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Echolocation on Manifolds

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ECHOLOCATION ON MANIFOLDS

by

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Abstract

We consider the question asked by Wyman and Xi [WX23]: "Can you hear your location on a manifold?" In other words, can you locate a unique point x on a manifold, up to symmetry, if you know the Laplacian eigenvalues and eigenfunctions of the manifold? In [WX23], Wyman and Xi showed that echolocation holds on one- and two-dimensional rectangles with Dirichlet boundary conditions using the pointwise Weyl counting function. They also showed echolocation holds on ellipsoids using Gaussian curvature.

In this thesis, we provide full details for Wyman and Xi's proof for one- and two-dimensional rectangles and we show that echolocation also holds on many three-dimensional boxes. We also prove that echolocation holds on rectangles with certain mixed boundary conditions using a similar approach. Secondly, we explore echolocation via Gaussian curvature and we focus on two categories of manifolds, namely surfaces of revolution and minimal surfaces. We provide counterexamples to two conjectures of necessary conditions for echolocation to hold on surfaces of revolution. We also show that Gaussian curvature is not enough for us to echolocate on Enneper's

surface and Henneberg's surface by constructing pairs of non-symmetric points that have identical Gaussian curvatures.

Chapter 1

Introduction

In 1966, Mark Kac asked the famous question: "Can one hear the shape of a drum?" [Kac66] The question has become important in the field of spectral geometry while its origins were in physics and chemistry. Around the mid-19th century, the French philosopher Auguste Comte speculated that knowledge of the chemical composition of stars would be beyond science's grasp forever. However, spectroscopy was developed a few years later and enabled scientists to determine stellar atmospheric composition. Helium, a major universal element, was first discovered in the spectrum of the sun during a total solar eclipse via spectroscopy. By observing the natural vibration frequencies of the star, scientists were able to compare them to the frequencies of chemicals in the lab and determine the chemical composition of the star. The study of the frequencies of different systems has led to many theoretical questions, including Kac's question. More specifically, if people know the vibration frequencies (spectrum) of a drum, is it possible to determine the shape of the drum? In 1992, Gordon, Webb,

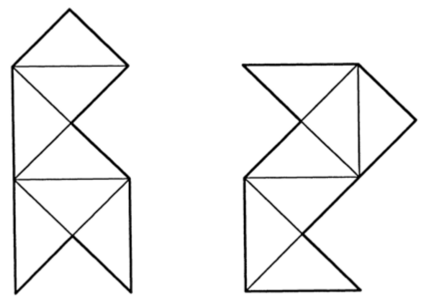

Figure 1.1: Two different drums that sound the same.

and Wolpert [GW96] gave a negative answer to the question by constructing a pair of concave polygons that produce the same vibration frequencies while having different shapes, as shown in Figure 1.1. In general, a concave polygon is a polygon that has at least one part that is "caved in" or indented. Although it seems that Kac's original question has been completely answered for concave polygons, there are still variations and related questions yet to be answered. For example, are there specific shapes of drums that one can hear?

Before discussing more specific questions, we shall first introduce a piece of important terminology: manifold. Essentially, a manifold is a mathematical object that might be somewhat "curvy" or might possess some other special properties, but if you zoom in close enough on any small part of it, it looks flat. For example, consider the earth as a manifold. A tiny piece of the Earth's surface appears flat when you stand on it, even though we know the Earth can be approximated by a sphere. In this thesis, we work under the assumption that our manifolds are smooth and compact; smooth means you can slide your finger along it without hitting any abrupt stops while compact means our manifolds don't stretch out to infinity.

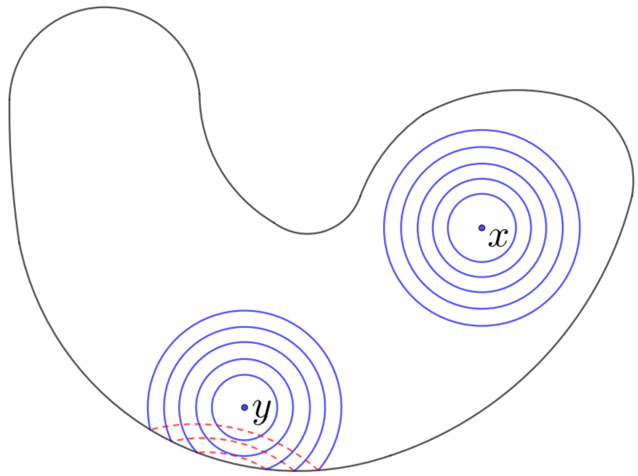

Figure 1.2: Echolocation in a concave drumhead. [WX23]

We focus on a variation of Kac's problem. Instead of trying to identify the shape of a drum, or in general, a manifold, Wyman and Xi asked "Can you hear your location on a manifold?" [WX23] Note that if a manifold is symmetric, then there are definitely pairs of points that sound the same, which makes the question uninteresting to some extent. Therefore, Wyman and Xi proposed

Question 1. *Suppose we know everything there is to know about the manifold M. Is it possible to identify a point x on the manifold up to symmetry?*

Here, we provide a physical interpretation of the question: "Can you hear where a drum is struck?" If you know the shape of a drum and if the drum gets struck at a point x, would it be possible to determine x, up to some symmetry, by listening to the sounds that the drum produces? Figure 1.2 represents a concave drumhead getting hit at points x and y, and the red dashed lines represent the sound reverberation. In the work of Buser, Conway, Doyle, and Semmler [BCDS94], they identified some pairs of drums that, if struck at some special points, would "sound the same." In other words, the waves that the drums produce will have the same frequency and the same intensity. Therefore, it would be natural to ask, can we identify those special points? Wyman and Xi's [WX23] showed echolocation is possible on one-dimensional strings and two-dimensional rectangles with fixed boundaries using eigenfunctions. Eigenfunctions contain rich geometric information that helps us study the vibration of the manifolds. Later, we will start with eigenfunctions to derive a formal definition of echolocation. Wyman and Xi argued if we consider higher dimensional hyper-rectangles, the problem will be more complicated. In this thesis, we extend from their examples and show that echolocation holds on three-dimensional boxes. Furthermore, we also look at rectangles with different boundary conditions. More specifically, when one side of the rectangle is no longer simply fixed, can we still use eigenfunctions to echolocate?

In Example 3.4 of their paper, Wyman and Xi also showed that you can echolocate on an ellipsoid via Gaussian curvature [WX23]. Gaussian curvature is a mathematical property that measures how curvy a manifold is. In this thesis, we use Gaussian curvature to explore echolocation on two different categories of manifolds, namely surfaces of revolution and minimal surfaces.

The thesis is organized as follows. In Chapter 2, we provide some mathematical definitions that are crucial for the understanding of this thesis and allow us to rephrase Question 1 more rigorously. Chapter 3 contains the results of echolocation on different rectangles via eigenfunctions. Chapter 4 starts with an introduction to Gaussian curvature and the derivation of its formula, then we discuss the results of echolocation on surfaces of revolution and minimal surfaces using Gaussian curvature. Finally, Chapter 5 concludes the thesis and contains some directions for future studies.

Chapter 2

Background

2.1 Physical Interpretation of "Can Be Heard"

Consider any 2-dimensional drumhead M and denote the boundary as ∂M. Suppose we hit the drum at some point; we may model the vertical displacement of any point (x, y) on the drumhead at any time t by a function $u(t, x, y)$. To further derive information about u, imagine the disturbance on the drumhead. This traveling disturbance is what we call a "wave." The wave equation is a mathematical model that helps us understand how this wave moves over time and space. There are two key components of the wave equation. First is the time evolution, or acceleration. Imagine hitting a drumhead. The drumhead will vibrate and move up and down rapidly. The drum is displaced a lot at the point where it was hit but not a lot near the boundary. As time goes on, the point of maximum displacement moves across the drumhead, creating a traveling wave pattern. Eventually, the vibration diminishes and the wave becomes smaller. The "time evolution" part of our equation helps us predict how the surface of the drum will move at every moment after it is hit. The second important part of the equation encodes the curvature of the drum. As we can imagine, when the membrane of the drumhead is more curved, more force would be needed to return the drumhead to equilibrium.

Mathematically, for any $(x, y) \in M, t > 0$, the wave equation is given by

$$u_{tt} = c^2 \Delta u, \tag{2.1}$$

where u_{tt} is the second derivative of $u(t, x, y)$ with respect to t and $\Delta u = u_{xx} + u_{yy}$. For $(x, y) \in \partial M$, the boundary condition will vary depending on our assumption. Here, we may assume that any point on the boundary must satisfy $u(t, x, y) = 0$ for any $t > 0$, so the boundary of the drumhead is always fixed. The positive constant c^2 depends on the properties of each drum such as materials and tension of the membrane; without loss of generality, we assume $c^2 = 1$. Even though there might be other solutions to the wave equation, we're particularly interested in solutions of the form

$$u(t, x, y) = v(x, y)e^{i\omega t}, \tag{2.2}$$

for which we can separate the space- and time-dependence of u. If we twice differentiate u with respect to t, we have

$$u_{tt} = -\omega^2 v(x, y)e^{i\omega t} = -\omega^2 u,$$

where the ω are different frequencies that the drum produces. Plugging this expression for u_{tt} into the wave equation, and rewriting it so that the right-hand-side equals 0 for further reference, we get u must satisfy

$$\Delta u + \omega^2 u = 0. \tag{2.3}$$

It was well-known [Tayl] that there exists a sequence of ω for which we can find non-trivial solution u to Equation (2.3). Here, we might interpret the famous question "Can you hear the shape of the drum" as "If you have complete information about the frequencies that a drum produces, can you determine the shape of the drum?" In other words, does the sequence of frequencies (ω_n) uniquely determine the shape of M? As we will see in the next chapter, solving the wave equation helps us extract useful information from the vibrations of the "drum."

2.2 Heat Kernel and Spectral Invariants

Now, we take a more mathematical approach toward the properties of a drum that can be "heard." Consider a certain manifold M with boundary ∂M. Imagine heating it at a single point $x \in M$ to an infinite temperature and keeping the rest of the manifold at temperature 0. This initial condition is modeled by the Dirac δ-function supported at the point $x \in M$, denoted by δ_x. Next, we let the heat flow within the manifold while keeping the temperature on the boundary ∂M at 0. Let u be a function that models this heat flow, so $u(t, x, y)$ gives the temperature of point y

at time t given that the manifold is initially heated at x. Note that here x, y each represents a unique point on M, where we use this notation because we don't restrict our manifold to $\mathbb{R}^2$.

Definition 1. *In the $d-$dimensional Euclidean space $\mathbb{R}^d$, let*

$$\Delta f = \sum_{j=1}^{d} \frac{\partial^2 f}{\partial x_j^2},$$

where $f = f(x_1, \ldots, x_d)$ is twice differentiable and ∂ denotes the partial derivative. The operator $-\Delta$ is called the Laplacian in $\mathbb{R}^d$.

With this notation, our temperature function $u(t, x, y)$ solves the heat flow model given by

$$\begin{cases} \Delta u = \dfrac{\partial u}{\partial t} \\ u(0, x, y) = \delta_x(y) \\ u(t, x, y) = 0, \qquad y \in \partial M, t > 0. \end{cases} \tag{2.4}$$

In order to understand the solution to this model, we first introduce the definition of Laplacian eigenvalues and eigenfunctions.

Definition 2. *Given a manifold M, the Laplacian eigenvalue problem is a set of equations of the form*

$$\Delta\phi(x) + \lambda\phi(x) = 0, \quad x \in M \tag{2.5}$$

$$\phi(x) = 0, \quad x \in \partial M. \tag{2.6}$$

If $\phi \neq 0$ in M satisfies the Laplacian eigenvalue problem, we say ϕ is an *eigenfunc-*

tion and the corresponding λ is called the *eigenvalue*. The condition where $\phi(x) = 0$ for $x \in \partial M$ is called the *Dirichlet boundary condition*.

Returning to our model (2.4), the *fundamental solution* of the heat equation, or the *heat kernel*, is represented by an infinite sum. The heat kernel on M is given by

$$u(t, x, y) = \sum_{j=0}^{\infty} e^{-\lambda_j t} \phi_j(x)\phi_j(y),$$

where $\{\lambda_j\}_{j=0}^{\infty}$, $\{\phi_j\}_{j=0}^{\infty}$ are the eigenvalues and eigenfunctions, respectively, for the Laplacian eigenvalue problem. The set $\{\phi_j\}_{j=0}^{\infty}$ also forms a basis for the space $L^2(M)$, meaning that any eigenfunction can be written as a unique sum of scalar multiples, or linear combination, of the ϕ_j's. Further on, we might omit 'Laplacian' in this thesis.

The function $u(t, x, y) \in \mathbb{R}_+ \times M \times M$ is continuous in all three variables. By the Strong Spectral Theorem for a Riemannian manifold given by Theorem 2.2.21 in [LMD23], the eigenfunctions $\{\phi_j\}_{j=0}^{\infty}$ form an orthonormal set where the norm on M is defined by

$$||\phi|| = \left(\int_M |\phi|^2 dA \right)^{1/2}.$$

Therefore, we have

$$\begin{aligned}
\int_M u(t, x, x) dA &= \int_M \sum_{j=0}^{\infty} e^{-\lambda_j t} \phi_j(x)\phi_j(x) dA \\
&= \sum_{j=0}^{\infty} e^{-\lambda_j t} \int_M \phi_j(x)^2 dA \\
&= \sum_{j=0}^{\infty} e^{-\lambda_j t}.
\end{aligned}$$

Definition 3. *The heat trace of a manifold M is defined by*

$$u_M(t) := \Sigma_{j=0}^{\infty} e^{-\lambda_j t} = \operatorname{Tr} e^{t\Delta_M},$$

where λ_n are the Laplacian eigenvalues.

Notice that in Equation 2.5, the eigenvalues λ_n are exactly the squares of the frequencies ω_n in Equation 2.3 if we order them so that the sequences $(\lambda_n), (\omega_n)$ are both increasing. This gives us the connection between Laplace eigenvalues and the physical vibration frequencies of a manifold.

Definition 4. *A property of the manifold M is a* spectral invariant *(can be "heard") if it's completely determined by the spectral data (eigenvalues and eigenfunctions).*

It has been shown that there exists a unique heat kernel $u_M(t, x, y)$ on every compact manifold M with or without boundary [DG04]. Notice that the Laplace eigenvalues of a manifold completely determine the heat trace, so the heat trace of a manifold is a spectral invariant. The asymptotic expansion of the heat trace will show that certain properties of a manifold, including the Gaussian curvature, are spectral invariants; we will use this fact in later chapters.

2.3 Audible Quantity and Echolocation

Now we introduce the pointwise Weyl counting function

$$N_x(\lambda) = \sum_{\lambda_j \leqslant \lambda} |\phi_j(x)|^2 \tag{2.7}$$

where λ and ϕ are the Laplacian eigenvalues and eigenfunctions, respectively. N_x is a spectral invariant as it's explicitly defined by λ and ϕ. Following Wyman and Xi, we give the following definition:

Definition 5. *An audible quantity is any function f on M satisfying $f(x) = f(y)$ whenever $N_x = N_y$ identically. In other words, any spectral invariant is an audible quantity.*

Given the definition of audible quantity, we can phrase the question of echolocation more rigorously.

Question 2. *Let M be a manifold with or without boundary. Are we able to locate any point x, up to symmetry, given a certain set of audible quantities?*

Chapter 3

Echolocation with Eigenfunctions

In this chapter, we will explore echolocation on some basic manifolds. Wyman and Xi showed that echolocation holds on 1-D strings and 2-D rectangles; we will start with these and add more details to their proof.

3.1 1-Dimensional and 2-Dimensional Examples

For our first example, we consider a simple string of length a. Suppose we fix the endpoints of the string on the x-axis. As the string is plucked (i.e., pulled away from the x-axis and released), the tension in the string will cause it to move up and down. Figure 3.1 provides an illustration of this problem, in which we wish to determine x where the string is plucked.

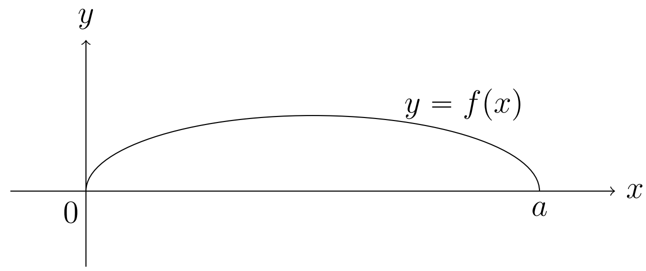

Figure 3.1: 1-D string with fixed endpoints at 0 and a.

Let $u(x,t)$ model the behavior of the string. Then u should satisfy the wave equation (2.1). In this one dimensional case, we have $\Delta u = u_{xx}$. Thus, we can rewrite the wave equation as

$$u_{tt} - c^2 u_{xx} = 0. \tag{3.1}$$

Without loss of generality, we may assume $c^2 = 1$. Since the two endpoints of the string are fixed, we have the boundary conditions

$$u(0,t) = u(a,t) = 0. \tag{3.2}$$

We have the initial condition

$$u(x,0) = f(x) \tag{3.3}$$

for some function f that describes the initial configuration of the string.

In order to solve the wave equation, we use the method of separation of variables. We express $u(x,t)$ as a product of two functions, one only depending on x and the other only depending on t. In other words,

$$u(x,t) = X(x)T(t).$$

Then, we have $u_{xx}(x,t) = X''(x)T(t)$ and $u_{tt}(x,t) = X(x)T''(t)$. Substituting into equation (3.1), we get

$$X(x)T''(t) - X''(x)T(t) = 0.$$

By assumption, $u(x,t) \neq 0$, so the functions X and T are nonzero. We can simplify the above equation and get

$$\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)}. \tag{3.4}$$

Note that the ratio is a constant since the left-hand side of equation (3.4) does not depend on x and the right-hand side doesn't depend on t; we denote the constant by k. Therefore, we have

$$\frac{X''(x)}{X(x)} = k, \text{ or } X''(x) - kX(x) = 0$$

which means X is a solution to the eigenvalue problem for 1-D string. We consider three cases for k.

Case 1, $\mathbf{k > 0}$: Note that the characteristic equation of the differential equation is

$$r^2 - k = 0,$$

so we get $r = \pm\sqrt{k}$. The solution is

$$X(x) = Ae^{\sqrt{k}x} + Be^{-\sqrt{k}x}$$

for some $A, B \in \mathbb{R}$. Recall that we have boundary conditions that $X(0) = X(a) = 0$, so by applying the boundary condition, we get $A = B = 0$, a contradiction to $u(x,t) \neq 0$. Thus, $k > 0$ is not feasible.

Case 2, k = 0: We have $X'' = 0$ which means

$$X(x) = Ax + B$$

for some $A, B \in \mathbb{R}$. By the boundary conditions, we get $X(x) = 0$ again.

Case 3, k < 0: In this case, the general solution is

$$X(x) = A\cos(\sqrt{-k}x) + B\sin(\sqrt{-k}x)$$

for some $A, B \in \mathbb{R}$. By the boundary conditions, we get $X(0) = 0$ which yields $A = 0$. Thus, $X(a) = B\sin(\sqrt{-k}a) = 0$, which gives $k = -(\frac{n\pi}{a})^2$ for $n \in \mathbb{N}$.

Although B can be arbitrary, here we follow Wyman and Xi's strategy of choosing B such that $||X(x)|| = 1$. In order to solve for B, we have

$$\int_0^a B^2 \sin^2\left(\frac{n\pi}{a}x\right) dx = 1$$
$$\frac{1}{4}aB^2\left(2 - \frac{\sin(2\pi n)}{\pi n}\right) = 1.$$

Notice that since $n \in \mathbb{N}$, $\sin(2\pi n) = 0$. Thus, we get $B = \sqrt{2/a}$.

Let $B = \sqrt{2/a}$ and define $\phi_n(x) := X_n(x)$; we have $||\phi_n(x)|| = 1$ for all $n \in \mathbb{N}$. Also, $\{\phi_n\}_{n\in\mathbb{N}}$ is the set of orthonormal eigenfunctions of a 1-D string and $-k =$

$(n\pi/a)^2$ are corresponding eigenvalues.

Therefore, the eigenvalues and eigenfunctions of the Laplacian eigenvalue problem of 1-D string are given by

$$\lambda_n = \left(\frac{n\pi}{a}\right)^2, \qquad \phi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$$

where $n \in \mathbb{N}$.

Example 3.1. For the 1-D string, the first eigenfunction is

$$\phi_1(x) = \sqrt{\frac{2}{a}} \sin(\pi x/a),$$

and we have the audible quantity

$$N_x(\lambda_1) = N_x\left(\frac{\pi^2}{a^2}\right) = |\phi_1(x)|^2 = \frac{2}{a}\sin^2\left(\frac{\pi x}{a}\right).$$

Notice that the function

$$\frac{\partial N_x(\lambda_1)}{\partial x} = \frac{4\pi}{a^2}\cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi x}{a}\right).$$

For $x \in (0, a/2)$, we have $\cos(\pi x/a) > 0$ and $\sin(\pi x/a) > 0$, so $\frac{\partial N_x(\lambda_1)}{\partial x} > 0$, suggesting $N_x(\lambda_1) = (2/a)\sin^2(\pi x/a)$ is increasing, and injective, for $x \in (0, a/2)$.

Notice that

$$\frac{2}{a}\sin^2\left(\frac{\pi x}{a}\right) = \frac{2}{a}\sin^2\left(\frac{\pi(a-x))}{a}\right),$$

and the string is symmetric with respect to the line $x = a/2$, so we say this audible quantity allows us to echolocate x up to the midpoint reflective symmetry of the 1-D string.

Example 3.2. We now extend the 1-D case to 2-D. Consider a rectangle $M = [0, a] \times [0, b]$ with Dirichlet boundary conditions, in other words, points on the boundary are fixed. Without loss of generality, we may assume $a = 1$ and $b \in (0, 1]$ as all other rectangles with fixed boundaries can be obtained by scaling and rotating this case. By solving the similar Laplacian eigenvalue problem, we get the eigenfunctions as

$$\phi_{n,m}(x, y) = \frac{2}{\sqrt{b}} \sin(\pi n x) \sin\left(\frac{\pi m y}{b}\right)$$

where $n, m \in \mathbb{N}$ with corresponding eigenvalues $\lambda_{n,m} = \pi^2 \left(n^2 + \frac{m^2}{b^2}\right)$.

Wyman and Xi [WX23] showed that echolocation is possible when $b \in (0, 1)$ by considering the audible quantity

$$\begin{aligned} \frac{N_{(x,y)}(\lambda_{2,1}) - N_{(x,y)}(\lambda_{1,1})}{N_{(x,y)}(\lambda_{1,1})} &= \frac{\left(\sin^2(2\pi x)\sin^2(\frac{\pi y}{b}) + \sin^2(\pi x)\sin^2(\frac{\pi y}{b})\right) - \sin^2(\pi x)\sin^2(\frac{\pi y}{b})}{\sin^2(\pi x)\sin^2(\frac{\pi y}{b})} \\ &= \frac{\sin^2(2\pi x)\sin^2(\frac{\pi y}{b})}{\sin^2(\pi x)\sin^2(\frac{\pi y}{b})} \\ &= \frac{4\sin^2(\pi x)\cos^2(\pi x)}{\sin^2(\pi x)} \\ &= 4\cos^2(\pi x). \end{aligned} \tag{3.5}$$

The penultimate line comes from the double-angle formula, i.e. $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ for all $\theta \in \mathbb{R}$. Since $\cos(\pi x) = \cos(\pi(1 - x))$ for $x \in (0, 0.5)$, as in the 1-d case, we can determine x up to symmetry with respect to the line $x = 0.5$. After identifying

x, consider the audible quantity

$$N_{(x,y)}(\lambda_{1,1}) = \frac{4}{b}\sin^2(\pi x)\sin^2\left(\frac{\pi y}{b}\right). \tag{3.6}$$

Recall that $\sin^2(\theta) + \cos^2(\theta) = 1$ for all $\theta \in \mathbb{R}$, so we can rewrite Equation (3.6) as

$$\sin^2\left(\frac{\pi y}{b}\right) = \frac{b}{4} \cdot \frac{N_{(x,y)}(\lambda_{1,1})}{1 - \cos^2(\pi x)}.$$

Since we've already identified x, the right-hand-side is a fixed number. Arguing as in Example 3.1, $\sin^2(\pi y/b)$ is injective for $y \in (0, \frac{b}{2})$, and since

$$\sin^2\left(\frac{\pi y}{b}\right) = \sin^2\left(\frac{\pi(b-y)}{b}\right),$$

we can determine the $y-$coordinate up to a reflection symmetry with respect to the line $y = b/2$.

Before discussing the case $b = a = 1$, we shall first introduce the definition of multiplicity.

Definition 6. *The multiplicity of an eigenvalue λ is the dimension of its corresponding eigenspace (set of eigenfunctions). If the dimension is one, we say λ is simple.*

In our case, if there exist j distinct pairs of number (n, m) such that $(n, m) \neq (a, b)$ and $\lambda_{n,m} = \lambda_{a,b}$, the multiplicity of $\lambda_{a,b}$ is $j + 1$. If $j = 0$, then the multiplicity of $\lambda_{a,b}$ is 1 and it is simple. Moreover, the first eigenvalue is always simple.

Now we will prove that echolocation holds on rectangles with $a = b = 1$ (squares);

Wyman and Xi also considered this case but their proof lacks some crucial details.

For the case $a = b = 1$, we have $\lambda_{2,1} = \lambda_{1,2}$ has multiplicity of 2. Consider again the audible quantities

$$\begin{aligned}
\frac{N_{(x,y)}(\lambda_{2,1}) - N_{(x,y)}(\lambda_{1,1})}{N_{(x,y)}(\lambda_{1,1})} &= \frac{\sin^2(2\pi x)\sin^2(\pi y) + \sin^2(\pi x)\sin^2(2\pi y)}{\sin^2(\pi x)\sin^2(\pi y)} \\
&= \frac{\sin^2(2\pi x)}{\sin^2(\pi x)} + \frac{\sin^2(2\pi y)}{\sin^2(\pi y)} \\
&= 4\left(\cos^2(\pi x) + \cos^2(\pi y)\right) \\
&= 4(1 - \sin^2(\pi x) + 1 - \sin^2(\pi y)) \\
&= 8 - 4\left(\sin^2(\pi x) + \sin^2(\pi y)\right)
\end{aligned}$$

and

$$N_{x,y}(\lambda_{1,1}) = 4\sin^2(\pi x) \cdot \sin^2(\pi y).$$

Therefore, we know that both the product and the sum of $\sin^2(\pi x)$ and $\sin^2(\pi y)$ are audible. Denote the sum by p and the product by q:

$$\begin{cases} \sin^2(\pi x) + \sin^2(\pi y) = p \\ \sin^2(\pi x)\sin^2(\pi y) = q. \end{cases} \tag{3.7}$$

We get $\sin^4(\pi x)(p - \sin^2(\pi x)) = q$ by plugging the first equation to the second one, and we can rewrite the equation as

$$-\sin^2(\pi x) + p\sin^2(\pi x) - q = 0.$$

By the root formula for quadratic equations, we get

$$\sin^2(\pi x) = \frac{-p \pm \sqrt{p^2 - 4q}}{-2}.$$

However, the root formula suggests that there exist two potential values for $\sin^2(\pi x)$ that will yield non-symmetric x values. To deal with this issue, we need to apply the following trick. Note that $\sin(\pi x)\sin(\pi y) = \sqrt{q}$ and since $\sin^2(\pi x) + \sin^2(\pi y) = p$, we have

$$\begin{aligned}(\sin(\pi x) - \sin(\pi y))^2 &= \sin^2(\pi x) + \sin^2(\pi y) - 2\sin(\pi x)\sin(\pi y) \\ &= p - 2\sqrt{q}.\end{aligned}$$

So,

$$\sin(\pi x) - \sin(\pi y) = \sqrt{p - 2\sqrt{q}}.$$

To simplify our notation, let

$$\ell = \sqrt{q} \quad \text{and} \quad k = \sqrt{p - 2\sqrt{q}}.$$

So, $\sin(\pi y) = \sin(\pi x) - k$, and since $\sin(\pi x)\sin(\pi y) = \ell$, we can do the similar substitution as Equation (3.7) and write

$$\sin^2(\pi x) - k\sin(\pi x) - \ell = 0.$$

By solving the quadratic again, we get

$$\sin(\pi x) = \frac{k \pm \sqrt{k^2 + 4\ell}}{2}.$$

Since $x, y \in (0,1)$ by assumption, we have $\sin(\pi x) > 0$ and $\sin(\pi y) > 0$. Thus, $\ell > 0$ and we have $\sqrt{k^2 + 4\ell} > k$. Therefore, we can conclude that

$$\sin(\pi x) = \frac{k + \sqrt{k^2 + 4\ell}}{2} \quad \text{and} \quad \sin^2(\pi y) = p - \sin^2(\pi x),$$

so we're able to determine x up to the left-right reflective symmetry about the midline. Similarly, y can be determined up to the up-down reflective symmetry about the midline.

3.2 Rectangular Boxes with Distinct Side Lengths

There's a jump in complexity when we consider 3-D rectangles (boxes). Before getting into the example, for convenience, we explicitly write out the general formula of eigenfunctions and eigenvalues for $k-$dimensional hyperrectangles. The formula can be derived via similar separation of variable techniques as we did in previous sections. Suppose we have a $k-$dimensional hyperrectangle with side lengths $c_1, \ldots, c_k$. Without loss of generality, let $1 = c_1 > c_2 > \cdots > c_k$. A complete orthonormal basis of eigenfunctions can be written as

$$\phi_{n_1,\ldots,n_k}(x_1, \ldots, x_k) = \prod_{i=1}^{k} \left\{ \sqrt{\frac{2}{c_i}} \sin\left(\frac{\pi n_i x_i}{c_i}\right) \right\}, \qquad n_1, \ldots, n_k \in \mathbb{N}$$

with respective eigenvalues

$$\lambda_{n_1,\ldots,n_k} = \pi^2 \sum_{i=1}^{k} \frac{n_i^2}{c_i^2}.$$

Now, consider a box $M = [0, a] \times [0, b] \times [0, c]$ with Dirichlet boundary conditions. Similar to previous examples, without loss of generality, we may assume $1 = a > b > c > 0$. Wyman and Xi mentioned in their paper that this case is similar to the previous lower dimensional cases and can be solved via similar arguments without providing any further detail. We will show that even though echolocation is possible, it's much more complicated.

Lemma 1. *$\lambda_{1,1,1}$ and $\lambda_{2,1,1}$ are always the two smallest eigenvalues and they are simple.*

Proof. Recall that our general form of eigenvalues is

$$\lambda_{n,m,\ell} = \pi^2 \left(n^2 + \frac{m^2}{b^2} + \frac{\ell^2}{c^2} \right)$$

for some positive integers n, m, ℓ. Thus, we can write

$$\begin{aligned} \lambda_{1,1,1} &= \pi^2 \left(1 + \frac{1}{b^2} + \frac{1}{c^2} \right) \\ \lambda_{2,1,1} &= \pi^2 \left(4 + \frac{1}{b^2} + \frac{1}{c^2} \right). \end{aligned}$$

Notice that if we fix two of n, m, ℓ and let only one vary, the order of eigenvalues will only depend on the varying index. For example, consider $\lambda_{n,1,1}$ where $n > 2$; clearly we have $\lambda_{n,1,1} > \lambda_{2,1,1} > \lambda_{1,1,1}$. Therefore, in order to show that $\lambda_{1,1,1}$ and $\lambda_{2,1,1}$ are the smallest eigenvalues, we only need to consider $\lambda_{1,2,1}$ and $\lambda_{1,1,2}$. If we suppose, for the sake of reaching a contradiction that there exist some b, c such that $\lambda_{1,2,1} \leqslant \lambda_{2,1,1}$, we will have

$$\pi^2\left(4+\frac{1}{b^2}+\frac{1}{c^2}\right)\geqslant\pi^2\left(1+\frac{4}{b^2}+\frac{1}{c^2}\right).$$

By solving the inequality, we get $b \geqslant 1$, which contradicts our assumption that $1 > b$. Thus, we have $\lambda_{2,1,1} < \lambda_{1,2,1}$. Similarly, we can show that $\lambda_{2,1,1} < \lambda_{1,1,2}$, so $\lambda_{1,1,1}$ and $\lambda_{2,1,1}$ are the smallest eigenvalues, and they are simple, as desired. □

As a result, if we consider the audible quantity and, similar to Equation 3.5 in Example 3.2, apply the double-angle formula, we have

$$\frac{N_{(x,y,z)}(\lambda_{2,1,1})-N_{(x,y,z)}(\lambda_{1,1,1})}{N_{(x,y,z)}(\lambda_{1,1,1})}=\frac{\sin^2(2\pi x)\sin^2(\frac{\pi y}{b})\sin^2(\frac{\pi z}{c})}{\sin^2(\pi x)\sin^2(\frac{\pi y}{b})\sin^2(\frac{\pi z}{c})}=4\cos^2(\pi x),$$

and we can identify the x-coordinate up to the reflective symmetry in the plane $x = 0.5$.

In order to identify y, let's consider

$$\lambda_{1,2,1}=\pi^2\left(1+\frac{4}{b^2}+\frac{1}{c^2}\right).$$

Note that $\lambda_{1,2,1}$ is not necessarily simple; it depends on the actual side lengths of the box. In other words, we want to identify some side lengths of b and c such that for the corresponding box $[0,1]\times[0,b]\times[0,c]$, there exist some positive integers n, m, ℓ that cause $\lambda_{1,2,1}$ to have multiplicity more than 1, in other words,

$$1+\frac{4}{b^2}+\frac{1}{c^2}=n^2+\frac{m^2}{b^2}+\frac{\ell^2}{c^2}. \tag{3.8}$$

We first consider ℓ. If $\ell > 2$, the right-hand-side is always greater than the left-hand-side because $b > c$. If $\ell = 2$, the only solution to Equation 3.8 is $b = c$ and $m = n = 1$, which contradicts our assumption that $b > c$. Thus, ℓ has to be 1. So we can write the above equation as

$$1 + \frac{4}{b^2} = n^2 + \frac{m^2}{b^2}. \tag{3.9}$$

By simplifying, we get

$$b^2 = \frac{m^2 - 4}{1 - n^2}. \tag{3.10}$$

Since when $n = 1$, b is undefined, we have $n \geqslant 2$, and the denominator can only be negative. Since $b^2 > 0$, we need $m^2 - 4 < 0$, so $m = 1$. So we can further simplify and get

$$b = \sqrt{\frac{3}{n^2 - 1}}. \tag{3.11}$$

Notice if $n = 2$, we have $b = 1$ which is impossible, so $n > 2$. Therefore, for an arbitrary cube with distinct side lengths a, b, c where $1 = a > b > c > 0$, if the side length $b = \sqrt{\frac{3}{n^2-1}}$ for some positive integer $n > 2$, the multiplicity of the eigenvalue $\lambda_{1,2,1}$ will be greater than 1. This multiplicity will not be affected by c as long as $c < b$. More specifically, since we have at most 1 such n for each b, the multiplicity will be at most 2. Figure 3.2 shows such behavior where the x, y-axes are b, c values, respectively. For visual clarity, we restrict our $n \in [3, 10] \cap \mathbb{Z}$; $n = 1$ and $n = 2$ are not included since b is undefined when $n = 1$, and $b = 1 = a$ when $n = 2$, which cannot be true. When n gets larger, there will be more parallel vertical lines that approach 0. On the vertical lines are the points where given that specific (b, c), there exists a unique n such that $\lambda_{n,1,1} = \lambda_{1,2,1}$, so the multiplicity of $\lambda_{1,2,1}$ is 2. The dashed line is $b = c$ for reference; all the points should be below the dashed line as $b > c$.

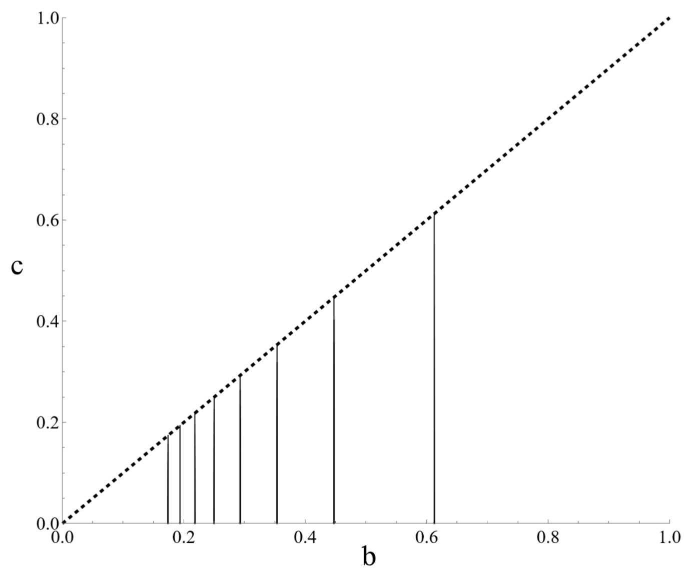

Figure 3.2: Pairs of (b, c) for which the multiplicity of $\lambda_{1,2,1}$ is 2

As long as we know the fact that the multiplicity is at most 2, we can try to identify the y-coordinate. Recall the "first" 3 eigenfunctions from before:

$$\begin{aligned} \phi_{1,1,1} &= \alpha \sin(\pi x) \sin\left(\frac{\pi y}{b}\right) \sin\left(\frac{\pi z}{c}\right) \\ \phi_{2,1,1} &= \alpha \sin(2\pi x) \sin\left(\frac{\pi y}{b}\right) \sin\left(\frac{\pi z}{c}\right) \\ \phi_{1,2,1} &= \alpha \sin(\pi x) \sin\left(\frac{2\pi y}{b}\right) \sin\left(\frac{\pi z}{c}\right), \end{aligned}$$

where $\alpha = \frac{2\sqrt{2}}{\sqrt{bc}}$. If all of their corresponding eigenvalues are simple, since we've already identified x-coordinate, we are left with a two-dimensional problem, then we can deal with the problem in the same way as in the 2-D case where the rectangle has distinct side lengths, by considering the audible quantity $(N(\lambda_{1,2,1}) - N(\lambda_{1,1,1}))/N(\lambda_{1,1,1})$.

But now, suppose $\lambda_{n,1,1} = \lambda_{1,2,1}$, so we want

$$\phi_{n,1,1} = \alpha \sin(n\pi x) \sin\left(\frac{\pi y}{b}\right) \sin\left(\frac{\pi z}{c}\right).$$

We consider the audible quantity

$$\begin{aligned}
&\frac{N_{(x,y,z)}(\lambda_{n,1,1}) - N_{(x,y,z)}(\lambda_{2,1,1})}{N_{(x,y,z)}(\lambda_{1,1,1})}\\
=&\frac{\sin^2(n\pi x)\sin^2(\pi y/b)\sin^2(\pi z/c) + \sin^2(\pi x)\sin^2(2\pi y/b)\sin^2(\pi z/c)}{\sin^2(\pi x)\sin^2(\pi y/b)\sin^2(\pi z/c)}\\
=&\frac{\sin^2(2\pi y/b)}{\sin^2(\pi y/b)} + \frac{\sin^2(n\pi x)}{\sin^2(\pi x)}\\
=&4\cos^2(\pi y/b) + \frac{\sin^2(n\pi x)}{\sin^2(\pi x)}.
\end{aligned}$$

Since we already have x based on the first two eigenfunctions, we can now derive y up to symmetry with respect to the plane $y = b/2$. The z- coordinate, up to symmetry of the plane $z = c/2$, follows immediately from the $N_{(x,y,z)}(\lambda_{1,1,1})$.

3.3 Rectangular Box with $a = b$ and $c > \frac{\sqrt{2}}{2}$

Consider the case where $1 = a = b > c > 0$. We will have

$$\lambda_{n,m,\ell} = \pi^2\left(n^2 + m^2 + \frac{\ell^2}{c^2}\right).$$

Then, by a similar argument as in Lemma 1, we can see that the first eigenvalue $\lambda_{1,1,1} = \pi^2 (2 + 1/c^2)$ will always be simple. The second eigenvalues are

$$\lambda_{2,1,1} = \lambda_{1,2,1} = \pi^2 \left(5 + 1/c^2\right).$$

Notice that $\lambda_{2,1,1} < \lambda_{1,1,2}$ because if $\lambda_{2,1,1} > \lambda_{1,1,2}$, we will derive $c^2 > 1$, which will be a contradiction. Now, we have

$$\begin{aligned}
\phi_{1,1,1} &= \alpha \sin(\pi x) \sin(\pi y) \sin\left(\frac{\pi z}{c}\right) \\
\phi_{2,1,1} &= \alpha \sin(2\pi x) \sin(\pi y) \sin\left(\frac{\pi z}{c}\right) \\
\phi_{1,2,1} &= \alpha \sin(\pi x) \sin(2\pi y) \sin\left(\frac{\pi z}{c}\right).
\end{aligned}$$

We determine the multiplicity of $\lambda_{1,1,2}$. Note that for all $n, m, \ell \in \mathbb{N}$, we will have

$$\lambda_{n,m,\ell} = \lambda_{m,n,\ell} = \pi^2 \left(n^2 + m^2 + \frac{\ell^2}{c^2}\right).$$

Suppose $\lambda_{n,m,\ell} = \lambda_{1,1,2}$. Notice that ℓ must equal 1, since for $\ell > 2$ the equation

$$n^2 + m^2 + \frac{\ell^2}{c^2} = 1 + 1 + \frac{4}{c^2}$$

will have no solution, and $m = n = 1$ is the only solution when $\ell = 2$. So we have

$$n^2 + m^2 + \frac{1}{c^2} = 2 + \frac{4}{c^2}.$$

Thus, when

$$c = \sqrt{\frac{3}{n^2 + m^2 - 2}}$$

for some integers n, m, $\lambda_{1,1,2}$ will have multiplicity of more than 1. If $n = m$, the multiplicity will be 2; otherwise, when $n \neq m$, the multiplicity will be 3 as n and m are symmetric. In this thesis, we focus on $c \geqslant \sqrt{2}/2$. In other words, the multiplicity of $\lambda_{1,1,2}$ is at most 2. When $n = m = 2$, $\lambda_{1,1,2}$ will have multiplicity of 2 and $c = \sqrt{2}/2$. Recall that we've shown that $\lambda_{2,1,1} = \lambda_{1,2,1} < \lambda_{1,1,2}$, and when one of n, m equals 1 and the other equals 3, $c = \sqrt{3/8} < \sqrt{2}/2$, a contradiction. Thus, when $c > \sqrt{2}/2$, we cannot find integers n, m such that $\lambda_{n,m,1} = \lambda_{1,1,2}$. As a result, we claim that when $c > \sqrt{2}/2$, $\lambda_{1,1,2}$ has multiplicity 1.

Theorem 1. *Echolocation is possible via the first three eigenfunctions on the rectangular box* $[0, 1] \times [0, 1] \times [0, c]$ *where* $c > \sqrt{2}/2$.

Proof. The first four eigenvalues are $\lambda_{1,1,1}, \lambda_{2,1,1} = \lambda_{1,2,1}$, and $\lambda_{1,1,2}$. Since $c > \sqrt{2}/2$, we know from the discussion above that $\lambda_{1,1,2}$ has multiplicity of 1. Consider

$$\frac{N(\lambda_{1,1,2}) - N(\lambda_{1,2,1})}{N(\lambda_{1,1,1})} = \frac{\sin^2(\pi x)\sin^2(\pi y)\sin^2(2\pi z/c)}{\sin^2(\pi x)\sin^2(\pi y)\sin^2(\pi z/c)} = 4\cos^2(\pi z),$$

which is enough for us to determine z up to symmetry of the plane $z = c/2$. From $N(\lambda_{1,1,1})$, we can then obtain $\sin^2(\pi x)\sin^2(\pi y)$. Note that $\frac{N(\lambda_{1,1,2}) - N(\lambda_{1,1,1})}{N(\lambda_{1,1,1})} = 4\cos^2(\pi x) + 4\cos^2(\pi y)$, by solving quadratic equations in a similar way as we did for (3.7), we can determine x and y up to reflective symmetry of the box. $\square$

3.4 Rectangular Box with $a = b$ and $c = \frac{\sqrt{2}}{2}$

We consider the case where our manifold is a $[0,1] \times [0,1] \times [0, \frac{\sqrt{2}}{2}]$ rectangular box. In this case, even though we know exactly the size of the box, there's no clear combination of the pointwise Weyl counting functions as in previous examples that gives us a direct formula for echolocation. Therefore, we shall suppose, for the sake of reaching a contradiction, that there exists a pair of points (x_1, y_1, z_1) and (x_2, y_2, z_2) with the same first three pointwise Weyl counting functions.

Theorem 2. *There exists an isometry between any two points with the same first three pointwise Weyl counting functions in the* $[0,1] \times [0,1] \times [0, \frac{\sqrt{2}}{2}]$ *rectangular box.*

Proof. Recall that our first 'three' eigenvalues are $\lambda_{1,1,1}, \lambda_{2,1,1} = \lambda_{1,2,1}$, and $\lambda_{1,1,2} = \lambda_{2,2,1}$. Since by assumption

$$N_{(x_1,y_1,z_1)}(\lambda_{1,1,2}) = N_{(x_2,y_2,z_2)}(\lambda_{1,1,2}) \quad \text{and} \quad N_{(x_1,y_1,z_1)}(\lambda_{1,2,1}) = N_{(x_2,y_2,z_2)}(\lambda_{1,2,1}),$$

we have

$$N_{(x_1,y_1,z_1)}(\lambda_{1,1,2}) - N_{(x_1,y_1,z_1)}(\lambda_{1,2,1}) = N_{(x_2,y_2,z_2)}(\lambda_{1,1,2}) - N_{(x_2,y_2,z_2)}(\lambda_{1,2,1}).$$

If we explicitly write out the terms, we have

$$\begin{aligned} &\sin^2(\pi x_1)\sin^2(\pi y_1)\sin^2(2\sqrt{2}\pi z_1) + \sin^2(2\pi x_1)\sin^2(2\pi y_1)\sin^2(\sqrt{2}\pi z_1) \\ =&\sin^2(\pi x_2)\sin^2(\pi y_2)\sin^2(2\sqrt{2}\pi z_2) + \sin^2(2\pi x_2)\sin^2(2\pi y_2)\sin^2(\sqrt{2}\pi z_2) \end{aligned} \tag{3.12}$$

To simplify our notation, define

$$\begin{aligned} a &= \sin^2(\pi x_1), \quad b = \sin^2(\pi y_1), \quad c = \sin^2(\sqrt{2}\pi z_1) \\ d &= \sin^2(\pi x_2), \quad e = \sin^2(\pi y_2), \quad f = \sin^2(\sqrt{2}\pi z_2). \end{aligned} \tag{3.13}$$

By the fact that for any $\theta \in \mathbb{R}$, $\sin^2(\theta) + \cos^2(\theta) = 1$ and $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$, we get $\sin^2(2\theta) = 4\sin^2(\theta)(1 - \sin^2(\theta))$. Thus, if we start with the left-hand-side of Equation 3.12, we can write $\sin^2(2\sqrt{2}\pi z_1)$ as $4c(1-c)$. Similarly, we have $\sin^2(2\pi x_1) = 4a(1-a)$ and $\sin^2(2\pi y_1) = 4b(1-b)$. We can simplify the right-hand-side using the same approach. Therefore, we get

$$4abc(1-c) + 16(1-a)(1-b)abc = 4def(1-f) + 16(1-d)(1-e)def. \tag{3.14}$$

Recall that by assumption, we have

$$N_{(x_1,y_1,z_1)}(\lambda_{1,1,1}) = N_{(x_2,y_2,z_2)}(\lambda_{1,1,1}), \text{ and } N_{(x_1,y_1,z_1)}(\lambda_{2,1,1}) = N_{(x_2,y_2,z_2)}(\lambda_{2,1,1}),$$

Thus, the audible quantity

$$\begin{aligned} &\frac{N(\lambda_{2,1,1}) - N(\lambda_{1,1,1})}{N(\lambda_{1,1,1})} \\ =&\frac{\sin^2(2\pi x)\sin^2(\pi y)\sin^2(\sqrt{2}\pi z) + \sin^2(\pi x)\sin^2(2\pi y)\sin^2(\sqrt{2}\pi z)}{\sin^2(\pi x)\sin^2(\pi y)\sin^2(\sqrt{2}\pi z)} \\ =&4\cos^2(\pi x) + 4\cos^2(\pi y) \end{aligned} \tag{3.15}$$

should be the same for (x_1, y_1, z_1) and (x_2, y_2, z_2). Therefore, we have

$$(1 - a) + (1 - b) = (1 - d) + (1 - e).$$

By simplifying, we get $e = a + b - d$. Also, since the two points have the same $N(\lambda_{1,1,1})$, we get $abc = def$, so we can write

$$f = \frac{abc}{d(a + b - d)}. \tag{3.16}$$

Therefore, if we divide both sides of Equation 3.14 by abc and substitute the first f on the right-hand-side with (3.16), the equation can be simplified as

$$1 - c + 4(1 - a)(1 - b) = (1 - \frac{abc}{d(a + b - d)}) + 4(1 - d)(1 - a - b + d). \tag{3.17}$$

By solving this equation for d using Mathematica, we get four potential solutions for d:

$$d = \begin{cases} \frac{1}{2}\left(a + b - \sqrt{a^2 + b^2 + \frac{1}{2}(c - \sqrt{(4ab + c)^2})}\right) \\ \frac{1}{2}\left(a + b + \sqrt{a^2 + b^2 + \frac{1}{2}(c - \sqrt{(4ab + c)^2})}\right) \\ \frac{1}{2}\left(a + b - \sqrt{a^2 + b^2 + \frac{1}{2}(c + \sqrt{(4ab + c)^2})}\right) \\ \frac{1}{2}\left(a + b + \sqrt{a^2 + b^2 + \frac{1}{2}(c + \sqrt{(4ab + c)^2})}\right) \end{cases} \tag{3.18}$$

By (3.13), we have $x_2 = \frac{\sin^{-1}(\sqrt{d})}{\pi}$; if we substitute a, b, c into (3.18) and simplify x_2

by setting the constraint that $a, b, c \in (0,1]$, we get four corresponding solutions:

$$x_2 = \begin{cases} y_1, & \text{if } \cos(2\pi x_1) < \cos(2\pi y_1); \\ x_1, & \text{otherwise}; \end{cases} \tag{3.19}$$

$$x_2 = \begin{cases} y_1, & \text{if } \cos(2\pi x_1) \geqslant \cos(2\pi y_1); \\ x_1, & \text{otherwise}; \end{cases} \tag{3.20}$$

$$x_2 = \frac{1}{\pi}\arcsin\left(\frac{\sqrt{\sin^2(\pi x_1) + \sin^2(\pi y_1) - \sqrt{(\sin^2(\pi x_1) + \sin^2(\pi y_1))^2 + \sin^2(\sqrt{2}\pi z_1)}}}{\sqrt{2}}\right); \tag{3.21}$$

$$x_2 = \frac{1}{\pi}\arcsin\left(\frac{\sqrt{\sin^2(\pi x_1) + \sin^2(\pi y_1) + \sqrt{(\sin^2(\pi x_1) + \sin^2(\pi y_1))^2 + \sin^2(\sqrt{2}\pi z_1)}}}{\sqrt{2}}\right). \tag{3.22}$$

Notice that in solution (3.21),

$$\sqrt{(\sin^2(\pi x_1) + \sin^2(\pi y_1))^2 + \sin^2(\sqrt{2}\pi z_1)} > \sin^2(\pi x_1) + \sin^2(\pi y_1),$$

so the numerator is undefined. If x_2 takes the solution (3.22), we would have

$$\begin{aligned} d &= \sin^2(\pi x_2) \\ &= \frac{1}{2}\left(\sin^2(\pi x_1) + \sin^2(\pi y_1) + \sqrt{(\sin^2(\pi x_1) + \sin^2(\pi y_1))^2 + \sin^2(\sqrt{2}\pi z_1)}\right) \\ &> \sin^2(\pi x_1) + \sin^2(\pi y_1). \end{aligned}$$

We would also have

$$\begin{aligned} e &= \sin^2(\pi y_2) \\ &= a + b - d \\ &= \sin^2(\pi x_1) + \sin^2(\pi y_1) - d \\ &< 0, \end{aligned}$$

a contradiction.

Therefore, given x_1, y_1, z_1, the solution for x_2 would be either $x_2 = y_1$ or $x_2 = x_1$. Then, we can identify y_2 and z_2 up to reflective symmetry from e and f. It's straightforward that the two points (x_1, y_1, z_1) and (x_2, y_2, z_2) are identical up to either reflective symmetry or rotational symmetry of the box.

Thus, we conclude that there exists an isometry between any two points with the same first three pointwise Weyl counting functions in the $[0, 1] \times [0, 1] \times [0, \frac{\sqrt{2}}{2}]$ rectangular box. $\square$

For cases where $c < \sqrt{2}/2$, and $c = \sqrt{\frac{3}{n^2+m^2-2}}$ for some $n, m \in \mathbb{N}$, if $n = m$, echolocation holds and can be shown using similar techniques. However, when $n \neq m$, the multiplicity of $\lambda_{1,1,2}$ will be 3. Also, the cases will be different for each pair of (n, m); we won't discuss those in this thesis.

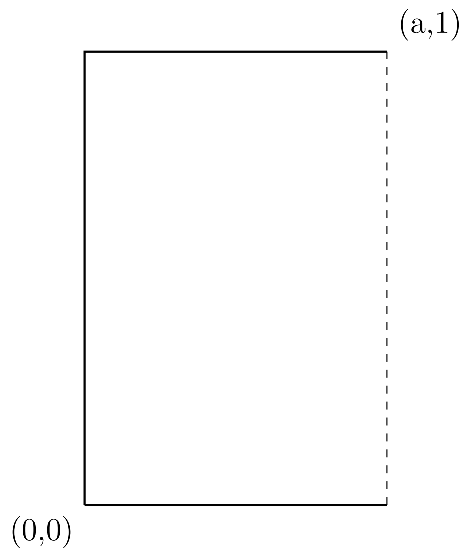

Figure 3.3: Rectangle with mixed boundary conditions, where the dashed lines denote the Neumann boundary conditions and the thick lines denote the Dirichlet boundary condition.

3.5 Rectangles with Mixed Boundary Conditions

Up to this point, all our computations assume the Dirichlet boundary condition, in other words, all boundaries are fixed. Now we introduce a new boundary condition. As usual, let our manifold be M with boundary ∂M.

Definition 7. *Suppose $u : M \to \mathbb{R}$ is a function defined on M. The Neumann boundary condition is given by*

$$\left.\frac{\partial u}{\partial v}\right|_{\partial M} = 0,$$

where $\frac{\partial u}{\partial v}$ denotes the derivative of u in the direction normal to the boundary ∂M.

This condition essentially states that the rate of change of the function u in the normal direction at the boundary is 0. In our case, u is the eigenfunction on the manifold and v is always parallel to the Euclidean axes.

Now we consider a rectangle. Without loss of generality, let $M = (0, a) \times (0, 1)$ with Neumann boundary condition on the side $\{1\} \times (0, 1)$ and Dirichlet on the other sides, as shown in Figure 3.3.

To study echolocation on this rectangle, we first need to find a basis for its eigenfunctions and the corresponding eigenvalues. We follow a similar process as in the 1-D string example. Recall that we need to solve the wave equation

$$u_{tt} = c^2(u_{xx} + u_{yy})$$

and we may assume $c = 1$. Suppose that $u(t, x, y)$ is a solution to the wave equation and u can be separated such that

$$u(t, x, y) = T(t)X(x)Y(y), \tag{3.23}$$

We are particularly interested in the parts involving x and y. Since the normal vector on the side $\{1\} \times (0, 1)$ is in the positive x direction, the boundary conditions are given by

$$u(t, 0, y) = 0 \tag{3.24}$$

$$u(t, x, 0) = u(t, x, a) = 0 \tag{3.25}$$

$$u_x(t, a, y) = 0. \tag{3.26}$$

By substituting equation 3.23 into the wave equation and simplifying, we get

$$\frac{T''}{T} = \frac{X''}{X} + \frac{Y''}{Y} = A \tag{3.27}$$

for some $A \in \mathbb{R}$. By reordering, we get

$$X'' = BX, \quad Y'' = (A - B)Y \tag{3.28}$$

for some $B \in \mathbb{R}$. From the boundary condition 3.25, we know $Y(0) = Y(a) = 0$. Following a similar process as we did for the 1-D string, we get

$$Y(y) = \beta_1 \sin(n\pi y), \qquad \text{for } n = 1, 2, \ldots, \quad \beta_1 \in \mathbb{R}. \tag{3.29}$$

Now we consider $X(x)$. We have $X'' = BX$ and the boundary conditions 3.24 and 3.26 tell us $X(0) = 0$ and $X'(a) = 0$, respectively. Similar to 1-D, we can exclude the cases where $B \geqslant 0$. Thus, we have $B < 0$ and the general solution of $X'' = BX$ is given by

$$X(x) = c_1 \cos(\sqrt{-B}x) + c_2 \sin(\sqrt{-B}x).$$

The condition $X(0) = 0$ yields $c_1 = 0$. The condition $X'(a) = 0$ implies

$$c_2\sqrt{-B}\cos(\sqrt{-B}a) = 0.$$

Therefore, we have

$$c_2 \in \mathbb{R}, \quad \sqrt{-B} = \frac{(m + \frac{1}{2})\pi}{a}, \qquad \text{for } m = 0, 1, 2, \ldots$$

and the solution for $X(x)$ is given by

$$X(x) = \beta_2 \sin\left(\frac{(m+\frac{1}{2})\pi}{a}x\right), \quad \text{for } m = 0,1,2,\dots, \quad \beta_2 \in \mathbb{R}. \tag{3.30}$$

Combining equation 3.29 and equation 3.30, a basis of eigenfunctions can be written as

$$e_{m,n} = \sin\left(\frac{(m+\frac{1}{2})}{a}\pi x\right)\sin(n\pi y), \quad m = 0,1,2,\dots, \quad n = 1,2,\dots \tag{3.31}$$

with corresponding eigenvalues

$$\lambda_{m,n} = \frac{\pi^2}{4a^2}\left((2m+1)^2 + 4a^2n^2\right). \tag{3.32}$$

Theorem 3. *We can always echolocate on rectangles $(0,a)\times(0,1)$ where $a \leqslant 1$ with Neumann boundary conditions on a side of length a and Dirichlet on the other three.*

Proof. As always, in order to use the pointwise Weyl counting function, Equation 2.7, we need to find the ordering of the eigenvalues. Since $\frac{\pi^2}{4a^2}$ is a common factor for all eigenvalues, we denote it by k for simplicity. Note that $\lambda_{0,1}$ is clearly always the smallest eigenvalue. We divide the proof into two cases.

Case 1: $a \neq \sqrt{2/3}$.

In this case, we have

$$\lambda_{1,1} = (9 + 4a^2)k$$
$$\lambda_{0,2} = (1 + 16a^2)k.$$

Since $a \neq \sqrt{2/3}$, we see that $\lambda_{1,1} \neq \lambda_{0,2}$. More specifically, when $a < \sqrt{2/3}$, $\lambda_{0,2}$ is always the second smallest eigenvalue, and $\lambda_{0,1} < \lambda_{0,2} < \lambda_{1,1}$. When $a > \sqrt{2/3}$, note that since $a \leqslant 1$, $\lambda_{2,1} > \lambda_{0,2}$. It's also obvious that $\lambda_{0,3} > \lambda_{0,2}$. Also, when $a > \sqrt{2/3}$, we have $a^2 > 2/3$, so $1 + 16a^2 > 9 + 4a^2$ and $\lambda_{1,1} < \lambda_{0,2}$. Therefore, $\lambda_{0,1} < \lambda_{1,1} < \lambda_{0,2}$ are the three smallest eigenvalues.

When $a < \sqrt{2/3}$, consider the audible quantity

$$\frac{N(\lambda_{0,2}) - N(\lambda_{0,1})}{N(\lambda_{0,1})} = \frac{\sin^2(\frac{1}{2a}\pi x)\sin^2(2\pi y)}{\sin^2(\frac{1}{2a}\pi x)\sin^2(\pi y)} = \frac{4\sin^2(\pi y)\cos^2(\pi y)}{\sin^2(\pi y)} = 4\cos^2(\pi y);$$

When $a > \sqrt{2/3}$, consider the audible quantity

$$\frac{N(\lambda_{0,2}) - N(\lambda_{1,1})}{N(\lambda_{0,1})} = \frac{\sin^2(\frac{1}{2a}\pi x)\sin^2(2\pi y)}{\sin^2(\frac{1}{2a}\pi x)\sin^2(\pi y)} = 4\cos^2(\pi y).$$

In either case, the audible quantity is always enough for us to echolocate y up to reflective symmetry with respect to the line $y = 0.5$. Then we can deduce x, up to symmetry, from $N(\lambda_{0,1})$ directly.

Case 2: $a = \sqrt{2/3}$.

When $a = \sqrt{2/3}$, our eigenvalue $\lambda_{1,1}$, or equivalently $\lambda_{0,2}$, has multiplicity of 2, so

$$N(\lambda_{1,1}) = N(\lambda_{0,2}) = N(\lambda_{0,1}) + \sin^2\left(\frac{3}{2a}\pi x\right)\sin^2(\pi y) + \sin^2\left(\frac{1}{2a}\pi x\right)\sin^2(2\pi y).$$

Instead of looking at the first three eigenvalues, we consider $\lambda_{0,3}$ and $\lambda_{0,6}$. We claim that $\lambda_{0,n}$ is simple when $n = 3$ and $n = 6$.

Recall that the general formula for $\lambda_{m,n}$ is given by Equation 3.32. Suppose $\lambda_{0,n}$ is not simple, so there exist $a, b \in \mathbb{N}$ such that $\lambda_{a,b} = \lambda_{0,n}$, and we have

$$(2a+1)^2 + 4\left(\frac{2}{3}b^2\right) = 1 + \frac{8}{3}n^2.$$

By reordering and simplifying the equation, we get

$$n = \sqrt{\frac{3}{2}a^2 + \frac{3}{2}a + b^2}. \tag{3.33}$$

Note that when $n = 6$, if we let $a = 1, \ldots, 5$, and for each value of a we find the corresponding b that solves the Equation 3.33, we have the following possibilities:

$$\begin{aligned}
a = 1 \Rightarrow &\quad b = \sqrt{33} \\
a = 2 \Rightarrow &\quad b = \sqrt{27} \\
a = 3 \Rightarrow &\quad b = \sqrt{18} \\
a = 4 \Rightarrow &\quad b = \sqrt{6} \\
a \geqslant 5 \Rightarrow &\quad b < 0.
\end{aligned}$$

So, no $a, b \in \mathbb{N}$ exists such that $\sqrt{\frac{3}{2}a^2 + \frac{3}{2}a + b^2} = 6$. Therefore, $\lambda_{0,3}$ is simple. Using a similar approach, we can show that $\lambda_{0,6}$ is also simple.

In fact, by explicitly computing the eigenvalues, we have

$$\boldsymbol{\lambda_{0,1}} < \lambda_{1,1} = \lambda_{0,2} < \lambda_{1,2} < \boldsymbol{\lambda_{0,3}} < \cdots < \lambda_{2,5} = \lambda_{3,4} = \lambda_{4,2} < \boldsymbol{\lambda_{0,6}}.$$

Recall the double- and triple-angle formulas: $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ and $\sin(3\theta) =$

$3\sin(\theta) - 4\sin^3(\theta)$ for all $\theta \in \mathbb{R}$. Consider functions φ and ψ defined in terms of the audible quantities by

$$\varphi(x,y) := \frac{N(\lambda_{0,3}) - N(\lambda_{1,2})}{N(\lambda_{0,1})} = \frac{\sin^2(3\pi y)}{\sin^2(\pi y)} = (3 - 4\sin^2(\pi y))^2,$$
$$\psi(x,y) := \frac{N(\lambda_{0,6}) - N(\lambda_{4,2})}{N(\lambda_{0,3}) - N(\lambda_{1,2})} = \frac{\sin^2(6\pi y)}{\sin^2(3\pi y)} = 4\cos^2(3\pi y).$$

Assume there exists $y_1 \in (0, 0.5)$ such that $y_2 = \alpha y_1$ for some $\alpha \in \mathbb{R}$ and that

$$\begin{cases} \varphi(x,y_1) = \varphi(x,y_2) = \varphi(x,\alpha y_1) \\ \psi(x,y_1) = \psi(x,y_2) = \psi(x,\alpha y_1) \end{cases}. \tag{3.34}$$

By solving the system of equations (3.34) through Mathematica, we obtain

$$\alpha = \begin{cases} \dfrac{-1+y_1}{y_1} \\ -1 + \dfrac{1}{y_1} \\ -1 \\ 1 \end{cases} \tag{3.35}$$

Notice that since $y_2 \in (0,1)$, $\alpha > 0$, so we are left with

$$\alpha = -1 + \frac{1}{y_1} \quad \text{or} \quad \alpha = 1.$$

When $\alpha = -1 + 1/y_1$, we would have $y_2 = \alpha y_1 = 1 - y_1$, which is the reflection of y_1 across the line $y = 0.5$. When $\alpha = 1$, we have $y_2 = y_1$. Therefore, y_1 and y_2 are either identical or symmetric, so we're able to echolocate up to the reflective symmetry of

Figure 3.4: Rectangle with mixed boundary conditions where $a > 1$, the dashed lines denote the Neumann boundary conditions and the thick lines denote the Dirichlet boundary condition.

$y = 0.5$. Thus, one can uniquely echolocate the $y-$coordinate up to symmetry, and we can determine $\sin^2\left(\frac{1}{2a}\pi x\right)$ by considering $N(\lambda_{0,1})$. Note that $\sin^2\left(\frac{1}{2a}\pi x\right)$ is injective for $x \in [0, a)$, so we can uniquely identify the $x-$coordinate. □

Notice that we've only considered rectangles $(0, a) \times (0, 1)$ where $a \in (0, 1)$. The Neumann boundary condition is set on one side of length 1. We propose several potential directions for future work.

If we allow $a > 1$ and keep the Neumann boundary condition on the same side, as shown in Figure 3.4, the multiplicity of $\lambda_{m,n}$ will vary depending on a and becomes much more complicated. Furthermore, what if two sides of the rectangle are equipped with the Neumann boundary condition: can we echolocate if the two sides are adjacent, or opposite, to each other?

Besides rectangles, the question will be more exciting while complicated if we consider boxes, as we have more freedom to specify different boundary conditions on different sides. These can be interesting questions for further exploration.

Chapter 4

Echolocation with Gaussian Curvature

4.1 Intuition for Gaussian curvature

Gaussian curvature is a concept from differential geometry that measures how curved a surface is at some certain point. For some familiar examples, one can imagine a flat sheet of paper that has a constant Gaussian curvature of zero; the curvature of a sphere is also constant and can be thought of as the amount by which the sphere deviates from being flat. A nice property about Gaussian curvature is that it depends only on the shape of the surface itself, and if you bend the surface without tearing or stretching it, the Gaussian curvature at each point will not change. For example, even if you roll or bend the flat sheet of paper into a cone or something else, as long as you can flatten it back, the Gaussian curvature stays zero everywhere. Gaussian curvature is also independent of our coordinate system, which is a useful fact for our later computation.

A more mathematical intuition for Gaussian curvature is as follows: Pick any point on a surface S and we can take the vector perpendicular to the surface at that point. Any plane containing this *normal vector* is called a *normal plane*. Each normal plane will intersect the surface and the intersection will be a 2-dimensional curve, and we can measure the curvature of the curve. Specifically, suppose the curve is parametrized by $x = x(t)$ and $y = y(t)$; we can compute the curvature of the curve by

$$\kappa = \frac{x'y'' - y'x''}{(x'^2 + y'^2)^{3/2}}.$$

Note that since we have infinitely many normal planes, we have infinitely many such curvatures. The maximum and minimum of these curvatures are called *principal curvatures*, denoted by κ_1, κ_2. The *Gaussian curvature* is given by

$$K = \kappa_1 \kappa_2.$$

4.2 The Two Fundamental Forms and Computation of Gaussian Curvature

A surface patch is usually used to study the local behavior of a surface.

Definition 8. *A surface patch is a smooth mapping from an open subset* Ω *of* $\mathbb{R}^2$ *into* $\mathbb{R}^3$ *defined by* $\sigma(u, v)$*, where* $\sigma(u, v)$ *represents a point of the surface in* $\mathbb{R}^3$*, for each pair of* $(u, v) \in \Omega$*. That is,* $\sigma(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$ *.*

A surface patch is closely related to the *parametrization* of a surface. For surfaces that cannot be represented by a single continuous function over the entire domain, one can construct multiple surface patches to describe different regions. These patches together constitute the parametrization of the surface.

Suppose we have a surface S with surface patch $\sigma(u, v)$. Pick any point p on S and let σ_u and σ_v denote the derivatives of σ evaluated at the point $(u_0, v_0) \in \mathbb{R}^2$ such that $\sigma(u_0, v_0) = p$. Note that σ_u and σ_v form a basis of the tangent plane, assume it's well-defined, to S at p, denoted by T_pS, so any tangent vector to S at p can be expressed uniquely as a linear combination of σ_u and σ_v. Suppose $\vec{v} = \lambda\sigma_u + \mu\sigma_v$ where λ and μ are constants. Define maps $du : T_pS \to \mathbb{R}$ and $dv : T_pS \to \mathbb{R}$ by

$$du(\vec{v}) = \lambda, \quad dv(\vec{v}) = \mu.$$

Consider first the dot product of v with itself. We have the square of the length of $\vec{v}$ given by

$$||\vec{v}||^2 = \lambda^2 \langle\sigma_u, \sigma_u\rangle + 2\lambda\mu \langle\sigma_u, \sigma_v\rangle + \mu^2 \langle\sigma_v, \sigma_v\rangle . \tag{4.1}$$

Letting

$$E = ||\sigma_u||^2, \quad F = \sigma_u \cdot \sigma_v, \quad G = ||\sigma_v||^2, \tag{4.2}$$

we can rewrite Equation 4.1 as

$$||\vec{v}||^2 = E\lambda^2 + 2F\lambda\mu + G\mu^2 = Edu^2 + 2Fdudv + Gdv^2.$$

Definition 9. *The expression*

$$Edu^2 + 2Fdudv + Gdv^2$$

is called the first fundamental form of the surface patch $\sigma(u, v)$*.*

The first fundamental form provides a way to measure distances and angles, and to compute areas, on the surface.

In order to study the curvature, we also need to consider the second fundamental form which measures how much the surface deviates from the tangent plane at a point $\sigma(u, v)$. The deviation is represented by

$$(\sigma(u + \delta u, v + \delta v) - \sigma(u, v)) \cdot \vec{N},$$

where $\vec{N}$ is the standard unit normal vector. We can expand this expression via Taylor's theorem for two variables. The full derivation can be found in Chapter 7.1 of [Pre10] and finally we have the deviation written as

$$\frac{1}{2}(L(\delta u)^2 + 2M\delta u\delta v + N(\delta v)^2) + \text{ remainder},$$

where

$$L = \sigma_{uu} \cdot \vec{N}, \quad M = \sigma_{uv} \cdot \vec{N}, \quad N = \sigma_{vv} \cdot \vec{N}, \tag{4.3}$$

and the remainder tends to zero as δu and δv tend to zero. Note that δu and δv essentially represent the same meaning as du and dv, respectively, as we defined earlier.

Definition 10. *The expression*

$$Ldu^2 + 2Mdudv + Ndv^2$$

is called the second fundamental form of a surface patch σ.

The Weingarten map, also known as the shape operator, is a linear operator that provides another way to think about the curvature of a surface. It encodes the changes in the normal vector to the surface as one moves along the surface. Mathematically, let S be an oriented surface and let $\vec{N} : S \to \mathbb{R}^3$ denote the orientation. In other words, $\vec{N}$ is the unit normal field on S. Thus, $\vec{N}$ maps any point $p \in S$ to a vector in S^2. For $p \in S$, we consider the derivative of $\vec{N}$ at p, denoted by $d\vec{N}_p : T_pS \to T_pS$, and we give the following definition:

Definition 11. *For every $p \in S$, the linear transformation $W_{p,S} : T_pS \to T_pS$ given by*

$$W_{p,S} = -d\vec{N}_p$$

is called the Weingarten map of S at p.

The Gaussian curvature K at point p is defined by the determinant of W, denoted by $\det(W)$. Recall that in the last section, we defined Gaussian curvature as the product of principal curvatures κ_1 and κ_2. In fact, κ_1 and κ_2 are two eigenvalues of the Weingarten map, so $K = \det(W) = \kappa_1\kappa_2$. We will see in the following computation that the curvature of a surface can be understood both through the behavior of normal vectors (as captured by the Weingarten map) and through the curvature extremes.

Now we can derive the explicit formula for K. Define two symmetric matrices $\mathcal{F}_I$ and $\mathcal{F}_{II}$ by

$$\mathcal{F}_I = \begin{pmatrix} E & F \\ F & G \end{pmatrix}, \quad \mathcal{F}_{II} = \begin{pmatrix} L & M \\ M & N \end{pmatrix}.$$

Recall that $\{\sigma_u, \sigma_v\}$ is a basis for T_pS. The matrix representation of the Weingarten map $W_{p,S}$ with respect to the basis $\{\sigma_u, \sigma_v\}$ is $\mathcal{F}_I^{-1}\mathcal{F}_{II}$. The proof is given in Chapter 8.1 in [Pre10]. Therefore, we have

$$K = \det(\mathcal{F}_I^{-1}\mathcal{F}_{II}) = \frac{\det(\mathcal{F}_{\mathcal{I}\mathcal{I}})}{\det(\mathcal{F}_I)} = \frac{LN - M^2}{EG - F^2}. \tag{4.4}$$

We will use this formula to compute Gaussian curvature from now on.

Recall from Chapter 2 that every manifold has a unique heat kernel, and the heat kernel is a spectral invariant. Here, we sketch the proof from Chapter 6.1.2 in [LMD23] that the Gaussian curvature is a spectral invariant.

Theorem 4. *Let M be a surface. The Gaussian curvature of M is a spectral invariant.*

Proof. By Minakshisundaram-Pleijel asymptotic expansion of the heat kernel for $t \to$

0^+, we have

$$e(t,x,x) = (4\pi t)^{-1} \sum_{j=0}^{k} a_j(x) t^j + O(t^{k+1})$$

for all $k > 0$, where $a_j(x)$ are called the *local heat invariants*. The heat invariants contain rich geometric information about the surface. For example, a_0 is the volume of the surface and a_1 involves the scalar curvature. They are 'local' because they can be computed in terms of the geometry of M near x. $O(t^{k+1})$ is the big O notation that describes the error term in the asymptotic expansion of the heat kernel as $t \to 0^+$, this part of the formula will become negligible at the rate of at least t^{k+1} as t gets closer to zero.

Note that we also have $e(t,x,x) = \sum_{j=0}^{\infty} e^{-\lambda_j t} \phi_j(x)^2$. Therefore, as $t \to 0^+$, we have

$$\sum_{j=0}^{\infty} e^{-\lambda_j t} \sim (4\pi t)^{-1} \sum_{j=0}^{\infty} a_j t^j.$$

Note that the heat trace on the left-hand side is completely determined by the Laplace spectrum. This implies that the local heat invariants $a_j(x)$ are also determined by the Laplace spectrum. More importantly, according to calculation in Chapter 3.3 of [RS97], $a_1(x) = \frac{1}{6}\tau(x)$, where $\tau(x)$ is the scalar curvature of M at point x, so the scalar curvature is also a spectral invariant. Since scalar curvature is twice the Gaussian curvature, the Gaussian curvature is also a spectral invariant. □

4.3 Echolocation on Surfaces of Revolution

A surface of revolution is a surface in Euclidean space $\mathbb{R}^3$ created by rotating a curve around an axis of rotation. The curve being rotated is called the *profile curve.* We could consider the earth as a surface of revolution where the profile curve is any longitude. Without loss of generality, we may assume our profile curve is any curve in the $xz-$plane and the axis of rotation is the $z-$axis. If the profile curve is parametrized by

$$\gamma(u) = (f(u), 0, g(u)),$$

by rotating it around the $z-$axis, the parametrization of the surface of revolution will be given by

$$\sigma(u, v) = (f(u)\cos v, f(u)\sin v, g(u)),$$

where usually $v \in [0, 2\pi]$.

The Gaussian curvature of a surface of revolution has a simple form if the profile curve has a certain parametrization.

Definition 12. *A curve $\gamma(u) = (f(u), 0, g(u))$ has unit-speed parametrization if*

$$||\dot{\gamma}|| = \dot{f}^2 + \dot{g}^2 = 1,$$

where $\dot{f}, \dot{g}$ are the first derivative of f and g with respect to u, respectively.

Lemma 2. *[Pre10, Example 8.1.4] The Gaussian curvature of a surface of revolution is $-\frac{\ddot{f}}{f}$ if the profile curve has unit-speed parametrization.*

Proof. Consider the surface of revolution parametrized by

$$\sigma(u, v) = (f(u)\cos v, f(u)\sin v, g(u)),$$

with $\dot{f}^2 + \dot{g}^2 = 1$ by assumption. Note that by differentiating both sides with respect to u, we get $\ddot{f}\dot{f} + \dot{g}\ddot{g} = 0$. By computing the first and second fundamental forms of this surface using Equations 4.2 and 4.3 (the full computation, which we omit here, can be found in Example 6.1.3 and Example 7.1.2 in [Pre10]), we will find that

$$E = 1,\ F = 0,\ G = f^2,\ \ L = \dot{f}\ddot{g} - \ddot{f}\dot{g},\ M = 0,\ N = f\dot{g}.$$

Thus, we can compute the Gaussian curvature using Equation 4.4 and get

$$K = \frac{LN - M^2}{EG - F^2} = \frac{(\dot{f}\ddot{g} - \ddot{f}\dot{g})f\dot{g}}{f^2}. \tag{4.5}$$

Note that the numerator can be written as

$$\begin{aligned}
(\dot{f}\ddot{g} - \ddot{f}\dot{g})f\dot{g} &= (\dot{f}\ddot{g}\dot{g} - \ddot{f}\dot{g}^2)f \\
&= (\dot{f}(-\ddot{f}\dot{f}) - \ddot{f}\dot{g}^2)f \\
&= (-\dot{f}^2\ddot{f} - \ddot{f}\dot{g}^2)f \\
&= -\ddot{f}f(\dot{f}^2 + \dot{g}^2) \\
&= -\ddot{f}f.
\end{aligned}$$

Therefore, $K = \frac{-\ddot{f}f}{f^2} = -\frac{\ddot{f}}{f}$. □

Wyman and Xi [WX23] stated that echolocation holds on the torus but they didn't provide proof for that. We will provide a proof. A torus has rotational symmetry around its central axis. It also has reflective symmetry when bisected along a plane that contains its central axis.

Theorem 5. *Echolocation on a torus is possible, up to symmetry, with Gaussian curvature.*

Proof. Consider a general torus. It is a surface of revolution where the profile curve is a circle or an ellipsoid. We consider the case for the circle. Suppose the circle has center $(c, 0, 0)$ and radius R. Without loss of generality, we assume $c > 0$ and $R \in \mathbb{R}$. Note that we don't necessarily assume $c - R > 0$ as whether the torus intersects itself will not affect our computation of its Gaussian curvature.

The circle will be parameterized as

$$\gamma(\theta) = \left(c + R\cos\left(\frac{\theta}{R}\right), 0, R\sin\left(\frac{\theta}{R}\right)\right)$$

where $\theta \in [0, 2R\pi]$. We have $\dot{\gamma}(\theta) = \left(-\sin\left(\frac{\theta}{R}\right), 0, \cos\left(\frac{\theta}{R}\right)\right)$, so

$$||\dot{\gamma}(\theta)|| = \sqrt{\left(-\sin\left(\frac{\theta}{R}\right)\right)^2 + \left(\cos\left(\frac{\theta}{R}\right)\right)^2} = 1,$$

and γ is a unit-speed parametrization. Therefore, the parametrization of the torus is given by

$$\sigma(\theta, \phi) = \left(\left(c + R\cos\left(\frac{\theta}{R}\right)\right)\cos(\phi), \left(c + R\cos\left(\frac{\theta}{R}\right)\right)\sin(\phi), R\sin\left(\frac{\theta}{R}\right)\right)$$

where $\theta \in [0, 2R\pi)$ and $\phi \in [0, 2\pi)$. Define

$$f(\theta) = c + R\cos\left(\frac{\theta}{R}\right), \quad g(\theta) = R\sin\left(\frac{\theta}{R}\right).$$

Since we've already shown that $\dot{f}^2 + \dot{g}^2 = 1$,

$$K = -\frac{\ddot{f}}{f} = \frac{\frac{1}{R}\cos\left(\frac{\theta}{R}\right)}{c + R\cos\left(\frac{\theta}{R}\right)} = \frac{\cos\left(\frac{\theta}{R}\right)}{R\left(c + R\cos\left(\frac{\theta}{R}\right)\right)}.$$

Notice that

$$\frac{\partial K}{\partial \theta} = -\frac{c\sin\left(\frac{\theta}{R}\right)}{R^2\left(c + R\cos\left(\frac{\theta}{R}\right)\right)^2}. \tag{4.6}$$

For $\theta \in [0, R\pi)$, $\sin\left(\frac{\theta}{R}\right) > 0$ and the denominator is clearly greater than zero, so $\frac{\partial K}{\partial \theta}$ is always less than zero. In other words, $K(\theta)$ is decreasing when θ goes from 0 to $R\pi$. Thus, $K(\theta)$ is injective for $\theta \in [0, R\pi)$. Therefore, we can identify θ up to symmetry with respect to the $xy-$plane on a torus. Since the torus is rotationally symmetric about the $z-$ axis, we can say echolocation holds on the torus by Gaussian curvature without identifying ϕ. $\square$

Now we take a look at a second surface of revolution called a catenoid. A catenoid has rotational symmetry around its central axis. It also has reflective symmetry when bisected along a plane that contains its central axis. It's also symmetric with respect to reflection in the $xz-$plane.

Theorem 6. *Gaussian curvature is enough for echolocation on a catenoid up to symmetry.*

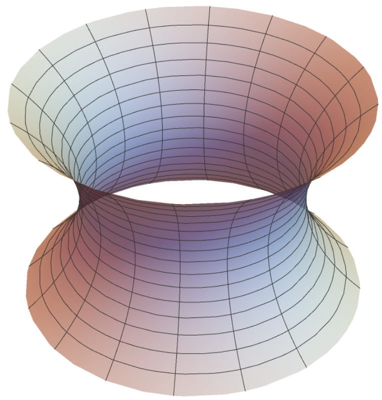

Figure 4.1: Catenoid with $c = 1$ and $\theta \in [0, R\pi)$

Proof. Let's consider a general catenoid parametrized by

$$x(u,\theta) = c\cosh\left(\frac{u}{c}\right)\cos(\theta)$$
$$y(u,\theta) = c\cosh\left(\frac{u}{c}\right)\sin(\theta)$$
$$z(u,\theta) = u,$$

$\theta \in [0, 2\pi)$, and $u \in [-c, c]$ for some $c > 0$. A catenoid with $c = 1$ is shown in Figure 4.1. The coefficients for the first and the second fundamental forms are given by

$$E = c^2\cosh^2\left(\frac{u}{c}\right),\ F = 0,\ G = \cosh^2\left(\frac{u}{c}\right),$$
$$L = -c, \qquad M = 0, \qquad N = \frac{1}{c}.$$

By equation 4.4, the Gaussian curvature of a catenoid is given by

$$K = -\frac{1}{c^2}\operatorname{sech}^4\left(\frac{u}{c}\right).$$

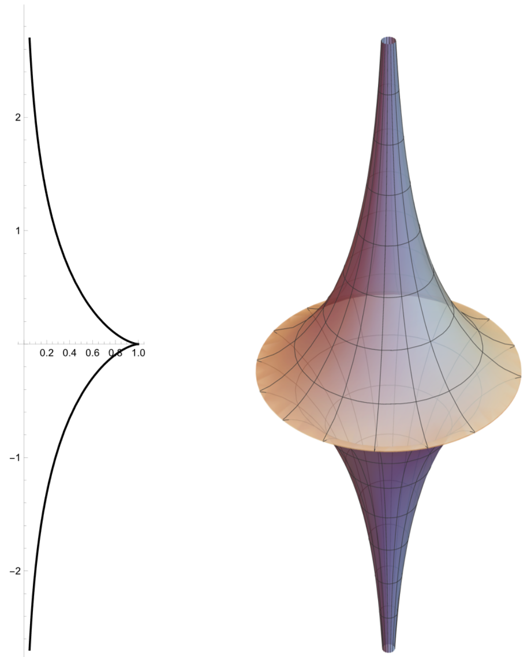

Figure 4.2: Tractrix and tractroid

Note that

$$\frac{\partial K}{\partial u} = -\frac{4u}{c^3}\left(\operatorname{sech}^4\left(\frac{u}{c}\right)\tanh\left(\frac{u}{c}\right)\right).$$

For $u \in (0, c]$, since $\tanh\left(\frac{u}{c}\right) > 0$, $\frac{\partial K}{\partial u}$ is always less than zero, so K is injective for $u \in (0, c]$; we can uniquely identify u up to symmetry of the $xy-$plane. Similar to a torus, since a catenoid is a surface of revolution, we don't need to identify θ. Therefore, we can echolocate on a catenoid up to symmetry using Gaussian curvature. $\square$

Next, we show that there are also some surfaces of revolution for which Gaussian curvature is not enough for echolocation.

For example, consider a tractroid, or a pseudo-sphere, generated by rotating a tractrix about its asymptote ($z-$axis in our case), as shown in Figure 4.2.

The tractroid is parametrized by

$$x(u,\theta) = \operatorname{sech}(u)\cos(\theta)$$

$$y(u,\theta) = \operatorname{sech}(u)\sin(\theta)$$

$$z(u,\theta) = u - \tanh(u),$$

where $u \in [-10, 10]$ and $\theta \in [0, 2\pi)$. We restrict $u \in [-10, 10]$ so that the surface is compact. The coefficients for the first and second fundamental forms are given by

$$E = \tanh^2(u), \quad F = 0, \quad G = \operatorname{sech}^2(u)$$

and

$$L = -\operatorname{sech}(u)\tanh(u), \quad M = 0, \quad N = \operatorname{sech}(u)\tanh(u),$$

respectively. We find that the tractroid has a constant Gaussian curvature $K = -1$. Clearly, it does not help us echolocate.

Therefore, it's natural for us to consider potential necessary and sufficient conditions to echolocate using Gaussian curvature; we propose two conjectures for surfaces of revolution $(f(u)\cos(v), f(u)\sin(v), g(u))$ such that $\dot{f}^2 + \dot{g}^2 = 1$.

Conjecture 1: If f is injective, then Gaussian curvature K is enough for echolocation.

The conjecture is false. Let $f(u) = \sin(u)$ and $g(u) = \cos(u)$; f is injective for $u \in [0, \pi/2)$ while $K = -\ddot{f}/f = 1$ is not enough for echolocation.

Conjecture 2: If f is injective and the Gaussian curvature K is not a constant, it's enough for echolocation.

This conjecture ends up being much harder to disprove. We first provide an example to show why certain examples would not work as a counterexample. In the following two examples, our strategy is to first consider $f(u)$, then compute the Gaussian curvature. If the Gaussian curvature is not injective on the interval I, in other words, f is a promising candidate, we also need to check if $g(u)$ is well-defined over I where $g(u)$ is computed from the unit-speed parametrization assumption.

Example 1: First, consider $f(u) = \ln(u) + u$; then

$$K = -\frac{\frac{\partial}{\partial u}(\frac{1}{u} + 1)}{\ln(u) + u} = \frac{1/u^2}{\ln(u) + u}.$$

If we allow $u \in \mathbb{R}$, K is not injective up to some symmetry. Then we compute the corresponding $g(u)$ such that our parametrization is unit-speed; we need $\dot{f}^2 + \dot{g}^2 = 1$. Therefore,

$$g = \int \sqrt{-\frac{1}{u^2} - \frac{2}{u}}\, du.$$

Since $g \in \mathbb{R}$, we need $-\frac{1}{u^2} - \frac{2}{u} \geqslant 0$ and so $u \in [-\frac{1}{2}, 0)$. However, f is undefined for $u \in [-\frac{1}{2}, 0)$, so this doesn't work as a counterexample.

Example 2: Let $f(u) = \sin(u) - u$; then $\dot{f} = \cos(u) - 1$, so

$$K = -\frac{\partial}{\partial u}(\cos(u) - 1) \cdot \frac{1}{\sin(u) - u} = \frac{\sin(u)}{\sin(u) - u}.$$

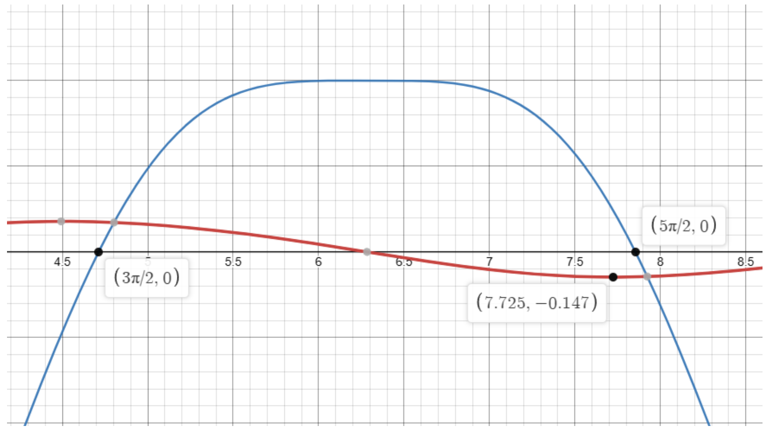

Figure 4.3: Graphs of K and $-\cos^2(u) + 2\cos(u)$

By the unit-speed parameterization, we follow a similar procedure as in the above example to get

$$g = \int \sqrt{-\cos^2(u) + 2\cos(u)}\, du.$$

Notice that g is a real-valued function when u is defined on a certain union of disjoint intervals. In Figure 4.3, the function K is shown in the red curve, and the function $-\cos^2(u) + 2\cos(u)$ is given in blue. We want the blue curve to be above the $u-$axis so that g is real.

If we pick $u \in [\frac{3\pi}{2}, \frac{5\pi}{2}]$, K is not injective over that interval. For example,

$$K(7.7) = K(7.75) = -0.14723$$

To visualize the parametrized surface, we compute g numerically. The surface is shown in Figure 4.4 where the thick line and dashed line are parametrized curve $\{(f(7.7)\cos(v), f(7.7)\sin(v), g(7.7))\ |v \in (0, 2\pi)\}$ and $\{(f(7.75)\cos(v), f(7.75)\sin(v), g(7.75))\ |v \in (0, 2\pi)\}$, respectively.

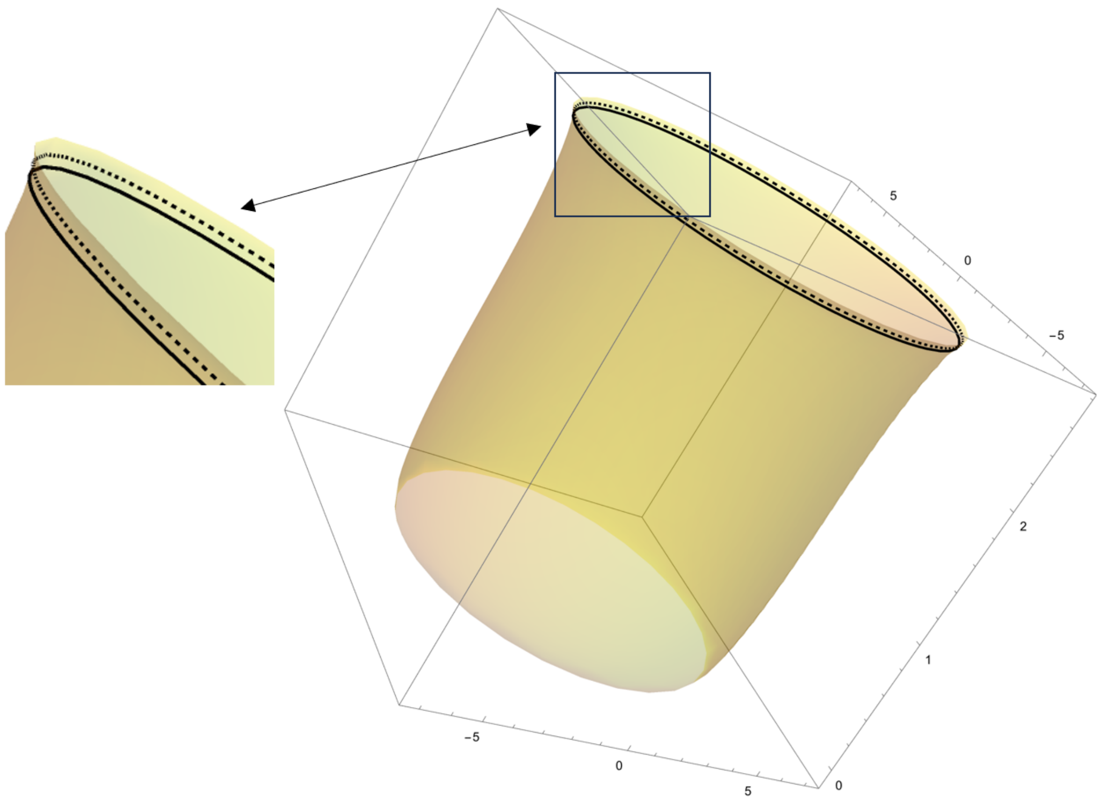

Figure 4.4: Visualization of the counterexample

4.4 Echolocation on Minimal Surfaces

4.4.1 Introduction to Minimal Surfaces

Now we consider another category of surfaces, namely minimal surfaces. Minimal surfaces are fascinating objects in mathematics and physics with both beauty and complexity. The physical properties of minimal surfaces can be explained by a soap film experiment. Imagine dipping a wireframe of any shape into a soap solution and then pulling it out. The soap film that spans the frame tends to minimize its area, forming a minimal surface. This phenomenon occurs because the surface tension of the soap film seeks to reduce the surface area to the smallest possible value for the given boundary conditions. In other words, the energy is minimized for the soap film.

Mathematically, minimal surfaces are defined as surfaces with zero mean curvature, which means the average of the two principal curvatures is zero. Minimal surfaces also have connections with complex analysis. Many minimal surfaces can be parametrized in terms of two complex-valued functions via the Enneper-Weierstrass Parameterization [CK05]. The first but trivial example of a minimal surface is a piece of flat paper. Clearly, it has zero curvature everywhere. The catenoid we discussed in Section 4.3 is also a minimal surface. In fact, the catenoid and the plane are the only connected surfaces of revolution which are also minimal surfaces[MP12].

In this section, we consider two more complicated and interesting minimal surfaces, namely Enneper's minimal surface and Henneberg's minimal surface.

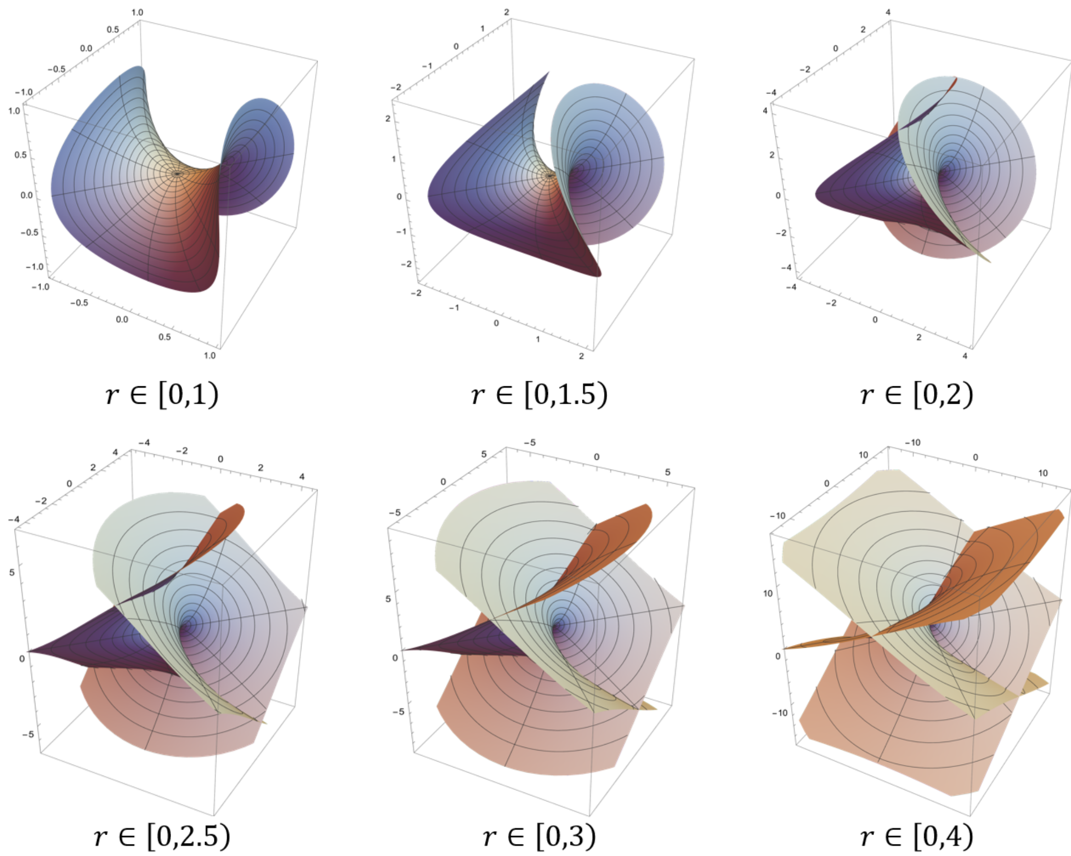

Figure 4.5: Enneper's minimal surface with different range of r

4.4.2 Enneper's Minimal Surface

Let's first consider Enneper's minimal surface. The surface is parametrized [WEEn] by $\sigma(r, \phi) = (x, y, z)$ where x, y, z are functions defined as

$$
\begin{aligned}
x(r, \phi) &= r\cos\phi - \frac{1}{3}r^3\cos(3\phi), \\
y(r, \phi) &= -\frac{1}{3}r[3\sin\phi + r^2\sin(3\phi)], \\
z(r, \phi) &= r^2\cos(2\phi),
\end{aligned}
$$

where $\phi \in [0, 2\pi)$ and $r \in [0, z]$. One has the flexibility to choose z and the surface will look different, as shown in Figure 4.5. In this thesis, we focus on the case where $r \in [0, 1]$ so that the surface does not intersect itself.

The Enneper's minimal surface has two obvious reflective symmetries with respect to the $xz-$plane and $yz-$plane. It also has a less obvious rotational symmetry: if we first rotate the surface about the $x-$axis by $180°$ and then rotate the surface about the $y-$axis by $180°$, the surface will look the same. Mathematically, we can define a rotational map R as

$$R = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

If we apply R to Enneper's minimal surface, the overall shape of the surface won't be changed. For each point $p = (x, y, z)$ on Enneper's minimal surface, it will be mapped to

$$Rp = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -x \\ -y \\ z \end{pmatrix}.$$

The Gaussian curvature of Enneper's minimal surface [WEEn] is given by

$$K = -\frac{4}{(1 + r^2)^4}.$$

With this information, we are able to identify r, but the Gaussian curvature contains no information about ϕ. Therefore, given any fixed r_0, the Gaussian curvature on the

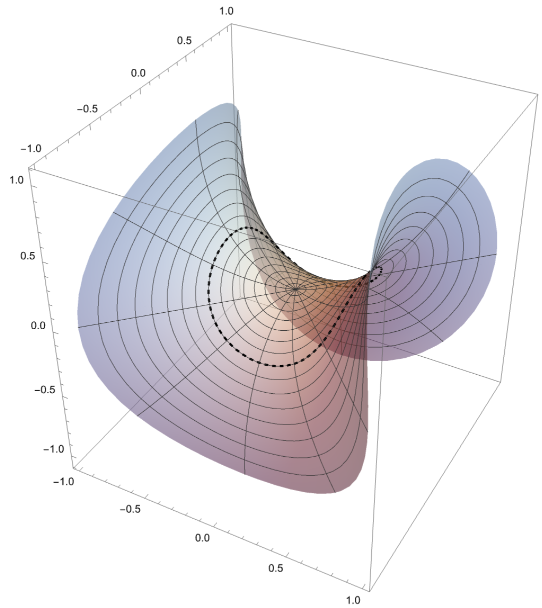

Figure 4.6: Set of points on Enneper's Minimal Surface that have the same Gaussian curvature shown by the dashed line

curve $\sigma(r_0, \phi)$ will be constant for all $\phi \in [0, 2\pi)$. Figure 4.6 shows a set of points on the curve $\sigma(0.5, \phi)$ that have the same Gaussian curvature. Note that the surface is not rotationally symmetric with respect to the $z-$axis, so Gaussian curvature is not enough for us to echolocate on Enneper's surface up to symmetry.

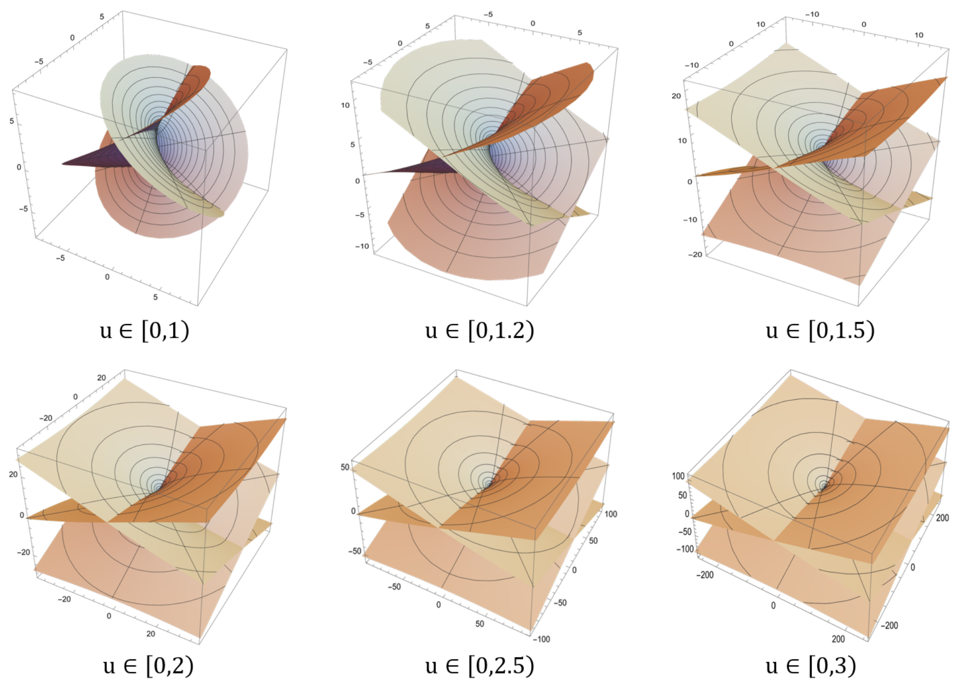

Figure 4.7: Henneberg's Minimal Surface with different range of u

4.4.3 Henneberg's Minimal Surface

We look at Henneberg's minimal surface. The minimal surface is parametrized by [WEHe] $\sigma(u, v) = (x, y, z)$ where x, y, z are functions defined as

$$\begin{aligned} x(u, v) &= 2\sinh u \cos v - \frac{2}{3}\sinh(3u)\cos(3v) \\ y(u, v) &= 2\sinh u \sin v + \frac{2}{3}\sinh(3u)\sin(3v) \\ z(u, v) &= 2\cosh(2u)\cos(2v) \end{aligned}$$

where $u \in [0, z)$ and $v \in [0, 2\pi)$; similar to Enneper's minimal surface, one has the flexibility to choose z, as shown in Figure 4.7. In our thesis, we focus on the case where $z = 1$.

The coefficients [WEHe] of the first fundamental form are given by

$$
\begin{aligned}
E &= 8\cosh^2 u\,[\cosh(4u) - \cos(4v)]\\
F &= 0\\
G &= 8\cosh^2 u\,[\cosh(4u) - \cos(4v)]\,,
\end{aligned}
$$

and the coefficients of the second fundamental form are given by

$$
\begin{aligned}
L &= -4\cos(2v)\sinh(2u)\\
M &= 4\cosh(2u)\sin(2v)\\
N &= 4\sinh(2u)\cos(2v).
\end{aligned}
$$

Therefore, we can compute the Gaussian curvature by

$$
\begin{aligned}
K(u,v) = \frac{LN - M^2}{EG - F^2} &= \frac{-16\cos^2(2v)\sinh^2(2u) - 16\cosh^2(2u)\sin^2(2v)}{64\cosh^4(u)(\cos(4v) - \cosh(4u))^2}\\
&= \frac{-4e^{-4u} - 4e^{4u} + 4e^{-4iv} + 4e^{4iv}}{64\cosh^4(u)(\cos(4v) - \cosh(4u))^2}\\
&= \frac{8(\cos(4v) - \cosh(4u))}{64\cosh^4(u)(\cos(4v) - \cosh(4u))^2}\\
&= \frac{\operatorname{sech}^4 u}{8(\cos(4v) - \cosh(4u))}.
\end{aligned}
$$

Henneberg's minimal surface is a non-orientable minimal surface, meaning it only has one side. The lines $\{x = \pm y, z = 0\}$ lie on the surface, rotation of 180° around any of these straight lines is a symmetry of the surface. In other words, it has a finite group of symmetry [OB16].

Theorem 7. *Echolocation on Henneberg's minimal surface is not possible via Gaus-*

sian curvature.

Proof. Let u_1, v_1 be arbitrary; the Gaussian curvature at the point $\sigma(u_1, v_1)$ is given by

$$K = \frac{\operatorname{sech}^4(u_1)}{8(\cos(4v_1) - \cosh(4u_1))}.$$

Let $u_2 = u_1 + a$ where $a \in \mathbb{R}$ and $u_2 \in [0, 1]$; we want to find a v_2 such that $K(u_1, v_1) = K(u_2, v_2)$. We can rewrite this as

$$8(\cos(4v_2) - \cosh(4(u_1 + a)))\operatorname{sech}^4(u_1) = 8\operatorname{sech}^4(u_1 + a)(\cos(4v_1) - \cosh(4u_1)),$$

and further

$$v_2 = \pm\frac{1}{4}\arccos[\cosh[4(a + u_1)] + \cosh^4(u_1)\operatorname{sech}^4(a + u_1)(\cos(4v_1) - \cosh(4u_1))].$$

Since $v_2 \in (0, 2\pi)$ and arccos has range $[0, \pi]$, we take the positive sign. If we obtain two points $p = \sigma(u_1, v_1)$ and $q = \sigma(u_2, v_2)$, where p and q are not related by any kind of symmetry as described above, then we've shown that Gaussian curvature is insufficient for echolocation on Hennegerg's minimal surface. For example, choose $u_1 = 0.2, v_1 = 4$, and let $a = 0.2$, so $u_2 = 0.4$. By the formula above, we can compute that $v_2 = 0.177562$. Thus, we get $P = (-0.621366, -0.532484, -0.314592)$ and $Q = (-0.0582912, 0.656155, 2.50797)$, as shown in Figure 4.8. We believe that these two points are not related by a symmetry.

To complete a rigorous proof of Theorem 8, we will show the following:

Corollary 1. *Given any* $(u_1, v_1) \in (0, 0.5) \times (0, 2\pi)$*, there is always an* $a \in (0, 1 - u_1)$

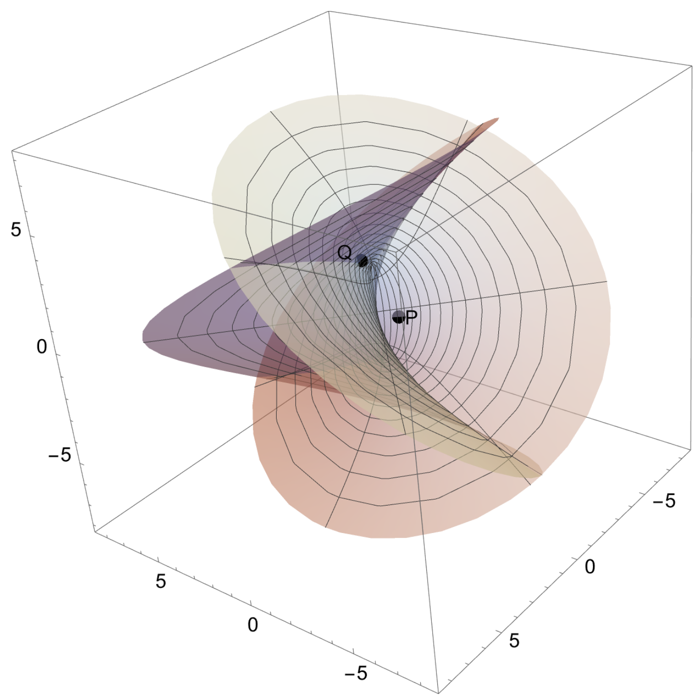

Figure 4.8: Two non-symmetric points with the same Gaussian curvature on Henneberg's surface

such that the point parametrized by $\sigma(u_2, v_2) = \sigma(u_1 + a, \pi/8)$ will have the same Gaussian curvature as the point parametrized by $\sigma(u_1, v_1)$.

We will show the existence via the Intermediate Value Theorem (IVT). Given any u_1, v_1, let $u_2 = u_1 + a$; by fixing K, as shown above, we have

$$v_2 = \frac{1}{4}\arccos[\cosh[4(a+u_1)] + \cosh^4(u_1)\operatorname{sech}^4(a+u_1)(\cos(4v_1) - \cosh(4u_1))]. \tag{4.7}$$

To simplify notation, we write u_1, v_1 as u, v in this proof. We can rewrite v_2 in terms of hyperbolic trigonometric functions and simplify using Mathematica; more specifically, we have

$$\begin{aligned} v_2 &= \frac{1}{4}\arccos\left\{\frac{e^{4(a+u)} + e^{-4(a+u)}}{2}\right. \\ &\quad \left. + \left(\left(\frac{e^u + e^{-u}}{2}\right)\left(\frac{2}{e^{a+u} - e^{-a-u}}\right)\right)^4\left(\frac{e^{4iv} + e^{-4iv}}{2} - \frac{e^{4u} + e^{-4u}}{2}\right)\right\} \\ &= \frac{1}{8}(\pi + 2i\log\left\{\frac{1}{2}i\left(e^{-4(a+u)} + e^{4(a+u)}\right) + \frac{ie^{4(a+u)}(e^{-u} + e^u)^4\left(-e^{-4u} - e^{4u} + e^{-4iv} + e^{4iv}\right)}{2\left(-1 + e^{2(a+u)}\right)^4}\right. \\ &\quad \left. + \sqrt{1 - \left(\frac{1}{2}\left(e^{-4(a+u)} + e^{4(a+u)}\right) + \frac{e^{4(a+u)}(e^{-u} + e^u)^4\left(-e^{-4u} - e^{4u} + e^{-4iv} + e^{4iv}\right)}{2\left(-1 + e^{2(a+u)}\right)^4}\right)^2}\right\}) \\ &= \frac{\pi}{8} + \frac{1}{4}i\log\left\{i\left[\cosh(4(a+u)) + \cosh^4(u)(\cos(4v) - \cosh(4u))\operatorname{csch}^4(a+u)\right]\right. \\ &\quad \left. + \sqrt{1 - \left[\cosh(4(a+u)) + \cosh^4(u)(\cos(4v) - \cosh(4u))\operatorname{csch}^4(a+u)\right]}\right\}. \end{aligned} \tag{4.8}$$

Define a function

$$\Gamma(a, u, v) = \cosh(4(a+u)) + \cosh^4(u)(\cos(4v) - \cosh(4u))\operatorname{csch}^4(a+u).$$

Recall that $u \in (0, 0.5), v \in (0, 2\pi)$, and $a \in [0, 1-u]$. In order to use IVT, first consider $\Gamma(a, u, v)$ when $a = 0$; then we have

$$\Gamma(0, u, v) = \cosh(4u) + (\cos(4v) - \cosh(4u)) \coth^4(u).$$

We want to find the maximum of $\Gamma(0, u, v)$. Notice that for any u, $\cosh(4u) > 0$ and $\coth^4(u) > 0$. Therefore, in order to make Γ large, we need to maximize $\cos(4v) - \cosh(4u)$. Clearly, for any u, since u is fixed,

$$\arg\max_{v \in [0, 2\pi)} \{\cos(4v) - \cosh(4u)\} = \arg\max_{v \in [0, 2\pi)} \Gamma(0, u, v) = \frac{n\pi}{2}, \quad n = 0, 1, 2, 3.$$

Without loss of generality, let $n = 0$, and thus $v = 0$. Therefore, we have

$$\Gamma(0, u, 0) = \cosh(4u) + (1 - \cosh(4u)) \coth^4(u),$$

and we can find $\arg\max_{u \in (0, 0.5)} \Gamma$ by taking the partial derivative. Note that

$$\frac{\partial \Gamma(0, u, 0)}{\partial u} = -2(-6\cosh(u) - 3\cosh(3u) + \cosh(5u)) \operatorname{csch}^3(u) > 0$$

for all $u \in (0, 0.5)$. Therefore, $\arg\max_{u \in (0, 0.5)} \Gamma(0, u, 0) = 0.5$, and we have for all $(u, v) \in (0, 0.5) \times (0, 2\pi)$,

$$\Gamma(0, u, v) < \Gamma(0, 0.5, 0) < -45 < 0.$$

Now consider $a = 1 - u$; we have

$$\Gamma(a, u, v) = \Gamma(1 - u, u, v) = \cosh(4) + \cosh^4(u)(\cos(4v) - \cosh(4u)) \operatorname{csch}^4(1).$$

Note that when $(u, v) \in (0, 0.5) \times (0, 2\pi)$,

$$\cosh(4u) \leqslant 5, \text{ and } \cos(4v) \geqslant -1.$$

Thus,

$$|\cos(4v) - \cosh(4u)| \leqslant 4 < \frac{\cosh(4)}{\cosh^4(1) \operatorname{csch}^4(1)}$$

and

$$\Gamma(1 - u, u, v) > 0.$$

Notice that Γ is continuous with respect to all three variables, so by the Intermediate Value Theorem, there exists $a \in (0, 1 - u)$ such that $\Gamma(a, u, v) = 0$. Therefore, with such an a, we can write

$$v_2 = \frac{\pi}{8} + \frac{1}{4} i \log(i\Gamma + \sqrt{1 - \Gamma}) = \frac{\pi}{8}.$$

Note that there are only finitely many symmetries for points on the curve parametrized by $v_2 = \pi/8$, but there are infinitely many points with $v_1 \in (0, 2\pi)$. Thus, given two points with the same Gaussian curvature, they are not necessarily symmetric. Therefore, we cannot echolocate on Henneberg's minimal surface via Gaussian curvature. $\square$

Chapter 5

Conclusion

In this thesis, we studied the question of echolocation asked by Wyman and Xi [WX23]. In particular, given a manifold M with boundary ∂M, are we able to locate any point $x \in M$, up to symmetry, given a certain set of audible quantities? To answer this question, we studied two types of audible quantities, namely the pointwise Weyl counting function and the Gaussian curvature.

We first explored echolocation by the pointwise Weyl counting function

$$N_x(\lambda) = \sum_{\lambda_j \leqslant \lambda} |\phi_j(x)|^2, \tag{5.1}$$

where λ_j and $\phi_j(x)$ are the Laplacian eigenvalues and eigenfunctions on M, respectively. Besides providing full details for proofs of echolocation on one-dimensional strings and two-dimensional rectangles, we explored the three-dimensional rectangles, on which we proved that echolocation up to symmetry is possible. We generalized the problem to echolocation on rectangles equipped with mixed (Dirichlet and Neumann) boundary conditions. By explicitly solving the Laplace eigenvalue problem via the wave equation, we showed that echolocation is possible for rectangles in the form $[0, a] \times [0, 1]$ with $a \in (0, 1)$ when the side $\{a\} \times [0, 1]$ is equipped with the Neumann boundary conditions while the rest have Dirichlet boundary conditions.

Secondly, we studied echolocation via Gaussian curvature on some more complicated surfaces. We first confirmed that echolocation is possible on the torus, and proved it for the new example of the catenoid. By studying the Gaussian curvature on surfaces of revolution, we provided counterexamples to two conjectures we made about the necessary and sufficient conditions for us to echolocate on a surface of revolution. We left this as an open problem. We also briefly explored echolocation on minimal surfaces. We showed that Gaussian curvature is not enough for echolocation on both Enneper's and Henneberg's minimal surfaces. Since minimal surfaces are themselves an interesting category of surfaces both for mathematicians and physicists, these two examples could provide insights for further studies on minimal surfaces.

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Appendix A: Mathematica Codes

.1 Equations from (3.18) to (3.22)

```
equation =
1 - c + 4 (1 - a) (1 - b) == 1 - abc/(d (a + b - d)) + 4 (1 - d) (1 - a - b + d);
solution = Solve[equation, d]

a = Sin[Pi*x1]^2;
b = Sin[Pi*y1]^2;
c = Sin[Sqrt[2]*Pi*z1]^2;

expr1 = 1/
   2 (a + b - Sqrt[
     2 a^2 + 2 b^2 - Sqrt[16 a b c + (4 a b - c)^2] + c]/Sqrt[2]);
expr2 = 1/
   2 (a + b + Sqrt[
```

```
    2 a^2 + 2 b^2 - Sqrt[16 a b c + (4 a b - c)^2] + c]/Sqrt[2]);
expr3 = 1/
   2 (a + b - Sqrt[
    2 a^2 + 2 b^2 + Sqrt[16 a b c + (4 a b - c)^2] + c]/Sqrt[2]);
expr4 = 1/
   2 (a + b + Sqrt[
    2 a^2 + 2 b^2 + Sqrt[16 a b c + (4 a b - c)^2] + c]/Sqrt[2]);


simplifiedExpr1 =
 FullSimplify[
  ArcSin[Sqrt[expr1]]/Pi, {0 < x1 < 0.5, 0 < y1 < 0.5,
   0 < z1 < Sqrt[2]/4}]
simplifiedExpr2 =
 FullSimplify[
  ArcSin[Sqrt[expr2]]/Pi, {0 < x1 < 0.5, 0 < y1 < 0.5,
   0 < z1 < Sqrt[2]/4}]
simplifiedExpr3 =
 FullSimplify[
  ArcSin[Sqrt[expr3]]/Pi, {0 < x1 < 0.5, 0 < y1 < 0.5,
   0 < z1 < Sqrt[2]/4}]
simplifiedExpr4 =
 FullSimplify[
  ArcSin[Sqrt[expr4]]/Pi, {0 < x1 < 0.5, 0 < y1 < 0.5,
   0 < z1 < Sqrt[2]/4}]
```

.2 Equation 3.35

```
Solve[{Cos[3 Pi y]^2 ==
   Cos[3 a Pi y]^2, (1 + 2 Cos[2 Pi y])^2 == (1 + 2 Cos[2 a Pi y])^2},
  a]
```

```
FullSimplify[
 ArcTan[-(Cos[3 [Pi] y]/(-1 + 2 Cos[2 [Pi] y])), -(Sqrt[
   1 - Cos[2 [Pi] y]]/Sqrt[2])]/([Pi] y), 0 < y < 0.5]
```

```
FullSimplify[
 ArcTan[-(Cos[3 [Pi] y]/(-1 + 2 Cos[2 [Pi] y])), Sqrt[
  1 - Cos[2 [Pi] y]]/Sqrt[2]]/([Pi] y), 0 < y < 0.5]
```

```
FullSimplify[
 ArcTan[Cos[3 [Pi] y]/(-1 + 2 Cos[2 [Pi] y]), -(Sqrt[
   1 - Cos[2 [Pi] y]]/Sqrt[2])]/([Pi] y), 0 < y < 0.5]
```

```
FullSimplify[
 ArcTan[Cos[3 [Pi] y]/(-1 + 2 Cos[2 [Pi] y]), Sqrt[
  1 - Cos[2 [Pi] y]]/Sqrt[2]]/([Pi] y), 0 < y < 0.5]
```

.3 Equations (4.7) and (4.8)

```
f =ArcCos[ (Exp[4 (a + u)] + Exp[-4 (a + u)])/
   2 + ((((Exp[u] + Exp[-u])/
         2) (2/(Exp[a + u] - Exp[-a - u])))^4)*(((Exp[4 I v] +
         Exp[-4 I v])/2) - ((Exp[4 u] + Exp[-4 u])/2))]/4;

f2=Simplify[TrigToExp[f]];

Simplify[ExpToTrig[f2]/. power_Power:>ExpToTrig[power]]
```