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Characterizing Linearizable QAPs by the Level-1 Reformulation-Linearization Technique

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Abstract

The quadratic assignment problem (QAP) is an extremely challenging NP-hard combinatorial optimization program. Due to its difficulty, a research emphasis has been to identify special cases that are polynomially solvable. Included within this emphasis are instances which are *linearizable*; that is, which can be rewritten as a linear assignment problem having the property that the objective function value is preserved at all feasible solutions. Various known sufficient conditions for identifying linearizable instances have been explained in terms of the continuous relaxation of a weakened version of the level-1 reformulation-linearization-technique (RLT) form that does not enforce nonnegativity on a subset of the variables. Also, conditions that are both necessary and sufficient have been given in terms of decompositions of the objective coefficients. The main contribution of this paper is the identification of a relationship between polyhedral theory and linearizability that promotes a novel, yet strikingly simple, necessary and sufficient condition for identifying linearizable instances; specifically, an instance of the QAP is linearizable if and only if the continuous relaxation of the same weakened RLT form is bounded. In addition to providing a novel perspective on the QAP being linearizable, a consequence of this study is that every linearizable instance has an optimal solution to the (polynomially-sized) continuous relaxation of the level-1 RLT form that is binary. The converse, however, is not true so that the continuous relaxation can yield binary optimal solutions to instances of the QAP that are not linearizable. Another consequence follows from our defining a maximal linearly independent set of equations in the lifted RLT variable space; we answer a recent open question that the theoretically best possible linearization-based bound cannot improve upon the level-1 RLT form.

Key Words: quadratic assignment problem, linearizable, RLT

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1 Introduction

The quadratic assignment problem (QAP) can be formulated as

$$\text{P: minimize}\left\{\sum_k\sum_\ell c_{k\ell}x_{k\ell}+\sum_i\sum_j\sum_{k\neq i}\sum_{\ell\neq j}C_{ijk\ell}x_{ij}x_{k\ell}:\boldsymbol{x}\in\boldsymbol{X},\boldsymbol{x}\text{ binary}\right\},$$

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where

$$\boldsymbol{X} \equiv \left\{ \boldsymbol{x} \geq \boldsymbol{0} : \sum_j x_{ij} = 1 \;\forall\, i,\; \sum_i x_{ij} = 1 \;\forall\, j \right\} \tag{1}$$

is the assignment set. Unless otherwise stated, we assume that all indices and summations run from 1 to n. Since it is desired to minimize a quadratic objective function over the assignment set $\boldsymbol{X}$, the problem name results. The QAP is NP-hard, and first arose in a facility location scenario [32]. In this context, the goal is to situate n facilities on n location sites, with each site housing exactly one facility, in such a manner as to minimize the combined cost of construction and material flow. Here, $C_{ijk\ell} = f_{ik}d_{j\ell}$ to represent the product of the flow f_{ik} between pairs of facilities i and k with the distance $d_{j\ell}$ between pairs of location sites j and ℓ. Then each coefficient $C_{ijk\ell}$ represents the cost of material flow between facilities i and k, given that facility i is located on site j *and* facility k is located on site ℓ. Each coefficient $c_{k\ell}$ represents a construction cost for situating facility k on site ℓ. This version of P is known as the Koopmans-Beckmann form. For this form, if the flows satisfy $f_{ik} = f_{ki}$ for all $(i,k), i < k$, and the distances satisfy $d_{j\ell} = d_{\ell j}$ for all $(j,\ell), j < \ell$, then the problem is called *symmetric*; otherwise it is *asymmetric*. More general forms that do not have $C_{ijk\ell} = f_{ik}d_{j\ell}$ have also been studied. The QAP boasts many applications [13, 20, 21, 24, 29, 33, 43, 44], with surveys in [11, 14, 35, 37]. The most efficient exact solution strategies [1, 8, 25, 26, 27] are limited to problems having $n \leq 40$.

Due to the problem's difficulty, a strategy for solving Problem P has been to seek out special objective function structures that allow optimal solutions to be obtained in polynomial time. For Koopmans-Beckmann forms, various works [10, 12, 15, 16, 19, 22, 34] exploit specific flow and distance structures for this purpose. Another contribution [18] provides a variation of Problem P that can be solved in pseudo-polynomial time. Of particular interest in our study is the identification of objective coefficients that allow P to be transformed into an equivalent *linear* assignment problem; such problems are referred to as being *linearizable*. Formally stated, an instance of Problem P with objective coefficients $c_{k\ell}$ and $C_{ijk\ell}$ is defined to be *linearizable* if there exists a scalar κ and coefficients $\hat{c}_{k\ell}$ so that

$$\kappa + \sum_k \sum_\ell \hat{c}_{k\ell} x_{k\ell} = \sum_k \sum_\ell c_{k\ell} x_{k\ell} + \sum_i \sum_j \sum_{k \neq i} \sum_{\ell \neq j} C_{ijk\ell} x_{ij} x_{k\ell} \quad \forall\, \boldsymbol{x} \in \boldsymbol{X},\; \boldsymbol{x} \text{ binary}. \tag{2}$$

Clearly, every instance of P that is expressible in the form (2) is polynomially solvable because each is reducible to a linear assignment problem. Although not explicitly stated in these terms, the works [10, 14, 23] provide sufficient conditions for recognizing such instances of Problem P. The paper [31] extends this body of work by providing, in terms of decompositions of the objective function coefficients, conditions that are both necessary and sufficient, and also gives an $O(n^4)$ algorithm for checking whether these conditions are satisfied. The paper [38] shows that, for Koopmans-Beckmann forms, the conditions can be checked in $O(n^2)$ time. Combinatorial characterizations of both symmetric [38] and asymmetric [17] linearizable Koopmans-Beckmann instances of Problem P are also available.

The primary contribution of this paper is a linkage between the notion of linearizability and a known linear reformulation of the QAP that has demonstrated utility in exact solution algorithms. This linkage provides a novel and simply-stated necessary and sufficient condition for identifying an instance of Problem P as being linearizable. Specifically, we show that the QAP is linearizable if and only if a certain linear program is bounded; this linear program is the continuous relaxation of a weakened version of the level-1 reformulation-linearization-technique (RLT) representation of Problem P [2, 30] that relaxes nonnegativity on a subset of the variables. Earlier work [7] characterized several sufficient conditions found in [10, 14, 23] by this representation, but no mention was made as to the more challenging direction of necessity. A consequence of our study is that the continuous relaxation of the non-weakened level-1 RLT form identifies a richer family of solvable instances of the QAP than those which are linearizable in the sense that an optimal binary solution will exist to this linear program for every linearizable instance, but the program can have an optimal binary solution without the QAP being linearizable.

Our approach is as follows. We first recall the level-1 RLT form in a known extended variable space, and then use a suggestion of [2, 30] to obtain a compact representation via a substitution of variables. We subsequently identify, in the new space, a maximal linearly independent (LI) set of equations that is valid for the n-factorial feasible solutions associated with the QAP. This set is obtained via a two-step procedure; the first step computes LI equations that are implied by the compact form and the second step shows that every other valid equation is a linear combination of those already computed. Specifically, the first step consists of deriving three sets of implied LI equations, and then showing that the aggregation of these sets is itself LI. The three sets are obtained, respectively, by: (i) judiciously selecting a subset of the constraints of the compact form, (ii) computing implied equations associated with a complete bipartite graph, and (iii) choosing any $2n-1$ of the equations defining $\boldsymbol{X}$. The second step uses the n-factorial feasible solutions to the QAP to identify coefficient structures that prohibit the existence of other LI equations that are valid in this space, thereby establishing our combined LI set of equations as being maximal. The derived maximal LI set of equations then allows us to obtain our main result that an instance of the QAP is linearizable if and only if a weakened version of our compact level-1 RLT form is bounded. As a byproduct of identifying a maximal LI set of equations, we give the dimension of the convex hull of feasible solutions for the compact level-1 RLT form, as well as for certain relaxations. Furthermore, our results answer an open question of [28] as to the acquisition of a spanning set of linearizable matrices for the quadratic expressions, establishing that the RLT gives their strongest possible linearization-based bound.

The remainder of the paper details our approach. In the next section, we briefly review the level-1 RLT form and present the more compact representation. We then provide a roadmap for exploiting the RLT structure in order to compute our maximal LI set of equations. This roadmap includes a pictorial description of the structure of our "to-be derived" LI constraints, including the number of equations and a partitioning of the variables. Section 3 follows the roadmap in deriving our three sets of implied LI equations and in explaining that the combined set is LI. Section 4 shows this combined LI set of equations to be maximal, and then uses this fact to prove our main result that an instance of the QAP is linearizable if and only if the continuous relaxation of a weakened level-1 RLT form is bounded. Section 5 provides concluding remarks and future research directions.

2 Level-1 RLT and Roadmap for Computing Linearly Independent Equations

In this section, we review the level-1 RLT form of Problem P, give a compact representation, and provide a roadmap for computing a maximal LI set of equations that is valid for this formulation. The roadmap details are presented in Sections 3 and 4, with Section 3 identifying the LI equations and Section 4 showing that the set is maximal.

2.1 Level-1 RLT Form

The RLT methodology was introduced in [4, 5, 6] to reformulate linearly-constrained quadratic 0-1 optimization problems into mixed-binary linear programs that afford tight linear programming relaxations. It was later extended (see [3, 40, 41, 42] and their references) into a broader theory for computing polyhedral outer-approximations of general discrete and nonconvex sets. Given a mixed-discrete optimization problem, the RLT constructs a hierarchy of successively tighter linear programming approximations, culminating at the highest level with a linear program whose feasible region gives an explicit algebraic description of the convex hull of feasible solutions.

The levels are obtained using the two steps of *reformulation* and *linearization*. For mixed-binary programs, the reformulation step multiplies "product factors" of the problem variables with the constraints, and enforces the binary identity that $x^2 = x$ for binary x. The linearization step then substitutes a continuous variable for each resulting product term.

Relative to the QAP, various authors [1, 25, 26, 27] have reported success in using different RLT levels to compute bounds within branch-and-bound schemes. Our specific interest here is with the level-1 form, as detailed in [2, 30] and given below.

$$\text{RLT1: minimize} \quad \sum_k \sum_\ell c_{k\ell} x_{k\ell} + \sum_i \sum_j \sum_{k\neq i} \sum_{\ell \neq j} C_{ijk\ell} y_{ijk\ell}$$

$$\text{subject to} \quad \sum_{j\neq \ell} y_{ijk\ell} = x_{k\ell} \qquad \forall\,(i,k,\ell),\ i\neq k \tag{3}$$

$$\sum_{i\neq k} y_{ijk\ell} = x_{k\ell} \qquad \forall\,(j,k,\ell),\ j\neq \ell \tag{4}$$

$$y_{ijk\ell} = y_{k\ell ij} \qquad \forall\,(i,j,k,\ell),\ i<k,\ j\neq \ell \tag{5}$$

$$y_{ijk\ell} \geq 0 \qquad \forall\,(i,j,k,\ell),\ i\neq k,\ j\neq \ell \tag{6}$$

$$\boldsymbol{x} \in \boldsymbol{X},\ \boldsymbol{x} \text{ binary} \tag{7}$$

Problem RLT1 derives from Problem P as follows. The reformulation step multiplies every constraint defining $\boldsymbol{X}$ in (1), including the $\boldsymbol{x} \geq \boldsymbol{0}$ inequalities, by each binary variable $x_{k\ell}$. Then $x_{k\ell} = x_{k\ell}x_{k\ell}$ is substituted for each (k,ℓ). The structure of $\boldsymbol{X}$ enforces that $x_{ij}x_{k\ell} = 0$ for all (i,j,k,ℓ) with $i=k, j\neq \ell$, or with $i\neq k, j=\ell$. For each remaining $x_{ij}x_{k\ell}$ product, the linearization step then substitutes a continuous variable $y_{ijk\ell}$. Specifically, for every (k,ℓ), the $n-1$ equations found within each of (3) and (4) are computed by multiplying the restrictions $\sum_j x_{ij} = 1\ \forall\ i\neq k$ and $\sum_i x_{ij} = 1\ \forall\ j\neq \ell$ of $\boldsymbol{X}$, respectively, by the variable $x_{k\ell}$. (Multiplication of either equation $\sum_j x_{kj} = 1$ or $\sum_i x_{i\ell} = 1$ by $x_{k\ell}$ reduces to $x_{k\ell} = x_{k\ell}$.) Equations (5) recognize that $x_{ij}x_{k\ell} = x_{k\ell}x_{ij}$ for all $(i,j,k,\ell), i<k, j\neq \ell$. Inequalities (6) result from multiplying the $\boldsymbol{x} \geq \boldsymbol{0}$ inequalities of $\boldsymbol{X}$ by the variables $x_{k\ell}$.

The RLT theory gives us that Problems P and RLT1 are equivalent in that an optimal solution to either problem yields an optimal solution to the other. This result [2, 30] follows because, for every $\hat{\boldsymbol{x}} \in \boldsymbol{X}$, $\hat{\boldsymbol{x}}$ binary, a point $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$ is feasible to constraints (3)–(6) if and only if $\hat{y}_{ijk\ell} = \hat{x}_{ij}\hat{x}_{k\ell}$ for all $(i,j,k,\ell), i\neq k, j\neq \ell$.

As suggested in [2, 30], we can use (5) to remove, via substitution, all variables $y_{ijk\ell}$ having $i > k, j\neq \ell$, from Problem RLT1, and then discard constraints (5). The following compact formulation results, where $C'_{ijk\ell} = C_{ijk\ell} + C_{k\ell ij}$ for all $(i,j,k,\ell), i<k, j\neq \ell$.

$$\text{RLT1}'\text{: minimize} \quad \sum_k \sum_\ell c_{k\ell} x_{k\ell} + \sum_i \sum_j \sum_{k>i} \sum_{\ell \neq j} C'_{ijk\ell} y_{ijk\ell}$$

$$\text{subject to} \quad \sum_{j\neq \ell} y_{ijk\ell} = x_{k\ell} \qquad \forall\,(i,k,\ell),\ i<k \tag{8}$$

$$\sum_{j\neq \ell} y_{k\ell ij} = x_{k\ell} \qquad \forall\,(i,k,\ell),\ i>k \tag{9}$$

$$\sum_{i<k} y_{ijk\ell} + \sum_{i>k} y_{k\ell ij} = x_{k\ell} \qquad \forall\,(j,k,\ell),\ j\neq \ell \tag{10}$$

$$y_{ijk\ell} \geq 0 \qquad \forall\, (i,j,k,\ell),\ i<k,\ j\neq \ell \tag{11}$$
$$\boldsymbol{x} \in \boldsymbol{X},\ \boldsymbol{x} \text{ binary}$$

Here, equations (3) become (8) and (9), equations (4) become (10), and inequalities (6) reduce to (11).

Observe the difference in size between Problems RLT1 and RLT1$'$. Problem RLT1 has n^2 variables $\boldsymbol{x}$ and $n^2(n-1)^2$ variables $\boldsymbol{y}$, for a total of $n^2(n^2-2n+2)$. Relative to constraints, RLT1 has $2n^2(n-1)$ equations in (3) and (4) combined, $\frac{n^2(n-1)^2}{2}$ equations in (5), and $2n$ equations in $\boldsymbol{X}$ of (7), for a total of $\frac{n^2(n-1)(n+3)}{2}+2n$, in addition to the nonnegativity restrictions on $\boldsymbol{x}$ and $\boldsymbol{y}$. Due to the variable substitution, Problem RLT1$'$ has $\frac{n^2(n-1)^2}{2}$ fewer variables and constraints than does RLT1. In addition, (11) has half the number of nonnegativity restrictions found in (6). Specifically,

$$\text{RLT1}' \text{ has } n^2+\frac{n^2(n-1)^2}{2} \text{ variables and } 2n^2(n-1)+2n \text{ equations.} \tag{12}$$

A variant of Problem RLT1$'$ that is obtained by removing the nonnegativity restrictions on $\boldsymbol{y}$ in (11), call this variant RLT1$''$, will be important in this study. For future reference, we let $\overline{\text{RLT1}}'$ and $\overline{\text{RLT1}}''$ be the linear programming relaxations of Problems RLT1$'$ and RLT1$''$, respectively, that are obtained by removing the binary restrictions on $\boldsymbol{x}$. The paper [7] established boundedness of $\overline{\text{RLT1}}''$ as a sufficient condition for an instance of Problem P to be linearizable. In this paper, we show the more challenging direction that boundedness of $\overline{\text{RLT1}}''$ is also a necessary condition for linearizability.

2.2 Roadmap for Computing a Maximal Linearly Independent Set of Equations

In this section, we present a roadmap for computing a maximal LI set of equations in the variables $(\boldsymbol{x},\boldsymbol{y})$ that is valid for the n-factorial solutions to RLT1$'$. Each equation is implied by (8)–(10) and the equality restrictions of $\boldsymbol{X}$. The roadmap is followed in Section 3, and the maximal property is shown in Section 4.

By the RLT process, restrictions (8)–(10) and the equations of $\boldsymbol{X}$ form a linearly dependent set. Certain redundant constraints have been pointed out in prior works [7, 39] for RLT1, and these dependencies transfer to RLT1$'$. The paper [7] eliminates $n(n-1)+1$ equations by observing that any single equation in $\boldsymbol{X}$ is redundant, so that the $n(n-1)$ equations present in either (3) or (4) which are computed by multiplying the chosen equation with the associated variables $x_{k\ell}$, are also redundant. In a different study, the paper [39] explains that either the first or second set of n equations of $\boldsymbol{X}$ is implied by the other constraints of RLT1, so that the selected n equations can be removed. We more generally observe in the next two sections that RLT1$'$ contains a maximal LI set of $2(n-1)^3+n(n-1)$ equations, so that $3n^2-3n+2$ of the $2n^2(n-1)+2n$ total equations reported in (12) are implied. For future reference, this number of LI equations is computed by

$$\Big(2(n-1)^3+n(n-1)\Big)=\Big(2n^2(n-1)+2n\Big)-\Big(3n^2-3n+2\Big), \tag{13}$$

with the right side of the expression subtracting the number of implied equations from the total. In particular, Section 3 provides this number of LI constraints that are implied by the equations of RLT1$'$, and Section 4 shows the number to be maximal. The tool we use is an exploitation of the coefficient structure that must hold true within every equation valid for RLT1$'$.

To illustrate our desired maximal LI set of equations pictorially, consider the linear system of equations in matrix form found in Figure 1, which is expressed in terms of the parameters m_1, m_2, m_3, and n_1, n_2, n_3, n_4, as given, where $m_1 = n_2$ and $m_2 = n_3$, with $n_1 + n_2 + n_3 + n_4 = n^2 + \frac{n^2(n-1)^2}{2}$

$$
\begin{array}{r}
m_1\{ \\ m_2\{ \\ m_3\{
\end{array}
\overset{\begin{array}{cccc} \overbrace{}^{n_1} & \overbrace{}^{n_2} & \overbrace{}^{n_3} & \overbrace{}^{n_4} \end{array}}{\left[\begin{array}{c:c:c:c}
\boldsymbol{A} & \boldsymbol{B} & \boldsymbol{C} & \boldsymbol{D} \\ \hdashline
\boldsymbol{E} & \mathbf{0} & \boldsymbol{F} & \boldsymbol{G} \\ \hdashline
\boldsymbol{H} & \mathbf{0} & \mathbf{0} & \mathbf{0}
\end{array}\right]}
\left[\begin{array}{c}
\boldsymbol{x} \\ \hdashline \boldsymbol{y}_1 \\ \hdashline \boldsymbol{y}_2 \\ \hdashline \boldsymbol{y}_3
\end{array}\right]
=
\left[\begin{array}{c}
\mathbf{0} \\ \hdashline \mathbf{0} \\ \hdashline \mathbf{1}
\end{array}\right]
$$

with

- $m_1 = 2(n-1)^3$, $m_2 = (n-1)(n-2) - 1$, $m_3 = 2n - 1$
- $n_1 = n^2$, $n_2 = 2(n-1)^3$, $n_3 = (n-1)(n-2) - 1$, $n_4 = \frac{(n-3)(n-2)(n-1)n}{2} + 1$

Figure 1: Pictorial depiction of linearly independent equations implied by (8)–(10) and the equations of $\boldsymbol{X}$ in RLT1$'$

equaling the number of variables in RLT1$'$ as noted in (12), and with $m_1 + m_2 + m_3 = 2(n-1)^3 + n(n-1)$ as found in the left expression of (13). As indicated by the matrix multiplication, the first set of n_1 variables corresponds to $\boldsymbol{x}$ while the last three sets of $n_2 + n_3 + n_4$ variables comprise a partition of $\boldsymbol{y}$ into $\boldsymbol{y}_1$, $\boldsymbol{y}_2$, and $\boldsymbol{y}_3$. The first $m_1 + m_2$ equations will be constructed to be implied by (8)–(10) so as to satisfy the following properties: the first m_1 equations will associate an invertible matrix $\boldsymbol{B}$ with the set of n_2 variables $\boldsymbol{y}_1$ and the second m_2 equations will contain none of these variables $\boldsymbol{y}_1$, but will associate an invertible matrix $\boldsymbol{F}$ with the set of n_3 variables $\boldsymbol{y}_2$. The final m_3 equations will represent any selection of $2n-1$ equality constraints from $\boldsymbol{X}$ of (1). The last set of n_4 variables $\boldsymbol{y}_3$ are those variables $\boldsymbol{y}$ not present in either $\boldsymbol{y}_1$ or $\boldsymbol{y}_2$. The invertibility of the matrices $\boldsymbol{B}$ and $\boldsymbol{F}$, and the linear independence of the rows of $\boldsymbol{H}$ will make the overall system LI. Here, the notation $\mathbf{0}$ and $\mathbf{1}$ is used to denote compatibly-sized matrices (vectors) of all zeroes and ones, respectively. Also, the matrix $\boldsymbol{C}$ found in the first set of m_1 equations is distinct from, and should not be confused with, the quadratic objective coefficients $C_{ijk\ell}$ introduced in Problem P, with $\boldsymbol{C}$ in bold and having no subscripts.

Sections 3 and 4 relate to Figure 1 as follows. Section 3 details the system formation from the constraints of RLT1$'$, emphasizing the construction of the invertible matrices $\boldsymbol{B}$ and $\boldsymbol{F}$. It then states that the total collection of $m_1 + m_2 + m_3$ equations is LI. Section 4 establishes that these LI equations are maximal for the set of solutions to RLT1$'$. A key argument to show the maximal property is that the vector $\boldsymbol{\beta}$ found in any chosen equation $\boldsymbol{\alpha x} + \boldsymbol{\beta y} = \kappa$ that is valid for all $(\boldsymbol{x}, \boldsymbol{y})$ feasible to RLT1$'$ has those entries $\beta_{ijk\ell}$ associated with the variables $\boldsymbol{y}_3$ uniquely defined in terms of the $\beta_{ijk\ell}$ associated with the variables $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$.

3 Identification of Linearly Independent Equations

In this section, and consistent with equation (13) and Figure 1, we identify $2(n-1)^3 + n(n-1) = m_1 + m_2 + m_3$ LI equations that are implied by the equations of RLT1$'$. Section 3.1 chooses the first m_1 equations as a subset of those found within (8)–(10). Section 3.2 derives the second m_2 equations from (8)–(10) and shows that these equations, when combined with those of Section 3.1, form a LI set. Within these sections, we emphasize the invertibility of the matrices $\boldsymbol{B}$ and $\boldsymbol{F}$, and the corresponding partitioning of the variables $\boldsymbol{y}$ into $\boldsymbol{y}_1$, $\boldsymbol{y}_2$, and $\boldsymbol{y}_3$. Section 3.3 shows that these $m_1 + m_2$ equations, when augmented with any $m_3 = 2n-1$ equations of $\boldsymbol{X}$ to describe $\boldsymbol{Hx} = \boldsymbol{1}$, collectively form a LI set.

3.1 First Set of Linearly Independent Equations

Choose the first m_1 equations from (8)–(10) as follows, where we find it convenient for the proof of upcoming Lemma 3.1 to express each equation (14)–(18) with a single variable in the left side.

- Select the $\frac{(n-1)^2(n-2)}{2}$ equations of (8) having $k < n$ and $\ell < n$ to obtain (14).
- Select the $\frac{(n-1)^2(n-2)}{2}$ equations of (9) having $i < n$ and $\ell < n$ to obtain (15).
- Select the $(n-1)^2(n-2)$ equations of (10) having $j < n$, $k < n$, and $\ell < n$ to obtain (16).
- Select the $(n-1)^2$ equations of (10) having $k < n$ and $\ell = n$ to obtain (17).
- Select the $(n-1)^2$ equations of (10) having $j = n$ and $k < n$ to obtain (18).

$$y_{ink\ell} = x_{k\ell} - \sum_{j \neq \ell, n} y_{ijk\ell} \quad \forall\, (i,k,\ell),\ i < k < n,\ \ell < n \tag{14}$$

$$y_{k\ell in} = x_{k\ell} - \sum_{j \neq \ell, n} y_{k\ell ij} \quad \forall\, (i,k,\ell),\ k < i < n,\ \ell < n \tag{15}$$

$$y_{k\ell nj} = x_{k\ell} - \sum_{i<k} y_{ijk\ell} - \sum_{k<i<n} y_{k\ell ij} \quad \forall\, (j,k,\ell),\ j \neq \ell,\ j < n,\ k < n,\ \ell < n \tag{16}$$

$$y_{knnj} = x_{kn} - \sum_{i<k} y_{ijkn} - \sum_{k<i<n} y_{knij} \quad \forall\, (j,k),\ j < n,\ k < n \tag{17}$$

$$y_{k\ell nn} = x_{k\ell} - \sum_{i<k} y_{ink\ell} - \sum_{k<i<n} y_{k\ell in} \quad \forall\, (k,\ell),\ k < n,\ \ell < n \tag{18}$$

Consider the lemma below.

Lemma 3.1. *Equations (14)–(18) form a LI set.*

Proof. No equation in (14)–(16) contains a variable $y_{ijk\ell}$ in the right side having an index i, j, k, ℓ equal to n. But every such equation has that single variable found on the left side with an index equal to n, and these $2(n-1)^2(n-2)$ variables are distinct. Thus, (14)–(16) form a LI set. No equation in (14)–(18) contains a variable $y_{ijk\ell}$ in the right side having two indices i, j, k, ℓ equal to n. But every equation in (17) and (18) has that single variable found on the left side with two indices equal to n, and these $2(n-1)^2$ variables are distinct. Thus, (14)–(18) form a LI set. □

Via a simple counting argument, we have that every variable $y_{ijk\ell}$ present in RLT1′ with at least one subscript equal to n is found in the left side of an expression of (14)–(18). The total number of such variables is computable from (12) by subtracting the $\frac{(n-1)^2(n-2)^2}{2}$ variables $\boldsymbol{y}$ in the formulation of RLT1′ that models the QAP of size $n-1$ from the $\frac{n^2(n-1)^2}{2}$ variables $\boldsymbol{y}$ in the formulation of RLT1′ that models the QAP of size n. The result is $2(n-1)^3$, which is the number of equations in (14)–(18).

Relative to the roadmap of Figure 1, the vector $\boldsymbol{y}_1$ denotes those variables $y_{ijk\ell}$ such that

$$y_{ijk\ell} \text{ has at least one index } j, k, \ell \text{ equal to } n, \tag{19}$$

and the vectors $\boldsymbol{y}_2$ and $\boldsymbol{y}_3$ correspond to those variables $y_{ijk\ell}$ having no index i, j, k, ℓ equal to n. The matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ correspond to the coefficients on the variables $\boldsymbol{x}$ and $\boldsymbol{y}_1$, respectively, in (14)–(18), with $\boldsymbol{B}$ invertible. The matrices $\boldsymbol{C}$ and $\boldsymbol{D}$ correspond to the coefficients on the vectors $\boldsymbol{y}_2$ and $\boldsymbol{y}_3$, with $\boldsymbol{y}_2$ and $\boldsymbol{y}_3$ further described in Section 3.2 following Theorem 3.3.

3.2 Second Set of Linearly Independent Equations

Given any $(j, \ell), j < \ell < n$, restrictions (10) enforce the n equations

$$-x_{k\ell} + \sum_{i<k} y_{ijk\ell} + \sum_{i>k} y_{k\ell ij} = 0 \;\; \forall\, k < n \quad \text{and} \quad x_{nj} - \sum_{i<n} y_{i\ell nj} = 0.$$

Summing these equations for each such (j, ℓ) gives

$$x_{nj} - \sum_{k<n} x_{k\ell} + \sum_{k<n} \left(\sum_{i<k} y_{ijk\ell} + \sum_{k<i<n} y_{k\ell ij} \right) = 0 \;\; \forall\, (j, \ell), j < \ell < n.$$

Interchanging indices i and k in the last terms of each above equation, we get that restrictions (10) imply the following $\frac{(n-1)(n-2)}{2}$ equations.

$$x_{nj} - \sum_{k<n} x_{k\ell} + \sum_{i=1}^{n-2} \sum_{k=i+1}^{n-1} (y_{ijk\ell} + y_{i\ell kj}) = 0 \;\; \forall\, (j, \ell), j < \ell < n \tag{20}$$

Similarly, given any $(i, k), i < k < n$, restrictions (8) enforce the left $n-1$ equations and restrictions (9) enforce the right equation of

$$-x_{k\ell} + \sum_{j \neq \ell} y_{ijk\ell} = 0 \;\; \forall\, \ell < n \quad \text{and} \quad x_{in} - \sum_{j<n} y_{inkj} = 0.$$

Summing these equations for each such (i, k) gives

$$x_{in} - \sum_{\ell<n} x_{k\ell} + \sum_{\ell<n} \left(\sum_{j<\ell} y_{ijk\ell} + \sum_{\ell<j<n} y_{ijk\ell} \right) = 0 \;\; \forall\, (i, k), i < k < n.$$

Interchanging indices j and ℓ in the last terms of each above equation, we get that restrictions (8) and (9) imply the following $\frac{(n-1)(n-2)}{2}$ equations.

$$x_{in} - \sum_{\ell<n} x_{k\ell} + \sum_{j=1}^{n-2} \sum_{\ell=j+1}^{n-1} (y_{ijk\ell} + y_{i\ell kj}) = 0 \;\; \forall\, (i, k), i < k < n \tag{21}$$

The $(n-1)(n-2)$ equations (20) and (21), less any single restriction, will form our second set of $m_2 = (n-1)(n-2) - 1$ LI equations as explained below.

Observe that each of the $\frac{(n-1)^2(n-2)^2}{4}$ variables $y_{ijk\ell}$ having $i < k < n$ and $j < \ell < n$ appears (with coefficient 1) only in that equation of (20) indexed by (j, ℓ) and in that equation of (21) indexed by (i, k). Therefore, the structure of (20) and (21) has the columns corresponding to these $\frac{(n-1)^2(n-2)^2}{4}$ variables forming the node-edge incidence matrix of a complete bipartite graph consisting of edges between $\frac{(n-1)(n-2)}{2}$ nodes associated with the ordered pairs (j, ℓ) indexing the equations (20) and $\frac{(n-1)(n-2)}{2}$ nodes associated with the ordered pairs (i, k) indexing the equations (21). This graph is depicted in Figure 2, where the left and right columns of nodes are labeled with

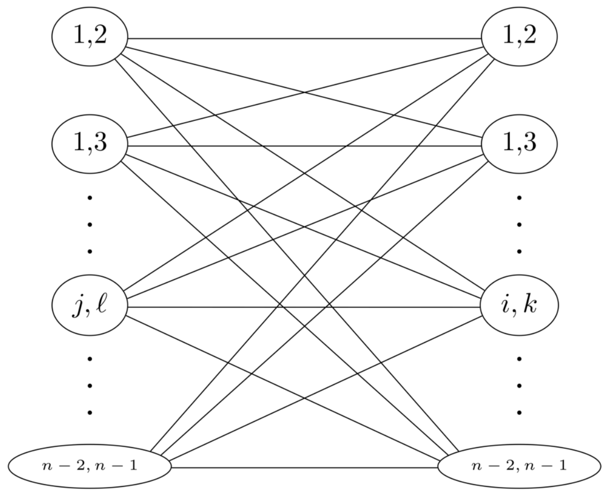

Figure 2: Complete bipartite graph associated with equations (20) and (21)

the ordered pairs $(j,\ell), j<\ell<n$, and $(i,k), i<k<n$, consistent with (20) and (21), respectively. In this manner, an edge incident with a left node (j,ℓ) and a right node (i,k) symbolizes the variable $y_{ijk\ell}$.

Since the variables $y_{ijk\ell}$ associated with any spanning tree of Figure 2 (see chapters 9 and 10 of [9] for example) form a linearly independent set, the rank of the node-edge incidence matrix is $(n-1)(n-2)-1$ and, furthermore, the removal of any single equation does not lessen this rank. Then Lemma 3.2 follows without proof.

Lemma 3.2. *Every selection of* $m_2=(n-1)(n-2)-1$ *equations from* (20) *and* (21) *contains* m_2 *columns in the variables* $y_{ijk\ell}$ *that form a LI set.*

Theorem 3.3 below shows that the m_1 LI equations of Section 3.1 combined with any m_2 LI equations from (20) and (21) form a LI set.

Theorem 3.3. *The* $m_1=2(n-1)^3$ *equations of* (14)–(18) *together with any* $m_2=(n-1)(n-2)-1$ *equations of* (20) *and* (21) *form a LI set.*

Proof. Lemma 3.1 established that the m_1 equations (14)–(18) form a LI set and Lemma 3.2 showed that any collection of m_2 equations from (20) and (21) has m_2 variables $y_{ijk\ell}$ that form a LI set. Each of the m_1 variables $y_{ijk\ell}$ found in $\boldsymbol{y}_1$ was noted in (19) to have at least one of the indices j,k,ℓ equal to n. None of these variables appear in (20) or (21), completing the proof. □

Equations (20) and (21), less any single restriction, relate to Figure 1 as follows. The matrix $\boldsymbol{E}$ of Figure 1 corresponds to the variables $\boldsymbol{x}$ of (20) and (21), and the $m_2\times n_2$ matrix $\mathbf{0}$ follows from the observation in the proof of Theorem 3.3 that no variables in $\boldsymbol{y}_1$ appear in either (20) or (21). The invertible matrix $\boldsymbol{F}$ is composed of the columns associated with any $m_2=n_3$ variables $y_{ijk\ell}$ whose edges form a spanning tree in Figure 2, as these columns are LI. Those variables $y_{ijk\ell}$ selected to define the spanning tree are denoted as $\boldsymbol{y}_2$. Then $\boldsymbol{y}_3$ corresponds to the $n_4=\frac{(n-1)^2(n-2)^2}{2}-\Big((n-1)(n-2)-1\Big)$ variables $y_{ijk\ell}$ found in (20) and (21) which are not in $\boldsymbol{y}_2$, and the matrix $\boldsymbol{G}$ corresponds to the coefficients on $\boldsymbol{y}_3$ in the chosen m_2 equations. These descriptions of $\boldsymbol{y}_2$ and $\boldsymbol{y}_3$ define the matrices $\boldsymbol{C}$ and $\boldsymbol{D}$ in terms of (14)–(18).

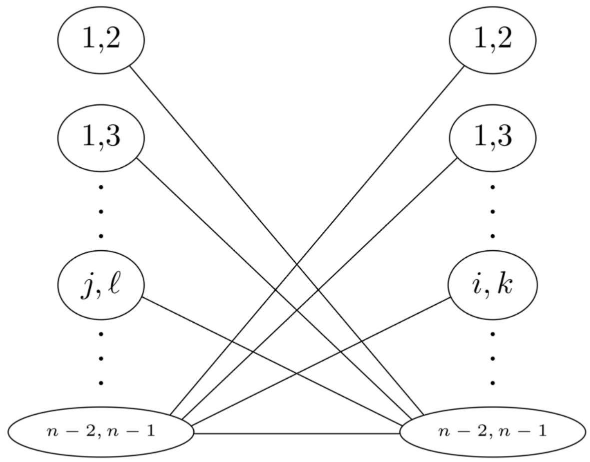

Figure 3: Set of n_3 basic variables from Figure 2 associated with matrix $\boldsymbol{F}$ of Figure 1

For future reference, we choose $\boldsymbol{y}_2$ to be those variables $y_{ijk\ell}$ having either

$$y_{(n-2)j(n-1)\ell} \text{ with } j < \ell < n \quad \text{or} \quad y_{i(n-2)k(n-1)} \text{ with } i < k < n. \tag{22}$$

Figure 3 graphically depicts the variables of (22), using the same sets of nodes as Figure 2, and a subset of the edges.

3.3 Augmented Set of Linearly Independent Equations

Theorem 3.3 showed that the first $m_1 + m_2$ equations of Figure 1 form a LI set. Any $m_3 = 2n - 1$ equations of the assignment set $\boldsymbol{X}$, represented by $\boldsymbol{Hx} = \boldsymbol{1}$ in Figure 1, form a LI set (again see chapters 9 and 10 of [9]). The below theorem shows that the collection of $m_1 + m_2 + m_3$ equations obtained by appending any such m_3 equations to the above-described $m_1 + m_2$ equations forms a LI set.

Theorem 3.4. *The $m_1 = 2(n-1)^3$ equations of (14)–(18) together with any $m_2 = (n-1)(n-2)-1$ equations of (20) and (21) and any $m_3 = 2n - 1$ equations of $\boldsymbol{X}$ form a LI set.*

Proof. The first $m_1 + m_2$ equations are LI by Theorem 3.3, and any m_3 equations of $\boldsymbol{X}$ are known to be LI. Since the matrices $\boldsymbol{B}$ and $\boldsymbol{F}$ of Figure 1 are invertible and every such m_3 equations of $\boldsymbol{X}$ involve no variables $\boldsymbol{y}_1$ or $\boldsymbol{y}_2$, the proof is complete. □

Theorem 3.4 shows that the $m_1 + m_2 + m_3$ equations of Figure 1 form a LI set, but it does not prove that this set is maximal in the sense that every valid equation for RLT1′ is implied by these equations. Section 4 establishes this maximality result and then uses it to derive our necessary and sufficient condition for Problem P to be linearizable.

4 Necessary and Sufficient Condition for Linearizability

Section 3 set the stage for establishing our necessary and sufficient condition for identifying an instance of Problem P as being linearizable. Our argument is a two-step approach, with each of Sections 4.1 and 4.2 devoted to a step. The first step is to show that the $m_1 + m_2 + m_3$ LI equations of Figure 1 that were presented in Section 3 form a maximal LI set in that every linear equation $\boldsymbol{\alpha x} + \boldsymbol{\beta y} = \kappa$ which is valid for the n-factorial solutions to RLT1$'$ can be expressed as a linear combination of these equations of Figure 1. This result is given in Theorem 4.1. Following the discussion at the end of Section 2.2, the step is accomplished by showing that for every such $\boldsymbol{\alpha x} + \boldsymbol{\beta y} = \kappa$, the vector $\boldsymbol{\beta}$ has those entries $\beta_{ijk\ell}$ associated with the variables $\boldsymbol{y}_3$ uniquely defined in terms of the $\beta_{ijk\ell}$ associated with the variables $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$. Notably, this maximal LI set of equations will allow us to trivially compute the dimension of the convex hull of feasible solutions to RLT1$'$, as well as the dimensions of the feasible regions to $\overline{\text{RLT1}}'$ and $\overline{\text{RLT1}}''$. The second step, given in Theorem 4.3, relates equations (8)–(10) of $\overline{\text{RLT1}}''$, as well as the $m_1 + m_2 + m_3$ equations of Figure 1 that are implied by the constraints of $\overline{\text{RLT1}}''$, to the definition of linearizable found in (2). The last portion of the proof of Theorem 4.3 uses the maximum LI set argument from Step 1 to show that, if Problem P is linearizable, then there exists a linear combination of the $m_1 + m_2 + m_3$ equations of Figure 1 that allows the quadratic objective coefficients of $\overline{\text{RLT1}}''$ to be replaced by linear expressions, so that $\overline{\text{RLT1}}''$ can be reduced to a (bounded) assignment problem over the set $\boldsymbol{X}$.

4.1 Characterizing all Equations Valid for Problem RLT1$'$

We begin by defining index sets $\mathcal{S}$ and $\mathcal{S}_{pq}$ of 4-tuples associated with the objective function coefficients $C'_{ijk\ell}$ of Problem RLT1$'$. These sets will appear in the upcoming proof of Theorem 4.1.

Definition 4.1. *Let $\mathcal{N} \equiv \{1, 2, \ldots, n\}$ and, for $n \geq 5$, let $\mathcal{N}^a \equiv \mathcal{N} - \{a\}$ for each $a \in \{1, \ldots, n-3\}$.*

- *Define $\mathcal{S}$ as the set of all 4-tuples $(i, j, k, \ell) \in \mathcal{N} \times \mathcal{N} \times \mathcal{N} \times \mathcal{N}$ having $i < k, j \neq \ell$, that satisfy at least one of conditions (i)-(iii) below.*

  (i) j, k, or ℓ is equal to n

  (ii) $i = n-2$ and $k = n-1$ with $j < \ell < n$

  (iii) $j = n-2$ and $\ell = n-1$ with $k < n$

- *For $n \geq 5$, define $\mathcal{S}_{pq} \subseteq \mathcal{S}$ for $p \in \{1, \ldots, n-3\}$ and $q \in \{1, \ldots, n-3\}$ as the set of all 4-tuples $(i, j, k, \ell) \in \mathcal{N}^p \times \mathcal{N}^q \times \mathcal{N}^p \times \mathcal{N}^q$ having $i < k, j \neq \ell$, that satisfy at least one of conditions (i)-(iii).*

The sets $\mathcal{N}^a$, $\mathcal{S}$, and $\mathcal{S}_{pq}$ are defined in the given manner for three reasons. First, we have from (19) and (22) that $\mathcal{S}$ is the set of indices (i, j, k, ℓ) of the variables $y_{ijk\ell} \in \boldsymbol{y}_1 \cup \boldsymbol{y}_2$ within RLT1$'$; that is, $\mathcal{S} = \{(i, j, k, \ell) : y_{ijk\ell} \in \boldsymbol{y}_1 \cup \boldsymbol{y}_2\}$. Second, when $n \geq 5$, for any chosen p and q with $p \in \{1, \ldots, n-3\}$ and $q \in \{1, \ldots, n-3\}$, we can apply the logic of Section 3 to that smaller instance of RLT1$'$ defined over the size $n-1$ assignment set in the variables $x_{k\ell}$ having $k \in \mathcal{N}^p$ and $\ell \in \mathcal{N}^q$ to obtain that $\mathcal{S}_{pq}$ is the set of indices (i, j, k, ℓ) of the variables $y_{ijk\ell} \in \boldsymbol{y}_1 \cup \boldsymbol{y}_2$ within that smaller instance. Specifically, $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ for this smaller instance of RLT1$'$ are defined exactly as in (19) and (22), respectively, with the additional restriction that $(i, j, k, \ell) \in \mathcal{N}^p \times \mathcal{N}^q \times \mathcal{N}^p \times \mathcal{N}^q$ instead of the more general $(i, j, k, \ell) \in \mathcal{N} \times \mathcal{N} \times \mathcal{N} \times \mathcal{N}$. Third, the sets $\mathcal{N}^a$ and $\mathcal{S}_{pq}$ are needed in our proof only for $n \geq 5$, and are thus accordingly defined.

Now consider Theorem 4.1, the main result of this section.

Theorem 4.1. *The $m_1 + m_2 + m_3$ LI equations associated with Figure 1 (and explicitly stated in Theorem 3.4) form a maximal LI set in that every linear equation which is valid for the n-factorial solutions to RLT1$'$ is implied. Equivalently, every equation of the form*

$$\sum_k \sum_\ell \alpha_{k\ell} x_{k\ell} + \sum_i \sum_j \sum_{k>i} \sum_{\ell \neq j} \beta_{ijk\ell} y_{ijk\ell} = \kappa \tag{23}$$

that is satisfied for all $(\boldsymbol{x}, \boldsymbol{y})$ *feasible to RLT1'* *can be computed as a linear combination of the* $m_1 + m_2 + m_3$ *equations of Figure 1.*

Proof. Consider any equation of the form (23), say

$$\sum_k \sum_\ell \bar{\alpha}_{k\ell} x_{k\ell} + \sum_i \sum_j \sum_{k>i} \sum_{\ell \neq j} \bar{\beta}_{ijk\ell} y_{ijk\ell} = \bar{\kappa} \tag{24}$$

that is satisfied for all $(\boldsymbol{x}, \boldsymbol{y})$ feasible to RLT1'. Given that $\mathcal{S} = \{(i,j,k,\ell) : y_{ijk\ell} \in \boldsymbol{y}_1 \cup \boldsymbol{y}_2\}$ and $\boldsymbol{B}$ and $\boldsymbol{F}$ are invertible matrices, there exists a (unique) linear combination of the first $m_1 + m_2$ equations of Figure 1 that yields an equation of the form

$$\sum_k \sum_\ell \hat{\alpha}_{k\ell} x_{k\ell} + \sum_{(i,j,k,\ell) \in \mathcal{S}} \bar{\beta}_{ijk\ell} y_{ijk\ell} + \sum_{(i,j,k,\ell) \notin \mathcal{S}} \hat{\beta}_{ijk\ell} y_{ijk\ell} = 0 \tag{25}$$

such that the coefficients $\bar{\beta}_{ijk\ell}$ having $(i,j,k,\ell) \in \mathcal{S}$ are as found in (24).

The proof consists of two parts that combine to compute (24) as a linear combination of (25) and the m_3 equations $\boldsymbol{Hx} = \boldsymbol{1}$ of Figure 1. The first part shows that, given any equation of the form (23) that is satisfied for all $(\boldsymbol{x}, \boldsymbol{y})$ feasible to RLT1', each $\beta_{ijk\ell}$ having $(i,j,k,\ell) \notin \mathcal{S}$ is uniquely defined in terms of the $\beta_{ijk\ell}$ having $(i,j,k,\ell) \in \mathcal{S}$. Because (24) and (25) are both of the form (23) and have coefficients of $\bar{\beta}_{ijk\ell}$ for all $(i,j,k,\ell) \in \mathcal{S}$, then $\bar{\beta}_{ijk\ell}$ of (24) and $\hat{\beta}_{ijk\ell}$ of (25) must have $\bar{\beta}_{ijk\ell} = \hat{\beta}_{ijk\ell}$ for all $(i,j,k,\ell) \notin \mathcal{S}$, so that equation (25) is expressible as

$$\sum_k \sum_\ell \hat{\alpha}_{k\ell} x_{k\ell} + \sum_i \sum_j \sum_{k>i} \sum_{\ell \neq j} \bar{\beta}_{ijk\ell} y_{ijk\ell} = 0. \tag{26}$$

The second part of the proof shows that there exists a linear combination of the m_3 equations $\boldsymbol{Hx} = \boldsymbol{1}$ of Figure 1 that gives the equation

$$\sum_k \sum_\ell (\bar{\alpha}_{k\ell} - \hat{\alpha}_{k\ell}) x_{k\ell} = \bar{\kappa}. \tag{27}$$

The sum of (26) and (27) is (24), so (24) will then be computable as a linear combination of the $m_1 + m_2 + m_3$ equations of Figure 1.

The two parts of the proof follow.

1. As the constraints of RLT1' enforce $\boldsymbol{x} \in \boldsymbol{X}$, $\boldsymbol{x}$ binary, and $y_{ijk\ell} = x_{ij} x_{k\ell}$ for all $(i,j,k,\ell), i < k, j \neq \ell$, the first part of the proof is to show that, given any equation of the form

$$\sum_k \sum_\ell \alpha_{k\ell} x_{k\ell} + \sum_i \sum_j \sum_{k>i} \sum_{\ell \neq j} \beta_{ijk\ell} x_{ij} x_{k\ell} = \kappa \tag{28}$$

   that is satisfied for all $\boldsymbol{x} \in \boldsymbol{X}$, $\boldsymbol{x}$ binary, each $\beta_{ijk\ell}$ having $(i,j,k,\ell) \notin \mathcal{S}$ is uniquely defined in terms of the $\beta_{ijk\ell}$ having $(i,j,k,\ell) \in \mathcal{S}$. The proof is inductive on the size n of the set $\boldsymbol{X}$ and exploits the fact that fixing any variable $x_{pq} = 1$ within $\boldsymbol{X}$ produces a size $n-1$ assignment set. As we will see, part of the inductive argument requires that $n \geq 6$, necessitating the base cases of $n = 3$, $n = 4$, and $n = 5$ found in the Appendix.

   For the inductive step, consider an equation of the form (28) that is satisfied for all $\boldsymbol{x} \in \boldsymbol{X}$, $\boldsymbol{x}$ binary, where $\boldsymbol{X}$ is a size $n \geq 6$ assignment set and suppose that the result holds true for all such equations that are valid for the size $n-1$ assignment set. Specifically, then, for any (p,q) having $p \in \{1, \ldots, n-3\}$ and $q \in \{1, \ldots, n-3\}$, the result holds true for the equation

$$\sum_{k<p} \sum_{\ell \neq q} (\alpha_{k\ell} + \beta_{k\ell pq}) x_{k\ell} + \sum_{k>p} \sum_{\ell \neq q} (\alpha_{k\ell} + \beta_{pqk\ell}) x_{k\ell} + \sum_{i \neq p} \sum_{j \neq q} \sum_{\substack{k>i \\ k \neq p}} \sum_{\substack{\ell \neq j \\ \ell \neq q}} \beta_{ijk\ell} x_{ij} x_{k\ell} = \kappa - \alpha_{pq} \tag{29}$$

that is formed by fixing $x_{pq} = 1$ within (28). This equation is valid for all binary $\boldsymbol{x}$ feasible to the size $n-1$ assignment set in the variables $x_{k\ell}$ having $k \in \mathcal{N}^p$ and $\ell \in \mathcal{N}^q$. By the second reason following Definition 4.1, the inductive hypothesis gives us that each coefficient $\beta_{ijk\ell}$ in (29) having $(i,j,k,\ell) \notin \mathcal{S}_{pq}$ is uniquely defined in terms of the $\beta_{ijk\ell}$ having $(i,j,k,\ell) \in \mathcal{S}_{pq} \subseteq \mathcal{S}$.

Now choose any β_{rstu} found in (28) for which $(r,s,t,u) \notin \mathcal{S}$. Since $n \geq 6$, we can select $p \in \{1,\ldots,n-3\}$ and $q \in \{1,\ldots,n-3\}$ so that neither r nor t is equal to p and so that neither s nor u is equal to q. Such a selection ensures that β_{rstu} appears in (29). Since $(r,s,t,u) \notin \mathcal{S}$, we have that $(r,s,t,u) \notin \mathcal{S}_{pq}$ and therefore that β_{rstu} is uniquely defined in terms of the $\beta_{ijk\ell}$ having $(i,j,k,\ell) \in \mathcal{S}_{pq} \subseteq \mathcal{S}$.

2. Since (24) and (26) hold for all $(\boldsymbol{x},\boldsymbol{y})$ feasible to RLT1$'$, equation (27) holds for all $\boldsymbol{x} \in \boldsymbol{X}$, $\boldsymbol{x}$ binary. Additionally, as the extreme points of $\boldsymbol{X}$ are all binary, equation (27) holds for all $\boldsymbol{x} \in \boldsymbol{X}$. In particular, since $\tilde{\boldsymbol{x}}$ defined by $\tilde{x}_{ij} = \frac{1}{n}$ for all (i,j) has $\tilde{\boldsymbol{x}} \in \boldsymbol{X}$, every $\boldsymbol{d} \in \mathbb{R}^{n^2}$ satisfying $\boldsymbol{Hd} = \boldsymbol{0}$ must have $\displaystyle\sum_k \sum_\ell (\bar{\alpha}_{k\ell} - \hat{\alpha}_{k\ell}) d_{k\ell} = 0$ because, for sufficiently small $\epsilon > 0$, $\tilde{\boldsymbol{x}} + \epsilon \boldsymbol{d} \in \boldsymbol{X}$, and must therefore satisfy (27). Then no solution exists to $\boldsymbol{Hd} = \boldsymbol{0}$ having $\displaystyle\sum_k \sum_\ell (\bar{\alpha}_{k\ell} - \hat{\alpha}_{k\ell}) d_{k\ell} > 0$, so that Farkas' Lemma (see [9, Section 5.3, pp. 234-237], for example) shows the existence of a linear combination of the m_3 equations $\boldsymbol{Hx} = \boldsymbol{1}$ of Figure 1 yielding (27).

This completes the proof. □

A consequence of Theorem 4.1 is the corollary below which gives the dimension of the convex hull of feasible solutions to Problem RLT1$'$ and, as a result of the second statement of the corollary, the dimensions of the feasible regions to $\overline{\text{RLT1}}'$ and $\overline{\text{RLT1}}''$.

Corollary 4.2. *The dimension of the convex hull of feasible solutions to Problem RLT1$'$ is $n + \frac{(n-1)^2(n-2)^2}{2}$. Moreover, every relaxation of the convex hull that enforces the $m_1 + m_2 + m_3 = 2(n-1)^3 + n(n-1)$ equations of Figure 1 has this same dimension.*

Proof. Since the number of variables in RLT1$'$ is $n^2 + \frac{n^2(n-1)^2}{2}$ as noted in (12), the dimension of the convex hull of feasible solutions to Problem RLT1$'$ is

$$n + \frac{(n-1)^2(n-2)^2}{2} = \left(n^2 + \frac{n^2(n-1)^2}{2}\right) - \left(2(n-1)^3 + n(n-1)\right),$$

as follows from Theorem 4.1 combined with, for example, [36, Proposition 2.4, p. 87]. Regarding the second statement, every relaxation of the convex hull contains the convex hull itself, so its dimension is at least that of the convex hull. On the other hand, for every relaxation that enforces the equations of Figure 1, [36, Proposition 2.4, p. 87] gives us that the dimension of the relaxation is at most that of the convex hull. □

4.2 The Necessary and Sufficient Condition for Linearizability

Our necessary and sufficient condition for Problem P to be linearizable, which links the concept of linearizability to polyhedral theory, follows.

Theorem 4.3. *An instance of Problem P is linearizable if and only if the linear program $\overline{\text{RLT1}}''$ is bounded.*

Proof. Suppose that $\overline{\text{RLT1}}''$ is bounded. Since $\overline{\text{RLT1}}''$ is feasible, a dual solution exists. Compute a linear combination of equations (8)–(10) using any chosen dual solution to obtain

$$\sum_k \sum_\ell \tilde{c}_{k\ell} x_{k\ell} + \sum_i \sum_j \sum_{k>i} \sum_{\ell \neq j} C'_{ijk\ell} y_{ijk\ell} = 0 \ \ \forall\, (\boldsymbol{x}, \boldsymbol{y}) \text{ feasible to } \overline{\text{RLT1}}'', \tag{30}$$

so that

$$\sum_k \sum_\ell \tilde{c}_{k\ell} x_{k\ell} + \sum_i \sum_j \sum_{k>i} \sum_{\ell \neq j} C'_{ijk\ell} x_{ij} x_{k\ell} = 0 \ \ \forall\, \boldsymbol{x} \in \boldsymbol{X},\ \boldsymbol{x} \text{ binary}.$$

In computing (30), we used the fact that any chosen dual solution to $\overline{\text{RLT1}}''$ must yield the objective coefficient $C'_{ijk\ell}$ of $\overline{\text{RLT1}}''$ for each variable $y_{ijk\ell}$, but can yield values $\tilde{c}_{k\ell}$ which differ from the objective coefficients $c_{k\ell}$ of $\overline{\text{RLT1}}''$ corresponding to the variables $x_{k\ell}$. Rewriting this last equation with $\hat{c}_{k\ell} = c_{k\ell} - \tilde{c}_{k\ell}$ for all (k, ℓ), we obtain (2) with $\kappa = 0$ upon recalling that $x_{ij}x_{k\ell} = x_{k\ell}x_{ij}$ and $C'_{ijk\ell} = C_{ijk\ell} + C_{k\ell ij}$ for all $(i, j, k, \ell), i < k, j \neq \ell$.

Now suppose that Problem P is linearizable, so that there exist coefficients $\hat{c}_{k\ell}$ for all (k, ℓ) and a scalar κ satisfying (2). It follows that

$$\kappa + \sum_k \sum_\ell \hat{c}_{k\ell} x_{k\ell} = \sum_k \sum_\ell c_{k\ell} x_{k\ell} + \sum_i \sum_j \sum_{k>i} \sum_{\ell \neq j} C'_{ijk\ell} y_{ijk\ell} \quad \forall\, (\boldsymbol{x}, \boldsymbol{y}) \text{ feasible to RLT1}'. \tag{31}$$

Theorem 4.1 gives us that (31) can be computed as a linear combination of the $m_1 + m_2 + m_3$ equations of Figure 1, which are implied by the constraints of $\overline{\text{RLT1}}''$. Thus, (31) holds for all $(\boldsymbol{x}, \boldsymbol{y})$ feasible to $\overline{\text{RLT1}}''$ so that the objective function to $\overline{\text{RLT1}}''$ can be replaced with $\kappa + \sum_k \sum_\ell \hat{c}_{k\ell} x_{k\ell}$ without changing value at any feasible point. The resulting linear program is an assignment problem over the set $\boldsymbol{X}$ so that $\overline{\text{RLT1}}''$ is bounded. □

Theorem 4.3 is our main result that establishes a connection between linearizability and boundedness of $\overline{\text{RLT1}}''$. Notably, a consequence of this theorem is an algebraic check via the matrices $\boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}, \boldsymbol{F}$, and $\boldsymbol{G}$ of Figure 1 for determining in $O(n^4)$ time whether an instance of Problem P is linearizable. This $O(n^4)$ complexity reaffirms, via a different approach, a result of [31]. Our approach consists of two steps presented below as corollaries to Theorem 4.3. The first step, found in Corollary 4.4, equates the determination of the linearizability of an instance of Problem P to checking the consistency of a linear system of equations. The second step, found in Corollary 4.5, shows that the consistency of these equations can be checked in $O(n^4)$ time.

Begin by rewriting Problem $\overline{\text{RLT1}}''$ as Problem R below

$$\begin{aligned} \text{R: minimize}\{\boldsymbol{c}^T\boldsymbol{x} + \boldsymbol{c}_1^T\boldsymbol{y}_1 + \boldsymbol{c}_2^T\boldsymbol{y}_2 + \boldsymbol{c}_3^T\boldsymbol{y}_3 :\ & \boldsymbol{y}_1 = -\boldsymbol{B}^{-1}(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{C}\boldsymbol{y}_2 + \boldsymbol{D}\boldsymbol{y}_3), \\ & \boldsymbol{y}_2 = -\boldsymbol{F}^{-1}(\boldsymbol{E}\boldsymbol{x} + \boldsymbol{G}\boldsymbol{y}_3), \boldsymbol{H}\boldsymbol{x} = \boldsymbol{1},\ \boldsymbol{x} \geq \boldsymbol{0}\}, \end{aligned}$$

where $\boldsymbol{c}$, $\boldsymbol{c}_1$, $\boldsymbol{c}_2$, and $\boldsymbol{c}_3$ are column vectors of sizes n_1, n_2, n_3, and n_4, respectively, and where we used the implication from Theorem 4.1 that the $m_1 + m_2 + m_3$ equations of Figure 1 form a maximal LI set of restrictions for $\overline{\text{RLT1}}''$. We also used the invertibility of the matrices $\boldsymbol{B}$ and $\boldsymbol{F}$ to isolate the variables $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ in the constraints.

Now consider the following corollary to Theorem 4.3.

Corollary 4.4. *Given an instance of Problem P, rewrite $\overline{\text{RLT1}}''$ in terms of the matrices $\boldsymbol{A}$ through $\boldsymbol{H}$ of Figure 1 as in Problem R. Then P is linearizable if and only if*

$$\boldsymbol{c}_3^T = \boldsymbol{c}_1^T\boldsymbol{B}^{-1}\boldsymbol{D} + (\boldsymbol{c}_2^T - \boldsymbol{c}_1^T\boldsymbol{B}^{-1}\boldsymbol{C})\boldsymbol{F}^{-1}\boldsymbol{G}. \tag{32}$$

Proof. Use the first two families of equality constraints in R to sequentially substitute the variables $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ out of the objective function to obtain R$'$ below.

$$\begin{aligned}\text{R}'\text{: minimize}\{&[\boldsymbol{c}^T - \boldsymbol{c}_1^T\boldsymbol{B}^{-1}\boldsymbol{A} - (\boldsymbol{c}_2^T - \boldsymbol{c}_1^T\boldsymbol{B}^{-1}\boldsymbol{C})\boldsymbol{F}^{-1}\boldsymbol{E}]\boldsymbol{x}\\ &+[\boldsymbol{c}_3^T - \boldsymbol{c}_1^T\boldsymbol{B}^{-1}\boldsymbol{D} - (\boldsymbol{c}_2^T - \boldsymbol{c}_1^T\boldsymbol{B}^{-1}\boldsymbol{C})\boldsymbol{F}^{-1}\boldsymbol{G}]\boldsymbol{y}_3 :\\ &\boldsymbol{y}_1 = -\boldsymbol{B}^{-1}(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{C}\boldsymbol{y}_2 + \boldsymbol{D}\boldsymbol{y}_3), \boldsymbol{y}_2 = -\boldsymbol{F}^{-1}(\boldsymbol{E}\boldsymbol{x} + \boldsymbol{G}\boldsymbol{y}_3),\\ &\boldsymbol{H}\boldsymbol{x} = \boldsymbol{1},\ \boldsymbol{x} \geq \boldsymbol{0}\}\end{aligned}$$

Problem $\overline{\text{RLT1}}''$ is bounded if and only if Problem R$'$ is bounded, and Problem R$'$ is clearly bounded if and only if equations (32) hold true. The result follows from Theorem 4.3. □

The below result shows that the complexity of determining whether equations (32) are satisfied is $O(n^4)$.

Corollary 4.5. *The linearizability of an instance of Problem P can be checked in $O(n^4)$ time.*

Proof. It is sufficient to show that each of the three matrix products $\boldsymbol{B}^{-1}\boldsymbol{C}$, $\boldsymbol{B}^{-1}\boldsymbol{D}$, and $\boldsymbol{F}^{-1}\boldsymbol{G}$ of (32) contains at most $O(n^4)$ nonzero entries. We first show that the total number of nonzero entries in $\boldsymbol{B}^{-1}\boldsymbol{C}$ and $\boldsymbol{B}^{-1}\boldsymbol{D}$ combined is $4(n-1)^2(n-2)^2$, and then show that the number of nonzero entries in $\boldsymbol{F}^{-1}\boldsymbol{G}$ is $\frac{3n^2(n-3)^2}{2} + n(n-3) + 1$.

1. For $\boldsymbol{B}^{-1}\boldsymbol{C}$ and $\boldsymbol{B}^{-1}\boldsymbol{D}$, compute the representation $\boldsymbol{y}_1 = -\boldsymbol{B}^{-1}(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{C}\boldsymbol{y}_2 + \boldsymbol{D}\boldsymbol{y}_3)$ from (14)–(18) via the following steps.

   - Leave (14)–(16) unchanged.
   - Use (14) and (15) to substitute out of (17) all variables $y_{ijk\ell}$ having a single index j, k, or ℓ equal to n to obtain (33).
   - Use (14) and (15) to substitute out of (18) all variables $y_{ijk\ell}$ having a single index j, k, or ℓ equal to n to obtain (34).

$$y_{knnj} = x_{kn} - \sum_{i<k}\Big[x_{ij} - \sum_{\ell\neq j,n} y_{ijk\ell}\Big] - \sum_{k<i<n}\Big[x_{ij} - \sum_{\ell\neq j,n} y_{k\ell ij}\Big] \quad \forall\,(j,k),\ j<n,\ k<n \tag{33}$$

$$y_{k\ell nn} = x_{k\ell} - \sum_{i<k}\Big[x_{k\ell} - \sum_{j\neq \ell,n} y_{ijk\ell}\Big] - \sum_{k<i<n}\Big[x_{k\ell} - \sum_{j\neq \ell,n} y_{k\ell ij}\Big] \quad \forall\,(k,\ell),\ k<n,\ \ell<n \tag{34}$$

   Equations (14)–(16), (33), and (34) take the form $\boldsymbol{y}_1 = -\boldsymbol{B}^{-1}(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{C}\boldsymbol{y}_2 + \boldsymbol{D}\boldsymbol{y}_3)$. Count the number of nonzero entries in $\boldsymbol{B}^{-1}\boldsymbol{C}$ and $\boldsymbol{B}^{-1}\boldsymbol{D}$ combined. Each of the $(n-1)^2(n-2)$ total equations of (14) and (15) contains $n-2$ such entries and each of the $(n-1)^2(n-2)$ equations of (16) contains $n-2$ such entries. Also, each of the $2(n-1)^2$ total equations of (33) and (34) contains $(n-2)^2$ such entries. Summing, the total number of such entries is $4(n-1)^2(n-2)^2$.

2. For $\boldsymbol{F}^{-1}\boldsymbol{G}$, compute the representation $\boldsymbol{y}_2 = -\boldsymbol{F}^{-1}(\boldsymbol{E}\boldsymbol{x} + \boldsymbol{G}\boldsymbol{y}_3)$ from (20) and (21) via the following steps. Here, $\boldsymbol{y}_2$ is as defined in (22) and graphically depicted in Figure 3.

   - Select the $\frac{(n-1)(n-2)}{2} - 1$ equations of (20) having $j < n-2$ to obtain (35).
   - Select the $\frac{(n-1)(n-2)}{2} - 1$ equations of (21) having $i < n-2$ to obtain (36).

- Select the equation of (20) having $(j,\ell) = (n-2, n-1)$. Use (36) to substitute out variables $y_{ijk\ell}$ with $(j,\ell) = (n-2,n-1), i < k < n$, and $i < n-2$ to obtain (37).

$$y_{(n-2)j(n-1)\ell} = -x_{nj} + \sum_{k<n} x_{k\ell} - y_{(n-2)\ell(n-1)j} - \sum_{i=1}^{n-3}\sum_{k=i+1}^{n-1}(y_{ijk\ell} + y_{i\ell kj}) \quad \forall\, (j,\ell),\ j<\ell<n,\ j<n-2 \tag{35}$$

$$y_{i(n-2)k(n-1)} = -x_{in} + \sum_{\ell<n} x_{k\ell} - y_{i(n-1)k(n-2)} - \sum_{j=1}^{n-3}\sum_{\ell=j+1}^{n-1}(y_{ijk\ell} + y_{i\ell kj}) \quad \forall\, (i,k),\ i<k<n,\ i<n-2 \tag{36}$$

$$y_{(n-2)(n-2)(n-1)(n-1)} = -x_{n(n-2)} + \sum_{k<n} x_{k(n-1)} - y_{(n-2)(n-1)(n-1)(n-2)} - \sum_{i=1}^{n-3}\sum_{k=i+1}^{n-1}\left[-x_{in} + \sum_{\ell<n} x_{k\ell} - \sum_{j=1}^{n-3}\sum_{\ell=j+1}^{n-1}(y_{ijk\ell} + y_{i\ell kj})\right] \tag{37}$$

Equations (35)–(37) take the form $\boldsymbol{y}_2 = -\boldsymbol{F}^{-1}(\boldsymbol{Ex} + \boldsymbol{Gy}_3)$. Count the number of nonzero entries in $\boldsymbol{F}^{-1}\boldsymbol{G}$. Each of the $(n-1)(n-2)-2$ total equations of (35) and (36) contains $(n-1)(n-2)-1$ such entries, and the single equation of (37) contains $\frac{n^2(n-3)^2}{2}+1$ such entries. Summing, the total number of such entries is $\frac{3n^2(n-3)^2}{2} + n(n-3) + 1$.

□

We conclude this section with three remarks. First, we answer an open question of [28] relative to obtaining the best possible "linearization-based" bound for the QAP. The work of [28] presents a methodology for using linearizable objective structures to obtain linearization-based bounds for quadratic combinatorial optimization problems. In general, to obtain the best possible such bound, a *spanning set* of linearizable objective coefficients (see [28, Definition 1]) is needed. For the QAP, this paper reported generating such a spanning set by brute force computation for instances having $n \leq 9$. For larger n, the spanning set remained unknown. The paper left as an open question whether the best linearization-based bound can exceed that of the level-1 RLT. Theorems 4.1 and 4.3 combine to give a spanning set for general n and answer the question negatively. In fact, Theorem 4.1 not only gives an explicit representation of such a spanning set for general n by showing that the equality constraints of $\overline{\text{RLT1}}''$ span the set of linearizable objective functions, but it also establishes the equations of Figure 1 as providing a maximal LI such set.

Second, Theorem 4.3 and Corollaries 4.4 and 4.5 explain the linearizable property in terms of the linear program $\overline{\text{RLT1}}''$, but the related linear program $\overline{\text{RLT1}}'$ can solve a richer family of instances of Problem P than those that are linearizable. The feasible region to $\overline{\text{RLT1}}'$ is contained within that of $\overline{\text{RLT1}}''$ so it is reasonable to expect that there exist instances where the former has an optimal binary solution but the latter does not. This is indeed the case, as [7, Example 2] presents an instance of Problem P where $\overline{\text{RLT1}}'$ gives an optimal binary solution, but $\overline{\text{RLT1}}''$ is unbounded. Using Theorem 4.3, we can now conclude that the given example is not linearizable.

Finally, if the linear program $\overline{\text{RLT1}}''$ is bounded, then it has an optimal binary solution, and consequently so does $\overline{\text{RLT1}}'$. This implication, which follows from [7, Theorem 2], is readily explained in terms of the proof of Theorem 4.3. As noted in the "if" direction, computing a linear combination of (8)–(10) using any dual solution gives (30). Then the objective function to $\overline{\text{RLT1}}''$ can be replaced with $\sum_k \sum_\ell (c_{k\ell} - \tilde{c}_{k\ell})x_{k\ell}$ without changing value at any feasible point. The resulting linear program is an assignment problem over the set $\boldsymbol{X}$ with all binary extreme points. (The implication also follows from the equivalence between Problems $\overline{\text{RLT1}}''$ and R$'$ found in the proof of Corollary 4.4 since $\overline{\text{RLT1}}''$ bounded reduces R$'$ to an assignment problem over $\boldsymbol{X}$.)

5 Conclusions and Future Research

This paper provides a simply-stated, polyhedral-based necessary and sufficient condition for identifying an instance of the QAP as being *linearizable*. The notion of QAP linearizability has garnered attention in the literature because, even though the QAP is known to be NP-hard in general, a linearizable instance is solvable as an assignment problem. Our analysis is novel in that it relies on the boundedness of a linear program. Notably, the form that we employ was used by [7] to explain several sufficient conditions for linearizability found in [10, 14, 23], but no mention was made as to the more challenging direction of necessity. Our approach is different than alternate methods found in [31], which use decompositions of the objective coefficients. It also allows us to answer an open question of [28] by giving the best possible linearization-based bound, and by doing so with a minimal number of constraints.

A consequence of our analysis is the potential for improved solution methods. Since we have available a maximal LI set of equations in the space of the level-1 RLT formulation, we have the option to construct reduced versions of $\overline{\text{RLT1}}'$ by replacing the $2n^2(n-1)+2n$ equality constraints with a maximal LI set of $2(n-1)^3+n(n-1)$ equations, lessening the number of equations by $3n^2-3n+2$. As all omitted equations are implied, the reduced form does not sacrifice relaxation strength. Given the recognized tightness [2, 30] of $\overline{\text{RLT1}}'$, this reduction may prove fruitful in branch-and-bound routines. The challenge now is to exploit the structure of the LI equations when solving the linear programming relaxations.

In addition to providing a polyhedral perspective on linearizability, Problem $\overline{\text{RLT1}}'$ gives insights into a role of linearizability for solving instances of the QAP which are *not* linearizable. Suppose that $\overline{\text{RLT1}}'$ is solved for some instance of the QAP. As noted earlier, if the instance is linearizable, then $\overline{\text{RLT1}}'$ will have an optimal binary solution. If the instance is not linearizable, then the reduced costs on the variables $\boldsymbol{y}$ within $\overline{\text{RLT1}}'$ cannot all be 0. In this latter case, the nonzero reduced costs can be used to guide branch-and-bound routines, as well as to serve as the basis for "hot starts" in computing bounds. Updated such costs with hot starts were used within a decomposition method [2, 30] to solve the continuous relaxation of RLT1 that relaxes the binary restrictions on $\boldsymbol{x}$ in (7), and formed the basis of a global solution procedure for Problem P based on the level-2 RLT form [1]. The branching on nonzero reduced costs within an enumerative routine can be envisioned as searching for sub-instances of a QAP that are linearizable, and thus permit fathoming at the identified partial binary solutions.

Additional research directions involve the level-1 and higher-level RLT forms, both for the QAP and related problems. For the QAP, we conjecture that higher-level forms can be used to identify new sets of conditions, other than linearizable, that permit solving in polynomial time. Known conditions that show promise in this regard are found in [10, 12, 15, 16, 19, 22, 34]. For related problems such as the quadratic shortest path and the quadratic semi-assignment, usage of the level-1 RLT to identify a maximal LI set of equations holds promise for novel characterizations of linearizable instances. Research in these directions is underway.

6 Appendix

Theorem 3, Part 1 - Base Cases Consider an assignment set $\boldsymbol{X}$ of size $n = 3$, $n = 4$, or $n = 5$. Given an equation of the form

$$\sum_k \sum_\ell \alpha_{k\ell} x_{k\ell} + \sum_i \sum_j \sum_{k>i} \sum_{\ell \neq j} \beta_{ijk\ell} x_{ij} x_{k\ell} = \kappa \tag{38}$$

that is satisfied for all $\boldsymbol{x} \in \boldsymbol{X}$, $\boldsymbol{x}$ binary, each $\beta_{ijk\ell}$ having $(i, j, k, \ell) \notin \mathcal{S}$ is uniquely defined in terms of the $\beta_{ijk\ell}$ having $(i, j, k, \ell) \in \mathcal{S}$.

Proof. For $n = 3$ and $n = 4$, the proof explicitly states the n-factorial equations obtained by individually inserting each solution to $\boldsymbol{x} \in \boldsymbol{X}$, $\boldsymbol{x}$ binary, within (38), and then computes linear combinations of these equations to suitably define the $\beta_{ijk\ell}$ having $(i, j, k, \ell) \notin \mathcal{S}$. For $n = 5$, the proof uses both an argument identical to the inductive step found in Part 1 of the proof of Theorem 4.1 and the approach of cases $n = 3$ and $n = 4$ to compute linear combinations of equations resulting from (38). For this last case, the n-factorial equations are not explicitly stated for the sake of brevity.

We first define useful notation for given n. Let $\mathcal{R}$ denote the set of all 4-tuples $(i, j, k, \ell) \in \mathcal{N} \times \mathcal{N} \times \mathcal{N} \times \mathcal{N}$ having $i < k, j \neq \ell$, for which $(i, j, k, \ell) \notin \mathcal{S}$. Consistent with Figure 1, we then have $\mathcal{R} = \{(i, j, k, \ell) : y_{ijk\ell} \in \boldsymbol{y}_3\}$ and $|\mathcal{R}| = n_4 = \frac{(n-3)(n-2)(n-1)n}{n} + 1$. Furthermore, distinguish each of the n-factorial solutions to $\boldsymbol{x} \in \boldsymbol{X}$, $\boldsymbol{x}$ binary, with a distinct row vector in $\mathbb{R}^n$ in the following manner: each row vector, say $\boldsymbol{\theta} \in \mathbb{R}^n$, consists of a permutation of the numbers $1, \ldots, n$, so that $\boldsymbol{x}_{\boldsymbol{\theta}}$ denotes the binary solution having $x_{i\theta(i)} = 1$ for all i and $x_{ij} = 0$ otherwise. For example, when $n = 3$ and $\boldsymbol{\theta} = (3, 1, 2)$, then $\boldsymbol{x}_{\boldsymbol{\theta}} \equiv \boldsymbol{x}_{(3,1,2)}$ has $x_{13} = x_{21} = x_{32} = 1$, and $x_{ij} = 0$ otherwise. Let $E(\boldsymbol{x}_{\boldsymbol{\theta}})$ denote the equation obtained by setting $\boldsymbol{x} = \boldsymbol{x}_{\boldsymbol{\theta}}$ within (38), and let $f(\boldsymbol{x}_{\boldsymbol{\theta}})$ denote the sum of the terms within $E(\boldsymbol{x}_{\boldsymbol{\theta}})$ that correspond to those $(i, j, k, \ell) \in \mathcal{S}$, so that $f(\boldsymbol{x}_{\boldsymbol{\theta}}) \equiv \sum_{(i,j,k,\ell) \in \mathcal{S}} \beta_{ijk\ell} x_{ij} x_{k\ell}$ evaluated at $\boldsymbol{x} = \boldsymbol{x}_{\boldsymbol{\theta}}$.

Base Case: $\boldsymbol{n = 3}$ There are 3-factorial = 6 solutions to $\boldsymbol{x} \in \boldsymbol{X}$, $\boldsymbol{x}$ binary, and $\mathcal{R} = \{(1, 2, 2, 1)\}$. The 6 equations resulting from (38) are listed below.

(1) $E(\boldsymbol{x}_{(1,2,3)}): \ \alpha_{11} + \alpha_{22} + \alpha_{33} + f(\boldsymbol{x}_{(1,2,3)}) = \kappa$
(2) $E(\boldsymbol{x}_{(1,3,2)}): \ \alpha_{11} + \alpha_{23} + \alpha_{32} + f(\boldsymbol{x}_{(1,3,2)}) = \kappa$
(3) $E(\boldsymbol{x}_{(2,1,3)}): \ \alpha_{12} + \alpha_{21} + \alpha_{33} + f(\boldsymbol{x}_{(2,1,3)}) + \beta_{1221} = \kappa$
(4) $E(\boldsymbol{x}_{(2,3,1)}): \ \alpha_{12} + \alpha_{23} + \alpha_{31} + f(\boldsymbol{x}_{(2,3,1)}) = \kappa$
(5) $E(\boldsymbol{x}_{(3,1,2)}): \ \alpha_{13} + \alpha_{21} + \alpha_{32} + f(\boldsymbol{x}_{(3,1,2)}) = \kappa$
(6) $E(\boldsymbol{x}_{(3,2,1)}): \ \alpha_{13} + \alpha_{22} + \alpha_{31} + f(\boldsymbol{x}_{(3,2,1)}) = \kappa$

Compute the linear combination

$$E(\boldsymbol{x}_{(1,2,3)}) - E(\boldsymbol{x}_{(1,3,2)}) - E(\boldsymbol{x}_{(2,1,3)}) + E(\boldsymbol{x}_{(2,3,1)}) + E(\boldsymbol{x}_{(3,1,2)}) - E(\boldsymbol{x}_{(3,2,1)})$$

of these equations to obtain that

$$\beta_{1221} = f(\boldsymbol{x}_{(1,2,3)}) - f(\boldsymbol{x}_{(1,3,2)}) - f(\boldsymbol{x}_{(2,1,3)}) + f(\boldsymbol{x}_{(2,3,1)}) + f(\boldsymbol{x}_{(3,1,2)}) - f(\boldsymbol{x}_{(3,2,1)}).$$

This completes the $n = 3$ base case.

Base Case: $\boldsymbol{n = 4}$ There are 4-factorial = 24 solutions to $\boldsymbol{x} \in \boldsymbol{X}$, $\boldsymbol{x}$ binary, and $\mathcal{R} = \{(1, 1, 2, 2),$ $(1, 1, 2, 3)$, $(1, 1, 3, 2)$, $(1, 1, 3, 3)$, $(1, 2, 2, 1)$, $(1, 2, 3, 1)$, $(1, 3, 2, 1)$, $(1, 3, 2, 2)$, $(1, 3, 3, 1)$, $(1, 3, 3, 2)$, $(2, 2, 3, 1)$, $(2, 3, 3, 1)$, $(2, 3, 3, 2)\}$. The 24 equations resulting from (38) are listed below.

(1) $E(\boldsymbol{x}_{(1,2,3,4)}): \alpha_{11} + \alpha_{22} + \alpha_{33} + \alpha_{44} + \beta_{1122} + \beta_{1133} + f(\boldsymbol{x}_{(1,2,3,4)}) = \kappa$

(2) $E(\boldsymbol{x}_{(1,2,4,3)}) : \alpha_{11} + \alpha_{22} + \alpha_{34} + \alpha_{43} + \beta_{1122} + f(\boldsymbol{x}_{(1,2,4,3)}) = \kappa$
(3) $E(\boldsymbol{x}_{(1,3,2,4)}) : \alpha_{11} + \alpha_{23} + \alpha_{32} + \alpha_{44} + \beta_{1123} + \beta_{1132} + \beta_{2332} + f(\boldsymbol{x}_{(1,3,2,4)}) = \kappa$
(4) $E(\boldsymbol{x}_{(1,3,4,2)}) : \alpha_{11} + \alpha_{23} + \alpha_{34} + \alpha_{42} + \beta_{1123} + f(\boldsymbol{x}_{(1,3,4,2)}) = \kappa$
(5) $E(\boldsymbol{x}_{(1,4,2,3)}) : \alpha_{11} + \alpha_{24} + \alpha_{32} + \alpha_{43} + \beta_{1132} + f(\boldsymbol{x}_{(1,4,2,3)}) = \kappa$
(6) $E(\boldsymbol{x}_{(1,4,3,2)}) : \alpha_{11} + \alpha_{24} + \alpha_{33} + \alpha_{42} + \beta_{1133} + f(\boldsymbol{x}_{(1,4,3,2)}) = \kappa$
(7) $E(\boldsymbol{x}_{(2,1,3,4)}) : \alpha_{12} + \alpha_{21} + \alpha_{33} + \alpha_{44} + \beta_{1221} + f(\boldsymbol{x}_{(2,1,3,4)}) = \kappa$
(8) $E(\boldsymbol{x}_{(2,1,4,3)}) : \alpha_{12} + \alpha_{21} + \alpha_{34} + \alpha_{43} + \beta_{1221} + f(\boldsymbol{x}_{(2,1,4,3)}) = \kappa$
(9) $E(\boldsymbol{x}_{(2,3,1,4)}) : \alpha_{12} + \alpha_{23} + \alpha_{31} + \alpha_{44} + \beta_{1231} + \beta_{2331} + f(\boldsymbol{x}_{(2,3,1,4)}) = \kappa$
(10) $E(\boldsymbol{x}_{(2,3,4,1)}) : \alpha_{12} + \alpha_{23} + \alpha_{34} + \alpha_{41} + f(\boldsymbol{x}_{(2,3,4,1)}) = \kappa$
(11) $E(\boldsymbol{x}_{(2,4,1,3)}) : \alpha_{12} + \alpha_{24} + \alpha_{31} + \alpha_{43} + \beta_{1231} + f(\boldsymbol{x}_{(2,4,1,3)}) = \kappa$
(12) $E(\boldsymbol{x}_{(2,4,3,1)}) : \alpha_{12} + \alpha_{24} + \alpha_{33} + \alpha_{41} + f(\boldsymbol{x}_{(2,4,3,1)}) = \kappa$
(13) $E(\boldsymbol{x}_{(3,1,2,4)}) : \alpha_{13} + \alpha_{21} + \alpha_{32} + \alpha_{44} + \beta_{1321} + \beta_{1332} + f(\boldsymbol{x}_{(3,1,2,4)}) = \kappa$
(14) $E(\boldsymbol{x}_{(3,1,4,2)}) : \alpha_{13} + \alpha_{21} + \alpha_{34} + \alpha_{42} + \beta_{1321} + f(\boldsymbol{x}_{(3,1,4,2)}) = \kappa$
(15) $E(\boldsymbol{x}_{(3,2,1,4)}) : \alpha_{13} + \alpha_{22} + \alpha_{31} + \alpha_{44} + \beta_{1322} + \beta_{1331} + \beta_{2231} + f(\boldsymbol{x}_{(3,2,1,4)}) = \kappa$
(16) $E(\boldsymbol{x}_{(3,2,4,1)}) : \alpha_{13} + \alpha_{22} + \alpha_{34} + \alpha_{41} + \beta_{1322} + f(\boldsymbol{x}_{(3,2,4,1)}) = \kappa$
(17) $E(\boldsymbol{x}_{(3,4,1,2)}) : \alpha_{13} + \alpha_{24} + \alpha_{31} + \alpha_{42} + \beta_{1331} + f(\boldsymbol{x}_{(3,4,1,2)}) = \kappa$
(18) $E(\boldsymbol{x}_{(3,4,2,1)}) : \alpha_{13} + \alpha_{24} + \alpha_{32} + \alpha_{41} + \beta_{1332} + f(\boldsymbol{x}_{(3,4,2,1)}) = \kappa$
(19) $E(\boldsymbol{x}_{(4,1,2,3)}) : \alpha_{14} + \alpha_{21} + \alpha_{32} + \alpha_{43} + f(\boldsymbol{x}_{(4,1,2,3)}) = \kappa$
(20) $E(\boldsymbol{x}_{(4,1,3,2)}) : \alpha_{14} + \alpha_{21} + \alpha_{33} + \alpha_{42} + f(\boldsymbol{x}_{(4,1,3,2)}) = \kappa$
(21) $E(\boldsymbol{x}_{(4,2,1,3)}) : \alpha_{14} + \alpha_{22} + \alpha_{31} + \alpha_{43} + \beta_{2231} + f(\boldsymbol{x}_{(4,2,1,3)}) = \kappa$
(22) $E(\boldsymbol{x}_{(4,2,3,1)}) : \alpha_{14} + \alpha_{22} + \alpha_{33} + \alpha_{41} + f(\boldsymbol{x}_{(4,2,3,1)}) = \kappa$
(23) $E(\boldsymbol{x}_{(4,3,1,2)}) : \alpha_{14} + \alpha_{23} + \alpha_{31} + \alpha_{42} + \beta_{2331} + f(\boldsymbol{x}_{(4,3,1,2)}) = \kappa$
(24) $E(\boldsymbol{x}_{(4,3,2,1)}) : \alpha_{14} + \alpha_{23} + \alpha_{32} + \alpha_{41} + \beta_{2332} + f(\boldsymbol{x}_{(4,3,2,1)}) = \kappa$

The 13 linear combinations of equation numbers (1)–(24) listed below use the functions $f(\boldsymbol{x_\theta})$ to recursively express the $|\mathcal{R}| = 13$ coefficients $\beta_{ijk\ell}$ having $(i,j,k,\ell) \in \mathcal{R}$ in terms of the $\beta_{ijk\ell}$ having $(i,j,k,\ell) \in \mathcal{S}$. For simplicity in presentation, we refer to the equations by their corresponding numbers.

- (1) − (2) − (3) + (4) + (5) − (6):
  $\beta_{2332} = f(\boldsymbol{x}_{(1,2,3,4)}) - f(\boldsymbol{x}_{(1,2,4,3)}) - f(\boldsymbol{x}_{(1,3,2,4)}) + f(\boldsymbol{x}_{(1,3,4,2)}) + f(\boldsymbol{x}_{(1,4,2,3)}) - f(\boldsymbol{x}_{(1,4,3,2)})$
- (7) − (8) − (9) + (10) + (11) − (12):
  $\beta_{2331} = f(\boldsymbol{x}_{(2,1,3,4)}) - f(\boldsymbol{x}_{(2,1,4,3)}) - f(\boldsymbol{x}_{(2,3,1,4)}) + f(\boldsymbol{x}_{(2,3,4,1)}) + f(\boldsymbol{x}_{(2,4,1,3)}) - f(\boldsymbol{x}_{(2,4,3,1)})$
- (13) − (14) − (15) + (16) + (17) − (18):
  $\beta_{2231} = f(\boldsymbol{x}_{(3,1,2,4)}) - f(\boldsymbol{x}_{(3,1,4,2)}) - f(\boldsymbol{x}_{(3,2,1,4)}) + f(\boldsymbol{x}_{(3,2,4,1)}) + f(\boldsymbol{x}_{(3,4,1,2)}) - f(\boldsymbol{x}_{(3,4,2,1)})$
- (7) − (8) − (13) + (14) + (19) − (20):
  $\beta_{1332} = f(\boldsymbol{x}_{(2,1,3,4)}) - f(\boldsymbol{x}_{(2,1,4,3)}) - f(\boldsymbol{x}_{(3,1,2,4)}) + f(\boldsymbol{x}_{(3,1,4,2)}) + f(\boldsymbol{x}_{(4,1,2,3)}) - f(\boldsymbol{x}_{(4,1,3,2)})$
- (9) − (11) − (15) + (17) + (21) − (23):
  $\beta_{1322} = f(\boldsymbol{x}_{(2,3,1,4)}) - f(\boldsymbol{x}_{(2,4,1,3)}) - f(\boldsymbol{x}_{(3,2,1,4)}) + f(\boldsymbol{x}_{(3,4,1,2)}) + f(\boldsymbol{x}_{(4,2,1,3)}) - f(\boldsymbol{x}_{(4,3,1,2)})$
- (1) − (2) − (7) + (8):
  $\beta_{1133} = -f(\boldsymbol{x}_{(1,2,3,4)}) + f(\boldsymbol{x}_{(1,2,4,3)}) + f(\boldsymbol{x}_{(2,1,3,4)}) - f(\boldsymbol{x}_{(2,1,4,3)})$
- (3) − (5) − (9) + (11):
  $\beta_{1123} = -f(\boldsymbol{x}_{(1,3,2,4)}) + f(\boldsymbol{x}_{(1,4,2,3)}) + f(\boldsymbol{x}_{(2,3,1,4)}) - f(\boldsymbol{x}_{(2,4,1,3)}) + \beta_{2331} - \beta_{2332}$

- (8) − (10) − (19) + (24):
  $\beta_{1221} = -f(\boldsymbol{x}_{(2,1,4,3)}) + f(\boldsymbol{x}_{(2,3,4,1)}) + f(\boldsymbol{x}_{(4,1,2,3)}) - f(\boldsymbol{x}_{(4,3,2,1)}) - \beta_{2332}$
- (7) − (12) − (13) + (18):
  $\beta_{1321} = f(\boldsymbol{x}_{(2,1,3,4)}) - f(\boldsymbol{x}_{(2,4,3,1)}) - f(\boldsymbol{x}_{(3,1,2,4)}) + f(\boldsymbol{x}_{(3,4,2,1)}) + \beta_{1221}$
- (1) − (6) − (15) + (17):
  $\beta_{1122} = -f(\boldsymbol{x}_{(1,2,3,4)}) + f(\boldsymbol{x}_{(1,4,3,2)}) + f(\boldsymbol{x}_{(3,2,1,4)}) - f(\boldsymbol{x}_{(3,4,1,2)}) + \beta_{1322} + \beta_{2231}$
- (11) − (12) − (21) + (22):
  $\beta_{1231} = -f(\boldsymbol{x}_{(2,4,1,3)}) + f(\boldsymbol{x}_{(2,4,3,1)}) + f(\boldsymbol{x}_{(4,2,1,3)}) - f(\boldsymbol{x}_{(4,2,3,1)}) + \beta_{2231}$
- (9) − (10) − (15) + (16):
  $\beta_{1331} = f(\boldsymbol{x}_{(2,3,1,4)}) - f(\boldsymbol{x}_{(2,3,4,1)}) - f(\boldsymbol{x}_{(3,2,1,4)}) + f(\boldsymbol{x}_{(3,2,4,1)}) + \beta_{1231} - \beta_{2231} + \beta_{2331}$
- (3) − (4) − (13) + (14):
  $\beta_{1132} = -f(\boldsymbol{x}_{(1,3,2,4)}) + f(\boldsymbol{x}_{(1,3,4,2)}) + f(\boldsymbol{x}_{(3,1,2,4)}) - f(\boldsymbol{x}_{(3,1,4,2)}) + \beta_{1332} - \beta_{2332}$

This completes the $n = 4$ base case.

Base Case: $n = 5$ For this case, $|\mathcal{R}| = 61$. We begin with an argument identical to the inductive step found in Part 1 of the proof of Theorem 4.1 to establish the result for those $\beta_{ijk\ell} \in \mathcal{R}$ having $(i,k) \neq (1,2), (j,\ell) \neq (1,2)$, and $(j,\ell) \neq (2,1)$. Specifically, these are the 41 $\beta_{ijk\ell}$ having $(i,j,k,\ell) \in \mathcal{R} - \mathcal{R}_0$, where

$$\begin{aligned}\mathcal{R}_0 \equiv \{&(1,1,2,2),(1,1,2,3),(1,1,2,4),(1,1,3,2),(1,1,4,2),(1,2,2,1),\\ &(1,2,2,3),(1,2,2,4),(1,2,3,1),(1,2,4,1),(1,3,2,1),(1,3,2,2),\\ &(1,4,2,1),(1,4,2,2),(1,4,2,3),(2,1,3,2),(2,1,4,2),(2,2,3,1),\\ &(2,2,4,1),(3,2,4,1)\}.\end{aligned}$$

For any (p,q) having $p \in \{1,2\}$ and $q \in \{1,2\}$, consider the equation formed by fixing $x_{pq} = 1$ within (38). This equation, given by

$$\sum_{k<p}\sum_{\ell\neq q}(\alpha_{k\ell} + \beta_{k\ell pq})x_{k\ell} + \sum_{k>p}\sum_{\ell\neq q}(\alpha_{k\ell} + \beta_{pqk\ell})x_{k\ell} + \sum_{i\neq p}\sum_{j\neq q}\sum_{\substack{k>i\\k\neq p}}\sum_{\substack{\ell\neq j\\ \ell\neq q}}\beta_{ijk\ell}x_{ij}x_{k\ell} = \kappa - \alpha_{pq}, \quad (39)$$

is of the form (38) and is valid for all binary $\boldsymbol{x}$ feasible to the size $n = 4$ assignment set in the variables $x_{k\ell}$ having $k \in \mathcal{N}^p$ and $\ell \in \mathcal{N}^q$. By the second reason following Definition 4.1, the $n = 4$ base case gives us that each coefficient $\beta_{ijk\ell}$ in (39) having $(i,j,k,\ell) \notin \mathcal{S}_{pq}$ is uniquely defined in terms of the $\beta_{ijk\ell}$ having $(i,j,k,\ell) \in \mathcal{S}_{pq} \subseteq \mathcal{S}$.

Consider any β_{rstu} found in (38) for which $(r,s,t,u) \in \mathcal{R} - \mathcal{R}_0$. By definition of the set $\mathcal{R}_0$, we can select $p \in \{1,2\}$ and $q \in \{1,2\}$ so that neither r nor t is equal to p and so that neither s nor u is equal to q. Such a selection ensures that β_{rstu} appears in (39). Since $(r,s,t,u) \notin \mathcal{S}$, we have that $(r,s,t,u) \notin \mathcal{S}_{pq}$ and therefore that β_{rstu} is uniquely defined in terms of the $\beta_{ijk\ell}$ having $(i,j,k,\ell) \in \mathcal{S}_{pq} \subseteq \mathcal{S}$, as desired.

We now prove the result for the $\beta_{ijk\ell}$ having $(i,j,k,\ell) \in \mathcal{R}_0$ in a similar fashion as to the $n = 3$ and $n = 4$ base cases, but for brevity do not explicitly state the equations resulting from (38). We instead specify linear combinations that recursively express these $|\mathcal{R}_0| = 20$ $\beta_{ijk\ell}$ in terms of the $\beta_{ijk\ell}$ having $(i,j,k,\ell) \in \mathcal{S} \cup (\mathcal{R} - \mathcal{R}_0)$. These linear combinations establish the result because, as shown above, every $\beta_{ijk\ell}$ having $(i,j,k,\ell) \in (\mathcal{R} - \mathcal{R}_0)$ is uniquely defined in terms of the $\beta_{ijk\ell}$ having $(i,j,k,\ell) \in \mathcal{S}$.

- $E(\boldsymbol{x}_{(3,4,1,2,5)}) - E(\boldsymbol{x}_{(3,4,1,5,2)}) - E(\boldsymbol{x}_{(3,4,2,1,5)}) + E(\boldsymbol{x}_{(3,4,2,5,1)}) + E(\boldsymbol{x}_{(3,4,5,1,2)}) - E(\boldsymbol{x}_{(3,4,5,2,1)})$:
  $\beta_{3241} = f(\boldsymbol{x}_{(3,4,1,2,5)}) - f(\boldsymbol{x}_{(3,4,1,5,2)}) - f(\boldsymbol{x}_{(3,4,2,1,5)}) + f(\boldsymbol{x}_{(3,4,2,5,1)})$
  $+ f(\boldsymbol{x}_{(3,4,5,1,2)}) - f(\boldsymbol{x}_{(3,4,5,2,1)})$
- $E(\boldsymbol{x}_{(3,2,5,1,4)}) - E(\boldsymbol{x}_{(3,2,5,4,1)}) - E(\boldsymbol{x}_{(3,5,2,1,4)}) + E(\boldsymbol{x}_{(3,5,2,4,1)})$:
  $\beta_{2241} = -f(\boldsymbol{x}_{(3,2,5,1,4)}) + f(\boldsymbol{x}_{(3,2,5,4,1)}) + f(\boldsymbol{x}_{(3,5,2,1,4)}) - f(\boldsymbol{x}_{(3,5,2,4,1)})$
  $+ \beta_{2244} + \beta_{3241}$
- $E(\boldsymbol{x}_{(3,2,1,5,4)}) - E(\boldsymbol{x}_{(3,2,4,5,1)}) - E(\boldsymbol{x}_{(3,5,1,2,4)}) + E(\boldsymbol{x}_{(3,5,4,2,1)})$:
  $\beta_{2231} = -f(\boldsymbol{x}_{(3,2,1,5,4)}) + f(\boldsymbol{x}_{(3,2,4,5,1)}) + f(\boldsymbol{x}_{(3,5,1,2,4)}) - f(\boldsymbol{x}_{(3,5,4,2,1)})$
  $+ \beta_{2234} - \beta_{3442}$
- $E(\boldsymbol{x}_{(3,1,4,2,5)}) - E(\boldsymbol{x}_{(3,1,4,5,2)}) - E(\boldsymbol{x}_{(3,4,1,2,5)}) + E(\boldsymbol{x}_{(3,4,1,5,2)})$:
  $\beta_{2142} = -f(\boldsymbol{x}_{(3,1,4,2,5)}) + f(\boldsymbol{x}_{(3,1,4,5,2)}) + f(\boldsymbol{x}_{(3,4,1,2,5)}) - f(\boldsymbol{x}_{(3,4,1,5,2)})$
  $+ \beta_{2442} - \beta_{3442}$
- $E(\boldsymbol{x}_{(3,1,2,4,5)}) - E(\boldsymbol{x}_{(3,1,5,4,2)}) - E(\boldsymbol{x}_{(3,4,2,1,5)}) + E(\boldsymbol{x}_{(3,4,5,1,2)})$:
  $\beta_{2132} = -f(\boldsymbol{x}_{(3,1,2,4,5)}) + f(\boldsymbol{x}_{(3,1,5,4,2)}) + f(\boldsymbol{x}_{(3,4,2,1,5)}) - f(\boldsymbol{x}_{(3,4,5,1,2)})$
  $+ \beta_{2432} + \beta_{3241}$
- $E(\boldsymbol{x}_{(3,2,1,5,4)}) - E(\boldsymbol{x}_{(3,4,1,5,2)}) - E(\boldsymbol{x}_{(5,2,1,3,4)}) + E(\boldsymbol{x}_{(5,4,1,3,2)})$:
  $\beta_{1322} = -f(\boldsymbol{x}_{(3,2,1,5,4)}) + f(\boldsymbol{x}_{(3,4,1,5,2)}) + f(\boldsymbol{x}_{(5,2,1,3,4)}) - f(\boldsymbol{x}_{(5,4,1,3,2)})$
  $+ \beta_{2243} - \beta_{2443}$
- $E(\boldsymbol{x}_{(3,2,1,4,5)}) - E(\boldsymbol{x}_{(3,5,1,4,2)}) - E(\boldsymbol{x}_{(4,2,1,3,5)}) + E(\boldsymbol{x}_{(4,5,1,3,2)})$:
  $\beta_{1422} = f(\boldsymbol{x}_{(3,2,1,4,5)}) - f(\boldsymbol{x}_{(3,5,1,4,2)}) - f(\boldsymbol{x}_{(4,2,1,3,5)}) + f(\boldsymbol{x}_{(4,5,1,3,2)})$
  $+ \beta_{1322} - \beta_{2243} + \beta_{2244}$
- $E(\boldsymbol{x}_{(4,2,1,5,3)}) - E(\boldsymbol{x}_{(4,3,1,5,2)}) - E(\boldsymbol{x}_{(5,2,1,4,3)}) + E(\boldsymbol{x}_{(5,3,1,4,2)})$:
  $\beta_{1423} = f(\boldsymbol{x}_{(4,2,1,5,3)}) - f(\boldsymbol{x}_{(4,3,1,5,2)}) - f(\boldsymbol{x}_{(5,2,1,4,3)}) + f(\boldsymbol{x}_{(5,3,1,4,2)})$
  $+ \beta_{1422} - \beta_{2244}$
- $E(\boldsymbol{x}_{(2,4,1,3,5)}) - E(\boldsymbol{x}_{(2,5,1,3,4)}) - E(\boldsymbol{x}_{(3,4,1,2,5)}) + E(\boldsymbol{x}_{(3,5,1,2,4)})$:
  $\beta_{1224} = -f(\boldsymbol{x}_{(2,4,1,3,5)}) + f(\boldsymbol{x}_{(2,5,1,3,4)}) + f(\boldsymbol{x}_{(3,4,1,2,5)}) - f(\boldsymbol{x}_{(3,5,1,2,4)})$
  $+ \beta_{2442} - \beta_{2443}$
- $E(\boldsymbol{x}_{(2,3,1,4,5)}) - E(\boldsymbol{x}_{(2,5,1,4,3)}) - E(\boldsymbol{x}_{(4,3,1,2,5)}) + E(\boldsymbol{x}_{(4,5,1,2,3)})$:
  $\beta_{1223} = -f(\boldsymbol{x}_{(2,3,1,4,5)}) + f(\boldsymbol{x}_{(2,5,1,4,3)}) + f(\boldsymbol{x}_{(4,3,1,2,5)}) - f(\boldsymbol{x}_{(4,5,1,2,3)})$
  $+ \beta_{1423} + \beta_{2342}$
- $E(\boldsymbol{x}_{(1,3,2,4,5)}) - E(\boldsymbol{x}_{(1,3,5,4,2)}) - E(\boldsymbol{x}_{(4,3,2,1,5)}) + E(\boldsymbol{x}_{(4,3,5,1,2)})$:
  $\beta_{1132} = -f(\boldsymbol{x}_{(1,3,2,4,5)}) + f(\boldsymbol{x}_{(1,3,5,4,2)}) + f(\boldsymbol{x}_{(4,3,2,1,5)}) - f(\boldsymbol{x}_{(4,3,5,1,2)})$
  $+ \beta_{1432} + \beta_{3241}$
- $E(\boldsymbol{x}_{(1,3,4,2,5)}) - E(\boldsymbol{x}_{(1,3,4,5,2)}) - E(\boldsymbol{x}_{(4,3,1,2,5)}) + E(\boldsymbol{x}_{(4,3,1,5,2)})$:
  $\beta_{1142} = -f(\boldsymbol{x}_{(1,3,4,2,5)}) + f(\boldsymbol{x}_{(1,3,4,5,2)}) + f(\boldsymbol{x}_{(4,3,1,2,5)}) - f(\boldsymbol{x}_{(4,3,1,5,2)})$
  $+ \beta_{1442} - \beta_{3442}$
- $E(\boldsymbol{x}_{(2,3,1,5,4)}) - E(\boldsymbol{x}_{(2,3,4,5,1)}) - E(\boldsymbol{x}_{(5,3,1,2,4)}) + E(\boldsymbol{x}_{(5,3,4,2,1)})$:
  $\beta_{1231} = -f(\boldsymbol{x}_{(2,3,1,5,4)}) + f(\boldsymbol{x}_{(2,3,4,5,1)}) + f(\boldsymbol{x}_{(5,3,1,2,4)}) - f(\boldsymbol{x}_{(5,3,4,2,1)})$
  $+ \beta_{1234} - \beta_{3442}$
- $E(\boldsymbol{x}_{(2,3,5,1,4)}) - E(\boldsymbol{x}_{(2,3,5,4,1)}) - E(\boldsymbol{x}_{(5,3,2,1,4)}) + E(\boldsymbol{x}_{(5,3,2,4,1)})$:
  $\beta_{1241} = -f(\boldsymbol{x}_{(2,3,5,1,4)}) + f(\boldsymbol{x}_{(2,3,5,4,1)}) + f(\boldsymbol{x}_{(5,3,2,1,4)}) - f(\boldsymbol{x}_{(5,3,2,4,1)})$
  $+ \beta_{1244} + \beta_{3241}$
- $E(\boldsymbol{x}_{(3,1,2,5,4)}) - E(\boldsymbol{x}_{(3,4,2,5,1)}) - E(\boldsymbol{x}_{(5,1,2,3,4)}) + E(\boldsymbol{x}_{(5,4,2,3,1)})$:
  $\beta_{1321} = -f(\boldsymbol{x}_{(3,1,2,5,4)}) + f(\boldsymbol{x}_{(3,4,2,5,1)}) + f(\boldsymbol{x}_{(5,1,2,3,4)}) - f(\boldsymbol{x}_{(5,4,2,3,1)})$
  $+ \beta_{2143} - \beta_{2443}$

- $E(\boldsymbol{x}_{(3,1,2,4,5)}) - E(\boldsymbol{x}_{(3,5,2,4,1)}) - E(\boldsymbol{x}_{(4,1,2,3,5)}) + E(\boldsymbol{x}_{(4,5,2,3,1)})$:
  $\beta_{1421} = f(\boldsymbol{x}_{(3,1,2,4,5)}) - f(\boldsymbol{x}_{(3,5,2,4,1)}) - f(\boldsymbol{x}_{(4,1,2,3,5)}) + f(\boldsymbol{x}_{(4,5,2,3,1)})$
  $+ \beta_{1321} - \beta_{2143} + \beta_{2144}$
- $E(\boldsymbol{x}_{(1,3,2,4,5)}) - E(\boldsymbol{x}_{(1,5,2,4,3)}) - E(\boldsymbol{x}_{(4,3,2,1,5)}) + E(\boldsymbol{x}_{(4,5,2,1,3)})$:
  $\beta_{1123} = -f(\boldsymbol{x}_{(1,3,2,4,5)}) + f(\boldsymbol{x}_{(1,5,2,4,3)}) + f(\boldsymbol{x}_{(4,3,2,1,5)}) - f(\boldsymbol{x}_{(4,5,2,1,3)})$
  $+ \beta_{1423} + \beta_{2341}$
- $E(\boldsymbol{x}_{(1,4,2,3,5)}) - E(\boldsymbol{x}_{(1,5,2,3,4)}) - E(\boldsymbol{x}_{(3,4,2,1,5)}) + E(\boldsymbol{x}_{(3,5,2,1,4)})$:
  $\beta_{1124} = -f(\boldsymbol{x}_{(1,4,2,3,5)}) + f(\boldsymbol{x}_{(1,5,2,3,4)}) + f(\boldsymbol{x}_{(3,4,2,1,5)}) - f(\boldsymbol{x}_{(3,5,2,1,4)})$
  $+ \beta_{2441} - \beta_{2443}$
- $E(\boldsymbol{x}_{(1,2,3,4,5)}) - E(\boldsymbol{x}_{(1,5,3,4,2)}) - E(\boldsymbol{x}_{(4,2,3,1,5)}) + E(\boldsymbol{x}_{(4,5,3,1,2)})$:
  $\beta_{1122} = -f(\boldsymbol{x}_{(1,2,3,4,5)}) + f(\boldsymbol{x}_{(1,5,3,4,2)}) + f(\boldsymbol{x}_{(4,2,3,1,5)}) - f(\boldsymbol{x}_{(4,5,3,1,2)})$
  $+ \beta_{1422} + \beta_{2241} - \beta_{2244}$
- $E(\boldsymbol{x}_{(2,1,3,5,4)}) - E(\boldsymbol{x}_{(2,4,3,5,1)}) - E(\boldsymbol{x}_{(5,1,3,2,4)}) + E(\boldsymbol{x}_{(5,4,3,2,1)})$:
  $\beta_{1221} = -f(\boldsymbol{x}_{(2,1,3,5,4)}) + f(\boldsymbol{x}_{(2,4,3,5,1)}) + f(\boldsymbol{x}_{(5,1,3,2,4)}) - f(\boldsymbol{x}_{(5,4,3,2,1)})$
  $+ \beta_{1224} + \beta_{2142} - \beta_{2442}$

This completes the $n = 5$ base case. □

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