$

where

$$\bar{\mathcal{R}}_\infty \triangleq \sum_{i=1}^{n_p} \ln |p_i| + \frac{1}{2} \ln \left(\frac{2\pi e}{12} \right) + \ln 2.$$

(b) If $\{F_i\}_{i \in \mathbb{N}_0}$ is the family of filters in Theorem 6.1, then there exists $n \in \mathbb{N}_0$, $N \in \mathbb{R}^+$, and an EC-ED pair, such that the choice $F = F_i$, with $i \geq n$, and $\Delta \geq N$ guarantees MSS and $\mathcal{R} < \bar{\mathcal{R}}_\infty$, for any choice of $A \in \mathcal{U}_\infty$.

   Proof: Define $K \triangleq \frac{1}{2} \ln \left(\frac{2\pi e}{12} \right) + \ln 2$.

(a) Consider the case of ECDQs with unrestricted memory. Equation (22) and Theorem 5.2 yield

$$R_\infty < \inf_{\substack{\sigma_q^2 = \frac{\Delta^2}{12} \\ A \in \mathcal{U}_\infty, \, F \in \mathcal{RH}_2}} \frac{1}{4\pi} \int_{-\pi}^{\pi} \ln \frac{S_w}{\sigma_q^2} d\omega + K$$

$$= \inf_{\substack{\sigma_q^2 = \frac{\Delta^2}{12} \\ A \in \mathcal{U}_\infty, \, F \in \mathcal{RH}_2}} \frac{1}{2} \ln (1 + \gamma) + K. \tag{29}$$

The upper bound in (28) follows upon using Theorem 6.1 in (29). The lower bound follows from Theorem 2.1 in [34]. The case o memoryless ECDQs follows similarly.

(b) For any $A \in \mathcal{U}_\infty$, $\epsilon > 0$ and sufficiently large $i \in \mathbb{N}_0$ and $\Delta_o \in \mathbb{R}^+$, $\frac{1}{2} \ln \left(1 + \gamma|_{F = F_i, \sigma_q^2 = \Delta_o^2/12} \right) - \sum_{i=1}^{n_p} \ln |p_i| < \epsilon$. Also, irrespective of the memory in the ECDQ, there exists $\delta > 0$ such that $\mathcal{R} + \delta = \frac{1}{2} \ln (1 + \gamma) + K$ (see (17), (22) and (24)). Thus, $\mathcal{R}|_{F = F_i, \Delta = \Delta_o} < \sum_{i=1}^{n_p} \ln |p_i| + K + \epsilon - \delta$ and the result follows upon choosing $\epsilon < \delta$.

■

Corollary 6.2 establishes lower and upper bounds on the minimal average data-rate that is compatible with MSS in the considered NCS, when an independent source-coding scheme is employed. A feature of our proposal is that ECDQs with unrestricted memory do not provide any advantage over memoryless ones (at least from a MSS point of view; see also discussion at the end of Section V-B.2). This is relevant in practice since the implementation of ECDQs with unrestricted memory is computationally prohibitive. Indeed, in order to design an independent coding scheme that achieves stability at an average data-rate smaller than $\bar{\mathcal{R}}_\infty$, it suffices to use a *memoryless* ECDQ with sufficiently large quantization step Δ, an EC-ED pair designed using Huffman coding, and filters A and F that, for the situation studied in Section VI, guarantee MSS at SNRs sufficiently close to $\gamma_{\inf}$. By doing so, however, the performance of the NCS will be compromised (see Corollary 6.1). This is consistent with results in [34], [36].

The results of Corollary 6.2 show that, when an independent source-coding scheme is employed in the NCS of Figure 1, MSS can be achieved at average data-rates that are guaranteed to be no larger than the absolute bound identified in [34], [54] plus $\frac{1}{2} \ln \left(\frac{2\pi e}{12} \right) + \ln 2$ nats (≈ 1.254 bits) per sample. This extra rate is, in our view, a fair price to be paid if one constrains oneself to the conceptually simple source-coding schemes considered in this paper. We note however that the upper bound in (28) is a worst case upper bound. As mentioned earlier, in practice one can expect to achieve MSS at rates no larger than $\bar{\mathcal{R}}_\infty - \ln 2$ (see [16], [21]).

Our results can also be used to provide upper bounds on the average data-rate that is needed to achieve MSS, when memoryless source-coding schemes are employed in the considered NCS. Indeed, if one chooses $A = 1$, $F = 0$ and a memoryless ECDQ, then the resulting source-coding scheme has no memory and it is easy to show that Δ and the EC-ED pair can be chosen so as to achieve MSS at an average data-rate satisfying

$$\mathcal{R} < \frac{1}{2} \ln \left(1 + ||1 - S||_2^2 \right) + \frac{1}{2} \ln \left(\frac{2\pi e}{12} \right) + \ln 2. \tag{30}$$

Relation (30) can be contrasted with the results of Section VI in [55] (even though [55] focuses on a different notion of stability). That work shows that there exist memoryless encoders that guarantee stability with bounded (but otherwise unknown) data-rates, whist (30) provides a computable upper bound on the minimal average data-rate compatible with MSS.

VII. CONCLUSIONS

This paper has studied a control problem where an LTI system is designed assuming transparent feedback and, at a later design stage, a unity signal transfer function source-coding scheme is to be used so as to minimize the effects that data-rate limits in the feedback path have on closed loop performance. To address this problem, we have focused on a class of source-coding schemes and, by doing so, we have established a bridge between information and control theories. A key result of our work is that, for the considered class of coding schemes, average data-rate limits can be enforced by imposing signal-to-noise ratio constraints in a related additive noise communication channel. As an application of our results, we studied the interplay between stability and average data-rates in the considered setup. For that problem, the proposed class of coding schemes was shown to achieve mean square stability at average data-rates that are guaranteed to be less than 1.254 bits per sample above the absolute minimum established in [34].

We refer the reader to [48], [49] for applications of the framework presented in this paper to problems beyond stabilization. A key open problem not addressed in this work is how to incorporate average data-rate limits into control problems using causal (but otherwise unrestricted) source-coding schemes.

APPENDIX

The following definitions and results are standard and can be found in [8]. Unless otherwise stated, all variables are assumed to be random variables (RVs) with well defined (joint) probability density (or mass) functions (PFs). The PF of x (x, y) is denoted $f(x)$ $(f(x, y))$. $f(x|y)$ refers to the conditional PF of x, given y. $\mathcal{E}_x\{\cdot\}$ denotes mean with respect to x; $x \perp\!\!\!\perp y$ stands for x independent of y.

The differential entropy of x is defined via $h(x) \triangleq -\mathcal{E}_x\{\ln f(x)\}$. The conditional differential entropy of x, given y, is defined via $h(x|y) \triangleq -\mathcal{E}_{x,y}\{\ln f(x|y)\}$. Facts about h:

$(h1)$ $h(x|y) \leq h(x)$ (equality iff $x \perp\!\!\!\perp y$), $h(x|y, z) = h(x|y)$ iff $f(x, z|y) = f(x|y)f(z|y)$.

$(h2)$ $h(x_0, \cdots, x_{n-1}) = \sum_{i=0}^{n-1} h(x_i|x_0, \cdots, x_{i-1})$.

$(h3)$ If x is Gaussian with finite variance, $h(x) = \frac{1}{2}\ln 2\pi e\sigma_x^2$.

If x and y are discrete RVs, then we use $H(x)$ $(H(x|y))$ to denote the (conditional) entropy of x (given y). The definitions are analogue to the continuous case. A fact about H:

$(H1)$ $H(x) \geq H(x|y) \geq H(x|y, z) \geq 0$.

The (conditional) mutual information between x and y (given z) is defined via $I(x; y) \triangleq h(x) - h(x|y)$ $(I(x; y|z) = h(x|z) - h(x|y, z))$ Properties of I:

$(I1)$ $I(x; y) = I(y; x)$, $I(x; y|z) = I(y; x|z)$.

$(I2)$ $I(x; y) \geq 0$ (equality iff $x \perp\!\!\!\perp y$).

$(I3)$ $I(x, y; z) = I(x; z) + I(y; z|x)$.

$(I4)$ $I(x; z|y) = 0$ iff $f(x, z|y) = f(x|y)f(z|y)$.

(Conditional) mutual information between discrete RVs is defined as in the continuous RV case.

The relative entropy between x and y (or divergence of the distribution of x with respect to that of y) is defined via $D(x||y) \triangleq \mathcal{E}_x \{\ln f(x) f(y)^{-1}\}$. Given joint distributions $f(x,y)$ and $f(w,z)$, the conditional relative entropy is defined via $D(x|y||w|z) \triangleq \mathcal{E}_{x,y} \{\ln f(x|y) f(w|z)^{-1}\}$. Facts about D:

($D1$) $D(x||y) \geq 0$ (equality iff $f(x) \stackrel{(a.e.)}{=} f(y)$), $D(x|y||w|z) \geq 0$.

($D2$) If x_G is the Gaussian counterpart of x (see Definition 5.2), then $D(x||x_G) = h(x_G) - h(x)$. If x_G, y_G are the Gaussian counterparts of x, y, then $D(x|y||x_G|y_G) = h(x_G|y_G) - h(x|y)$.

($D3$) If x is uniformly distributed on $\left(-\frac{\Delta}{2}, \frac{\Delta}{2}\right)$, and y is zero mean Gaussian with variance $\sigma^2 = \Delta^2/12$, then $D(x||y) = \frac{1}{2} \ln \left(\frac{2\pi e}{12}\right)$.

The RVs x, y, z form a Markov chain (denoted $x - y - z$) iff $f(x, z|y) = f(x|y) f(z|y)$. If, conditioned upon w, $x - y - z$, then we write $x|_w - y|_w - z|_w$. Facts about Markov chains:

($M1$) If z is a deterministic function of y, then $x - y - z$.

($M2$) If $x - y - z$, then $I(x;y) \geq I(x;z)$ (equality iff $x - z - y$).

($M3$) If $x|_w - y|_w - z|_w$, then $I(x;y|w) \geq I(x;z|w)$ (equality iff $x|_w - z|_w - y|_w$ as well).

The entropy rate of a stochastic process x is defined via $h_\infty(x) \triangleq \lim_{k\to\infty} \frac{h(x^{k-1})}{k}$. A useful fact is the following:

($R1$) If x is an asymptotically wss process with stationary PSD S_x, then $h_\infty(x) \leq \frac{1}{4\pi} \int_{-\pi}^{\pi} \ln \left(2\pi e S_x\right) d\omega$ (equality iff, in addition, x is asymptotically Gaussian; see Lemma 4.3 in [30]).

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