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The Impact of Geometry Montessori Education on Students’ Skills and Mindsets

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Abstract

Montessori education has gained recognition due to its long-lasting positive effects on students. However, no studies have targeted its effectiveness on geometry education. This action research project investigated the impact and effectiveness of switching a non-sequential geometry curriculum with the Montessori method and curriculum on students' attitudes and geometrical skills. During six weeks, a population of 16 grade 6 students received weekly lessons that followed the Montessori method and curriculum while their skills were measured and compared against the British Columbia (BC) geometry curriculum. Students were interviewed before and after the intervention to track changes in their learning attitudes. Teacher observations and tracking of student work complemented the data. Results showed that the intervention had an overall positive impact, with a 13% increase in student's confidence in their geometrical skills. Likewise, 93% of students reported having an easier time understanding abstract concepts when previously demonstrated with Montessori materials. Also, 100% reported that geometric concepts became more evident in a curriculum with logically sequenced lessons, and 53% reported increased joy related to geometry learning. In sum, it took students only six weeks of following the Montessori curriculum to master 66% of the BC outcomes for geometry, on average, a percentage that makes sense considering the reduced geometrical content and gaps found in the BC Curriculum during the present work's literature review. Therefore, replacing the BC Curriculum with the Montessori method and curriculum would benefit students. Future similar research focused on larger, possibly younger, populations would further enrich the literature.

Keywords: Montessori curriculum, learning attitude, geometry, geometrical skills

Introduction

There has been contention about when and how it is best to teach geometry to students. For decades, Educators have expressed polarized opinions on this topic. Some believe geometry has a link with art since immemorial times and should be taught to children in a hands-on, engaging way, starting from early childhood. In contrast, others believe geometry relates more to mathematics. Due to this belief, this latter group sustains that it should be taught as an abstract concept, always connected to the proper calculations, and only to pupils who already possess enough mathematical knowledge to understand the complexities of geometry. These two points of view have dictated how today's pupils will learn geometry.

Most public schools in North America follow geometry curricula designed to fit the mathematical point of view. Linking geometry to advanced mathematics and postponing most of its content until middle and high school has been the norm since the last extensive educational reform in the 1960s. However, some schools, mostly private schools in the U.S.A. and independent schools in Canada, have taken a different approach to teaching geometry. They complement the public-school curricula with content that allows children to experience geometrical concepts with hands-on materials throughout their elementary years. Such is the case of Montessori schools. Despite this dichotomy, no specific academic studies help determine which method and timing is more effective.

Even though research is limited, looking closely, it is possible to find studies that point to other relevant factors. For example, researchers such as Peterson (1973), who studied public high school students, noticed increased anxiety and stress surrounding the topic of geometry. He stated that unless teachers are committed to a new approach to geometry, the inevitable result will be student frustration and an inferior program in geometry (Peterson, 1973). Similarly, other researchers have studied elementary students in schools that follow an abstract and mathematical-oriented approach to teaching geometry (Hwang et al., 2019). They found that students struggle to understand geometrical concepts and proposed solutions, such as digitizing textbooks or creating learning apps. Their main argument is that content is more engaging when presented on a tablet or computer. Although their approach addresses some of the consequences of the problem of student engagement, it does not address the root issue that the non-sequential and abstract approach to teaching combined with belatedly introduced concepts are at the base of what causes the diminishing of student engagement in geometrical studies during the middle school years. These results suggest that teaching students only sporadic topics during elementary and then loading all the rest of the geometry content in the middle and high school years does not work. More research is required to determine the most effective way and time to teach geometry.

This action research project seeks to determine the effect of changing abstract, non-sequential, teacher-led curriculum for a student-led, hands-on geometry curriculum. The study was conducted in a grade 5-6 classroom within a Montessori independent school (in Canada, any non-public school is considered independent, and, if they receive government funding, they must meet the learning outcomes of the BC Curriculum.) The study population was the grade 6 students, who were 10 and 11 years old at the time of the data collection. They received Montessori geometry lessons that addressed the topics in the British Columbia official curriculum. Instead of the sporadic geometry lessons the provincial curriculum dictates, these students received between one and two weekly hands-on geometry lessons following the Montessori curriculum. Students answered pre- and post-surveys as well as pre-and post-academic assessments. These results, combined with weekly classroom observations and a student work rubric summary, suggest a difference between the two types of geometry curricula, where Montessori is the most effective.

Theoretical Framework

The current curriculum in British Columbia, Canada, postpones a considerable percentage of the geometrical content until middle and high school, leaving only minimal and segmented content for the elementary years. Like reading, geometry is a skill that requires a step-by-step structure, and the current curriculum does not provide this. The present teaching process creates unnecessary stress and elevates students' frustration (Peterson, 1973). This project explores how replacing an abstract, non-sequential approach to geometry education with the sequential, hands-on Montessori method would impact students' learning attitudes and skills.

Two major theoretical frameworks will support the design of this project. They are constructivism (Montessori, 1934; Piaget et al., 1960; Van Hiele, 1999; Vygotsky, 1986) and sensitive periods (Montessori, 1917, 1934). These frameworks are appropriate because they show the importance of manipulatives in geometry learning and highlight that the elementary years are the prime time for children to learn geometrical concepts.

Let us take constructivism first; it is an educational theory that explains how a learner builds their reality through their experiences and that, at the same time, learners actively construct their knowledge based on the concepts they had before (Lillard, 2016; Piaget et al., 1960; Vygotsky, 1986). This theory applies to this research because the Montessori curriculum always takes students from the known to the unknown. For example, children first learn to name geometrical figures before learning to analyze their parts. At the same time, in this theory, using manipulatives when teaching new concepts to children is significantly important (Montessori, 1934; Van Hiele, 1986, 1999). In Montessori's words, "The hand is the instrument of the intelligence. The child needs to manipulate objects and to gain experience by touching and handling them." (1946, p. 48). Therefore, this theory applies to the study of the impact of replacing an abstract teaching method with one that uses manipulative materials.

Another theory that will guide this study is the Sensitive Periods. It is a theory that can be found both in biology and in education. It refers to transitory cycles of development that appear in perfect synchronization with the development of the body (Peterson, 1973). Let us take newly hatched caterpillars, a famous example Dr. Montessori used to define what they are. As soon as they hatch, they are attracted to light. This attraction leads them to where new and tender leaves are perfect for their tiny teeth. However, once their teeth are strong enough to feed from other things, their sensitivity to light disappears (Montessori, 1936). Some sensitive periods last weeks or months, while others will extend over several years. Sensitive periods are a critical theory to guide this research because, according to Montessori (1934), the sensitive period for learning geometry happens during the elementary school years, which is very different from the traditional high school experience offered within the current educational curriculum.

Authors like Felix Klein favoured abstract geometry over hands-on learning. Also, they advocated for the delay of geometrical concepts until middle and high school years, when pupils were ready to understand the complexities of mathematics (Doorman et al., 2020). Public schools have taught geometry this way for many years, but the present research suggests much room for improvement. Evidence shows that students are experiencing unnecessary frustration due to the late introduction of geometry concepts (Peterson, 1973). This action research intends to determine whether another approach would be more beneficial. It is, therefore, supported by the theories of constructivism and sensitive periods.

Literature Review

This action research explores how replacing an abstract, non-sequential approach to geometry education with the sequential, hands-on Montessori method would impact students' learning attitudes and skills. The literature reviews section will review the scholarly work done to define the current curriculum, the views of other educators who advocate for an alternative approach to teaching geometry, and the reasons behind their claims. The following headings discuss these topics: 1. History: How the current curriculum came to be, 2 Current practices in the teaching of geometry and 3. Why reinvent the wheel?

History: How the current curriculum came to be

Freudenthal (1981b), an influential mathematician, situated the origins of the problem of effective geometry education in the history of Mathematics teaching. He argued that math teachers need to understand where the contemporary curricula originate. He cautioned teachers to find sources and genuinely understand the why behind what they teach. If teachers do not do this, they may pass on flawed reasoning that previous authors have built upon by diffusing the original message. He stated, “Whoever is interested in the history of mathematics should study the processes rather than the products of mathematical creativity” (Freudenthal, 1981b, p. 33). Therefore, we must go back to the beginning to understand the problems our predecessors faced that resulted in the changes reflected in today’s curriculum.

The contention about how best to teach geometry to students has a long history. The current geometry curricula in North America have not changed much since its last major reform in the 1960s. This reform was the product of a movement named New Math. At its core, it states that geometry learning cannot happen without prior knowledge of advanced mathematics

Because of New Math and its changes, geometry was postponed until middle and high school and was only briefly mentioned before that (Cao, 2018).

Doorman wrote about the New Math movement, which started in North America around the 1960s and is responsible for the first versions of the mathematics and geometry curricula that even today serve as the basis for public school curricula across North America. After this moment, the norm became to teach mathematics based on a set of theories rather than practical and applicable demonstrations (Doorman et al., 2020).

This simplistic view of geometry is far from what it was if we go back in time further. When educators discussed the addition of geometry to the high school curriculum in 1921, William Betz, the seventh president of the National Council of Teachers of Mathematics (NCTM), said, "It might be well to inform inexperienced teachers that intuitive geometry is not primarily a textbook subject, that it requires constant contact with concrete material, and that the method of procedure is all-important" (Peterson, 1973, p. 57). Later, in 1932, the NCTM appointed a geometry committee where J. Shibli expressed that the geometry concepts taught in high school were too easy and proposed to move plane geometry, the study of angles and figures, as well as formulas for the area of rectilinear shapes and circles to elementary, allowing high school students to focus on more complex subjects (Peterson, 1973).

However, the evolution of the geometry curriculum stopped after World War II. The main concern was that high school students were not studying mathematics, and a more significant emphasis on that subject was necessary. In 1955, the College Entrance Examination Board (CEEB) asked for the curriculum to be modified so the students would meet the advanced mathematics requirement. Four years later, the Secondary School Curriculum Committee reduced the geometry content to 20-30% of the pre-World War II curriculum. At the same time, the CEEB expressed that teachers should mainly emphasize the topics of measurement and the relationship between geometrical elements (Peterson, 1973, pp. 58-59).

According to Doorman et al. 2020, despite the belief that New Math resulted from the aftermath tensions of the Cold War and Sputnik's launch in 1957, the urge to move mathematics towards a more abstract approach began in the 19th Century when Felix Klein inspired secondary schools in Germany to replace Euclidean geometry with a simplified version of transformation geometry that he called "motion geometry." In all those "modern views" proposed, advanced mathematics dominated the learning process and resulted in a view of geometry as a subproduct of algebra describing space, removing all the opportunities for students to become familiar with space (Doorman et al., 2020).

Trafton & LeBlanc (1973) described New Math's detrimental effects on North America's geometry curriculum. Since geometry became a subject reserved for secondary teachers (Doorman et al., 2020), elementary teachers were polarized into two categories: those who believed geometry was not an essential part of the mathematics program and would teach it only if there was time and those who saw the importance of teaching what was known as "informal geometry" to the elementary pupils (Trafton & LeBlanc, 1973).

Trafton & LeBlanc (1973) attempted to create a summary of the geometry curriculum in the 1970s. In doing so, they describe that the main difficulty was the lack of agreement among teachers on the topics and when they needed to be covered. They observed that geometry curricula varied greatly from one school to the other and decided to create an average of the curriculum at that time, finding that the contents of geometry accounted for approximately 15% of the mathematical content in each grade. Within this 15%, there were also variations in the quality of the information. In their findings, teaching children about measuring was an essential part of geometry, while, for teachers, it was less critical to sequence the development of the concepts carefully. In most cases, geometry was only a series of unrelated experiences taught in the classroom (Trafton & LeBlanc, 1973, p. 24).

Allied countries followed the New Math reforms. However, most European countries used this approach for only a couple of decades and then moved on to continue developing their mathematical and geometrical curricula. On the other hand, the current curricula in North America still reflect the 1960s New Math approach, which is why the European geometry curriculum is far superior to North American curricula (Doorman et al., 2020).

Although New Math was the dominant pedagogy in all the Allied countries, both North American and European, there was a minority of pedagogues that opposed the opinions of this movement. These scholars, instead, sought to inspire attention to spatial orientation education starting in the first year of elementary school and covering all the basic concepts of Euclidean geometry before secondary school (Doorman et al., 2020). Among these were Friedrich Fröbel, who used hands-on materials to develop geometry education for kids between 4 and 14 years old. Maria Montessori, who developed a detailed hands-on curriculum to teach geometry from preschool through elementary. Jean Piaget, who, like Montessori, regarded hands-on activities as the best way for children to experience learning. Hans Freudenthal, who inspired the modification of the curriculum in the Netherlands to switch from New Math to what he called "Realistic Geometry" and, Pierre Van Hiele who, along his wife, Dina Van Hiele, developed a curriculum based on levels of geometrical thinking (Freudenthal, 1971; Montessori, 1917; Piaget et al., 1960; Van Hiele, 1986, 1999). According to these pedagogues, pupils should learn geometry throughout elementary school. It should use hands-on and interactive teaching methods that allow the educator to capture young pupils' attention rather than postpone the subject's delivery until students become old enough to understand advanced mathematics.

Trafton & LeBlanc (1973), in the publication of the National Council of Teachers of Mathematics in Washington, stated that a proper definition of the elementary geometry school curriculum would greatly benefit mathematics education and pointed out that a significant obstacle to this goal is the diffuse nature of geometry when attaching it to the mathematics curriculum. They say such a program should be cohesive, applicable, and designed with the elementary child in mind. Just like European countries did, it is high time North American countries started improving their geometry teaching practices.

Current practices in the teaching of geometry

The literature shows two main streams of thoughts on how to cover the subject of geometry. On the one hand, public schools teach a curriculum usually made public on local websites. Since this research is happening in British Columbia, the Canadian British Columbia curriculum is emphasized throughout the study. However, some students attend independent schools in the same cities where the public-school curricula can be modified or complemented.

Public schools in the province follow the mandated curriculum published on the province's website. This document has no geometry stand-alone curriculum, only geometry pieces within the mathematics curriculum. For instance, consider British Columbia's curriculum, summarized in Appendix A (Province of British Columbia, n.d., 2016). This appendix is a compiled list of the geometry content taught in public schools. It has been taken from these pieces within the mathematics curriculum from kindergarten to grade 12 to facilitate the review of the information. Here, it is easy to see that children in elementary public schools are learning minimal geometry content, with a sudden increase in both the amount and difficulty of the geometry they learn in middle school. Evidence, such as found by Cheng & Lin (2007, p. 119), shows that this approach to teaching geometry is ineffective. Instead, it is a problem because it sets children up for a challenging learning experience.

The problem with teaching geometry as a dependent subject is that it becomes a low priority for teachers. According to PISA, students in elementary schools should only be concerned with understanding space and shapes, which is why the curriculum from K to grade 3 focuses on the visualization of 2D and 3D shapes (Organization for Economic Co-operation and Development, 2013). Teachers are busy professionals, and sometimes the low-priority items do not cut, reducing the covered subjects (Trafton & LeBlanc, 1973).

Likewise, the assessments reflect the low prioritization of geometry, further discouraging teachers from teaching past the curriculum boundaries. According to the Smarter Balanced Assessment Consortium (2015), this has been evident in classrooms in the United States, showing that the focus of educators has been mainly on mathematical computation (Smarter Balanced Assessment Consortium, 2015). Although curriculum designers aim to help children develop problem-solving skills to facilitate the application of school subjects (Pellegrino & Hilton, 2013), the current approach is not achieving it and needs to be revised.

Unlike North America, where geometry curricula still follow the path outlined in the New Math movement, the Netherlands moved past the curriculum created during the New Math movement in about two decades, thanks to Hans Freudenthal, and began teaching what, in the Netherlands, is known as “Realistic Geometry Education.” Dorman described it as an approach focused on spatial orientation where students are first introduced to tasks in 3D contexts that allow them to develop intuitive geometrical reasoning throughout elementary school. Then, children continue with deductive and axiomatic geometry in secondary (Doorman et al., 2020; Freudenthal, 1981a).

Doorman (2022) stated that the traditional approach introduces students to a mathematical world without first allowing them to develop their intuition and practical knowledge about said world. The reform to the educational approach in the Netherlands made geometry more meaningful to students, increased the number of students actively involved in the subject, and broadened the range of topics such as vision lines that children in the Netherlands now learn in primary school or earlier. He claimed that more students now view advanced axiomatic geometry as exciting and intriguing with a strong basis such as this. If complementing a curriculum can have a beneficial impact, such as the one explained by Doorman (2022), educators in Canada should follow in those footsteps.

Why reinvent the wheel?

Most schools following a non-sequential geometry curriculum inherited their curriculum, or a considerable part of it, from the 1960s New Math reform. The severely diminished geometry topics during elementary school and the recommendation for using an abstract teaching approach are the foci of several studies. Scholars have found detrimental effects of this combination of curriculum and teaching approach. Hwang et al. (2019) found that students often fail to understand geometry concepts and solve geometric problems due to a lack of experience applying these concepts in daily life situations. Likewise, Cheng & Lin (2007, p. 119) observed that 40% of middle school students in a Taiwan national sample could not construct an acceptable geometrical proof. Karp & Werner (2011) discuss similar findings in a study that, though made in 1994, they believe accurately reflects the present state of mathematical preparation by saving the complex tasks for the years past elementary. In the study, tenth-grade students attempted to solve ninth grade-level assignments. 40% of the students could not complete such assignments, 30% got top grades, and only 3.5% displayed outstanding results in solving more complex additional exercises. These studies demonstrate that postponing the bulk of geometry for the years of secondary or high school is an ineffective teaching approach. It has caused many students to struggle with learning geometry and meeting expectations. These studies show the negative repercussions of implementing the New Math curriculum and demonstrate the necessity of reinventing the teaching wheel.

On the other hand, there are several schools in North America that, instead of following the traditional approach, follow the educational method outlined by Montessori (1917). Although her curriculum is thorough, wholly sequenced and exceeds the current academic standards, very few scholarly studies measure the effectiveness of Montessori's educational method. One of the few available is that of Lillard (2016), who found that at age 5, Montessori-educated students performed significantly better than their traditionally educated peers in literacy, numeracy, executive function, and social skills. This advantage over the students educated in public traditional schools was also true for the 12-year-old Montessori-educated students with significantly higher scores in creativity, writing, and social skills. (Lillard, 2016, pp. 355–364). Although Lillard's study demonstrates that Montessori students have an advantage over their traditionally educated peers, no similar studies measure the effectiveness of the Montessori method of education when it comes to geometry education. This study will fill that gap in the research. The following paragraphs explain the importance of this work.

Student-led sequential curriculum

Children need to understand why they are learning what their teachers are teaching. This action research will look at the impact of changing a non-sequential curriculum to a student-led sequential one. A review of the literature found different educators who have also seen the importance of taking learners through a step-by-step difficulty-increasing logical learning journey and found it to be a successful practice (Montessori, 1917; Piaget et al., 1960; Van Hiele, 1986). In addition, the results of Lillard's study (2016) demonstrate that the most effective education requires logical sequencing and inclusion of student input. Therefore, the most practical and effective geometry curriculum would follow a logical sequence following the children's interests and abilities.

Piaget

Piaget is known for his vast research in education. In his geometry research, he discovered that children were more successful when beginning their studies with topological relations such as connectedness, enclosure, and continuity —later summarized under the “rubber band geometry” umbrella—. In Piaget’s curriculum, learners could move on to rectilinear and Euclidean studies only after solidifying basic concepts. In this curriculum, the learner focuses first on the essential characteristics of shapes, such as lines, closed and open regions, and others. After mastering the basic concepts, three levels of achievement comprise the lessons on Euclidean space. The first level concerns qualitative operations involving distance, length, area, and volume. At this level, the student learns the concepts only through practical manipulation. Then, the second level involves the previously mentioned concepts with metrical measures. Here, students measure length, distance, angles, and others. Finally, the third level involves formal operations. For example, students calculate area or volume (Piaget et al., 1960). This curriculum shows sequencing and is used often by many constructivist educators, but Piaget believed, like many educators of his time, that young children were biologically incapable of comprehending higher-skilled concepts (Lillard, 2016, p. 352); because of this, his curriculum focused only on the essential skills of geometry, meant to be introduced to young learners and does not connect to the higher geometrical concepts that require advanced mathematics. Due to this basic nature, it would not be a good fit for this action research as it does not meet the current educational standards of grade 6 students as outlined in the BC Curriculum, where the study will take place.

Van Hiele

Despite its gaps, Piaget's ideas did influence other educators who used some of his concepts as building blocks for their more complete methods, such as the one developed by Van Hiele (1986). This researcher considered student engagement important in learning (Van Hiele, 1999). In his levelled model, students begin with a visual study where they identify shapes and figures; then, pupils move to describe their properties in the second level. On the third level, students identify relationships between classes. Learners produce short geometrical statements on the fourth level and move to actual axioms on level 5 (Van Hiele, 1986). Sadly, his model stops there, leaving no connection with other higher-level approaches, such as algebraic geometry.

Montessori

Maria Montessori was the first constructivist to develop a hands-on geometry curriculum that contained the complete sequence of lessons, considered students' interests, and connected with axiomatic and transformational geometry (See Appendix B). Montessori believed the shortest route to a child's mind was through the hands. Because of this, her method does not just focus on the delivery of the subject; instead, she shines a bigger spotlight on the importance of experiencing geometry. To ensure this, she created several didactic materials to accompany her curriculum. Her geometry model requires practice with this sequence of materials. Each material allows the students to use their hands to test what were once only abstract concepts. In addition, Montessori connects this experiential approach to higher mathematical thinking and 3D geometry —that Piaget and Van Hiele missed— by taking the learner through the stages of the development of algebraic equations based on geometrical principles and their application to three-dimensional solids and volume (Montessori, 1917, 1934, 1946; Piaget et al., 1960; Van Hiele, 1986).

Very few scholars have attempted to measure the effectiveness of these geometry curricula. Pussey (2003), recognizing Van Hiele’s curriculum as an improvement from Piaget’s work, focused on testing Van Hiele’s. Pussey (2003) found that when applied to middle school, the curriculum left students poorly prepared for high school. Pussey’s study also had a sample of elementary students, with whom the application of Van Hiele’s curriculum improved the student’s geometrical thinking levels but resulted only in low-level reasoning (Pusey, 2003). Another study, including Van Hiele’s curriculum, was done by Olive (1991). He found the observation of students to be of utmost importance because, without it, students would be introduced to concepts before they were ready, causing them to resort to a mere imitation of the teacher rather than showing a proper understanding of a concept (Olive, 1991). Van Hiele’s geometry curriculum is an improvement from Piaget’s in that it gives a better sequence and includes student understanding as a prerequisite for lesson introduction. However, it leaves gaps in the instruction of middle schoolers (Pussey, 2003). These gaps in the curriculum do not make it ideal for the grade 6 students who will be the subjects of this action research.

In comparison, Montessori’s curriculum surpasses both Piaget’s and Van Hiele’s in detail and the inclusion of student input in the delivery. In addition, Montessori has the only curriculum fully paired with hands-on materials to allow learners to experience each lesson (Montessori, 1917, 1934). Although no scholarly research exists to test the effectiveness of her geometrical approach specifically, Lillard (2016) found that students who followed Montessori's curriculum performed significantly better in numeracy, creativity, and executive function skills tests than students educated with a non-sequential curriculum. Therefore, the lessons in this action research will follow Montessori's curriculum primarily as it is the most complete one and exceeds the educational standards of public schools in North America.

Hands-on learning

Scholars have discussed the importance of using hands-on materials when teaching young children. Piaget et al. (1960) explained that manipulation and interaction with the world build the mental representation of space. Therefore, learners can only internalize geometrical concepts through hands-on activities that allow them to experience them. Likewise, Montessori (1946) spoke about the importance of teaching through manipulatives, as children's hands are the instruments to construct their intelligence. Because of this, she designed a vast array of educational materials to allow children to experience concepts in her curriculum. Therefore, the most effective way to teach geometry concepts to elementary students is through hands-on materials.

Researchers have connected improved learning experiences with using manipulatives in the classroom. Satterthwait (2010) asserted that hands-on activities are the best way to teach students. She explained that when gaining knowledge, neural networks interact and integrate experiences. Sensory input increases these networks, allowing students to better understand the world around them. In a different study, where students conducted mathematics laboratories with materials twice a week, Pedrotti & Chamberlain (1995) found an increase in the number of students who could connect mathematical concepts and their applicability. Likewise, Weber

(2003) supports this idea in his study, emphasizing the importance of allowing students to manipulate real-world geometry —flips, turns and slides— in his transformational geometry lessons. These researchers demonstrate the effectiveness of manipulatives in learning. However, the literature applied specifically to upper elementary geometry is scarce. Therefore, this action research will complement that body of literature.

The impact of joy on learning

The ease of learning a particular set of skills that the learner enjoys is the definition of natural talent. In his research with university students, Hopper (2002) found that even in students who did not report having a natural talent for sports, the interconnectedness of skill progression and skill practice considerably increased student enjoyment when learning and, in turn, allowed them to improve in skills they had already given up on. In contrast, when the content had more importance than the student's journey through the skill progression, the number of learners unable to achieve success increased. In addition, students could not apply skills learned in isolation to different contexts (Hopper, 2022, p. 44).

There is an interdependence between skills acquiring and positive. Students who have experienced success in a particular area will approach the subject with a natural feeling of confidence, which allows them to enjoy learning a new related skill. Light (2003) found that joy's impact also works the other way around. When students with less confidence are taught a progression of skills through games or hands-on learning, their feelings of empowerment rise and, in turn, increase their self-esteem (Light, 2003). Although Light's research is not concerned with the acquisition of geometrical skills, this action research will connect skill acquisition with the enjoyment of students who are engaged in hands-on geometry lessons, such as the "detective game" where the children find a triangle that has committed a crime while learning to identify and name all types of triangles.

Prepared Environment

A prepared environment is not just about the place where students learn. Montessori's work explains that a learning environment should be beautiful, organized, and decluttered. However, it also explains that the children's routine should be just as important. With the diminishing geometry curriculum, its study lost its place in the children's routine. It became a sporadic topic, removing the opportunity for continuous practice, such as they would have for writing or multiplication skills, leading to a decline in learner's geometrical skills (Cao, 2018).

One of the most famous slogans of the Western New Math movement was "Down with Euclid!" (Doorman et al., 2020). However, according to Karp & Werner (2011), Euclid did not go anywhere in Russia. After students learn visual geometry in elementary school, they move on to plane and three-dimensional geometry in middle and high school. Russian curriculum seems similar to the one followed in British Columbia. However, the critical difference is the intensity and, by extension, the depth given to the subject. In Russia, children receive 2 hours per week of geometry instruction (Karp & Werner, 2011). During the present study, students will have constant geometry practice, allowing them to strengthen their skills and keep them fresh enough to apply to the following concepts.

Methodology

Before the study

This study sought to determine how switching from an abstract non-sequential (BC Curriculum based) method to the hands-on Montessori method impacts students' learning attitude and geometry skills. The study started on par with the school year. Before starting the formal research, students received an introduction to the study during the first days of school.

The introduction happened in a conversation in which they received an overview of the geometry course, how lessons would happen, what kinds of one-on-one support would be available, and that they would have a teaching emphasis to help them apply previous concepts to new lessons. This last part would include small reminders, questions, and participative lessons. We also discussed how connecting concepts are critical to developing the confidence and joy of learning.

Next, we discussed where lessons would appear in the weekly schedule, how they would look in their individualized work plans, expectations for follow-up work and where to hand it in. With that part clear, we played a scavenger hunt game within the classroom so they could find individual and communal supplies necessary for the lessons' development, such as paper, duo tangs, compasses, and rulers, as well as the geometry Montessori materials for their use during lessons and practice. The baseline measurement happened after completing the introduction.

Population

The population of this study was composed of 15 grade 6 students, nine males and six females. Since they were underage, parents received a passive consent letter previously approved by the Internal Review Board (IRB). This information went out one week before school started, during the last week of August 2023, and parents had the opportunity to ask questions during the opening meet and greet of the school year. Parents provided many encouraging comments, and nobody asked to have their child's data removed from the study.

During the study

In the baseline phase, students answered a survey and an assessment (See Appendix C). The survey inquired about students' confidence levels and attitudes toward learning geometry, while the assessment showed their academic skills that served to personalize their lesson plans.

The information was digitized in a spreadsheet and compared with the student's answers in the post-study survey and assessment.

The initial intention was to create subgroups of work to present small group lessons. However, the assessment showed that only two students required more practice with basic concepts. Given the importance of social interactions within this age group, I concluded that dividing the group by academic levels would harm their learning attitude as those students may develop hindering beliefs about their learning capacity. Instead, I introduced the initial lessons as a whole grade 6 grouping. Afterwards, during work periods, the students could choose small groups to practice with the material organically while creating their written products. Those students who showed a lower level of knowledge in the initial baseline or required support to stay with tasks (regardless of the topic) received additional one-on-one review lessons with me to clarify the concepts they lacked.

Lessons

The study's lesson plans followed the Montessori curriculum but met the requirements of the British Columbia curriculum. For example, if the curriculum requirement was measuring angles, the students were offered those relevant Montessori lessons, including any pre-requisite concepts or missing links, even if such bases are missing from the BC curriculum. Children had one 15-20-minute weekly lesson, following the Montessori geometry curriculum and method. This study was planned for six weeks, but the total time was extended more because there were two weeks when the combination of school events and holidays did not allow enough time in the classroom for a new lesson and practice time. These weeks were not counted toward the six weeks of the study. The lessons in the study were planned and executed as presented in Figure 1.

Figure 1

Weekly lesson plans for the study

Week	Topic	Concept / Lesson
1	Review of basic concepts - lines, angles	Geometry Pre-Assessment
		Story of the Montessori protractor and review of types of angles (e.g., whole, straight, right, acute, obtuse, reflex, concave)
2		Measuring angles and the sum of interior angles - acute-angled scalene triangle, right-angled scalene triangle, and obtuse-angled scalene triangle
3	Study of figures	Review of parts quadrilaterals and other shapes and their geometrical analysis (square, rectangle, rhombus, parallelogram, trapezoid, common quadrilateral, and polygons).
4	Congruency, similarity, equivalence	Relationship of angles formed between parallel lines cut by a transversal.
5	Advanced study of polygons	Interior angles of polygons, derivation of formulas
6	Review of concepts	Geometry Post-Assessment

Data Collected

Apart from the pre-and post-surveys and assessments, data was collected from a weekly observation form (See Appendix D), in which I reflected on whether the students used the materials to gain understanding, preferred to work with or without materials, what their attitude towards the work was and their academic skill level. These observations happened throughout the work periods of the week when the children were engaged in geometry activities. In addition, pictures of student's work were inputted into a student's work rubric (See Appendix E) form and graded from 1-4 according to these parameters:

- Showed understanding/engagement during the lesson.
- Work presentation meets the standards.
- Practiced the required concept.
- Applied knowledge from previous lessons.

Protocols planned to analyze the data

Pre- and Post-Student Survey

The survey information (See Appendix C) was digitized in a spreadsheet to create a double-bar graph that reflects the changes, if any, in the student's view of themselves as learners after experiencing the intervention.

Pre- and Post-Assessments

I digitized and analyzed the assessments in a spreadsheet where each student had information on proficiency level per topic and the grade level of the topics covered according to the Geometry BC Curriculum. Since the student pre-assessment (Appendix F) contained all the geometry topics learned by the students in the previous school year, this exercise allowed for a comparison of a percentage of achievement in terms of lessons covered and level of proficiency as measured in a student post-assessment (Appendix G).

Observation Forms and Student Rubrics

Academic assessments provided critical information for the study. However, the enjoyment of learning is an essential piece of the Montessori method of education. Therefore, the student observation forms assessed the student's attitude toward learning. The six weeks were digitized in a spreadsheet to analyze this information and create multi-line graphs showing student engagement levels throughout the study.

Data Analysis

Data collection for this study was originally planned from mid-September to the end of October. However, two weeks of interruption due to school events extended the data collection time from mid-September to mid-November.

Lessons and follow-up

The lessons happened in a grades 5-6 classroom, with detailed data tracking applied only to the grade 6 lessons. Most lessons happened on Tuesday mornings when the grade 5 students received Music and French classes outside the classroom. Lessons were given as a whole group, using a cam recorder and a smart board for the children to see the demonstration with the materials. Children could choose to practice with the materials before completing the follow-up practice work. Many of them chose to practice first, while some preferred to practice in tandem with the resolution of their practice work.

In the first weeks of the study, students took most of the week to complete the practice. As the week progressed, they came up with new clarification inquiries that enriched the learning process. However, their work pace increased considerably throughout the study, and their clarification inquiries were slowly replaced, in part, by insightful geometrical analysis of the exercises we were doing. By the end of the study, most students could complete several geometrical exercises on the same day of the lesson.

Before the study, the intention was to divide the group depending on their skill level and work speed. However, only two students marked slightly lower than their peers, and I determined that it would be better to teach the lessons as a whole group rather than isolate these two students. Towards the beginning of the study, both students required much 1:1 organizational and academic support. However, one of them, a visual learner who reported enjoying the graphic and hands-on aspects of the lessons, gradually decreased the need for frequent academic support and only needed organizational cues to be successful in the lesson practices.

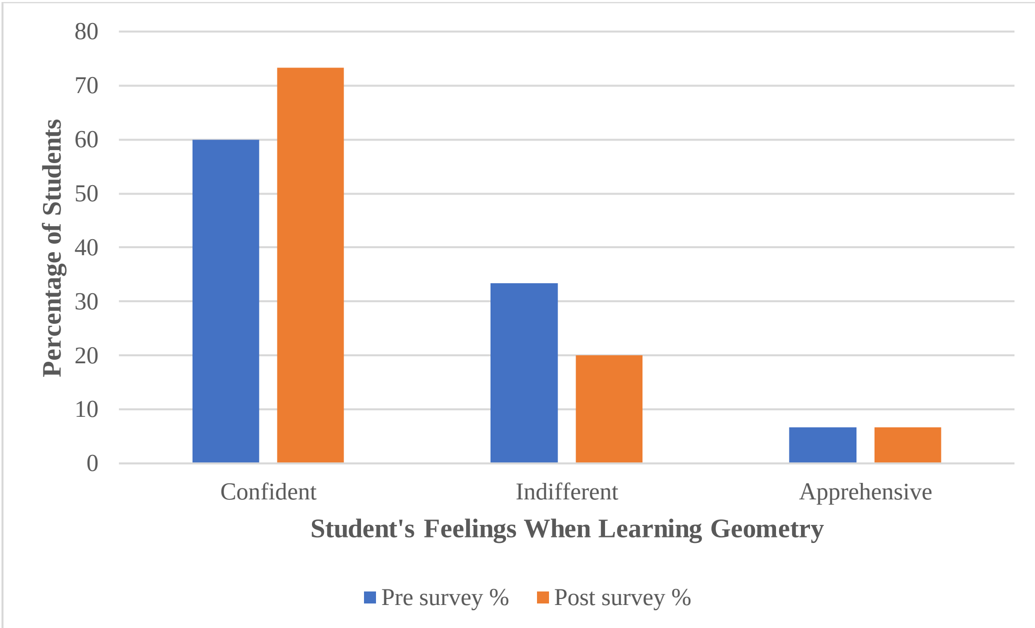
Confidence

Students completed individual self-assessments at the beginning and end of the study. In these documents, they reflected on their feelings regarding learning geometry. These surveys accurately show the student's feelings when called to receive a geometry lesson. They show whether the children found materials helpful in understanding abstract concepts, whether they saw any value in the way the Montessori curriculum follows a logical sequence in the lesson order, and whether their level of joy related to learning geometry lessons increased during the implementation of the study.

Figure 2 displays the percentage of children feeling confident, indifferent, or apprehensive when called to receive a Geometry lesson. Blue columns indicate the students' responses before the study, and orange marks their responses after the intervention. The data indicates a beneficial impact on student confidence, marked by a 13% increase in students who reported feeling confident and a 7% decrease in students who reported indifference. The percentage of students who reported apprehension towards the subject remained unchanged.

Figure 2

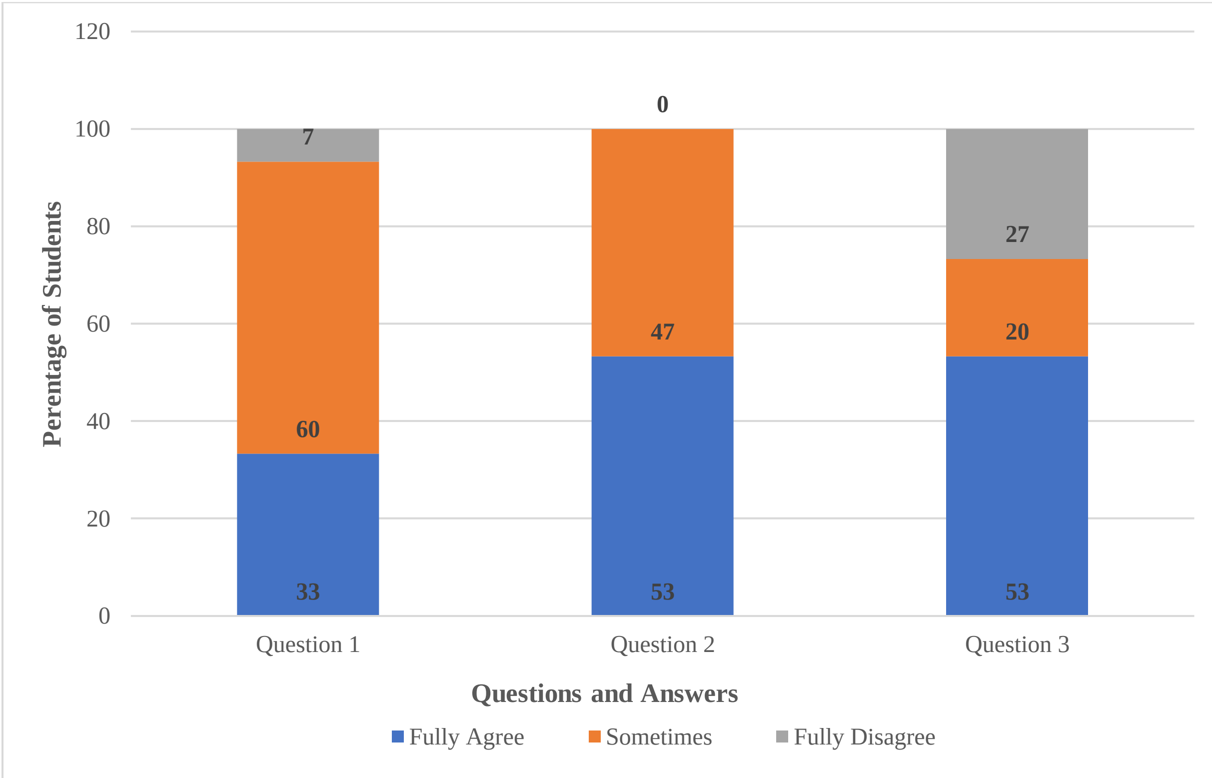
Students' responses to "How do you feel when called to receive a Geometry lesson?"



Similarly, Figure 3 shows 93% of students reported that most or all of the time, they had an easier time understanding abstract concepts after they saw it demonstrated with materials. Likewise, the totality of students reported that it was easier to understand geometric concepts with logically organized lessons. All students agreed with this statement, with 53% answering that this was the case all the time and the remaining 47% saying the previous statement was true most of the time. None of the students said that the lesson's logical organization was unimportant. Similarly, 53% reported that during the intervention, they felt an increase in joy related to geometry learning, 20% reported that the increase in joy towards geometry learning happened sometimes, and 27% reported no change in their level of joy compared to previous years of learning.

Figure 3

Student Survey Post-Intervention



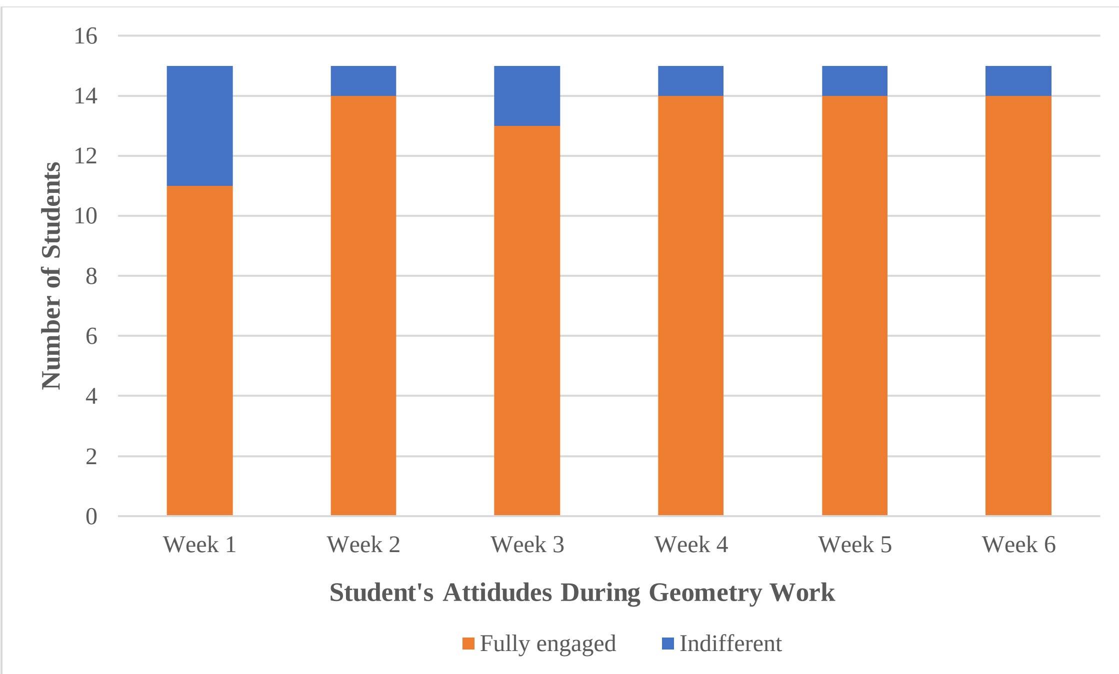
Note: Question 1: Abstract concepts were easier to understand once I had seen them with materials. Question 2: Having lessons that connected with the previous one in a logical way made geometry easier to understand. Question 3: During the past 6 weeks, I've felt more joy in learning geometry and felt more successful at it than in previous years.

These results are consistent with the teacher's observations (See Appendix D). Figure 4 illustrates the weekly recorded observations. This tool assessed students' learning attitudes during independent Geometry practice. It classified them as follows: Students classified as indifferent did the work but did not seem to enjoy it. They were classified as fully engaged when asking questions and excited about learning new concepts. Finally, they were classified as frustrated when they showed frustration during individual practice times.

Along the same lines, Figure 4 shows an increase in the percentage of fully engaged students during independent Geometry practices that began at 73% in week 1, and, except for week 3, where it changed to 86%, the percentage of fully engaged students remained at 93% for the duration of the study. Contrasted with the percentage of indifferent students starting at 27%, decreased to 7%, remaining there throughout the study and increasing only in the week 3 to 13%. No students were not engaged or frustrated, which is why these categories don't appear in the graph.

Figure 4

Students' learning attitude throughout weekly independent practices



As explained above, the initial lessons happened in a group format where students could ask questions as needed. However, the individualization of the learning happened during independent practice, where students could choose to access different supports, such as 1:1 teacher guidance with the geometric concepts or breaking tasks into smaller steps. Different learners benefit from different supports.

BC Curriculum Comparison

Students’ work output and in-classroom practice determined the assessment of their academic growth. Each lesson contained a list of expectations to meet in their work and practice. For example, to use conventions to write geometrical statements. They had access to a control of error or answer key to check their work, which they usually did with a different pen colour than the one they used for regular writing. Students’ work assessment followed the rubric provided during lessons. They were then classified as meeting most of the expectations, meeting some of the expectations or not meeting expectations. No student ended up not meeting expectations, as children always had the choice to go back and review their work.

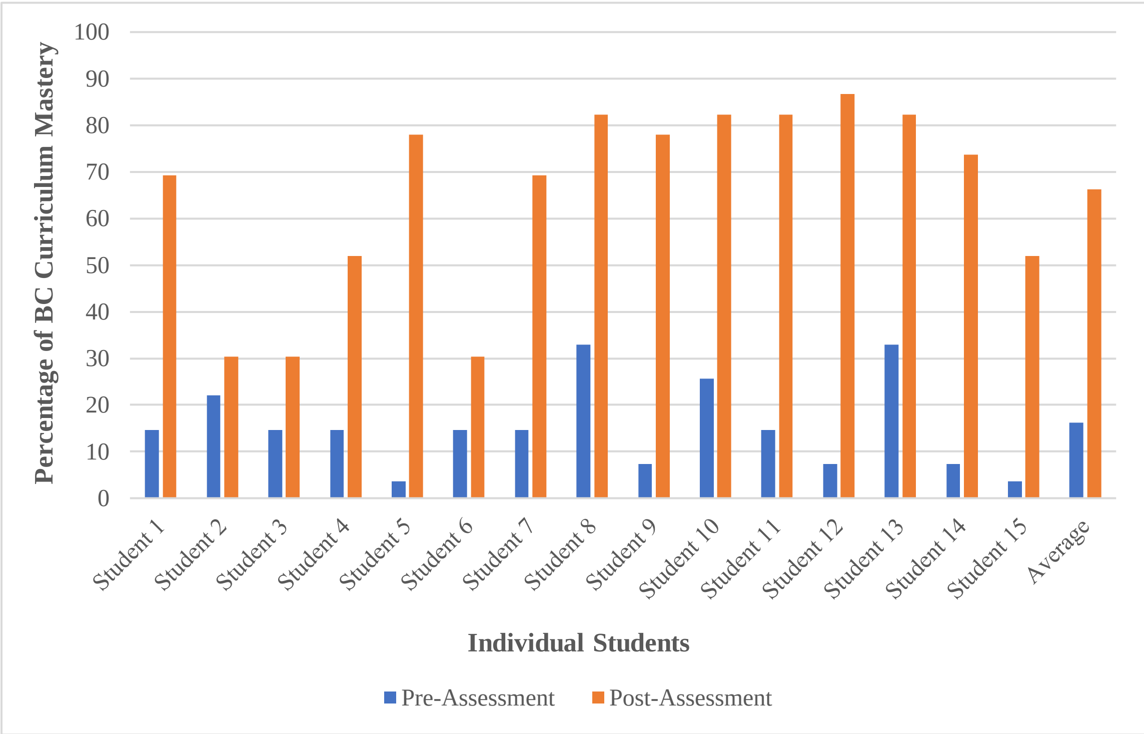
However, the delivery of the pre-and post-assessments worked differently. Students were asked to work individually and could only access Montessori materials, a pencil, and a ruler. Children could ask questions to clarify instructions but not to explain geometrical concepts. This was done so the assessments could truly reflect what the children remembered.

The academic pre- and post-assessments of the students measured their mastery over the learning outcomes outlined in the BC curriculum for grade 6. Each learning outcome for grade 6 has a percentage. Only some outcomes were used in the pre-and post-assessments, considering the six-week length of this study. The pre-assessment contained 73% of the BC outcomes for grade 6 geometry, whereas the post-assessment contained 87%. If students completed the post-assessment perfectly, they would have demonstrated mastery over 87% of the BC Curriculum outcomes for their grade level.

The results show academic growth concerning the BC Curriculum per student, illustrated in Figure 5. The blue bars show the percentage of BC Curriculum mastery demonstrated by students on the pre-assessment, which ranged from 4% to 33%, with an average of 16%. In contrast, the orange bar shows the post-assessment data, which ranged from 30% to 87%, with an average of 66%.

Figure 5

Percentage of BC Curriculum mastery



Action Plan

The main goal of this action research was to explore how replacing an abstract, non-sequential approach to geometry education with the sequential, hands-on Montessori method would impact students' learning attitudes and skills. The results indicate that implementing this project positively affected students' mastery of geometrical concepts, improved their confidence and learning attitude in future geometry lessons, and improved their enjoyment of the subject.

Given that government-funded independent schools must meet the learning outcomes of the BC Curriculum, regardless of the results of this study, implementation of the present intervention in public or independent schools would only be possible if the Montessori curriculum and method used throughout the study met or exceeded the BC Curriculum standards. Since the data proved that the Montessori curriculum exceeded BC geometry standards, I will continue to implement it moving forward. When extrapolating the percentage of mastery after the implementation, it can be predicted that if the implementation extends throughout the 35 weeks of an entire school year, children would likely cover the content of more than two grades as currently established in the BC Curriculum. I want to confirm whether this extrapolation aligns with the practice.

It is beyond this study's scope to address the impact a similar intervention would have on a larger scale. Future research on a larger population could deepen the understanding of the connections between different types of learners, and the classroom supports that lead to student success.

In addition, observations during the implementation of this research marked a differentiation between the supports used by self-identified auditory and visual learners. However, further study is needed to deepen the understanding of this area.

Likewise, this is the first study in the literature where the efficacy of the Montessori Method in geometry was measured. Similar studies could confirm or contrast the present findings.

Lastly, another recommendation considers that grade 6 students are already in middle school according to BC Standards. Therefore, their affinity with materials is lower than that of younger students. A future study could focus on a younger population, given that Montessori (1917, 1934) states that the sensitive period for geometry learning is the elementary years.

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Appendix A

K to 12 British Columbia Geometry curriculum

Grade	Topic	Lessons
K	direct comparative measurement	using a baseline for direct comparison in linear measurement
		linear height, width, length (e.g., longer than, shorter than, taller than, wider than)
		mass (e.g., heavier than, lighter than, same as)
		capacity (e.g., holds more, holds less)
	single attributes	sorting 2D shapes and 3D objects using a single attribute
	single attributes	using positional language, such as besides, on top of, under, and in front of
1	direct measurement	using a baseline for direct comparison in linear measurement
	comparison of 2D and 3D objects	comparing 2D shapes and 3D objects in the environment
		using positional language, such as besides, on top of, under, and in front of
		replicating composite 2D shapes and 3D objects (e.g., two triangles make a square)
2	multiple attributes of 2D and 3D objects	sorting 2D shapes and 3D objects, using two attributes, and explaining the sorting rule
		describing, comparing, and constructing 2D shapes (e.g. triangles, squares, rectangles, circles)
		identifying 2D shapes as part of 3D objects
3	measurement with standard units	linear measurements, using standard units (e.g., centimetre, metre, kilometre)
		capacity measurements, using standard units (e.g., millilitre, litre)
		Introduction of concepts of perimeter, area, and circumference (no calculations)
		area measurement, using square units (standard and non-standard)
		mass measurements, using standard units (e.g., gram, kilogram)
		estimation of measurements, using standard referents\
	construction of 3D objects	identifying 3D objects according to the 2D shapes of the faces and the number of edges and vertices (e.g., construction of nets, skeletons)

Grade	Topic	Lessons
		describing the attributes of 3D objects (e.g., faces, edges, vertices)
		identifying 3D objects by their mathematical terms (e.g., sphere, cube, prism, cone, cylinder)
		comparing 3D objects (e.g., How are rectangular prisms and cubes the same or different?)
		understanding the preservation of shape (properties remain regardless of orientation)
4	regular and irregular polygons	describing and sorting regular and irregular polygons based on multiple attributes
		investigating polygons (polygons are closed shapes with similar attributes)
	perimeter	using geoboards and grids to create, represent, measure, and calculate perimeter
	line symmetry	using concrete materials such as pattern blocks to create designs that have a mirror image within them
5	classification	investigating 3D objects and 2D shapes, based on multiple attributes
		describing and sorting quadrilaterals
		describing and constructing rectangular and triangular prisms
		identifying prisms in the environment
	single transformations	single transformations (slide/translation, flip/reflection, turn/rotation)
		using concrete materials with a focus on the motion of transformations
6	perimeter	of complex shapes
	area	grid paper explorations
		deriving formulas
		making connections between area of parallelogram and area of rectangle
	angle	straight, acute, right, obtuse, reflex
		constructing and identifying; include examples from local environment
		estimating using 45°, 90°, and 180° as reference angles
		angles of polygons

Grade	Topic	Lessons
	volume and capacity	using cubes to build 3D objects and determine their volume
		referents and relationships between units (e.g., cm3, m3, mL, L)
	triangles	scalene, isosceles, equilateral
		right, acute, obtuse
		classified regardless of orientation
	combinations of transformations	translation(s), rotation(s), and/or reflection(s) on a single 2D shape
		transforming, drawing, and describing image
7	circumference	constructing circles given radius, diameter, area, or circumference
		finding relationships between radius, diameter, circumference, and area to develop $C = \pi \times d$ formula
		applying $A = \pi \times r \times r$ formula to find the area given radius or diameter
	volume	formula
	combinations of transformations	translation(s), rotation(s), and/or reflection(s) on a single 2D shape; combination of successive transformations of 2D shapes; tessellations
8	surface area and volume	surface area of solids = sum of the areas of each side
		volume of solids
	Pythagorean theorem	modeling and applying
	construction of 3D objects	drawing and interpreting top, front, and side views of 3D objects
9	spatial proportional reasoning	scale diagrams, enlarge or reduce
10	primary trigonometric	sine, cosine, and tangent ratios
		right-triangle problems: determining missing sides and/or angles using trigonometric ratios and the Pythagorean theorem
		contexts involving direct and indirect measurement
	metric and imperial conversions	with a focus on length as a means to increase computational fluency

Grade	Topic	Lessons
	surface area and volume	including prisms and cylinders, formula manipulation
		contextualized problems involving 3D shapes
11	trigonometry	use of sine and cosine laws to solve non-right triangles, including ambiguous cases
		contextual and non-contextual problems
		angles in standard position:
		use of sine and cosine laws to solve non-right triangles, including ambiguous cases
		contextual and non-contextual problems
		angles in standard position: degrees, special angles, as connected with the 30-60-90 and 45-45-90 triangles
	3D objects	creating and interpreting exploded diagrams and perspective diagrams
		drawing and constructing 3D objects
	scale models	enlargements and reductions of 2D shapes and 3D objects
		comparing the properties of similar objects (length, area, volume)
		square-cube law
12	measuring	precision and accuracy
		unit analysis
	similar triangles	application of the Pythagorean theorem
	2D and 3D shapes	area, surface area, volume and nets
	3D objects	creating and reading various types of technical drawings
	geometric constructions	angles, triangles, triangle centres, quadrilaterals
	parallel and perpendicular	angle bisector lines
	circles as tools	constructing equal segments, midpoints
		perpendicular bisector
	circle geometry	properties of chords, angles, and tangents to mobilize the proving process

Grade	Topic	Lessons
	constructing tangents	lines tangent to circles, circles tangent to circles, circles tangent to three objects (e.g., points [PPP], three lines [LLL])
	isometries	transformations that maintain congruence (translations, rotations, reflections)
		composition of isometries
		tessellations
	non-isometric transformations	dilations and shear
		topology
	non-Euclidean geometries	perspective, spherical, Taxicab, hyperbolic
		tessellations

Table 1. Extracted from (Province of British Columbia, n.d.) and (Province of British Columbia, 2016)

Appendix B

Montessori geometry scope and sequence

Grade	Topic	Concept / Lesson
Children's House (Preschool and Kindergarten)	inro to geometry	the gift of Egypt: Beginnings of geometry
	sensorial exploration	name and matching of perfect shapes (e.g. circle, triangle, square) and related activities
		sensorial differentiation between square and rectangle
		names and matching triangles (equilateral, isosceles, scalene)
		names and matching quadrilaterals (e.g. trapezoid, parallelogram)
		names and matching curved shapes (e.g. ellipse, oval, quatrefoil)
		names and matching polygons (e.g. pentagon to decagon)
		using triangles to build shapes and naming them
		identifying and tracing important lines within shapes
		names of 10 common solids
		comparison of measurement between solids (height, with, length, mass)
		identifying bases within the solids
		classifying solids according to bases (stacking)
		capacity experiential comparisons done within practical life activities
		positional language (next to, in front of, behind)
		classifying solids according to bases (stacking)
Elementar y (Grades 1 to 6)	sensorial study of triangles	reviewing names of polygons that can be built with triangles
		forming shapes and identifying lines
		constructing pinwheels with triangles
		constructing diaphragms with triangles
		constructing stars with triangles

Grade	Topic	Concept / Lesson
Elementary (Grades 1 to 6)	basic concepts - lines	from point to solid
		definition of lines
		kinds of lines
		parts of straight line
		positions of a straight line in relation to the Earth
		relationship between two lines
	basic concepts - angles	types of angles (e.g. whole, straight, right, acute, obtuse, reflex, concave)
		parts of an angle
		story of the Montessori protractor
		measuring angles
		construction of angles
		positions of two straight lines
		rays and line segments having a common point
		construct perpendicular lines
		construct a square
		find the midpoint of a line
		draw and equilateral triangle
		draw a bisector
		the relationship between two angles
		vertical angles
		draw a sixty degree arc
		parallel lines cut by a transversal

Grade	Topic	Concept / Lesson
Elementary (Grades 1 to 6)		non- parallel lines cut by a transversal
		exploring the relationship between angles of same kind
		interior angles (alternate and same side)
		exterior angles (alternate and same side)
		alternate angles (interior and exterior)
		on the same side (interior and exterior)
		corresponding angles
		sum of interior angles - acute-angled scalene triangle, right-angled scalene triangle and obtuse-angled scalene triangle
		tessellation
	introduction to study of figures	open and closed regions and the concept of a polygon
	study of figures - triangles	review of names according to sides
		names according to angles
		full names of triangles (combining sides and angles)
		seven triangles of reality
		parts of the triangle - geometrical analysis
		special parts of a right-angled triangle
		other parts of triangles (sensorial exploration of axis, median, and bisector)
	study of figures - quadrilaterals	construction of quadrilaterals (trapezoid, parallelogram, rectangle, rhombus, square)
		parts of squares and rectangles - geometrical analysis
		parts of parallelograms - geometrical analysis
		types of trapezoids and parts of a trapezoid - geometrical analysis
		parts of a common quadrilateral - geometrical analysis
		formation of diagonals in quadrilaterals

Grade	Topic	Concept / Lesson
		nine quadrilaterals of reality
	study of figures - polygons	review of convex and reflex angles
		regular and irregular polygons
		examination of polygons with more than four sides
		diagonals in a polygon
		parts of a polygon - geometrical analysis
	advanced study of polygons	interior angles of polygons
		exterior angles of polygons
		3 more lines on triangles (orthocenter, incenter, barycenter, circumcenter)
	advanced sensorial study	congruency, similarity and equivalence
		forming new shapes with triangles
		relationship of figures according to parts
		relationship of figures according to lines
		comparison of all figures known
	arithmetic and sensorial analysis	ratio between different triangles
		ratio between different hexagons
		ratio of triangles using mediators (rhombus , trapezoids and deltoids)
		ratio between inscribed and circumbscribed figures
		sensorial introduction to the Pythagorean theorem
		equivalence of triangle to rectangle
		equivalence of rhombus to rectangle
		equivalence of common parallelogram to rectangle
		equivalence of trapezoid to rectangle
		equivalence of trapezoid to another rectangle

Grade	Topic	Concept / Lesson
Elementary (Grades 1 to 6)		equivalence of rectangle to pentagon
		equivalence of decagon to rectangle
	Pythagorean theorem	arithmetical proof
		concept of projection
		Euclid's demonstration
		nomenclature and relationship of lines
		algebraic demonstration
	perimeter	perimeter of polygons
	area	how area is measured
		formula and practice with area of rectangles
		derivation of the area formula of a parallelogram
		derivation of the area formula of an acute-angled triangle
		derivation of the area formula of an right-angled triangle
		derivation of the area formula of an obtuse-angled triangle
		derivation of the area formula of a square
		derivation of the area formula of a rhombus
		derivation of the area formula of a trapezoid
		derivation of the area formula of a regular polygon
		calculation of area with common quadrilaterals
	study of figures - circle	polygon with infinite sides
		parts of a circle
		computing circumference

Grade	Topic	Concept / Lesson
		area of a circle
		relationship between a circle and a straight line (tangent, secant)
		relationship between two circles
	study of figures - ellipse	parts of the ellipse
		area of an ellipse
	volume	surfaces become solids
		review of naming three dimensional solids
		derivation of the volume formula of a rectangular prism
		comparing volumes of rectangular prisms with other solids
		derivation of the volume formula of a regular triangle prism
		derivation of the volume formula of a regular hexagonal right prism
		derivation of the volume formula of a square based pyramid
		derivation of the volume formula of a cylinder
		derivation of the volume formula of a cone
		derivation of the volume formula of a sphere
		derivation of the volume formula of an ellipsoid
		derivation of the volume formula of an ovoid

Table 2. Extracted and summarized from (Montessori, 1917, 1934)

Appendix C

Student Pre- Survey

Mark each answer with an X

1. When I am learning something new, I prefer to…
    a) Have someone show me how to do it
    b) Have someone tell me how to do it
    c) Figure it out myself
2. I remember things best when I…
    a) Write or draw them
    b) Say them over and over
    c) Move around and use hand gestures while repeating them in my head
3. In the previous year, I've mainly learned Geometry…

    a) Abstractly (pen/pencil and paper only).
    b) I first use materials to help me visualize the concepts before writing/drawing things down on paper.

4. Generally, my first thought/feeling when called to receive a Geometry lesson is…

    a) Confident: It will be easy.
    b) Unsure: I can't remember the last topic I learned, and I will need a reminder before fully understanding a new one.
    c) Apprehensive: I am not too excited because I struggled with a topic in the past.

5. Think about your previous experiences learning Geometry to answer the following questions. To what extent do you agree or disagree with the following statements:

Statement	**I fully agree**	**Maybe**	**I fully disagree**
a) I prefer to know why I need to know a lesson before putting my energy into the lesson.			

b) I normally understand how I will use the concepts that I learn by the end of the lesson			

Post- Survey

Mark each answer with an X

1. In the last 6 weeks, I've mainly learned Geometry…

a) Abstractly (pen/pencil and paper exclusively).

b) I first use materials to help me visualize the concepts before writing/drawing things down on paper.

2. Generally, my first thought/feeling when called to receive a Geometry lesson is…

a) Confident: It will be easy.

b) Unsure: I can't remember the last topic I learned, and I will need a reminder before fully understanding a new one.

c) Apprehensive: Because I struggled with a topic in the past.

Statement	**I fully agree**	**Sometimes**	**I fully disagree**
a) I found it easier to learn a new concept when I saw them with materials first.			
b) Abstract concepts, such as formula applications, were easier to understand once I had seen them with materials.			
c) Having lessons that connected with the previous one in a logical way made geometry easier to understand.			
d) The logical sequence of the lessons made it easier to apply concepts from one week to the next.			

e) During the past 6 weeks, I've felt more joy in learning geometry and felt more successful at it than in previous years.			

Appendix D

Weekly Observation Form

GEOMETRY WEEKLY OBSERVATION FORM			
Week:		**Date:**	
Student	Observations		
	Used materials to gain understanding	Work preference (abstractly, materials, individually, group)	Attitude towards learning
Student 1			☺ 😐 ☹ 😣
Student 2			☺ 😐 ☹ 😣
Student 3			☺ 😐 ☹ 😣
Student 4			☺ 😐 ☹ 😣
Student 5			☺ 😐 ☹ 😣
Student 6			☺ 😐 ☹ 😣
Student 7			☺ 😐 ☹ 😣
Student 8			☺ 😐 ☹ 😣
Student 9			☺ 😐 ☹ 😣
Student 10			☺ 😐 ☹ 😣
Student 11			☺ 😐 ☹ 😣
Student 12			☺ 😐 ☹ 😣
Student 13			☺ 😐 ☹ 😣
Student 14			☺ 😐 ☹ 😣
Student 15			☺ 😐 ☹ 😣
Other comments:			

Appendix E

Student's Work Rubric

Student's identification number here

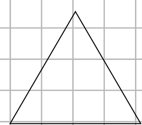
Week/Lesson	Miniature Picture of follow-up work	Understanding and Engagement during lesson	Work presentation meets standards	Practiced the required concept	Applied knowledge from previous lessons
Week # – Title of lesson	Picture of student's work	Observation notes from the form and others. Record supports used by students here.	Numerical grade out of 4. If not 4, record the reason.	Yes/No	Yes/No

Appendix F

Student Pre-Assessment

What do you remember?
Geometry

1. Write the names by sides and by angles of the following triangles.



Name by sides:
Name by angles:

Name by sides:
Name by angles:

Name by sides:
Name by angles:

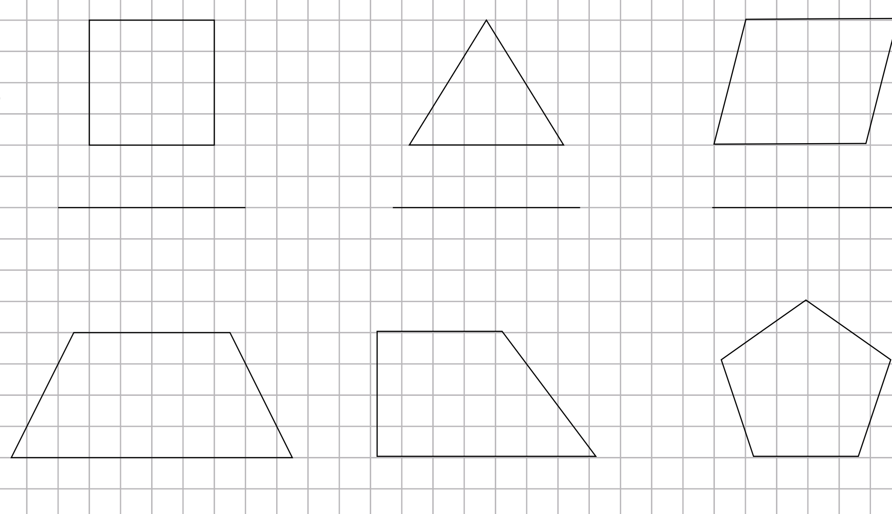
Name by sides:
Name by angles:

Name by sides:
Name by angles:

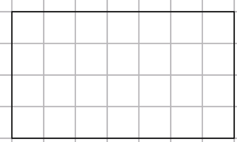
Name by sides:
Name by angles:

2. Under each shape, write their name. Then, use the colours indicated in the conventions to colour their corresponding parts.

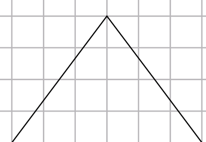
Navy	Perimeter
Red	Base
Green	Vertices
Purple	Altitude
Orange	Diagonal
Yellow	Surface



3. Measure the necessary parts to calculate the perimeter (P) and the area (A) of the following shapes. Use millimetres for your measurements and write down all of your calculations.
Note: Each line of the squares in the grid measures 5 mm.

a)  P = A =

b)  P = A =

c)  P = A =

Appendix G

Student Post-Assessment

1. Draw a triangle that meets the following criteria:
   - The three sides are different lengths.
   - 1 angle is greater than 90°

Then, circle the words that define this triangle:

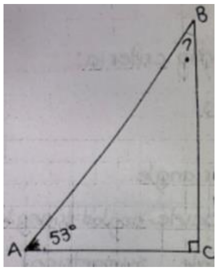
Scalene – Isosceles – Equilateral – Acute-angled triangle – Obtuse-angled triangle – Right-angled triangle – Symmetrical – Non-symmetrical

2. Draw a triangle that meets the following criteria:
   - Two sides are the same length, and the third is a different length.
   - 1 angle is exactly 90°

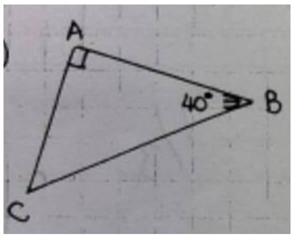
Then, circle the words that define this triangle:

Scalene – Isosceles – Equilateral – Acute-angled triangle – Obtuse-angled triangle – Right-angled triangle – Symmetrical – Non-symmetrical

3. Use what you've learned about the inner angles of triangles to figure out the amplitude of the angle B.



4. Find the amplitude of angle C.



5. Draw and name a polygon with eight sides. Then, find out the amplitude of one of its angles.