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Nielsen, Søren R.K.; Andersen, Lars

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Vehicle moving on a continuously supported beam with irregular surface

L. Andersen & S.R.K. Nielsen
Department of Building Technology and Structural Engineering, Aalborg University, Aalborg, Denmark

R. Iwankiewicz
School of Mechanical Engineering, University of the Witwatersrand, Johannesburg, South Africa

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ABSTRACT: In this paper a stochastic analysis is performed for single-degree-of-freedom vehicle moving uniformly along an infinite Bernoulli-Euler beam with random surface irregularities and supported by a Kelvin foundation. Both the beam and the foundation are assumed to be homogeneous, and all the material parameters of the system are assumed to be deterministic. Initially the equations of motion for the vehicle and beam are formulated in a moving co-ordinate system following the vehicle, and the frequency response functions for the displacement of the vehicle mass and beam are determined. Subsequently the surface irregularities are modelled as a random process. The displacement variance of the vehicle mass as well as the displacement variance of the beam under the oscillator are determined in terms of the autospectrum of the surface irregularities.

1 INTRODUCTION

A system consisting of a beam supported by a Kelvin foundation has often been used to represent a road, runway or railway track. Both the load from moving vehicles and the parameters of the beam and support may be considered stochastic variables, and different aspects of stochastic analysis have been considered in a number of papers.

Sobczyk (1970) performed an analysis of a Bernoulli-Euler beam on a random Kelvin foundation due to random excitation. Later, Frýba *et al.* (1993) examined the behaviour of an infinitely long Bernoulli-Euler beam on a Kelvin foundation with randomly varying parameters along the beam. Chang and Liu (1996) performed an analysis of a single degree of freedom (SDOF) vehicle moving along a finite beam with random surface on a nonlinear deterministic Kelvin foundation using finite elements and Monte Carlo simulation. Usually, when a vehicle moving on a beam structure is considered, the vehicle-beam interaction is disregarded. Examples include the work by Lombaert *et al.* (2000) and Mertrikine and Vrouwenvelder (2000).

In reality, parameters of the support may be frequency dependent, (Dieterman and Metrikine 1996; Metrikine and Popp 2000). In the present paper the parameters of the foundation are, nevertheless, assumed to be frequency independent and deterministic. However, the surface of the beam is irregular, described by a weakly homogeneous random process. An analytical method is presented for the analysis of an SDOF vehicle moving uniformly along the beam. The problem is formulated in a local co-ordinate frame following the vehicle, and interaction between the vehicle and the beam is taken into account. No Monte Carlo simulation is necessary since the system is assumed to be linear. A numerical example is given for surface irregularities with an autospectrum, which is often used to describe irregularities of a road surface. For certain configurations of the system, the interaction is shown to have a significant influence on the dynamic amplification of the displacement response of both the vehicle mass and the beam.

2 THEORY

A single-degree-of-freedom (SDOF) vehicle moves at the constant velocity v along a homogeneous Bernoulli-Euler beam supported by a homogeneous Kelvin foundation, thus having the along beam position $x = vt$ at the time t, see Fig. 1. The vehicle mass, spring stiffness, and viscous damping coefficient are denoted m_0, k_0, and c_0, respectively, whereas the beam has the bending stiffness EI and the mass per unit length μ. The Kelvin foundation has the stiffness κ and the viscous damping γ per unit length. All material properties of the system are assumed to be deterministic.

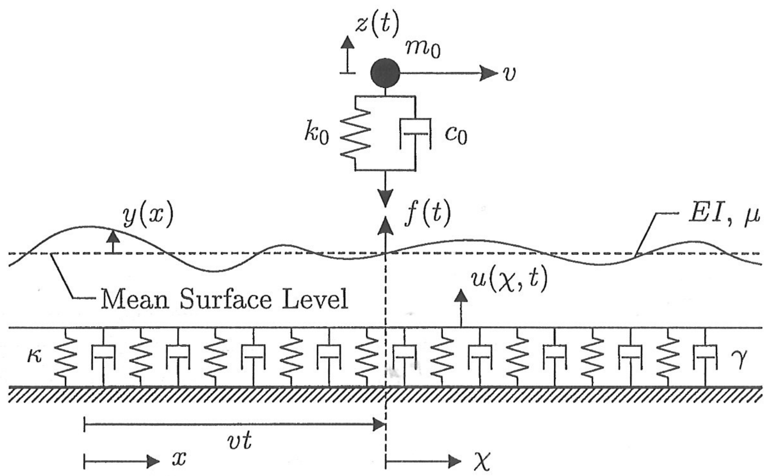

Figure 1: SDOF vehicle moving along a Bernoulli-Euler beam with irregular surface and supported by a Kelvin foundation.

The vehicle is in constant contact with the beam surface, which has the irregularities $y = y(x) = y(vt)$ measured from the mean level of the surface perpendicular to the beam axis. In a local moving co-ordinate system defined by the transformation $\chi = x - vt$, the equation of motion for the vehicle due to the surface irregularities becomes

$$m_0 \frac{d^2 z}{dt^2} + c_0 \left(\frac{dz}{dt} - \dot{u}(0,t) - v y'(vt) \right) + k_0 \left(z - u(0,t) - y(vt) \right) = 0, \tag{1}$$

where the prime denotes differentiation with respect to the argument. The vertical displacements of the vehicle mass and beam relative to the respective positions in the state of static equilibrium are denoted $z = z(t)$ and $u = u(\chi, t)$,respectively. Furthermore, the notation $\dot{u} = \left.\frac{\partial u}{\partial t}\right|_{\chi}$ has been introduced.

In the local co-ordinate frame, the equation of motion for the beam may be written

$$EI \frac{\partial^4 u}{\partial \chi^4} + \mu \left(\ddot{u} - 2v \frac{\partial \dot{u}}{\partial \chi} + v^2 \frac{\partial^2 u}{\partial \chi^2} \right) + \gamma \left(\dot{u} - v \frac{\partial u}{\partial \chi} \right) + \kappa u = f(t) \delta(\chi). \tag{2}$$

Here, $\ddot{u} = \left.\frac{\partial^2 u}{\partial t^2}\right|_{\chi}$ is the local acceleration whereas $f(t)$ is the moving load and $\delta(\chi)$ is the delta function. The force on the beam originates from the SDOF vehicle, i.e. $f(t) = -m_0 \frac{d^2 z}{dt^2} - m_0 g$, g being the gravitational acceleration. In the following analysis only the stochastic response will be considered, hence $f(t) = -m_0 \frac{d^2 z}{dt^2}$ will be used.

2.1 *Frequency Response Functions for Vehicle and Beam Displacements*

In fixed coordinates, harmonically varying surface irregularities with wavelength L are given by

$$y(x) = Ye^{ikx}, \tag{3}$$

where Y is the amplitude, $i = \sqrt{-1}$ is the imaginary unit and $k = \frac{2\pi}{L}$ is the wavenumber. By insertion of Eq. (3) with $x = vt$ into Eq. (1), the equation of motion for the vehicle becomes

$$m_0 \frac{d^2 z}{dt^2} + c_0 \left(\frac{dz}{dt} - \dot{u}(0,t) - i\omega Y e^{i\omega t} \right) + k_0 \left(z - u(0,t) - Ye^{i\omega t} \right) = 0, \tag{4}$$

in which $\omega = kv$ denotes the apparent circular frequency of the surface irregularities as seen from the vehicle in the local coordinate system.

Solutions to Eqs. (4) and (2) with $f(t) = -m_0 \frac{d^2 z}{dt^2}$ are on the form,

$$z(t) = Z(\omega)e^{i\omega t} \ , \ u_j(\chi, t) = U_j(\chi, \omega)e^{i\omega t} \ , \ j = 1,2,3,4, \tag{5}$$

where $Z(\omega)$ is the vehicle amplitude. $U_j(\chi,\omega) = m_0\omega^2 Z(\omega)\tilde{U}_j(\omega)e^{iK_j\chi}$ are the amplitude functions for the four bending wave components in the beam. Here $\tilde{U}_j(\omega)$ are the amplitudes at $\chi = 0$ for harmonic excitation with unit amplitude, i.e. $f(t) = e^{i\omega t}$, and the wavenumbers $K_j = K_j(\omega)$ correspond to the roots of the characteristic polynomial

$$K_j^4 - \frac{\mu v^2}{EI} K_j^2 - \frac{i\gamma v - 2\mu\omega v}{EI} K_j + \frac{\kappa - \mu\omega^2 + i\gamma\omega}{EI} = 0. \tag{6}$$

For $v = \gamma = 0$ a *cut-off frequency*, $\omega_c = \sqrt{\kappa/\mu}$, exists. When $\omega < \omega_c$, no travelling waves without attenuation will propagate. Furthermore, only two of the solutions to equation (6) are physically valid on either side of the load, (Andersen, Nielsen, and Kirkegaard 2001). The subscripts $j = 1,2$ will be used for the components existing at $\chi \leq 0$ (i.e. *behind* and *under* the load), whereas the components existing at $\chi > 0$ (i.e. *in front* of the load) have subscripts $j = 3,4$.

At $\chi = 0$ the displacement, rotation and bending moment of the beam must be continuous functions of χ. A unit amplitude load implies a jump of 1 in the shear force at the point where the load is applied. Hence, with the positive directions defined in figure 1 the amplitudes $\tilde{U}_j(\omega)$ are obtained by the following system of equations,

$$\begin{bmatrix} 1 & 1 & -1 & -1 \\ iK_1 & iK_2 & -iK_3 & -iK_4 \\ (iK_1)^2 & (iK_2)^2 & -(iK_3)^2 & -(iK_4)^2 \\ (iK_1)^3 & (iK_2)^3 & -(iK_3)^3 & -(iK_4)^3 \end{bmatrix} \begin{bmatrix} \tilde{U}_1 \\ \tilde{U}_2 \\ \tilde{U}_3 \\ \tilde{U}_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1/EI \end{bmatrix}. \tag{7}$$

From Eqs. (4) and (5) the amplitude of the SDOF mass, $Z(\omega)$, is calculated as

$$Z(\omega) = H_{ZY}(\omega)Y \ , \ H_{ZY}(\omega) = (i\omega c_0 + k_0)/D(\omega), \tag{8}$$

where the denominator, $D(\omega)$, is given by

$$D(\omega) = \left(-\omega^2 m_0 + i\omega c_0 + k_0\right) - \left(i\omega^3 m_0 c_0 + \omega^2 m_0 k_0\right) \sum_{j=1}^{2} \tilde{U}_j(\omega). \tag{9}$$

Analogously, the beam displacement may be found as

$$U(\chi,\omega) = H_{UY}(\chi,\omega)Y \ , \ H_{UY}(\chi,\omega) = H_{UZ}(\chi,\omega)H_{ZY}(\omega). \tag{10}$$

Here the frequency response function $H_{UZ}(\chi, \omega)$ is given by

$$H_{UZ}(\chi, \omega) = m_0 \omega^2 \sum_{j=j_1}^{j_2} \tilde{U}_j(\omega) e^{iK_j \chi} , \quad \begin{cases} \{j_1, j_2\} = \{1, 2\} \text{ for } \chi \leq 0 \\ \{j_1, j_2\} = \{3, 4\} \text{ for } \chi > 0 \end{cases} . \tag{11}$$

2.2 *Random Surface Irregularities*

In practice the surface irregularities are not harmonically varying, but will instead be described by a weakly homogeneous stochastic process. Given the one-sided autospectral density $S_Y(\omega)$ for the surface irregularities, the one-sided autospectral density $S_Z(\omega)$ for the displacement of the SDOF vehicle can be found. Also, the one-sided cross-spectral density $S_{UU}(\chi_1, \chi_2, \omega)$ for the beam displacement at two points χ_1 and χ_2 on the beam axis can be calculated. Thus, see e.g. Lin (1967),

$$S_Z(\omega) = |H_{ZY}(\omega)|^2 \, S_Y(\omega), \tag{12}$$

$$S_{UU}(\chi_1, \chi_2, \omega) = H^*_{UY}(\chi_1, \omega) H_{UY}(\chi_2, \omega) S_Y(\omega), \tag{13}$$

where $H_{ZY}(\omega)$ and $H_{UY}(\chi, \omega)$ are defined previously and $H^*_{UY}(\chi_1, \omega)$ is the complex conjugate of $H_{UY}(\chi_1, \omega)$. The principle of superposition is valid, because the governing equations are all linear.

From the Wiener-Khintchine relation the auto-covariance function $\kappa_{ZZ}(\tau)$ for the vehicle displacement and the cross-covariance function $\kappa_{UU}(\chi_1, \chi_2, \tau)$ for the beam displacement may be expressed as

$$\kappa_{ZZ}(\tau) = \int_0^\infty \cos(\omega\tau) S_Z(\omega) d\omega, \tag{14}$$

$$\kappa_{UU}(\chi_1, \chi_2, \tau) = 2 \int_0^\infty \left(\cos(\omega\tau) S^{\Re}_{UU} - \sin(\omega\tau) S^{\Im}_{UU} \right) d\omega, \tag{15}$$

respectively, where $S^{\Re}_{UU}$ and $S^{\Im}_{UU}$ are the real and imaginary parts of $S_{UU}(\chi_1, \chi_2, \omega)$, respectively.

2.3 *Non-dimensional Parameter Description*

The dynamic response of the vehicle and beam is self-induced, i.e. no external load acts on the system. Therefore, only the relative size of the system parameters is of importance. Dimension analysis and a further study of the system equations would imply that the following non-dimensional identities govern the problem,

$$\Omega_0 = \frac{\omega_0}{\omega_r}, \; \Omega_c = \frac{\omega_c}{\omega_0}, \; M_0 = \frac{m_0}{\mu L_c}, \; \nu = \frac{v}{v_c}, \tag{16}$$

$$\zeta_0 = \frac{c_0}{2\sqrt{m_0 k_0}}, \; \zeta_c = \frac{\gamma}{2\sqrt{\mu\kappa}}, \tag{17}$$

Here $\omega_0 = \sqrt{k_0/m_0}$ is the circular eigenfrequency of the SDOF vehicle and $\omega_r = v k_r$ is a characteristic circular frequency of the surface roughness, k_r being the corresponding characteristic wavenumber. L_c is a characteristic wavelength for bending waves in the beam, and v_c is the corresponding characteristic phase velocity. They are defined as

$$L_c = \frac{2\pi v_c}{\omega_c}, \; v_c = \sqrt[4]{\frac{EI\kappa}{\mu^2}}, \tag{18}$$

respectively. It should be noticed that ν serves as a kind of Mach number putting the vehicle velocity relative to the velocity of the bending waves.

3 NUMERICAL EXAMPLES

Irregularities of a road surface are often modelled as a weakly homogeneous process $\{Y(x)\ ,\ x \in R\}$ having the one-sided autospectral density

$$S_Y(k) = \begin{cases} 0 & , \quad k \notin [k_1, k_2] \\ \sigma_Y^2 \frac{k_1 k_2}{k_2 - k_1} \frac{1}{k^2} & , \quad k \in [k_1, k_2] \end{cases} , \tag{19}$$

where σ_Y^2 is the variance of the surface irregularities. The spectrum has a slope of 1:2 in double-logarithmic mapping and is valid for wavelengths between $L_2 = 0.1$ m and $L_1 = 10$ m corresponding to $k_1 = \frac{2\pi}{10}\ \mathrm{m}^{-1}$ and $k_2 = \frac{2\pi}{0.1}\ \mathrm{m}^{-1}$. The characteristic wavenumber for the surface roughness is chosen as $k_r = \sqrt{k_1 k_2}$, which lies in the middle of the $[k_1, k_2]$ interval on a logarithmic scale. A wavenumber domain representation is obtained by substituting $k = \omega / v$ into Eq. (19),

$$S_Y(\omega) = \begin{cases} 0 & , \quad \omega \notin [\omega_1, \omega_2] \\ \sigma_Y^2 \frac{\omega_1 \omega_2}{\omega_2 - \omega_1} \frac{v}{\omega^2} & , \quad \omega \in [\omega_1, \omega_2] \end{cases} , \tag{20}$$

where $\omega_1 = k_1 v$ and $\omega_2 = k_2 v$. Subsequently $S_Z(\omega)$ is found by insertion into Eq. (12).

In the following an analysis will be carried out for the variance of the SDOF mass displacement response, $\sigma_Z^2 = \kappa_{ZZ}(0)$, and the variance of the beam displacement response directly under the vehicle, $\sigma_U^2 = \kappa_{UU}(0,0,0)$. Due to the linearity of the problem σ_Z^2 and σ_U^2 are proportional to σ_Y^2. Hence, it is convenient to describe the respective variances by the *dynamic variance amplification factors*,

$$s_Z = \frac{\sigma_Z^2}{\sigma_Y^2}, \quad s_U = \frac{\sigma_U^2}{\sigma_Y^2}. \tag{21}$$

3.1 *Influence of Track Stiffness*

A beam with the mass per unit length $\mu = 100$ kg/m is considered. The vehicle has the circular eigenfrequency $\omega_0 = 2\pi\ \mathrm{s}^{-1}$ and the damping ratio $\zeta_0 = 1$, which are assumed to be typical values. The analysis is performed for the point mass $m_0 = 1000$ kg, and the characteristic wavelength of bending waves and the circular cut-off frequency are both varied. The values $L_c = 1, 10, 100$ m and $\Omega_c = 0.1, 1, 10$ are used. It should be noticed that with the definitions (16) to (18) an increase of Ω_c will imply an increase in both κ and EI for constant L_c, whereas an increase in L_c only leads to an increase in EI for constant Ω_c.

Figure 2 shows the dynamic amplification factors s_Z and s_U as functions of the vehicle velocity in the $v \in [1\ \mathrm{m/s}, 100\ \mathrm{m/s}]$ interval. Further to the results obtained using the indicated theory (*the continuous curves*), reference results (*the dashed curves*) are shown for a situation where the interaction between the beam and the vehicle is not taken into consideration, i.e. the vehicle does not feel the displacements of the beam, but only the surface roughness. In this case the denominator $D(\omega) = (-\omega^2 m_0 + i\omega c_0 + k_0)$ is used instead of the original denominator (9).

In the reference case with no interaction the response of the point mass decreases for increasing values of v beyond 10 m/s. The reason is that $\Omega_0 = 1$ when $v = 1$ m/s and ω_0 move to the bottom of the frequency range for the autospectra when $v = 10$ m/s. For velocities of more than 10 m/s, ω_0 lies outside the frequency range. When interaction of the vehicle and beam is taken into account the response of both the SDOF mass and the beam is almost unchanged, if the spring stiffness of the Kelvin foundation and the bending stiffness of the beam is sufficiently high. This also applies to the SDOF mass response when the stiffness of the beam and support is very low. In this case the estimate of the beam response is, however, incorrect in the reference calculation at low velocities.

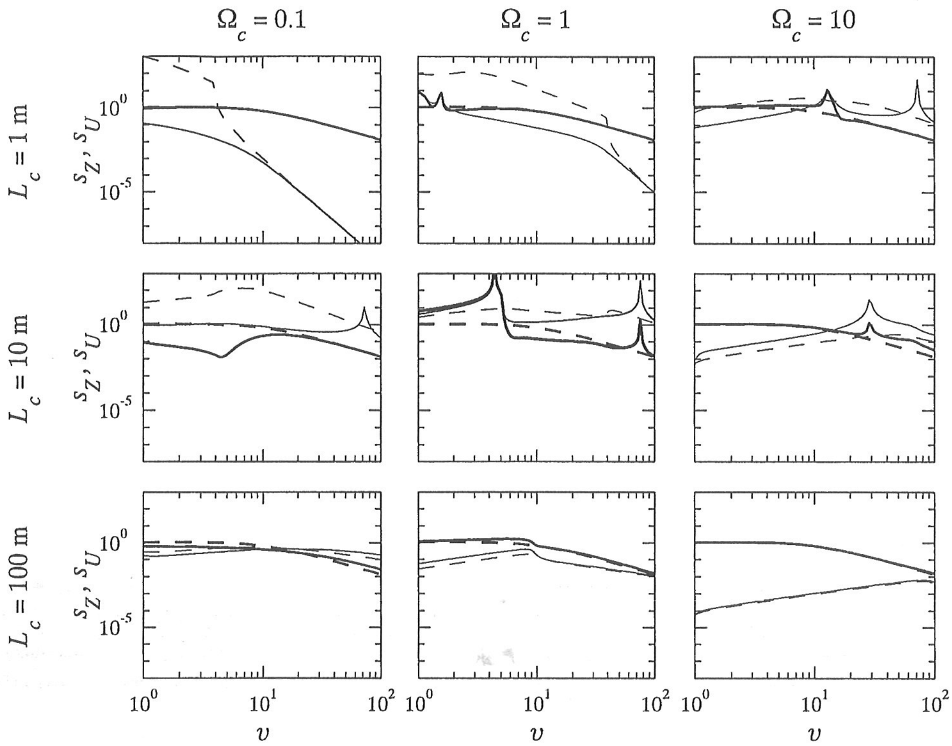

Figure 2: Dynamic amplification of vehicle mass response (—) and beam response (—) at $\chi = 0$. The dashed lines indicate the response when interaction between vehicle and beam is neglected.

For certain combinations of beam configurations and velocities, the beam response, and in some cases also the SDOF mass response, is amplified drastically. Values of both amplification factors as high as 10^3 are found with the most dramatic reinforcement taking place for $\Omega_c = 1$ and $L_c = 10$ m. In a real structure, this may of course be modified by inherent mechanisms not accounted for in the adopted model. Further, real road structures are typically very stiff in proportion to the suspension of the vehicles, i.e. Ω_c and L_c are very large. Analyses show that in such cases the local peaks in the amplification curves will move to higher velocities and the interaction between the vehicle and the beam structure becomes less important, as indicated by Cebon (1993). However, for heavy freight trains or high-speed passenger trains running on soft soil layers or rubber damping devices in metro tunnels, a low stiffness of the beam and support relative to the stiffness of the vehicle may be expected. Hence, the interaction may be of significant importance in the analysis of such problems.

3.2 *Influence of Damping*

Subsequently the system response is analysed for various combinations of the damping ratios ζ_0 and ζ_c. The following values have been used for the remaining system parameters: $m_0 = 1000$ kg, $\omega_0 = 2\pi$ s^{-1}, $\mu = 100$ kg/m, $\Omega_c = 10$, and $L_c = 10$ m. The results of the analysis are illustrated in Fig. 3. As it could be expected, the dynamic amplification of the SDOF mass displacement increases significantly when the damping ratio ζ_0 is decreased. There is almost a one-to-one correspondence between $1/\zeta_0$ and s_Z for velocities close to 10 m/s. Here the eigenfrequency ω_0 lies near the lower bound of the frequency interval for the autospectra where the major part of the variation in the surface irregularities is present.

However, the curves in Fig. 3 indicate that for velocities beyond 10 m/s an increase of the damping ratio of the vehicle is a disadvantage, especially when the damping in the support is relatively low.

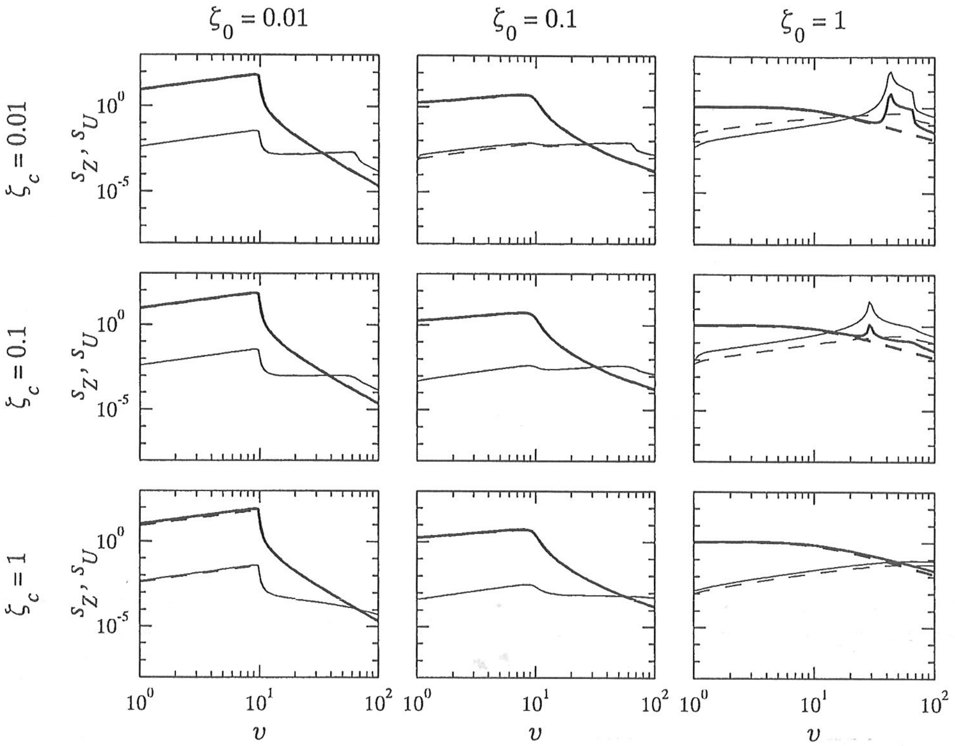

Figure 3: Dynamic amplification of vehicle mass response (—) and beam response (—) at $\chi = 0$. The dashed lines indicate the response when interaction between vehicle and beam is neglected.

Thus, a quite significant peak arises for the combination $\zeta_0 = 1$, $\zeta_c = 0.01$ and $v \approx 40$ m/s. At this velocity the eigenfrequency of the vehicle lies way below the bottom frequency of the autospectrum for the surface roughness. Hence, the origin of the peak must be resonance in the beam. For $\zeta_c = 0.01$ the peak is still present, though it is less pronounced. However, when both the vehicle and the beam are critically damped, the peak vanishes. Here the interaction has no influence on the displacement response of neither the SDOF mass nor the beam for the entire velocity range considered (the dashed and continuous curves coincide).

In practice vehicles are (close to) being critically damped. However, depending on the material of the underlying structure it seems likely that ζ_c will be of the order 0.01 to 0.1. This means that a strong displacement amplification will actually occur at a certain critical velocity of the vehicle.

4 CONCLUSIONS

The response of a single-degree-of-freedom vehicle moving with constant speed along a Bernoulli-Euler beam resting on a viscoelastic foundation of Kelvin type has been investigated. Only the stochastic part of the response due to random surface irregularities has been considered.

The analysis shows that when the beam and the supporting structure are relatively stiff compared to the suspension of the vehicle, the influence of interaction between the beam and vehicle is insignificant. This is in agreement with assumptions made by Lombaert *et al.* (2000). When the beam and the supporting structure are relatively flexible, the vehicle response is not significantly affected by the interaction terms. Their influence on the beam response is, however, essential.

For the given autospectrum of the surface roughness, some coincidences of the parameters of the beam, foundation and of the vehicle velocity may prove critical. For a critically damped vehicle a significant displacement variance amplification of the order 10^3 relative to the surface roughness may take place. Resonance of the vehicle as an isolated system is not the main problem in this case, which indicates that for a vehicle moving on a structure, which may interact with the vehicle, the mechanical design of both vehicle and structure is of great importance. Actually, it has been found in the analysis that a reduction of the damping in the vehicle may prove beneficial at high velocities. Nevertheless, when the damping of the vehicle is reduced, the dynamic amplification of the response is increased drastically at velocities where the eigenfrequency of the SDOF system lies within the frequency range of the autospectrum for the surface roughness.

The analysis shows that the interaction between the vehicle and the supporting structure should not be neglected in the analysis of supporting structures with a low bending stiffness and a low cut-off frequency. Also it should be noticed that neglecting the interaction in the devised analytical method does not make the calculation of the displacement variance amplification any simpler. In any case, the terms of the beam displacement that influence the vehicle mass displacement must be found, if the beam displacement under the vehicle is to be determined.

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