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On Recovery of the Singular Differential Laplace—Bessel Operator from the Fourier–Bessel Transform

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Abstract: This paper is devoted to the problem of the best recovery of a fractional power of the B-elliptic operator of a function on $\mathbb{R}^N_+$ by its Fourier–Bessel transform known approximately on a convex set with the estimate of the difference between Fourier–Bessel transform of the function and its approximation in the metric L_∞. The optimal recovery method has been found. This method does not use the Fourier–Bessel transform values beyond a ball centered at the origin.

Keywords: Bessel operator; extremal problem; optimal recovery; Fourier–Bessel transform

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1. Introduction

The problem of recovering fractional powers of operators was developed in the works [1–3] by G.G. Magaril-Ilyaev, K.Y. Osipenko and E.O. Sivkova. In these papers a Laplace operator was considered and the main instrument was a classical Fourier transform. We further developed these results for a more general Laplace—Bessel singular differential operator based on Fourier–Bessel transform. According to these authors, the formulation of the question of finding the error of optimal recovery and the optimal method on a class of elements ideologically goes back to Kolmogorov's work on the cross-sections of functional classes [4]. The formulation of the optimal recovery problem (but in a much simpler case than the one given here) belongs to Smolyak [5].

There are well-known situations when special attributes of the Laplace operator and Fourier transform can be carried onto elliptic singular differential operators comprising the Bessel operator. In such cases, the Fourier–Bessel transform is used for research. The theory of singular differential operators accommodating the Bessel operator, as well as functional spaces generated with such operators and the Fourier–Bessel transformation, has taken complete shape in the works of I.A. Kipriyanov and their disciples (see [6–12]). In this paper, we transfer the technique and results of [3] to the case of the Fourier–Bessel transform and the singular B-elliptic operator. We base our results on L_∞–estimates for a difference between Fourier–Bessel transform and its approximation on a convex subset with an error. The method, based on L_2^γ–estimates, was constructed in [13].

2. Necessary Definitions

We consider a part of the Euclidean space $\mathbb{R}^N$

$$\mathbb{R}^N_+ = \{\chi = (\chi', \chi''),$$

$$\chi' = (\chi_1, \ldots, \chi_n),\ \chi'' = (\chi_{n+1}, \ldots, \chi_N),\ \chi_1 \geqslant 0, \ldots, \chi_n \geqslant 0\},$$

where $1 \leqslant n \leqslant N$.

Let $\Xi^+ \subset \mathbb{R}^N_+$ be a domain abutting to the hyperplanes $x_1 = 0, \ldots, x_n = 0$. Let the boundary of Ξ^+ be a union of two parts: Γ^+ in $\mathbb{R}^N_+$ and Γ_0 in the hyperplanes $x_1 = 0, \ldots, x_n = 0$. We assume that $\Gamma_0 \subset \Xi^+$; however, we treat Ξ^+ as a domain.

Let Ξ^+_δ be a subset of Ξ^+. Let us assume that all points of Ξ^+_δ are located at a distance not more than δ from Γ^+. In this case we call Ξ^+_δ a *symmetrically interior (s-interior)* sub-domain of the set Ξ^+.

Let Ξ^- be a set constructed from Ξ^+ with symmetry in relation to $x' = 0$, $\Xi = \Xi^+ \cup \Xi^- \subset \mathbb{R}^N$.

Let us denote by $C^\ell_{ev}(\Xi^+)$ the set of all functions obeying the below listed properties. They are necessary for correct definitions of the Laplace–Bessel differential operator and Fourier–Bessel transform on proper functions.

1. Any function $\psi \in C^\ell_{ev}(\Xi^+)$, and all of its partial derivatives of every order not more than ℓ, are continuous in Ξ^+.

2. Even continuations of any function $\psi \in C^\ell_{ev}(\Xi^+)$ in relation to $x' = 0$ still belongs to the class $C^\ell(\Xi)$.

In addition, $C^\infty_{ev}(\Xi^+) = \bigcap\limits_{l=0}^{\infty} C^\ell_{ev}(\Xi^+)$.

We say that functions admitting symmetrical even continuation in relation to the corresponding variables while keeping smoothness are *even* with respect to these variables (see [6]).

Let us denote by $C^\ell_{ev,0}(\Xi^+)$ the set of all functions $\psi \in C^\ell_{ev}(\Xi^+)$ that equal zero beyond some s-interior sub-domain of Ξ^+. Let $\gamma = (\gamma_1, \ldots, \gamma_n)$, $(\chi')^\gamma = \prod\limits_{j=1}^{n} \chi_j^{\gamma_j}$, where $\gamma_j > 0$, $j = 1, \ldots, n$. We will write sometimes χ^γ instead of $(\chi')^\gamma$, if this does not cause misunderstandings.

Let us denote by $L^\gamma_p(\Xi^+)$ a closure of $C_{ev}(\Xi^+)$ by the norm

$$\|g(\bullet)\|_{L^\gamma_p(\Xi^+)} = \left[\int\limits_{\Xi^+} |g(\chi)|^p\ (\chi')^\gamma\, d\chi\right]^{1/p}.$$

If $\Xi^+ = R^N_+$, we can write L^γ_p without the symbol R^N_+. Let $L^\gamma_\infty(\Xi^+)$ be a closure of $C_{ev}(\Xi^+)$ by the norm

$$\|g\|_{L^\gamma_\infty(\Xi^+)} = \|g(x)\ (x')^\gamma\|_{L_\infty(\Xi^+)} = \operatorname{esssup}|g(x)\ x^\gamma|.$$

Let $L^\gamma_{p,loc}(\Xi^+)$ be the set of all such functions f, that

$$\int\limits_{\Xi^+_\delta} |f(\chi)|^p\ (\chi')^\gamma\, d\chi < +\infty$$

for all s-interior sub-domains Ξ^+_δ of the domain Ξ^+.

Let us introduce the spaces below.

Let $\mathcal{D}_{ev}(\Xi^+)$ $(\mathcal{E}_{ev}(\Xi^+))$ be the set of all constrictions of even functions with respect to $x' = 0$ in the space $\mathcal{D}(\Xi)$ $(\mathcal{E}(\Xi))$ to the set Ξ^+ (test functions) with this topology, induced by the topology in $\mathcal{D}(\Xi)$ $(\mathcal{E}(\Xi))$, $\mathcal{D}_{ev} = \mathcal{D}_{ev}(\mathbb{R}^N_+)$.

Let $\mathcal{S}_{ev}$ be the linear space of all functions (also test functions) $\psi(x) \in C^\infty_{ev}(\mathbb{R}^N_+)$ that tend to zero, as well as all their derivatives of all order, faster than any power of $|x|^{-1}$ as $|x| \to \infty$. The topology in $\mathcal{S}_{ev}$ is introduced as being the same as in the space $\mathcal{S}$ (see [14–19]).

We define the space of distributions $\mathcal{D}'_{ev}(\Xi^+)$ $(\mathcal{E}'_{ev}(\Xi^+), \mathcal{S}'_{ev})$ as the dual space of $\mathcal{D}_{ev}(\Xi^+)$ $(\mathcal{E}_{ev}(\Xi^+), \mathcal{S}_{ev})$ with the weak topology. The designation

$$\langle g(x), \psi(x)\rangle_\gamma = \langle g(x), \psi(x)\rangle \tag{1}$$

means the action of a distribution g on a test function ψ.

We do not make any difference between a function $g(x) \in L^\gamma_{1,loc}(\Xi^+)$ and the functional $g \in \mathcal{D}'_{ev}(\Xi^+)$ called *regular*, acting by the formula

$$\langle g(x), \psi(x)\rangle = \int\limits_{\Xi^+} g(\chi)\psi(\chi)\ (\chi')^\gamma\, d\chi. \tag{2}$$

If a functional in $\mathcal{D}'_{ev}(\Xi^+)$ is not regular, we call it *singular*.

Let us introduce the direct and inverse *mixed Fourier–Bessel transforms* in $\mathcal{S}_{ev}$ by the formulas

$$F_\gamma\psi = F_\gamma[\psi(\chi', \chi'')](\xi) =$$

$$= \int\limits_{R^N_+} \psi(\chi) \prod_{\kappa=1}^{n} j_{\nu_\kappa}(\xi_\kappa \chi_\kappa) e^{-i\chi''\bullet\xi''} (\chi')^\gamma\, d\chi =$$

$$= (2\pi)^{N-n} 2^{2|\nu|} \prod_{\kappa=1}^{n} \Gamma^2(\nu_\kappa + 1) F_\gamma^{-1}[\psi(\chi', -\chi'')](\xi),$$

$$F_\gamma^{-1}[\psi](\chi) = \frac{1}{(2\pi)^{N-n} 2^{2|\nu|} \prod\limits_{\kappa=1}^{n} \Gamma^2(\nu_\kappa + 1)} \times$$

$$\times \int\limits_{R^N_+} \psi(\xi) \prod_{\kappa=1}^{n} j_{\nu_\kappa}(\xi_\kappa \chi_\kappa) e^{i\chi''\bullet\xi''} (\xi')^\gamma\, d\xi =$$

$$= \frac{1}{(2\pi)^{N-n} 2^{2|\nu|} \prod\limits_{\kappa=1}^{n} \Gamma^2(\nu_\kappa + 1)} F_\gamma[\psi(\xi', -\xi'')](\chi),$$

where

$$\chi' \bullet \xi' = \chi_1\xi_1 + \ldots + \chi_n\xi_n,\ \ \chi'' \bullet \xi'' = \chi_{n+1}\xi_{n+1} + \ldots + \chi_N\xi_N,$$

$$|\nu| = \nu_1 + \ldots + \nu_n,$$

$$j_{\nu_\kappa}(z_\kappa) = \frac{2^{\nu_\kappa}\Gamma(\nu_\kappa + 1)}{z_\kappa^{\nu_\kappa}} J_{\nu_\kappa}(z_\kappa) =$$

$$= \Gamma(\nu_\kappa + 1) \sum_{m=1}^{\infty} \frac{(-1)^m z_\kappa^{2m}}{2^{2m} m! \Gamma(m + \nu_\kappa + 1)},$$

$\Gamma(\bullet)$ is the Euler gamma function, $J_{\nu_k}(\bullet)$ is the Bessel function of the first kind, $\nu_k = (\gamma_k - 1)/2$, $k = 1, \ldots, n$.

Theorem 1 (The Parseval–Plancherel theorem for the Fourier–Bessel transform [7])**.** *The formula*

$$\|\psi\|_{L_2^\gamma} = (2\pi)^{N-n} 2^{2|\nu|} \prod_{\kappa=1}^{n} \Gamma^2(\nu_\kappa + 1) \|\widehat{\psi}\|_{L_2^\gamma}, \quad \widehat{\psi} = F_\gamma[\psi]$$

(the Parseval–Plancherel formula) holds.

We define the Fourier–Bessel transform of a distribution g by the formula

$$\langle F_\gamma[g],\psi\rangle_\gamma = \langle g, F_\gamma[\psi]\rangle_\gamma,$$

where $\psi \in S$.

The B-elliptic Laplace–Bessel operator Δ_B is defined by the formula (see [6])

$$\Delta_B u = \sum_{k=1}^{n}\left(\frac{\partial^2 u}{\partial x_k^2} + \frac{\gamma_k}{x_k}\frac{\partial u}{\partial x_k}\right) + \sum_{k=n+1}^{N}\frac{\partial^2 u}{\partial x_k^2}.$$

We will use the denotations below:

$$\mathbf{\Pi} = (2\pi)^{N-n}2^{2|\nu|}\prod_{\kappa=1}^{n}\Gamma^2(\nu_\kappa+1) = 2^{N+|\gamma|}\pi^{N-n}\prod_{\kappa=1}^{n}\Gamma^2((\gamma_\kappa+1)/2), \tag{3}$$

$$B(\zeta,\rho) = \{\chi \in \mathbb{R}^N : |\chi-\zeta| \leqslant \rho\}, \tag{4}$$

$$B_+(\zeta,\rho) = \{\chi \in \mathbb{R}^N_+ : |\chi-\zeta| \leqslant \rho\}, \tag{5}$$

$$r_\Xi = \sup\{\rho > 0 : B(0,\rho) \subset \Xi\}, \tag{6}$$

$$\widehat{r} = \widehat{r}(\alpha,\epsilon,\delta) =$$

$$= \left(\frac{(|\gamma|+N+2\alpha)\Gamma((|\gamma|+N)/2)\prod_{\kappa=1}^{n}\Gamma((\gamma_\kappa+1)/2)}{2^{-|\gamma|-N+n+1}\pi^{(n-N)/2}\delta^2}\right)^{\frac{1}{|\gamma|+N+2\alpha}}, \tag{7}$$

$$r_0 = \min\{\widehat{r}, r_\Xi\}. \tag{8}$$

3. Problem Statement

We define the fractional degree of the operator Δ_B using the equality (see [20])

$$(-\Delta_B)^{\alpha/2}f(x) = F_\gamma^{-1}(|\xi|^\alpha F_\gamma f(\xi))(x),$$

where $\alpha > 0$. Let us introduce the functional spaces

$$\mathcal{W}^{\gamma,\alpha}_{\infty,2}(\mathbb{R}^N_+) =$$

$$= \{f(\bullet) \in \mathcal{S}'_{ev} : (-\Delta_B)^{\alpha/2}f(\bullet) \in L_2^\gamma(\mathbb{R}^N_+), F_\gamma[f(\bullet)] \in L_\infty(\mathbb{R}^N_+)\},$$

$$W^{\gamma,\alpha}_{\infty,2}(\mathbb{R}^N_+) = \{f(\bullet) \in \mathcal{W}^{\gamma,\alpha}_{\infty,2}(\mathbb{R}^N_+) : \|(-\Delta_B)^{\alpha/2}f(\bullet)\|_{L_2(\mathbb{R}^N_+)} \leqslant 1\}.$$

Let us consider a measurable non-empty bounded subset G in $\mathbb{R}^N_+$. Let $0 < \epsilon < \alpha$, $\delta > 0$. Suppose that a function $F_\gamma f \in \mathcal{W}^{\gamma,\alpha}_{\infty,2}(\mathbb{R}^N_+)$ is known approximately, namely, we know only such a function $g(\bullet) \in L_\infty(G)$ that

$$\| F_\gamma f(\bullet) - g(\bullet) \|_{L_\infty(G)} \leq \delta.$$

Based on this information, we aim to recover the functions f and $(-\Delta_B)^{\epsilon/2}f$ in the best possible way. We consider the function $g(\bullet) \in L_\infty(G)$ as an approximation of the constriction $F_\gamma f(\bullet)\Big|_G$ with the error δ in the $L_\infty(G)$–metric.

As in the papers [2,3], we call any measurable mapping

$$\mu : L_\infty(G) \longrightarrow L_2^\gamma(\mathbb{R}_+^N)$$

recovering method. Its error we define as

$$e\Big((-\Delta_B)^{\epsilon/2}, W_{\infty,2}^{\gamma,\alpha}(\mathbb{R}_+^N), G, \delta, \mu\Big) =$$

$$= \sup_{\mathbf{U}(\alpha,G,\delta)} \| (-\Delta_B)^{\epsilon/2} f(\bullet) - \mu(g(\bullet))(\bullet) \|_{L_2^\gamma(\mathbb{R}_+^N)},$$

where

$$\mathbf{U}(\alpha, G, \delta) =$$

$$= \Big\{ \Big(f(\bullet) \in W_{\infty,2}^{\gamma,\alpha}(\mathbb{R}_+^N), g(\bullet) \in L_\infty^\gamma(G) \Big) : \| F_\gamma f(\bullet) - g(\bullet) \|_{L_\infty(G)} \leq \delta \Big\}.$$

The value

$$E\Big((-\Delta_B)^{\epsilon/2}, W_{\infty,2}^{\gamma,\alpha}(R_+^N), G, \delta\Big) = \inf_\mu e\Big((-\Delta_B)^{\epsilon/2}, W_{\infty,2}^{\gamma,\alpha}(\mathbb{R}_+^N), G, \delta, \mu\Big)$$

$$= \inf_\mu \sup_{\mathbf{U}(\alpha,G,\delta)} \| (-\Delta_B)^{\epsilon/2} f(\bullet) - \mu(g(\bullet))(\bullet) \|_{L_2^\gamma(\mathbb{R}_+^N)}$$

is called the optimal recovery error (supremum is over all methods $\mu : L_\infty(G) \longrightarrow L_2^\gamma(\mathbb{R}_+^N)$).

The mappings μ, on which the lower bound is reached, we call the *optimal recovery methods*.

4. Lower Estimate of the Optimal Recovery Error Value

The proofs of both lemmas in this section are carried out according to the scheme suggested in the papers [13,21], with the difference that the initial error estimate is given here in the L_∞–metric and in those papers it is given in the L_2^γ–metric.

Let us explore an auxiliary task.

Extremal task I_∞**.**

$$\begin{gathered} \| (-\Delta_B)^{\epsilon/2} f \|_{L_2^\gamma(R_+^N)} \longrightarrow \max, \ \| F_\gamma f \|_{L_\infty(G)} \leq \delta, \\ \| (-\Delta_B)^{\alpha/2} f \|_{L_2^\gamma(R_+^N)} \leq 1, \quad f \in W_{\infty,2}^{\gamma,\alpha}(R_+^N). \end{gathered} \tag{9}$$

Lemma 1. *The optimal recovery error* $E\Big((-\Delta)^{\epsilon/2}, W_{\infty,2}^{\gamma,\alpha}(R_+^N), G, \delta\Big)$ *is not lower than the value of Extremal task* I_∞*.*

Proof. Let $u_0(\bullet)$ be an acceptable function of extremal task I_∞; in other words, $u_0(\bullet)$ satisfies the constraints of this problem. Then the function $-u_0(\bullet)$ is also acceptable. Let $\mu : L_\infty(G) \longrightarrow L_2^\gamma(\mathbb{R}_+^N)$ be any fixed method and $\mu(0)$ be an image of the zero element of the space $L_\infty(G)$ with the mapping μ. Then

$$2 \| (-\Delta_B)^{\epsilon/2} u_0(\bullet) \|_{L_2^\gamma(\mathbb{R}_+^N)} = \| 2 (-\Delta_B)^{\epsilon/2} u_0(\bullet) \|_{L_2^\gamma(\mathbb{R}_+^N)} =$$

$$= \| (-\Delta_B)^{\epsilon/2} u_0(\bullet) + (-\Delta_B)^{\epsilon/2} u_0(\bullet) \|_{L_2^\gamma(\mathbb{R}_+^N)} =$$

$$= \| (-\Delta_B)^{\epsilon/2} u_0(\bullet) - (-\Delta_B)^{\epsilon/2} (-u_0(\bullet)) \|_{L_2^\gamma(\mathbb{R}_+^N)} =$$

$$= \| (-\Delta_B)^{\epsilon/2} u_0(\bullet) - \mu(0)(\bullet) + \mu(0)(\bullet) - (-\Delta_B)^{\epsilon/2} (-u_0(\bullet)) \|_{L_2^\gamma(\mathbb{R}_+^N)} \leqslant$$

$$\leqslant \| (-\Delta_B)^{\epsilon/2} u_0(\bullet) - \mu(0)(\bullet) \|_{L_2^\gamma(\mathbb{R}_+^N)} +$$

$$+ \parallel (-\Delta_B)^{\epsilon/2}(-u_0)(\bullet) - \mu(0)(\bullet) \parallel_{L_2^\gamma(\mathbb{R}^N_+)} \leqslant$$

$$\leqslant 2 \sup_{\substack{\parallel F_\gamma f(\bullet) \parallel_{L_\infty(G)} \leqslant \delta, \\ \parallel (-\Delta_B)^{\alpha/2} f(\bullet) \parallel_{L_{2,\gamma}(\mathbb{R}^N_+)} \leqslant 1}} \parallel (-\Delta_B)^{\epsilon/2} u_0(\bullet) - \mu(0)(\bullet) \parallel_{L_2^\gamma(\mathbb{R}^N_+)} \leqslant$$

$$\leqslant 2 \sup_{\substack{\parallel F_\gamma f(\bullet) - g(\bullet) \parallel_{L_\infty(G)} \leqslant \delta, \\ \parallel (-\Delta_B)^{\alpha/2} f(\bullet) \parallel_{L_2^\gamma(\mathbb{R}^N_+)} \leqslant 1}} \parallel (-\Delta_B)^{\epsilon/2} u_0(\bullet) - \mu(g)(\bullet) \parallel_{L_2^\gamma(\mathbb{R}^N_+)} .$$

Now we can pass to a supremum over all acceptable functions of extremal task I_∞ on the left side of this inequality and for all mappings (methods) μ on the right side. Thus, we complete the proof. □

Let us explore another extremal task.

Extremal task I^2_∞.

$$\mathbf{\Pi}^{-1} \int_{\mathbb{R}^N_+} \xi^\gamma \, |\xi|^{2\epsilon} \, |F_\gamma f(\xi)|^2 \, d\xi \longrightarrow \max, \ \ \|F_\gamma f\|_{L_\infty(G)} \leqslant \delta,$$
$$\mathbf{\Pi}^{-1} \int_{\mathbb{R}^N_+} \xi^\gamma \, |\xi|^{2\alpha} \, |F_\gamma f(\xi)|^2 \, d\xi \leq 1, \ \ f(\bullet) \in \mathcal{W}^{\gamma,\alpha}_{\infty,2}(\mathbb{R}^N_+).$$

Lemma 2. *The squared value of extremal task I_∞ is equal to the value of extremal task I^2_∞.*

Proof. The affirmation of the lemma follows from the Parseval–Plancherel theorem for the Fourier–Bessel transform.

Furthermore, we assume that $\Xi = \Xi^+ \cup \Xi^-$ is convex. □

Theorem 2. *If $0 \notin \Xi^+$, then*

$$E\Big((-\Delta_B)^{\epsilon/2}, W^{\gamma,\alpha}_{\infty,2}(\mathbb{R}^N_+), \Xi^+, \delta\Big) = +\infty.$$

Proof. Assume that $0 \notin \Xi^+$. In this case we can apply the finite dimensional separability theorem (for instance [22]) and separate the origin from a convex set Ξ and, as a consequence, from Ξ^+. It means that there exists such a vector $\eta = (\eta_1, \ldots, \eta_N) \in \mathbb{R}^N_+$, $|\eta| = \sqrt{\eta_1^2 + \cdots + \eta_N^2} = 1$, that

$$\sup_{\xi \in \Xi^+} (\eta, \xi) \leq 0.$$

For arbitrarily small $\varepsilon > 0$, we introduce a ball $B_\varepsilon = B(\varepsilon\eta, \varepsilon/2)$. If $\xi \in B_\varepsilon$, then

$$(\xi - \varepsilon\eta, \xi - \varepsilon\eta) = |\xi|^2 + \varepsilon^2 |\eta|^2 - 2\varepsilon(\xi, \eta) \leq \varepsilon^2/4$$

and, taking into account that $|\eta| = 1$, we get

$$(\xi, \eta) \geq \frac{|\xi|^2}{2\varepsilon} + \frac{3\varepsilon}{8} > 0.$$

Thus, $B_\varepsilon \cap \Xi^+ = \oslash$.

Let us introduce a function $f_\varepsilon(\bullet)$ such that

$$F_\gamma f_\varepsilon(\xi) = \begin{cases} \mathbf{\Pi}^{1/2} \left(\int_{B_\varepsilon} \xi^\gamma \, |\xi|^{2\alpha} \, d\xi \right)^{-1/2}, \text{ if } \xi \in B_\varepsilon \ , \\ 0, \text{ if } \xi \notin B_\varepsilon \ . \end{cases}$$

Obviously, the function $F_\gamma f_\varepsilon$ has a bounded support and, therefore, $F_\gamma f_\varepsilon \in L_2^\gamma(\mathbb{R}_+^N)$ and $F_\gamma f_\varepsilon \in L_\infty(\mathbb{R}_+^N)$. Thus, $f_\varepsilon \in L_2^\gamma(\mathbb{R}_+^N)$. Moreover, the function $\xi \to -|\xi|^\alpha F_\gamma f_\varepsilon(\xi)$ belongs to $L_2^\gamma(\mathbb{R}_+^N)$. Therefore, $f_\varepsilon \in \mathcal{W}_{\infty,2}^{\gamma,\alpha}(\mathbb{R}_+^N)$, because $(-\Delta)^{\alpha/2} f_\varepsilon(\bullet) \in L_2^\gamma(\mathbb{R}_+^N)$. It is also easy to show that the function $f_\varepsilon(\bullet)$ satisfies other conditions of Problem I^2. Let $\xi \in B_\varepsilon$. Then $|\xi| = |\xi - \varepsilon\eta + \varepsilon\eta| \leq |\xi - \varepsilon\eta| + |\varepsilon\eta| \leq 3\varepsilon/2$. Taking this fact into account, we get

$$\frac{1}{(2\pi)^{N-n} 2^{2|\nu|} \prod\limits_{k=1}^{n} \Gamma^2(\nu_k+1)} \int\limits_{\mathbb{R}_+^N} \xi^\gamma\, |\xi|^{2\epsilon}\, |F_\gamma f(\xi)|^2\, d\xi =$$

$$= \frac{\int\limits_{B_\varepsilon} \xi^\gamma\, |\xi|^{2\epsilon}\, d\xi}{\int\limits_{B_\varepsilon} \xi^\gamma\, |\xi|^{2\alpha}\, d\xi} = \frac{\int\limits_{B_\varepsilon} \xi^\gamma\, |\xi|^{2\epsilon+2\alpha-2\alpha}\, d\xi}{\int\limits_{B_\varepsilon} \xi^\gamma\, |\xi|^{2\alpha}\, d\xi} \geqslant$$

$$\geqslant \left(\frac{3}{2}\varepsilon\right)^{-2(\alpha-\epsilon)} \frac{\int\limits_{B_\varepsilon} \xi^\gamma\, |\xi|^{2\alpha}\, d\xi}{\int\limits_{B_\varepsilon} \xi^\gamma\, |\xi|^{2\alpha}\, d\xi} = \left(\frac{3}{2}\varepsilon\right)^{-2(\alpha-\epsilon)}.$$

Since ε is arbitrary small, we get that the value of the objective functional in extremal task I^2 and, therefore, in the original recovery problem, can be made arbitrarily large. The proof is completed. □

Lemma 3. *Let $r_\Xi < \hat{r}$. The lower estimate*

$$E\Big((-\Delta_B)^{\epsilon/2}, W_{\infty,2}^{\gamma,\alpha}(R_+^N), \Xi^+, \delta\Big) \geqslant$$

$$\geqslant \left(\frac{2^{2-N-2|\nu|} \pi^{(n-N)/2} \delta^2 (\alpha-\epsilon) r_\Xi^{|\gamma|+N+2\epsilon}}{(|\gamma|+N+2\epsilon)(|\gamma|+N+2\alpha)\Gamma((|\gamma|+N)/2) \prod\limits_{j=1}^{n} \Gamma\big((\gamma_j+1)/2\big)} + \right.$$

$$\left. + \frac{1}{r_\Xi^{2(\alpha-\epsilon)}} \right)^{1/2} \tag{10}$$

holds.

Proof. The intersection of the boundary of semi-ball $B_+(0, r_\Xi) \subset \Xi^+$ and the boundary of Ξ^+ is non-empty. Assume that ξ_0 belongs to this intersection. Then $\xi_0 \notin \operatorname{int}\Xi^+$. Hence, there is the ability to separate the point ξ_0 from the convex set $\operatorname{int}\Xi^+$, i.e., there is a vector $\eta \in \mathbb{R}_+^N$, such that $|\eta| = 1$ and $\sup\limits_{\xi \in \Xi^+} (\eta, \xi) \leq (\eta, \xi_0)$. Explore a sequence of points $\xi_\kappa = \xi_0 + (1/\kappa)\eta$ $\kappa = 1, 2, 3. \ldots$, and a sequence of balls $B(\xi_\kappa, 1/(2\kappa))$. Every one of these balls does not intersect with Ξ^+. Indeed, if $\xi \in B(\xi_\kappa, 1/(2\kappa)$, then $(\eta, \xi) > (\eta, \xi_0)$, that is $\xi \notin \Xi^+$. Let

$$V_\kappa = \int\limits_{B(\xi_\kappa, 1/(2\kappa))} \xi^\gamma\, d\xi.$$

Let us explore a sequence of functions $f_\kappa(\bullet) \in \mathcal{W}_{\infty,2}^{\gamma,\alpha}(\mathbb{R}_+^N)$ having the Fourier–Bessel transform of the form

$$F_\gamma f_\kappa(\xi) = \begin{cases} \sigma^{1/2} \mathbf{\Pi}^{1/2} V_\kappa^{-1/2} \left(r_\Xi + \frac{3}{2\kappa}\right)^{-\alpha}, \text{ if } \xi \in B(\xi_\kappa, 1/(2\kappa)), \\ \delta, \text{ if } \xi \in B(0, r_\Xi), \\ 0, \quad \text{else.} \end{cases}$$

where
$$\sigma = 1 - \mathbf{\Pi}^{-1}\delta \int\limits_{B_+(0,r_\Xi)} \xi^\gamma |\xi|^{2\alpha}\, d\xi.$$

From the last two conditions we get $\|F_\gamma f_\kappa\|_{L_\infty(\Xi)} \leqslant \delta$. Let us note that, if $\xi \in B(\xi_\kappa, 1/(2\kappa))$, then
$$|\xi| = |\xi - \xi_0 - \frac{1}{\kappa}\eta + \xi_0 + \frac{1}{\kappa} \le \frac{1}{2\kappa} + |\xi_0| + \frac{1}{\kappa} = r_\Xi + \frac{3}{2\kappa}.$$

Hence,
$$\mathbf{\Pi}^{-1} \int\limits_{\mathbb{R}^N_+} \xi^\gamma\, |\xi|^{2\alpha}\, |F_\gamma f_\kappa(\xi)|^2\, d\xi =$$
$$= \mathbf{\Pi}^{-1} \int\limits_{B_+(0,r_\Xi)} \xi^\gamma\, |\xi|^{2\alpha}\, |F_\gamma f_\kappa(\xi)|^2\, d\xi +$$
$$+ \mathbf{\Pi}^{-1} \int\limits_{B(\xi_\kappa,1/(2\kappa))} \xi^\gamma\, |\xi|^{2\alpha} |F_\gamma f_\kappa(\xi)|^2\, d\xi =$$
$$= \delta^2 \mathbf{\Pi}^{-1} \int\limits_{B_+(0,r_\Xi)} \xi^\gamma\, |\xi|^{2\alpha}\, d\xi +$$
$$+ \sigma \mathbf{\Pi}^{-1}\, \mathbf{\Pi}\, V_\kappa^{-1} \left(r_\Xi + \frac{3}{2\kappa} \right)^{-2\alpha} \int\limits_{B(\xi_\kappa,1/(2\kappa))} \xi^\gamma\, |\xi|^{2\alpha}\, d\xi \le$$
$$\leqslant \delta \mathbf{\Pi}^{-1} \int\limits_{B_+(0,r_\Xi)} \xi^\gamma\, |\xi|^{2\alpha}\, d\xi +$$
$$+ \sigma V_\kappa^{-1} \left(r_\Xi + \frac{3}{2\kappa} \right)^{-2\alpha} \left(r_\Xi + \frac{3}{2\kappa} \right)^{2\alpha} \int\limits_{B(\xi_\kappa,1/(2\kappa))} \xi^\gamma\, d\xi = 1.$$

It means that functions f_κ are acceptable in problem I^2_∞.

Let us match "the Lagrange function" $\mathcal{L}(f(\bullet))$ to the problem I^2_∞
$$\mathcal{L}(f) = \mathbf{\Pi}^{-1} \int\limits_{R^N_+} \xi^\gamma(-|\xi|^{2\epsilon} + p(\xi) + \lambda |\xi|^{2\alpha}) |F_\gamma f(\xi)|^2\, d\xi, \tag{11}$$

where $\lambda = r_\Xi^{-2(\alpha-\epsilon)}$,
$$0 \leqslant p(\xi) = \begin{cases} |\xi|^{2\epsilon} - \lambda|\xi|^{2\alpha}, \text{ if } \xi \in B_+(0, r_\Xi), \\ 0, \text{if } \xi \in \mathbb{R}^N_+ \setminus B_+(0, r_\Xi). \end{cases}$$

For every I^2_∞-acceptable function, we have
$$\mathbf{\Pi}^{-1} \int\limits_{R^N_+} \xi^\gamma |\xi|^{2\epsilon} |F_\gamma f(\xi)|^2\, d\xi = \mathbf{\Pi}^{-1} \int\limits_{R^N_+} \xi^\gamma |\xi|^{2\epsilon} |F_\gamma f(\xi)|^2\, d\xi -$$
$$- \mathbf{\Pi}^{-1} \int\limits_{\Xi^+} \xi^\gamma p(\xi) \left(|F_\gamma f(\xi)|^2 - \delta^2 \right) d\xi -$$
$$- \lambda \left(\mathbf{\Pi}^{-1} \int\limits_{R^N_+} \xi^\gamma |\xi|^{2\alpha} |F_\gamma f(\xi)|^2\, d\xi - 1 \right) =$$

$$= -\mathcal{L}(f) + \delta^2 \mathbf{\Pi}^{-1} \int\limits_{R_+^N} \xi^\gamma p(\xi)\, d\xi + \lambda \leqslant \delta^2 \mathbf{\Pi}^{-1} \int\limits_{R_+^N} \xi^\gamma p(\xi)\, d\xi + \lambda.$$

In particular, the value of extremal task I^2_∞ is not more than

$$\delta^2 \mathbf{\Pi}^{-1} \int\limits_{R_+^N} \xi^\gamma p(\xi)\, d\xi + \lambda.$$

For every κ, we have

$$\mathcal{L}(f_\kappa) = \mathbf{\Pi}^{-1} \int\limits_{R_+^N} \xi^\gamma (-|\xi|^{2\epsilon} + p(\xi) + \lambda|\xi|^{2\alpha}) |F_\gamma f_\kappa(\xi)|^2\, d\xi =$$

$$= \sigma V_\kappa^{-1} \left(r_\Xi + \frac{3}{2\kappa} \right)^{-2\alpha} \int\limits_{B(\xi_\kappa, 1/(2\kappa))} \xi^\gamma (-|\xi|^{2\epsilon} + \lambda|\xi|^{2\alpha})\, d\xi.$$

For $\xi \in B(\xi_\kappa, 1/(2\kappa))$, we have

$$r_\Xi = |\xi_0| = |\xi_0 - \xi + \xi - \frac{1}{\kappa}\eta + \frac{1}{\kappa}\eta| \leq |\xi| + |\xi_0 + \frac{1}{\kappa}\eta - \xi| + |\frac{1}{\kappa}\eta| \leq |\xi| + \frac{1}{2\kappa} + \frac{1}{\kappa},$$

whence we get

$$|\xi| \geq r_\Xi - \frac{3}{2\kappa}.$$

Therefore,

$$\int\limits_{B(\xi_\kappa, 1/(2\kappa))} \xi^\gamma (-|\xi|^{2\epsilon})\, d\xi \leqslant -\left(r_\Xi - \frac{3}{2\kappa} \right)^{2\epsilon} V_\kappa.$$

On the other hand, we showed earlier that for $\xi \in B(\xi_\kappa, 1/(2\kappa))$

$$|\xi| \leqslant r_\Xi + \frac{3}{2\kappa}.$$

Therefore,

$$\int\limits_{B(\xi_\kappa, 1/(2\kappa))} \xi^\gamma (|\xi|^{2\alpha})\, d\xi \leqslant \left(r_\Xi + \frac{3}{2\kappa} \right)^{2\alpha} V_\kappa.$$

> From the last two inequalities, we have

$$0 \leqslant \mathcal{L}(f_\kappa) \leqslant \sigma \left(-\left(r_\Xi + \frac{3}{2\kappa} \right)^{-2\alpha} \left(r_\Xi + \frac{3}{2\kappa} \right)^{2\epsilon} + \lambda \right).$$

Since $\lambda = r_\Xi^{2(\epsilon - \alpha)}$, we obtain

$$\lim_{\kappa \to \infty} \mathcal{L}(f_\kappa) = 0. \tag{12}$$

Let us return to the integral

$$\mathbf{\Pi}^{-1} \int\limits_{\mathbb{R}_+^N} \xi^\gamma\, |\xi|^{2\alpha}\, |F_\gamma f_\kappa(\xi)|^2\, d\xi = \mathbf{\Pi}^{-1} \int\limits_{B_+(0, r_\Xi)} \xi^\gamma\, |\xi|^{2\alpha}\, |F_\gamma f_\kappa(\xi)|^2\, d\xi +$$

$$+ \mathbf{\Pi}^{-1} \int\limits_{B(\xi_\kappa, 1/(2\kappa))} \xi^\gamma\, |\xi|^{2\alpha} |F_\gamma f_\kappa(\xi)|^2\, d\xi =$$

$$= \delta^2 \mathbf{\Pi}^{-1} \int\limits_{B_+(0,r_\Xi)} \xi^\gamma \, |\xi|^{2\alpha} \, d\xi +$$

$$+\sigma \mathbf{\Pi}^{-1} \, \mathbf{\Pi} \, V_\kappa^{-1} \left(r_\Xi + \frac{3}{2\kappa} \right)^{-2\alpha} \int\limits_{B(\xi_\kappa, 1/(2\kappa))} \xi^\gamma \, |\xi|^{2\alpha} \, d\xi \geqslant$$

$$\geqslant \delta^2 \mathbf{\Pi}^{-1} \int\limits_{B_+(0,r_\Xi)} \xi^\gamma \, |\xi|^{2\alpha} \, d\xi +$$

$$+\sigma \left(r_\Xi + \frac{3}{2\kappa} \right)^{-2\alpha} \left(r_\Xi - \frac{3}{2\kappa} \right)^{2\alpha} .$$

This sequence tends to 1 when $\kappa \to \infty$. Hence, since

$$\delta^2 \mathbf{\Pi}^{-1} \int\limits_{B_+(0,r_\Xi)} \xi^\gamma \, |\xi|^{2\alpha} \, d\xi + \sigma \left(r_\Xi + \frac{3}{2\kappa} \right)^{-2\alpha} \left(r_\Xi - \frac{3}{2\kappa} \right)^{2\alpha} \leqslant$$

$$\leqslant \mathbf{\Pi}^{-1} \int\limits_{\mathbb{R}^N_+} \xi^\gamma \, |\xi|^{2\alpha} \, |F_\gamma f_\kappa(\xi)|^2 \, d\xi \leqslant 1,$$

we have

$$\lim_{\kappa \to \infty} \mathbf{\Pi}^{-1} \int\limits_{\mathbb{R}^N_+} \xi^\gamma \, |\xi|^{2\alpha} \, |F_\gamma f_\kappa(\xi)|^2 \, d\xi = 1. \tag{13}$$

Based on Formulas (12) and (13), we get

$$\lim_{\kappa \to \infty} \mathbf{\Pi}^{-1} \int\limits_{\mathbb{R}^N_+} \xi^\gamma \, |\xi|^{2\epsilon} \, |F_\gamma f_\kappa(\xi)|^2 \, d\xi = \lambda + \lim_{\kappa \to \infty} \delta^2 \mathbf{\Pi}^{-1} \int\limits_{\mathbb{R}^N_+} \xi^\gamma \, p(\xi) \, d\xi +$$

$$+\lambda \lim_{\kappa \to \infty} \left(\mathbf{\Pi}^{-1} \int\limits_{\mathbb{R}^N_+} \xi^\gamma \, |\xi|^{2\alpha} \, |F_\gamma f_\kappa(\xi)|^2 \, d\xi - 1 \right) - \lim_{\kappa \to \infty} \mathcal{L}(f_\kappa) +$$

$$+ \lim_{\kappa \to \infty} \mathbf{\Pi}^{-1} \int\limits_{\mathbb{R}^N_+} \xi^\gamma \, p(\xi) (|F_\gamma f_\kappa(\xi)|^2 - \delta^2) \, d\xi =$$

$$= \lambda + \delta^2 \mathbf{\Pi}^{-1} \int\limits_{\mathbb{R}^N_+} \xi^\gamma \, p(\xi) \, d\xi.$$

This fact means that the value

$$\lambda + r_\Xi^{-2\gamma} \delta^2 \mathbf{\Pi}^{-1} \int\limits_{\mathbb{R}^N_+} \xi^\gamma \, p(\xi) \, d\xi \tag{14}$$

is the value of the problem I^2_∞, and the square root from (14) is the value of the problem I_∞.

The integral

$$\int\limits_{B_+(0,r_\Xi)} \xi^\gamma \, |\xi|^{2\epsilon} \, d\xi$$

is a special case of the integral calculated in the book [23] (p. 613, Formula (4.635.1)). In our case, we get

$$\int\limits_{B_+(0,r_\Xi)} \xi^\gamma\, |\xi|^{2\epsilon}\, d\xi = \frac{r_\Xi^{|\gamma|+N+2\epsilon} \prod\limits_{j=1}^{n} \Gamma\big((\gamma_j+1)/2\big)\pi^{(N-n)/2}}{2^{n-1}(|\gamma|+N+2\epsilon)\Gamma((|\gamma|+N)/2)}. \tag{15}$$

Now we can calculate the value of the problem I^2_∞, substituting (15) into (14):

$$\lambda + \delta^2 \mathbf{\Pi}^{-1} \int\limits_{\mathbb{R}^N_+} \xi^\gamma\, p(\xi)\, d\xi =$$

$$= \frac{2^{2-N-2|\nu|}\pi^{(n-N)/2}\delta^2(\alpha-\epsilon) r_\Xi^{|\gamma|+N+2\epsilon}}{(|\gamma|+N+2\epsilon)(|\gamma|+N+2\alpha)\Gamma((|\gamma|+N)/2)\prod\limits_{j=1}^{n}\Gamma\big((\gamma_j+1)/2\big)} +$$

$$+ \frac{1}{r_\Xi^{2(\alpha-\epsilon)}}. \tag{16}$$

> From here, using Lemmas 1 and 2, we get a lower estimate of the error of optimal recovery (10). The lemma is proved. □

Lemma 4. *Let $r_\Xi > \widehat{r}$. The lower estimate*

$$E\Big((-\Delta_B)^{\epsilon/2}, W^{\gamma,\alpha}_{\infty,2}(R^N_+), \Xi^+, \delta\Big) \geqslant$$

$$\geqslant \sqrt{\frac{|\gamma|+N+2\alpha}{|\gamma|+N+2\epsilon}} \left(\frac{2^{-|\gamma|-N+n+1}\pi^{(n-N)/2}\delta^2}{\Gamma(N+|\gamma|)/2)\prod\limits_{j=1}^{n}\Gamma(\nu_j+1)}\right)^{\frac{\alpha-\epsilon}{|\gamma|+N+2\alpha}} \tag{17}$$

holds.

Proof. Explore a functions $u_0(\bullet) \in \mathcal{W}^{\gamma,\alpha}_{\infty,2}(\mathbb{R}^N_+)$ whose Fourier–Bessel transform has the form

$$F_\gamma u_0(\xi) = \begin{cases} \delta, \text{ if } \xi \in B_+(0,\widehat{r}), \\ 0, \text{ else.} \end{cases}$$

Obviously, $\|F_\gamma u_0\|_{L_\infty(\Xi)} \leqslant \delta$.

Let us again match "the Lagrange function" $\mathcal{L}(f)(f(\bullet))$ to the problem I^2_∞

$$\mathcal{L}(f) = \mathbf{\Pi}^{-1} \int\limits_{R^N_+} \xi^\gamma(-|\xi|^{2\epsilon} + \widehat{p}(\xi) + \widehat{\lambda}|\xi|^{2\alpha})|F_\gamma f(\xi)|^2\, d\xi, \tag{18}$$

where $\widehat{\lambda} = \widehat{r}^{-2(\alpha-\epsilon)}$,

$$0 \leqslant \widehat{p}(\xi) = \begin{cases} |\xi|^{2\epsilon} - \widehat{\lambda}|\xi|^{2\alpha}, \text{ if } \xi \in B_+(0,\widehat{r}), \\ 0, \text{ if } \xi \in \mathbb{R}^N_+ \setminus B_+(0,\widehat{r}). \end{cases}$$

Obviously, $\mathcal{L}(u_0) = 0$. Moreover, taking into account (15), we get

$$\mathbf{\Pi}^{-1} \int\limits_{R^N_+} \xi^\gamma |\xi|^{2\alpha} |F_\gamma f(\xi)|^2\, d\xi = \delta^2 \mathbf{\Pi}^{-1} \int\limits_{R^N_+} \xi^\gamma |\xi|^{2\alpha}\, d\xi =$$

$$= \frac{\delta^2 \widehat{r}^{|\gamma|+N+2\alpha} \prod\limits_{j=1}^{n} \Gamma\big((\gamma_j+1)/2\big) \pi^{(N-n)/2}}{2^{n-1}(|\gamma|+N+2\alpha)\Gamma((|\gamma|+N)/2)(2\pi)^{N-n} 2^{2|\nu|} \prod\limits_{k=1}^{n} \Gamma^2(\nu_k+1)} = 1. \quad (19)$$

It means that function u_0 is acceptable in problem I^2_∞. Given these facts, for any I^2_∞-acceptable function, we have

$$\mathbf{\Pi}^{-1} \int\limits_{R^N_+} \xi^\gamma |\xi|^{2\epsilon} |F_\gamma f(\xi)|^2 \, d\xi \leqslant \mathbf{\Pi}^{-1} \int\limits_{R^N_+} \xi^\gamma |\xi|^{2\epsilon} |F_\gamma f(\xi)|^2 \, d\xi -$$

$$-\mathbf{\Pi}^{-1} \int\limits_{\Xi^+} \xi^\gamma \widehat{p}(\xi) \Big(|F_\gamma f(\xi)|^2 - \delta^2 \Big) \, d\xi -$$

$$-\widehat{\lambda} \left(\mathbf{\Pi}^{-1} \int\limits_{R^N_+} \xi^\gamma |\xi|^{2\alpha} |F_\gamma f(\xi)|^2 \, d\xi - 1 \right) =$$

$$= -\mathcal{L}(f) + \delta^2 \mathbf{\Pi}^{-1} \int\limits_{R^N_+} \xi^\gamma p(\xi) \, d\xi + \lambda \leqslant$$

$$\leqslant -\mathcal{L}(u_0) + \delta^2 \mathbf{\Pi}^{-1} \int\limits_{R^N_+} \xi^\gamma \widehat{p}(\xi) \, d\xi + \widehat{\lambda} =$$

$$= \mathbf{\Pi}^{-1} \int\limits_{R^N_+} \xi^\gamma |\xi|^{2\epsilon} |F_\gamma u_0(\xi)|^2 \, d\xi -$$

$$-\mathbf{\Pi}^{-1} \int\limits_{B_+(0,\widehat{r})} \xi^\gamma \widehat{p}(\xi) (|F_\gamma u_0(\xi)|^2 - \delta^2) \, d\xi +$$

$$+\widehat{\lambda} \left(\mathbf{\Pi}^{-1} \int\limits_{R^N_+} \xi^\gamma |\xi|^{2\alpha} |F_\gamma u_0(\xi)|^2 \, d\xi - 1 \right) =$$

$$= \mathbf{\Pi}^{-1} \int\limits_{R^N_+} \xi^\gamma |\xi|^{2\epsilon} |F_\gamma u_0(\xi)|^2 \, d\xi.$$

This means that u_0 is the solution to problem I^2_∞. The value of this problem is equal to

$$\mathbf{\Pi}^{-1} \int\limits_{R^N_+} \xi^\gamma |\xi|^{2\epsilon} |F_\gamma u_0(\xi)|^2 \, d\xi = \delta^2 \mathbf{\Pi}^{-1} \int\limits_{B_+(0,\widehat{r})} \xi^\gamma |\xi|^{2\epsilon} \, d\xi =$$

$$= \delta^2 \mathbf{\Pi}^{-1} \int\limits_{B_+(0,\widehat{r})} \xi^\gamma (|\xi|^{2\epsilon} - \widehat{\lambda} |\xi|^{2\alpha}) \, d\xi + \widehat{\lambda} =$$

$$= \delta^2 \mathbf{\Pi}^{-1} \left(\frac{\widehat{r}^{|\gamma|+N+2\epsilon} \prod\limits_{j=1}^{n} \Gamma\big((\gamma_j+1)/2\big) \pi^{(N-n)/2}}{2^{n-1}(|\gamma|+N+2\epsilon)\Gamma((|\gamma|+N)/2)} - \right.$$

$$-\widehat{\lambda}\frac{\widehat{r}^{|\gamma|+N+2\alpha}\prod_{j=1}^{n}\Gamma\big((\gamma_j+1)/2\big)\pi^{(N-n)/2}}{2^{n-1}(|\gamma|+N+2\alpha)\Gamma((|\gamma|+N)/2)}\Bigg)+\widehat{\lambda}=$$

$$=\frac{|\gamma|+N+2\alpha}{|\gamma|+N+2\epsilon}\left(\frac{\Gamma(N+|\gamma|)/2)\prod_{j=1}^{n}\Gamma(\nu_j+1)}{2^{-|\gamma|-N+n+1}\pi^{(n-N)/2}\delta^2}\right)^{\frac{2(\epsilon-\alpha)}{|\gamma|+N+2\alpha}}$$

> From here, using Lemmas 1 and 2, we get a lower estimate of the error of optimal recovery (17). The lemma is proved. □

5. Upper Error Estimation and Optimal Recovery Method

We explore one more auxiliary extremal task.

Extremal task E.

$$\|\,(-\Delta_B)^{\epsilon/2}f(\bullet)-\widehat{\mu}_r(g(\bullet))(\bullet)\,\|_{L_2^\gamma(R_+^N)}\longrightarrow \max,$$
$$\|\,F_\gamma f(\bullet)-g(\bullet)\,\|_{L_\infty(\Xi_+)}\leq\delta,$$
$$\|\,(-\Delta_B)^{\alpha/2}f(\bullet)\,\|_{L_2^\gamma(R_+^N)}\leq 1,\ g(\bullet)\in L_\infty(\Xi_+),\ f(\bullet)\in\mathcal{W}_{\infty,2}^{\gamma,\alpha}(\mathbb{R}_+^N).$$

The optimality of the method $\widehat{\mu}_r$ from the statement of the theorem means that the value of extremal task E is equal to $E\Big((-\Delta_B)^{\epsilon/2},W_{\infty,2}^{\gamma,\alpha}(R_+^N),\Xi_+,\delta\Big)$.

We will look for the optimal method among linear mappings, which in Fourier images act according to the rule $F_\gamma\mu(g)=ag$ with a vanishing outside the ball $B(0,r_0)$ function a. Let m be a mapping of this kind. Then

$$\|\,(-\Delta_B)^{\epsilon/2}f(\bullet)-\widehat{\mu}_r(g(\bullet))(\bullet)\,\|^2_{L_2^\gamma(R_+^N)}=$$

$$=\mathbf{\Pi}^{-1}\int\limits_{|\xi|\leqslant r_0}\xi^\gamma\ \big||\xi|^\epsilon F_\gamma f(\xi)-a(\xi)g(\xi)\big|^2\,d\xi+$$
$$+\mathbf{\Pi}^{-1}\int\limits_{|\xi|\geqslant r_0}\xi^\gamma\ \big||\xi|^\epsilon F_\gamma f(\xi)\big|^2\,d\xi. \tag{20}$$

Let $\widetilde{\lambda}=\lambda$, $\widetilde{p}=p$, when $r_\Xi<\widehat{r}$, $\widetilde{\lambda}=\widehat{\lambda}$ and $\widetilde{p}=\widehat{p}$, when $r_\Xi\geqslant\widehat{r}$. According to the Cauchy–Bunyakovsky inequality, we obtain for the integrand function in the second integral:

$$\Big||\xi|^{2\epsilon}\,F_\gamma f(\xi)-a(\xi)g(\xi)\Big|^2=$$

$$\left|\frac{a(\xi)\sqrt{\widetilde{p}(\xi)}}{\sqrt{\widehat{p}(\xi)}}(F_\gamma f(\xi)-g(\xi))+\frac{|\xi|^\epsilon-a(\xi)}{\sqrt{\widetilde{\lambda}}|\xi|^\alpha}\sqrt{\widetilde{\lambda}}|\xi|^\alpha F_\gamma f(\xi)\right|^2\leqslant$$

$$\leqslant\left(\frac{|a(\xi)|^2}{\widetilde{p}(\xi)}+\frac{|\,|\xi|^\epsilon-a(\xi)\,|^2}{\widetilde{\lambda}\,|\xi|^{2\alpha}}\right)\times$$
$$\times(\widetilde{p}(\xi)|F_\gamma f(\xi)-g(\xi)|^2+\widetilde{\lambda}|\xi|^{2\alpha}|F_\gamma f(\xi)|^2).$$

For any ξ the minimum value of the expression

$$J(a)=\frac{|a(\xi)|^2}{\widetilde{p}(\xi)}+\frac{|\,|\xi|^\epsilon-a(\xi)\,|^2}{\widetilde{\lambda}\,|\xi|^{2\alpha}}$$

is reached at the point

$$\widehat{a} = |\xi|^{\epsilon}\left(1 - \left(\frac{|\xi|}{r_0}\right)^{2(\alpha-\epsilon)}\right). \tag{21}$$

This minimum value is equal to 1. Substituting (21) into the first term of (20) and taking into account the first constraint of the problem E, we get

$$\mathbf{\Pi}^{-1} \int\limits_{|\xi|\leqslant r_0} \xi^{\gamma}\, ||\xi|^{\epsilon} F_{\gamma} f(\xi) - a(\xi) g(\xi)|^2\, d\xi \leqslant$$

$$\leqslant \mathbf{\Pi}^{-1} \int\limits_{|\xi|\leqslant r_0} \xi^{\gamma}\, (\widetilde{p}(\xi)|F_{\gamma} f(\xi) - g(\xi)|^2 + \widetilde{\lambda}|\xi|^{2\alpha}|F_{\gamma} f(\xi)|^2)\, d\xi \leqslant$$

$$\leqslant \delta^2 \mathbf{\Pi}^{-1} \int\limits_{|\xi|\leqslant r_0} \xi^{\gamma}\, \widetilde{p}(\xi)\, d\xi + \widetilde{\lambda} \mathbf{\Pi}^{-1} \int\limits_{|\xi|\leqslant r_0} \xi^{\gamma}\, |\xi|^{2\alpha}|F_{\gamma} f(\xi)|^2\, d\xi. \tag{22}$$

For the second term of (20), we have

$$\mathbf{\Pi}^{-1} \int\limits_{|\xi|\geqslant r_0} \xi^{\gamma}\, |\xi|^{2\epsilon}|F_{\gamma} f(\xi)|^2\, d\xi \leqslant$$

$$\leqslant \mathbf{\Pi}^{-1} \int\limits_{|\xi|\geqslant r_0} \xi^{\gamma}\, |\xi|^{2\epsilon-2\alpha}|\xi|^{2\alpha}|F_{\gamma} f(\xi)|^2\, d\xi \leqslant$$

$$\leqslant \widetilde{\lambda} \mathbf{\Pi}^{-1} \int\limits_{|\xi|\geqslant r_0} \xi^{\gamma}\, |\xi|^{2\alpha}|F_{\gamma} f(\xi)|^2\, d\xi. \tag{23}$$

Adding up (22) and (23), we obtain the next upper estimation

$$\|\, (-\Delta_B)^{\epsilon/2} f(\bullet) - \widehat{m}_r(g(\bullet))(\bullet)\, \|^2_{L_2^{\gamma}(R_+^N)} \leqslant$$

$$\leqslant \delta^2 \mathbf{\Pi}^{-1} \int\limits_{|\xi|\leqslant r_0} \xi^{\gamma}\, \widetilde{p}(\xi)\, d\xi + \widetilde{\lambda} \mathbf{\Pi}^{-1} \int\limits_{\mathbb{R}_+^N} \xi^{\gamma}\, |\xi|^{2\alpha}|F_{\gamma} f(\xi)|^2\, d\xi =$$

$$= \delta^2 \mathbf{\Pi}^{-1} \int\limits_{|\xi|\leqslant r_0} \xi^{\gamma}\, \widetilde{p}(\xi)\, d\xi + \widetilde{\lambda}\, \|\, (-\Delta_B)^{\alpha/2} f(\bullet)\, \|^2_{L_2^{\gamma}(R_+^N)} \leqslant$$

$$\leqslant \delta^2 \mathbf{\Pi}^{-1} \int\limits_{|\xi|\leqslant r_0} \xi^{\gamma}\, \widetilde{p}(\xi)\, d\xi + \widetilde{\lambda}.$$

The obtained upper estimate of the squared error for the constructed method does not exceed the squared error of the optimal recovery method. This means that the constructed method is optimal. Thus, we proved the next result.

Theorem 3. *If* $0 \in \Xi$ *and* $r_{\Xi} < \widehat{r}$, *then*

$$E\Big((-\Delta_B)^{\epsilon/2}, W^{\gamma,\alpha}_{\infty,2}(R^N_+), \Xi^+, \delta\Big) =$$

$$= \left(\frac{2^{2-N-2|\nu|} \pi^{(n-N)/2} \delta^2 (\alpha-\epsilon) r_{\Xi}^{|\gamma|+N+2\epsilon}}{(|\gamma|+N+2\epsilon)(|\gamma|+N+2\alpha)\Gamma((|\gamma|+N)/2) \prod\limits_{j=1}^{n} \Gamma((\gamma_j+1)/2)} + \right.$$

$$+\frac{1}{r_\Xi^{2(\alpha-\epsilon)}}\Bigg)^{1/2}.$$

If $r_\Xi > \widehat{r}$, *then*

$$E\Big((-\Delta_B)^{\epsilon/2}, W^{\gamma,\alpha}_{\infty,2}(R^N_+), \Xi^+, \delta\Big) =$$

$$= \sqrt{\frac{|\gamma|+N+2\alpha}{|\gamma|+N+2\epsilon}}\left(\frac{2^{-|\gamma|-N+n+1}\pi^{(n-N)/2}\delta^2}{\Gamma(N+|\gamma|)/2)\prod\limits_{j=1}^{n}\Gamma(\nu_j+1)}\right)^{\frac{\alpha-\epsilon}{|\gamma|+N+2\alpha}}.$$

The method

$$\mu(g) = \mathbf{\Pi}^{-1}\int\limits_{B(0,r_0)_+}|\xi|^\epsilon\left(1-\left(\frac{|\xi|}{r_0}\right)^{2(\alpha-\epsilon)}\right)\times$$

$$\times g(\xi)\prod_{k=1}^{n} j_{\nu_k}(\xi_k x_k)e^{ix''\bullet\xi''}(\xi')^\gamma\, d\xi$$

is optimal.

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