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Vissing, Steffen; Krenk, Steen

Published in:
Proceedings of "Dynamics of Structures"

Publication date:
1994

Document Version
Accepted author manuscript, peer reviewed version

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Steffen Vissing & Steen Krenk
Department of Building Technology and Structural Engineering
Aalborg University, DK–9000 Aalborg, Denmark

Eigenvalue problems arise in various engineering problems as for instance structural dynamics and stability analysis as well as in heat tranfer analysis. The solution of linear or quadratic eigenvalue problems is therefore an essentiel part of the analysis. Quadratic eigenvalue problems are often solved after reduction to an indefinite linear eigenvalue problem of double size and thus the fundamental problem is the generalized indefinite linear eigenvalue problem. In this paper Lanczos algorithms for the symmetric problem is developed, but the algorithms may be generalized to nonsymmetric problems by introducing biorthogonal sets of left and right vectors. The generalized linear eigenvalue problem is of the form

$$(\mathbf{A} + \lambda \mathbf{B})\mathbf{w} = \mathbf{0} \tag{1}$$

where $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ real symmetric matrices which may be indefinite, λ is an eigenvalue and $\mathbf{w}$ is the corresponding n-dimensional eigenvector.

A general strategy for the generalized linear indefinite eigenvalue problem (1) consists of setting up a Krylov sequence representing the required eigenvectors and defining a proper orthogonality condition and vector orientation. A Krylov subspace suitable for approximate representation of the eigenvectors corresponding to the numerically smallest or largest eigenvalues can be generated by iterating with the matrix $\mathbf{A}^{-1}\mathbf{B}$ or the matrix $\mathbf{B}^{-1}\mathbf{A}$. The eigenvalue problem may typically arise from a finite element formulation in which the discretization causes the largest errors for the numerically largest eigenvalues. Further more only a small fraction of the eigenvectors corresponding to the numerically smallest eigenvalues may approximate a response to sufficient accuracy. Therefore the following Krylov sequence is used

$$[\mathbf{w}_0,\ \mathbf{A}^{-1}\mathbf{B}\mathbf{w}_0,\ (\mathbf{A}^{-1}\mathbf{B})^2\mathbf{w}_0, \ldots,\ (\mathbf{A}^{-1}\mathbf{B})^{m-1}\mathbf{w}_0] \tag{2}$$

However, an efficient use of the Krylov sequence requires some kind of condition on the base vectors, e.g. a suitable orthogonality condition. Natural choices for orthogonalizing the Krylov vectors are either of the two system matrices $\mathbf{A}$ and $\mathbf{B}$ or a simple matrix like the identity matrix. This paper presents algorithms in which the vectors are orthogonalized with respect to either of the system matrices. Hereby the vectors need in principle only be orthogonalized to the previous two vectors in order to achieve orthogonality and only two parameters in the recurrence formulae are needed. However, in finite precision the numerical error increase in each iteration whereby non-orthogonal vectors may be generated. In order to prevent the loss of orthogonality a simple reorthogonalisation procedure is applied. Using either $\mathbf{A}$ or $\mathbf{B}$ orthogonality conditions the n-dimensional eigenvalue

problem is reduced to an m-dimensional eigenvalue problem with a symmetric tridiagonal matrix $\mathbf{T}$ and, on account of indefinite system matrices, a diagonal matrix $\mathbf{E}$ containing ± 1 in the diagonal. Alternative, orthogonality conditions with respect to a matrix without direct relation to the problem e.g. the identity matrix may lead to fewer matrix operations but the tridiagonal matrix $\mathbf{T}$ is then replaced by an upper Hessenberg matrix.

Linear Lanczos algorithms for the generalized symmetric but indefinite eigenvalue problem with $\mathbf{A}$ and $\mathbf{B}$ orthogonality conditions are given in pseudo code as Algorithm 1 and Algorithm 2. In the algorithms α_j, $j = 1, 2, \cdots, n$ are the diagonal elements of $\mathbf{T}$, β_j, $j = 2, 3, \cdots, n$ are the sub- and superdiagonal elements of $\mathbf{T}$ and e_j, $j = 1, 2, \cdots, n$ are the diagonal elements of $\mathbf{E}$. The efficiency of the algorithms depends on the two matrix operations $\mathbf{B}\mathbf{q}_j$ and $\mathbf{A}^{-1}\mathbf{p}_j$ which is directly related to the bandwidth of the system matrices $\mathbf{A}$ and $\mathbf{B}$.

ALGORITHM 1: LINEAR LANCZOS WITH $\mathbf{A}$ ORTHOGONALITY

Start vector $\mathbf{p}_1$
Initialize $\mathbf{p}_0 := \mathbf{0}$
for $(j := 1$ to $m)$
  $\mathbf{q}_j := \mathbf{A}^{-1}\mathbf{p}_j$
  $d := \mathbf{q}_j^T \mathbf{p}_j$
  $e_j := \text{sign}(d)$
  $\beta_j := |d|^{1/2}$
  $\mathbf{q}_j := \mathbf{q}_j / \beta_j$
  $\mathbf{p}_j := \mathbf{p}_j / \beta_j$
  $\mathbf{p}_{j+1} := \mathbf{B}\mathbf{q}_j$
  $\alpha_j := \mathbf{q}_j^T \mathbf{p}_{j+1}$
  $\mathbf{p}_{j+1} := e_j \mathbf{p}_{j+1} - \alpha_j \mathbf{p}_j - \beta_j \mathbf{p}_{j-1}$
  $\mathbf{p}_{j+1} := \text{ortho}(\mathbf{p}_k, \mathbf{q}_k),\ k = 1, 2, \cdots$

ALGORITHM 2: LINEAR LANCZOS WITH $\mathbf{B}$ ORTHOGONALITY

Start vector $\mathbf{q}_1$
Initialize $\mathbf{q}_0 := \mathbf{0}$
for $(j := 1$ to $m)$
  $\mathbf{p}_j := \mathbf{B}\mathbf{q}_j$
  $d := \mathbf{q}_j^T \mathbf{p}_j$
  $e_j := \text{sign}(d)$
  $\beta_j := |d|^{1/2}$
  $\mathbf{q}_j := \mathbf{q}_j / \beta_j$
  $\mathbf{p}_j := \mathbf{p}_j / \beta_j$
  $\mathbf{q}_{j+1} := \mathbf{A}^{-1}\mathbf{p}_j$
  $\alpha_j := \mathbf{q}_{j+1}^T \mathbf{p}_j$
  $\mathbf{q}_{j+1} := e_j \mathbf{q}_{j+1} - \alpha_j \mathbf{q}_j - \beta_j \mathbf{q}_{j-1}$
  $\mathbf{q}_{j+1} := \text{ortho}(\mathbf{q}_k, \mathbf{p}_k),\ k = 1, 2, \cdots$

A full reorthogonalization procedure is shown in Algorithm 1 and 2 in which all the $\mathbf{p}_j$ vectors are stored. A more efficient reorthogonalization algorithm may be applied.

After the generalized Lanczos reduction a reduced m-dimensional eigenvalue problem is obtained in the form

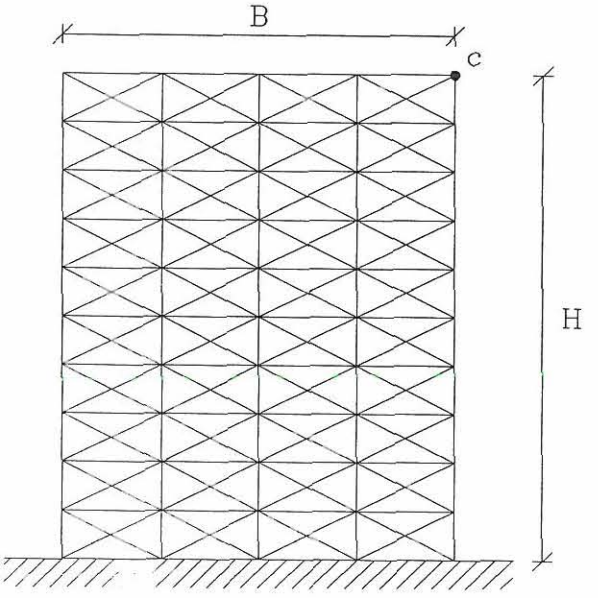
$$(\mathbf{E} + \lambda \mathbf{T})\mathbf{y} = \mathbf{0} \tag{3}$$

where $\mathbf{y}$ is an m-dimensional eigenvector corresponding to the eigenvalue λ. The eigenvalues of (3) approximate the numerically smallest eigenvalues of (1) while the corresponding eigenvectors are obtained by $\mathbf{w} = \mathbf{Q}\mathbf{E}\mathbf{y}$ where $\mathbf{Q} = [\mathbf{q}_1, \mathbf{q}_2, \cdots, \mathbf{q}_m]$ is a matrix storing the orthogonalized Krylov vectors known as Lanczos vectors. The eigenvalues of (3) may be

obtained by a generalization of a symmetric solver for the standard eigenvalue problem with a symmetric tridiagonal matrix e.g. the QR method or by an unsymmetric solver for the standard eigenvalue problem with the unsymmetric tridiagonal matrix **ET**. The unsymmetric solver will fill out the upper part of the matrix **T**. Additional work is needed on the stability of the symmetric solver.

Example

The test example shown in Figure 1 is a 10 stores building with a vertical viscous damper at the top. The physical properties are given in the figure. The building is modelled with 174 elements and has 150 free degrees of freedom. The damping matrix only contains one nonzero element corresponding to the damping coefficient c of the damper.



A	I	B	H	ρ	E
1	1/12	20	25	0.1	1

Figure 1: 10 stores building with a damper

In table 1 and 2 the eigenvalues for 10, 20 and 40 iterations with the algorithms for **A** and **B** orthogonality conditions are compared with the 20 smallest eigenvalues of the exact solution with $c = 0.5$. The eigenvalues for $c = 0$, $c = 0.5$, $c = 0.1$ are shown in Figure 2.

This example show no notable difference in the convergence rate for **A** and **B** orthogonality. Approximately half of the eigenvalues have converged.

Exact	10 iterations	20 iterations	40 iterations
$0.0014 \pm 0.0383i$	0.0014 ± 0.0383	$0.0014 \pm 0.0383i$	$0.0014 \pm 0.0383i$
$0.0049 \pm 0.0844i$	0.0049 ± 0.0844	$0.0049 \pm 0.0844i$	$0.0049 \pm 0.0844i$
$0.0108 \pm 0.1159i$	0.0108 ± 0.1160	$0.0108 \pm 0.1159i$	$0.0108 \pm 0.1159i$
$0.0040 \pm 0.2159i$	-0.1525	$0.0040 \pm 0.2159i$	$0.0040 \pm 0.2159i$
$0.0003 \pm 0.2311i$	0.0087 ± 0.2542	$0.0003 \pm 0.2312i$	$0.0003 \pm 0.2311i$
$0.0010 \pm 0.2610i$	0.9020	$-0.0059 \pm 0.2656i$	$0.0010 \pm 0.2610i$
$0.0277 \pm 0.2909i$	–	$0.0005 \pm 0.3364i$	$0.0277 \pm 0.2909i$
$0.0008 \pm 0.3249i$	–	$0.0084 \pm 0.4547i$	$0.0008 \pm 0.3249i$
$0.0022 \pm 0.3457i$	–	$0.0370 \pm 0.7749i$	$0.0022 \pm 0.3457i$
$0.0047 \pm 0.3644i$	–	$0.2590 \pm 1.9052i$	$0.0047 \pm 0.3646i$

Table 1: Eigenvalues of a 10 stores building, **A** orthogonality, $c = 0.5$.

Exact	10 iterations	20 iterations	40 iterations
$0.0014 \pm 0.0383i$	0.0014 ± 0.0383	$0.0014 \pm 0.0383i$	$0.0014 \pm 0.0383i$
$0.0049 \pm 0.0844i$	0.0049 ± 0.0844	$0.0049 \pm 0.0844i$	$0.0049 \pm 0.0844i$
$0.0108 \pm 0.1159i$	0.0108 ± 0.1159	$0.0108 \pm 0.1159i$	$0.0108 \pm 0.1159i$
$0.0040 \pm 0.2159i$	0.0102 ± 0.2398	$0.0040 \pm 0.2159i$	$0.0040 \pm 0.2159i$
$0.0003 \pm 0.2311i$	0.3499 ± 0.3230	$0.0003 \pm 0.2311i$	$0.0003 \pm 0.2311i$
$0.0010 \pm 0.2610i$	–	$0.0001 \pm 0.2631i$	$0.0010 \pm 0.2610i$
$0.0277 \pm 0.2909i$	–	$0.0034 \pm 0.3300i$	$0.0277 \pm 0.2909i$
$0.0008 \pm 0.3249i$	–	$0.0134 \pm 0.4285i$	$0.0008 \pm 0.3249i$
$0.0022 \pm 0.3457i$	–	$0.0624 \pm 0.7202i$	$0.0022 \pm 0.3457i$
$0.0047 \pm 0.3644i$	–	$0.7684 \pm 2.0851i$	$0.0048 \pm 0.3645i$

Table 2: Eigenvalues of a 10 stores building, **B** orthogonality, $c = 0.5$.

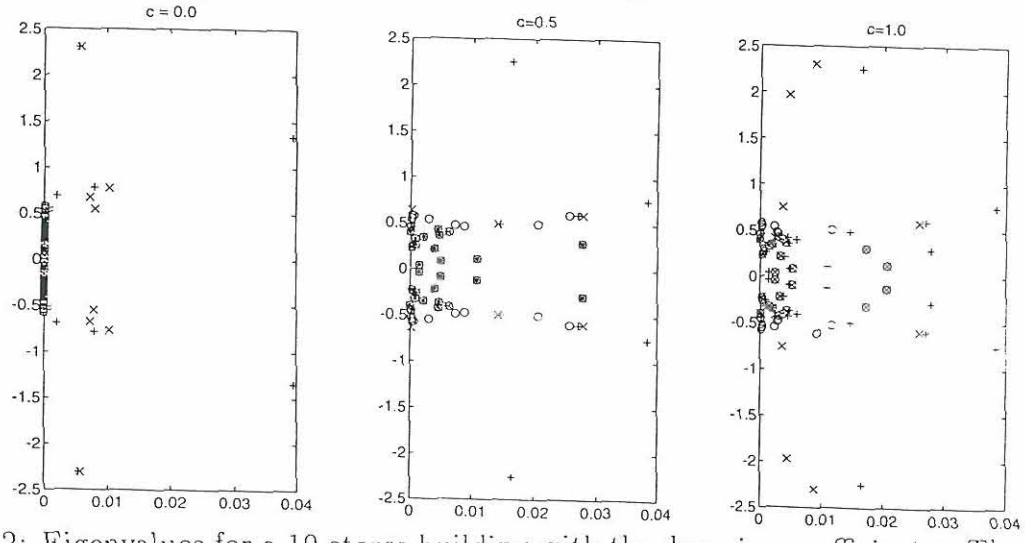


Figure 2: Eigenvalues for a 10 stores building with the damping coefficient c. The symbols o, + and × stands for, respectively, the exact solution, **A** orthogonality and **B** orthogonality.