Pseudo-Differential Equations in Spaces of Different Smoothness Exponents on Variables

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Abstract We study a model elliptic pseudo-differential equation and simplest boundary value problems for a half-space and a special cone in Sobolev–Slobodetskii spaces which have different smoothness with respect to separate variables. Sufficient conditions for a unique solvability for such boundary value problems are described

Keywords Sobolev–Slobodetskii space · Pseudo-differential equation · Cone · Boundary value problem

1 Introduction

The theory of pseudo-differential operators was appeared near a half-century ago, and it has taken attention of mathematicians for a long time [1–3]. More general Fourier integral operators and new functional spaces were studied in this context. As a rule the theory means constructing a symbolic calculus and an index formula. Such a theory is very convenient for generalization on smooth compact manifolds without a boundary, but for more complicated situations new constructions and new approaches were needed. More complicated situations mean presence of a smooth boundary, or more generally a non-smooth boundary. For manifolds with a smooth boundary a certain approach was suggested in [4], and it was based the factorization principle for an elliptic symbol at boundary points. This method is not applicable for manifolds with a non-smooth boundary, and it has initiated a lot of approaches for "non-smooth" situations [5, 6, 10–14].

This paper presents a future development of the second author's approach [17–20] to the theory of pseudo-differential equations and related boundary value problems in non-smooth domains.

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2 Elliptic Pseudo-differential Operators

2.1 Sobolev–Slobodetskii Spaces of Different Smoothness

Following to [15] (see also [16]) we introduce useful notations. A multidimensional Euclidean space $\mathbf{R}^M$ is represented as an orthogonal sum of subspaces in which only some of coordinates $x_1, x_2, \dots, x_M$ are nor vanishing. Namely, if $K \subset 1, ..., M$ is not empty set we put

$$\mathbf{R}^K = \{x \in \mathbf{R}^M : x = (x_1, \dots, x_M), x_j = 0, \forall j \notin K\} \subset \mathbf{R}^M.$$

Let $K_1, K_2, \dots, K_n \subset \{1, 2, \dots, M\}$ be a nonempty set so that

$$\bigcup_{j=1}^{n} K_j = \{1, 2, \dots, M\}, \quad K_i \cap K_j = \emptyset, i \neq j, \quad card\ K_j = k_j.$$

Thus, we obtain the representation

$$\mathbf{R}^M = \mathbf{R}^{K_1} \oplus \mathbf{R}^{K_2} \oplus \dots \oplus \mathbf{R}^{K_n},$$

where x_{K_j} is an element of the space $\mathbf{R}^{K_j}$. For functions defined in $\mathbf{R}^M$ we use the standard Fourier transform

$$\tilde{u}(\xi) = \int_{\mathbf{R}^M} e^{ix\cdot\xi} u(x) dx, \quad \xi = (\xi_1, \dots, \xi_M).$$

Let $S = (s_1, \dots, s_n)$. Now we introduce the Sobolev–Slobodetskii space $H^S(\mathbf{R}^M)$ as a Hilbert space with the inner product

$$(f, g) = \int_{\mathbf{R}^M} f(x)\overline{g(x)} dx$$

and the norm

$$||f||_S = \left(\int_{\mathbf{R}^M} (1 + |\xi_{K_1}|)^{2s_1} (1 + |\xi_{K_2}|)^{2s_2} \cdots (1 + |\xi_{K_n}|)^{2s_n} |\tilde{f}(\xi)|^2 d\xi \right)^{1/2}.$$

Such H^S-spaces have the same properties similar to usual Sobolev–Slobodetskii spaces [16]. Particularly, the usual space $H^s(\mathbf{R}^M)$ is obtained under the following choice of subsets K_j and parameters s_j:

$$K_1 = K_2 = \dots = K_{n-1} = \emptyset, \quad K_n = \{1, 2, \dots, M\}, \quad S = (0, 0, \dots, 0, s).$$

2.2 Model Operators and Equations

According to the local principle we will concentrate on studying a model pseudo-differential equation with operator with a symbol non-depending on a spatial variable.

Model pseudo-differential operators Let $\tilde{A}(\xi)$, $\xi \in \mathbf{R}^M$ be a measurable function. A model pseudo-differential operator A is defined as follows

$$(Au)(x0 = \frac{1}{(2\pi)^M} \int\limits_{\mathbf{R}^M} \int\limits_{\mathbf{R}^M} e^{i(x-y)\cdot\xi} \tilde{A}(\xi) u(y) dy d\xi,$$

and the function $\tilde{A}(\xi)$ is called a symbol of the pseudo-differential operator A.

We consider here the following class of symbols $A(\xi)$ satisfying the condition

$$c_1 \prod_{j=1}^{n} (1 + |\xi_{K_j}|)^{\alpha_j} \le |A(\xi)| \le c_2 \prod_{j=1}^{n} (1 + |\xi_{K_j}|)^{\alpha_j}, \quad \alpha_j \in \mathbf{R}, j = 1, 2, \dots, n, \tag{$*$}$$

with positive constants c_1, c_2.

Let $\alpha = (\alpha_1, \dots, \alpha_n)$

Lemma 1 *Let A be a pseudo-differential operator with the symbol $\tilde{A}(\xi)$ satisfying the condition $(*)$. Then $A : H^S(\mathbf{R}^M) \to H^{S-\alpha}(\mathbf{R}^M)$ is a linear bounded operator.*

Proof Indeed, we have

$$||Au||^2_{S-\alpha} = \int\limits_{\mathbf{R}^M} \prod_{j=1}^{n} (1 + |\xi_{K_j}|)^{2(s_j-\alpha_j)} \widetilde{Au}(\xi)|^2 d\xi =$$

$$= \int\limits_{\mathbf{R}^M} (1 + |\xi_{K_1}|)^{2(s_1-\alpha_1)} (1 + |\xi_{K_2}|)^{2(s_2-\alpha_2)} \cdots (1 + |\xi_{K_n}|)^{2(s_n-\alpha_n)} |\tilde{A}(\xi)\tilde{u}(\xi)|^2 d\xi \le$$

$$\le c_2 \int\limits_{\mathbf{R}^M} (1 + |\xi_{K_1}|)^{2s_1} (1 + |\xi_{K_2}|)^{2s_2} \cdots (1 + |\xi_{K_n}|)^{2s_n} |\tilde{u}(\xi)|^2 d\xi = c_2 ||u||^2_S,$$

and the proof is completed. □

Thus, we can start studying a solvability for the equation

$$(Au)(x) = v(x), \quad x \in \mathbf{R}^M, \tag{1}$$

where A is a pseudo-differential operator with the symbol $\tilde{A}(\xi)$ satisfying the condition $(*)$, and the right hand side $v \in H^{S-\alpha}(\mathbf{R}^M)$.

Corollary 1 *If A is a pseudo-differential operator with the symbol $\tilde{A}(\xi)$ satisfying the condition $(*)$ then the Eq. (1) with an arbitrary right hand side $v \in H^{S-\alpha}(\mathbf{R}^M)$ has unique solution $u \in H^S(\mathbf{R}^M)$. The a priori estimate*

$$||u||_S \leq C||v||_{S-\alpha}$$

holds.

Proof The operator A^{-1} with the symbol $\tilde{A}^{-1}(\xi)$ is a pseudo-differential operator. Its symbol satisfies the condition $(*)$ with order $-\alpha$ instead of α. Then we have

$$\tilde{u} = \tilde{A}^{-1}\tilde{v},$$

and therefore

$$||u||_S^2 = ||A^{-1}v||_S^2 =$$

$$\int\limits_{\mathbf{R}^M} (1+|\xi_{K_1}|)^{2s_1}(1+|\xi_{K_2}|)^{2s_2}\cdots(1+|\xi_{K_n}|)^{2s_n}|\tilde{A}^{-1}(\xi)\tilde{v}(\xi)|^2 d\xi \leq c_1^{-2}||v||_{S-\alpha}^2,$$

and the sentence is proved. □

Unfortunately, such a simple conclusion is possible for the space $\mathbf{R}^M$. If we will take a domain $D \subset \mathbf{R}^M$ and will try to study a solvability for similar equation then we will obtain a lot of difficulties related to invertibility of operators.

We extract special *canonical* domains D in Euclidean space $\mathbf{R}^M$. Such domains are conical domains and we will start from a standard convex cone in Euclidean space non-including a whole straight line. Let $C_{K_j} \subset \mathbf{R}^{K_j}$ and we would like to consider the equation

$$(Au)(x) = v(x), \quad x \in C_{K_j}. \tag{2}$$

Direct applying the Fourier transform does not give the required answer since we have no the convolution theorem. The Eq. (2) can be rewritten in the form

$$(P_{K_j}Au)(x) = v(x), \quad x \in C_{K_j},$$

where P_{K_j} is the restriction on C_{K_j},

$$(P_{K_j}u)(x) = \begin{cases} u(x), \ x \in C_{K_j}; \\ 0, \ x \notin \overline{C}_{K_j}. \end{cases}$$

and to use the Fourier transform we need to know what is the Fourier image of the operator P_{K_j}.

Structure of projectors For a general convex cone $C^m \subset \mathbf{R}^m$ one can define the Bochner kernel [7–9]

$$B_m(z) = \int_C e^{ix\cdot z}dx, \quad z = (z_1, \dots, z_m),$$

and the following representation in Fourier imaged

$$(FP_+u)(\xi) = \lim_{\tau\to 0+} \int_{\mathbf{R}^m} B_m(\xi' - \eta', \xi_m - \eta_m + i\tau)\tilde{u}(\eta', \eta_m)d\eta,$$

here P_+ is the projector on the cone C^m [14, 17]. There are certain concrete realizations in the latter formula.

Example 1 We consider here one-dimensional case in which we have only one cone, and this cone is $\mathbf{R}_+$ [4]. For this case it was proved for a function $u(x)$, $x \in \mathbf{R}$, that

$$(FP_+u)(\xi) = \frac{1}{2}\tilde{u}(\xi) + \frac{i}{2\pi} p.v. \int_{-\infty}^{+\infty} \frac{\tilde{u}(\eta)d\eta}{\xi - \eta}.$$

As a consequence we have for a function $u(x)$, $x \in \mathbf{R}^m$ and the cone $\mathbf{R}^m_+ = \{x \in \mathbf{R}^m : x = (x', x_m), x_m > 0\}$ the following result

$$(FP_+u)(\xi) = \frac{1}{2}\tilde{u}(\xi) + \frac{i}{2\pi} p.v. \int_{-\infty}^{+\infty} \frac{\tilde{u}(\xi', \eta_m)d\eta_m}{\xi_m - \eta_m}, \quad \xi = (\xi', \xi_m).$$

Example 2 Let $m = 2$, and

$$C^a_+ = \{x \in \mathbf{R}^2 : x = (x_1, x_2), x_2 > a|x_1|, a > 0\}.$$

Then we have [21]

$$(FP_{C^a_+}u)(\xi) = \frac{\tilde{u}(\xi_1 + a\xi_2, \xi_2) + \tilde{u}(\xi_1 - a\xi_2, \xi_2)}{2} +$$

$$+ v.p. \frac{i}{2\pi} \int_{-\infty}^{+\infty} \frac{\tilde{u}(\eta, \xi_2)d\eta}{\xi_1 + a\xi_2 - \eta} - v.p. \frac{i}{2\pi} \int_{-\infty}^{+\infty} \frac{\tilde{u}(\eta, \xi_2)d\eta}{\xi_1 - a\xi_2 - \eta}.$$

Example 3 Let $m = 3$, and $C^{a_1a_2}_+ = \{x \in \mathbf{R}^3 : x_2 > a_1|x_1| + a_2|x_2|\}$. Then

$$(FP_{C_+^{a_1a_2}}u)(\xi_1,\xi_2,\xi_3)=$$

$$=\frac{\tilde{u}(\xi_1-a_1\xi_3,\xi_2-a_2\xi_3,\xi_3)+\tilde{u}(\xi_1+a_1\xi_3,\xi_2-a_2\xi_3,\xi_3)}{4}+$$

$$+\frac{1}{2}(S_1\tilde{u})(\xi_1+a_1\xi_3,\xi_2-a_2\xi_3,\xi_3)-\frac{1}{2}(S_1\tilde{u})(\xi_1-a_1\xi_3,\xi_2-a_2\xi_3,\xi_3)+$$

$$+\frac{\tilde{u}(\xi_1-a_1\xi_3,\xi_2+a_2\xi_3,\xi_3)+\tilde{u}(\xi_1+a_1\xi_3,\xi_2+a_2\xi_3,\xi_3)}{4}+$$

$$+\frac{1}{2}(S_1\tilde{u})(\xi_1+a_1\xi_3,\xi_2+a_2\xi_3,\xi_3)-\frac{1}{2}(S_1\tilde{u})(\xi_1-a_1\xi_3,\xi_2+a_2\xi_3,\xi_3)+$$

$$+\frac{(S_2\tilde{u})(\xi_1-a_1\xi_3,\xi_2+a_2\xi_3,\xi_3)+(S_2\tilde{u})(\xi_1+a_1\xi_3,\xi_2+a_2\xi_3,\xi_3)}{2}+$$

$$+(S_1S_2\tilde{u})(\xi_1+a_1\xi_3,\xi_2+a_2\xi_3,\xi_3)-(S_1S_2\tilde{u})(\xi_1-a_1\xi_3,\xi_2+a_2\xi_3,\xi_3)-$$

$$-\frac{(S_2\tilde{u})(\xi_1-a_1\xi_3,\xi_2-a_2\xi_3,\xi_3)-(S_2\tilde{u})(\xi_1+a_1\xi_3,\xi_2-a_2\xi_3,\xi_3)}{2}-$$

$$-(S_1S_2\tilde{u})(\xi_1+a_1\xi_3,\xi_2-a_2\xi_3,\xi_3)+(S_1S_2\tilde{u})(\xi_1-a_1\xi_3,\xi_2-a_2\xi_3,\xi_3).$$

where

$$(S_1u)(\xi_1,\xi_2,\xi_3)=v.p\frac{i}{2\pi}\int_{-\infty}^{+\infty}\frac{u(\tau,\xi_2,\xi_3)d\tau}{\xi_1-\tau},$$

$$(S_2u)(\xi_1,\xi_2,\xi_3)=v.p\frac{i}{2\pi}\int_{-\infty}^{+\infty}\frac{u(\xi_1,\eta,\xi_3)d\eta}{\xi_2-\eta}.$$

This case was studied in [22]

Elliptic equations and complex variables This approach is related to the function theory of many complex variables, namely to functions which are holomorphic in radial tube domains [7–9].

Let $C_{K_j}\subset\mathbf{R}^{K_j}$, $j=1,\dots,n$, be convex cones non-including a whole straight line in $\mathbf{R}^{K_j}$. Let us compose the set $C=C_{K_1}\times\dots C_{K_n}$.

Lemma 2 *The set C is a cone in $\mathbf{R}^M$ non-including a whole straight line in $\mathbf{R}^M$.*

Proof Indeed, C is a cone since each C_j is a cone. If we will assume that C includes a certain line in $\mathbf{R}^M$ then we will conclude that each cone C_j includes a certain straight line. □

Now we will start studying a solvability of the equation

$$(Au)(x) = v(x), \quad x \in C, \tag{3}$$

and the solution is sought in the space $H^S(C)$.

Definition 1 The space $H^S(C)$ consists of functions (distributions) from $H^S(\mathbf{R}^M)$ with supports in $\overline{C}$.

The right-hand side v is chosen from the space $H_0^{S-\alpha}(C)$; by definition the space $H_0^S(C)$ is a space of distributions on C, admitting a continuation on $H^S(\mathbf{R}^M)$. The norm in the space $H_0^S(C)$ is defined as

$$||v||_S^+ = \inf ||\ell f||_S,$$

where the *infimum* is taken over all continuations ℓlf on the whole $\mathbf{R}^M$.

Fourier image of the space $H^S(C)$ will be denoted by $\tilde{H}^S(C)$

Definition 2 A radial tube domain over the cone C is called a domain in M-dimensional complex space $\mathbf{C}^M$ of the following type

$$T(C) \equiv \{z \in \mathbf{C}^M : z = x + iy, x \in \mathbf{R}^M, y \in C\}.$$

A conjugate cone $\overset{*}{C}$ is called such a cone in which for all points the condition

$$x \cdot y > 0, \quad \forall y \in C,$$

holds; $x \cdot y$ means inner product for x and y.

Definition 3 The wave factorization of an elliptic symbol $A(\xi)$ with respect to the cone C is called its representation in the form

$$A(\xi) = A_{\neq}(\xi)A_{=}(\xi),$$

where factors $A_{\neq}(\xi)$, $A_{=}(\xi)$ must satisfy the following conditions:

(1) $A_{\neq}(\xi)$, $A_{=}(\xi)$ are defined for all $\xi \in \mathbf{R}^M$ may be except the points $\xi \in \partial \overset{*}{C}$;

(2) $A_{\neq}(\xi)$, $A_{=}(\xi)$ admit an analytic continuation into radial tube domains $T(\overset{*}{C})$, $T(-\overset{*}{C})$ respectively with estimates

$$|A_{\neq}^{\pm 1}(\xi + i\tau)| \le c_1 \prod_{j=1}^{n} (1 + |\xi_{K_j}| + |\tau_{K_j}|)^{\pm æ_j},$$

$$|A_{=}^{\pm 1}(\xi - i\tau)| \le c_2 \prod_{j=1}^{n} (1 + |\xi_{K_j}| + |\tau_{K_j}|)^{\pm(\alpha_j - æ_j)}, \ \forall \tau \in \overset{*}{C}, \quad æ_j \in \mathbf{R}.$$

The vector $\text{æ} = (\text{æ}_1, \ldots, \text{æ}_n)$ is called an index of the wave factorization.

To apply the Fourier transform to the Eq. (3) we need to know what is FP_C; here F denotes the Fourier transform in M-dimensional space. Let us introduce the following notations. For every C_{K_j} we consider corresponding radial tube domain $T(\overset{*}{C}_{K_j})$ over the conjugate cone and an element of $T(\overset{*}{C}_{K_j})$ will be denoted by $\xi_{K_j} + i\tau_{K_j}$. Moreover, for ξ_{K_j} we will use the notation $\xi_{K_j} = (\xi'_{K_j}, \xi_{k_j})$, where ξ_{k_j} is the k_jth coordinate, and ξ'_{K_j} denotes left other coordinates. The same notations will be used for $x \in \mathbf{R}^{K_j}$, $x_{K_j} = (x'_{K_j}, x_{k_j})$.

As before we denote by P_C the restriction operator on C. Obviously,

$$P_C = \prod_{j=1}^{n} P_{K_j}.$$

and then

$$B_M(z) = \prod_{j=1}^{n} B_{k_j}(z_{K_j}), \quad z = (z_{K_1}, \ldots, z_{K_n}).$$

The last our observation is the following:

$$T(\overset{*}{C}) = \prod_{j=1}^{n} T(\overset{*}{C}_{K_j}),$$

and the Bochner kernel $B_M(z)$ will be a holomorphic function in $T(\overset{*}{C})$.

Theorem 1 *If the symbol $A(\xi)$ admits the wave factorization with respect to the cone C with the index æ such that $|\text{æ}_j - s_j| < 1/2$, $j = 1, \ldots, n$, then the Eq. (3) has unique solution in the space $H^S(C)$ for arbitrary right hand side $v \in H_0^{S-\alpha}(C)$.*

The a priori estimate

$$||u||_S \leq const\ ||v||^+_{S-\alpha}$$

holds.

Proof We use the Wiener–Hopf method [4, 14]. Let ℓv be an arbitrary continuation of v onto $\mathbf{R}^M$ then we put

$$u_-(x) = (\ell v)(x) - (Au)(x),$$

so that $v_-(x) = 0$ for $x \in C$. Further,

$$(Au)(x) + u_-(x) = (\ell v)(x),$$

and after applying the Fourier transform and the wave factorization we obtain

$$A_{\neq}(\xi)\tilde{u}(\xi) + A_{=}^{-1}(\xi)\tilde{u}_{-}(\xi) = A_{=}^{-1}(\xi)\widetilde{(\ell v)}(\xi) \tag{4}$$

Now we can use the following result (see [14]).

Property 1 *If* $\tilde{u}_{-} \in \widetilde{H}^{S}(\mathbf{R}^M \setminus C)$, $A_{=}^{-1}$ *is a factor of the wave factorization then* $A_{=}^{-1}\tilde{u}_{-} \in \widetilde{H}^{S+\alpha-æ}(\mathbf{R}^M \setminus C)$.

Obviously, the summand $A_{\neq}(\xi)\tilde{u}(\xi)$ belongs to $\widetilde{H}^{S-æ}(C)$ according to Lemma 1 and holomorphic properties, and $A_{=}^{-1}(\xi)\tilde{u}_{-}(\xi)$ belongs to $\widetilde{H}^{S-æ}(\mathbf{R}^M \setminus C)$ according to Property 1.

The right hand side $A_{=}^{-1}(\xi)\widetilde{(\ell v)}(\xi)$ belongs to the space $\widetilde{H}^{S-æ}(\mathbf{R}^M)$ (Lemma 1), and since $|æ_j - s_j| < 1/2$, $j = 1, \dots, n$, it can be uniquely represented as

$$A_{=}^{-1}(\xi)\widetilde{(\ell v)}(\xi) = \tilde{v}_{+}(\xi) + \tilde{v}_{-}(\xi), \tag{5}$$

where

$$\tilde{v}_{+}(\xi) = B_M\left(A_{=}^{-1}(\xi)\widetilde{(\ell v)}(\xi)\right), \quad \tilde{v}_{-}(\xi) = (I - B_M)\left(A_{=}^{-1}(\xi)\widetilde{(\ell v)}(\xi)\right).$$

The representation (5) is true since the operator $B_M : \widetilde{H}^{\delta}(\mathbf{R}^M) \to \widetilde{H}^{\delta}(\mathbf{R}^M)$ for $|\delta_J| < 1/2$, $j = 1, \dots, n$, and we remind that $|æ_j - s_j| < 1/2$, $j = 1, \dots, n$,.

Further, we rewrite the equality (4) in the form

$$A_{\neq}(\xi)\tilde{u}(\xi) - \tilde{v}_{+}(\xi) = \tilde{v}_{-}(\xi) - A_{=}^{-1}(\xi)\tilde{u}_{-}(\xi),$$

and we obtain that a distribution from $H^{\delta}(C)$ equals to a distribution from $H^{\delta}(\mathbf{R}^M \setminus \overline{C})$. But for such small δ this common distribution should be zero only [14]. Thus,

$$A_{\neq}(\xi)\tilde{u}(\xi) - \tilde{v}_{+}(\xi) = 0,$$

or in other words

$$\tilde{u}(\xi) = A_{\neq}^{-1}(\xi)B_M\left(A_{=}^{-1}(\xi)\widetilde{(\ell v)}(\xi)\right).$$

A priori estimate is based on Lemma 1 and boundedness property of the operator $B_M : \widetilde{H}^{\delta}(\mathbf{R}^M) \to \widetilde{H}^{\delta}(\mathbf{R}^M)$. Indeed,

$$||u||_S = ||\tilde{u}||_S = ||A_{\neq}^{-1}(\xi)B_M\left(A_{=}^{-1}(\xi)\widetilde{(\ell v)}(\xi)\right)||_S \leq$$

$$\leq const\ ||B_M\left(A_{=}^{-1}(\xi)\widetilde{(\ell v)}(\xi)\right)||_{S-æ} \leq const\ ||A_{=}^{-1}(\xi)\widetilde{(\ell v)}(\xi)||_{S-æ} \leq$$

$$\leq const\ ||\widetilde{(\ell v)}(\xi)||_{S-\alpha} = const\ ||\ell v||_{S-\alpha} \leq const\ ||v||^{+}_{S-\alpha},$$

and Theorem 1 is proved. □

Multiply solutions For the cone C_{K_j}, $j = 1, \ldots, n$, we suppose that a surface of this cone is given by the equation $x_{k_j} = \varphi_j(x'_{K_j})$, where $\varphi_j : \mathbf{R}^{k_j-1} \to \mathbf{R}$ is a smooth function in $\mathbf{R}^{k_j-1} \setminus \{0\}$, and $\varphi_j(0) = 0$.

Let us introduce the following change of variables

$$\begin{cases} t'_{K_j} = x'_{K_j} \\ t_{k_j} = x_{k_j} - \varphi_j(x'_{K_j}) \end{cases}$$

and we denote this operator by $T_{\varphi_j} : \mathbf{R}^{K_j} \to \mathbf{R}^{K_j}$. Since the cone is in one part of a half-space then points of the second part of a half-space will be fixed. Such change of variables can be defined for distributions also [22].

Below we will use notation F_m for the Fourier transform in m-dimensional space, so that the notation F_{K_j} will be the Fourier transform in $\mathbf{R}^{K_j}$.

Following to [22] we conclude

$$F_{K_j} T_{\varphi_j} = V_{\varphi_j} F_{K_j}.$$

Further, we introduce $T_\varphi : \mathbf{R}^M \to \mathbf{R}^M$ by the formula

$$T_\varphi = \prod_{j=1}^{n} T_{\varphi_j}$$

and construct the operator

$$V_\varphi = \prod_{j=1}^{n} V_{\varphi_j},$$

for which we have

$$F_M T_\varphi = V_\varphi F_M.$$

Let us introduce vectors $N = (n_1, \ldots, n_n)$, $L = (l_1, \ldots, l_n)$, $\delta{=}(\delta_1, \ldots, \delta_n)$, n_j, $l_j \in \mathbf{N}$, $|\delta_j| < 1/2$, $j = 1, \ldots, n$, and a polynomial $Q_N(\xi)$, $\xi \in \mathbf{R}^M$ satisfying the condition

$$|Q_N(\xi)| \sim \prod_{j=1}^{n} (1 + |\xi_{K_j}|)^{n_j}, \tag{6}$$

Theorem 2 *If the symbol $A(\xi)$ admits the wave factorization with the index* æ, æ $-$ $S = N + \delta$, *then a general solution of the Eq. (3) in Fourier images is given by the formula*

$$\tilde{u}(\xi) = A_{\neq}^{-1}(\xi) Q_N(\xi) B_M Q_N^{-1}(\xi) A_{=}^{-1}(\xi) \widetilde{(\ell v)}(\xi) +$$

$$+ A_{\neq}^{-1}(\xi) V_\varphi^{-1} \left(\sum_{l_1=1}^{n_1} \sum_{l_2=1}^{n_2} \cdots \sum_{l_n=1}^{n_n} \tilde{c}_L(\xi'_K) \xi_{k_1}^{l_1-1} \xi_{k_2}^{l_2-1} \cdots \xi_{k_n}^{l_n-1} \right),$$

where $c_L(x'_K) \in H^{S_L}(\mathbf{R}^{M-n})$ are arbitrary functions, $S_L=(s_1-æ_1+l_1-1/2,\dots,s_n-æ_n+l_n-1/2)$, $l_j=1,2,\dots,n_j)$, $j=1,2,...,n$, ℓv is an arbitrary continuation of v onto $H^{S-\alpha}(\mathbf{R}^M)$.

The a priori estimate

$$||u||_S \leq const \left(||v||^+_{S-\alpha} + \sum_{l_1=1}^{n_1}\sum_{l_2=1}^{n_2}\cdots\sum_{l_n=1}^{n_n} ||c_L||_{S_L} \right)$$

holds.

Proof Similar to the proof of Theorem 1 we obtain the equality (4). Further, let us note that the function $A_{=}^{-1}(\xi)\widetilde{(\ell v)}(\xi)$ belongs to the space $\tilde{H}^{S-æ}(\mathbf{R}^M)$. So, if take an arbitrary polynomial $Q_N(\xi)$ satisfying the condition (6) then the function $Q_N^{-1}(\xi)A_{=}^{-1}(\xi)\widetilde{(\ell v)}(\xi)$ will belong to the space $\tilde{H}^{-\delta}(\mathbf{R}^M)$.

Further, according to the theory of multidimensional Riemann problem [14] we can represent the latter function as a sum of two summands, this is so called a jump problem which can be solved by the operator B_M:

$$Q_N^{-1}A_{=}^{-1}\widetilde{(\ell v)} = f_+ + f_-,$$

where $f_+ \in \tilde{H}^{-\delta}(C)$, $f_- \in \tilde{H}^{-\delta}(\mathbf{R}^M \setminus C)$,

$$f_+ = B_M(A_{=}^{-1}\widetilde{(\ell v)}), \quad f_- = (I - B_M)(A_{=}^{-1}\widetilde{(\ell v)}).$$

Multiplying the equality (4) by $Q_N^{-1}(\xi)$ we rewrite it in the form

$$Q_N^{-1}A_{\neq}\tilde{u} + Q_N^{-1}A_{=}^{-1}\tilde{u}_- = f_+ + f_-,$$

or

$$Q_N^{-1}A_{\neq}\tilde{u} - f_+ = f_- - Q_N^{-1}A_{=}^{-1}\tilde{u}_-$$

In other words

$$A_{\neq}\tilde{u} - Q_N f_+ = Q_N f_- - A_{=}^{-1}\tilde{u}_-. \tag{7}$$

The left hand side of the equality (7) belongs to the space $\tilde{H}^{-N-\delta}(C)$, bur the right hand side belongs to the space $\tilde{H}^{-N-\delta}(\mathbf{R}^M \setminus C)$. Therefore, we have

$$F_M^{-1}(A_{\neq}\tilde{u} - Q_N f_+) = F_M^{-1}(Q_N f_- - A_{=}^{-1}\tilde{u}_-),$$

where the left hand side belongs to the space $H^{-N-\delta}(C)$, but right hand side belongs to the space $H^{-N-\delta}(\mathbf{R}^M \setminus C)$, from which we conclude immediately that this is a distribution supported on the surface ∂C.

The form for such a distribution is given in [22] for the cone C_{K_j} with help of the operator V_{φ_j}. Thus, we apply the operator T_φ to the latter equality and obtain

$$T_\varphi F_M^{-1}(A_{\neq}\tilde{u} - Q_N f_+) = T_\varphi F_M^{-1}(Q_N f_- - A_{=}^{-1}\tilde{u}_-),$$

so that both left hand side and right hand side is a distribution supported on the hyper-plane $x_{k_1} = 0, x_{k_2} = 0, \dots, x_{k_n} = 0$. Then

$$T_\varphi F_M^{-1}(A_{\neq}\tilde{u} - Q_N f_+) =$$

$$= \sum_{l_1=1}^{n_1}\sum_{l_2=1}^{n_2}\cdots\sum_{l_n=1}^{n_n} c_L(x'_K)\delta^{(l_1-1)}(x_{k_1})\delta^{(l_2-1)}(x_{k_2})\cdots\delta^{(l_n-1)}(x_{k_n}),$$

where $L = l_1, \dots, l_n)$, $x'_K = (x'_{K_1}, \dots, x'_{K_n}) \in \mathbf{R}^{M-n}$, δ is the Dirac mass-function.

Applying the Fourier transform we obtain

$$F_M T_\varphi F_M^{-1}(A_{\neq}\tilde{u} - Q_N f_+) =$$

$$= \sum_{l_1=1}^{n_1}\sum_{l_2=1}^{n_2}\cdots\sum_{l_n=1}^{n_n} \tilde{c}_L(\xi'_K)\xi_{k_1}^{l_1-1}\xi_{k_2}^{l_2-1}\cdots\xi_{k_n}^{l_n-1}, \tag{8}$$

Taking into account that $F_M T_\varphi F_M^{-1}$ we can write

$$A_{\neq}\tilde{u} - Q_N f_+ = V_\varphi^{-1}\left(\sum_{l_1=1}^{n_1}\sum_{l_2=1}^{n_2}\cdots\sum_{l_n=1}^{n_n} \tilde{c}_L(\xi'_K)\xi_{k_1}^{l_1-1}\xi_{k_2}^{l_2-1}\cdots\xi_{k_n}^{l_n-1}\right),$$

or finally

$$\begin{aligned}\tilde{u}(\xi) = A_{\neq}^{-1}(\xi)Q_N(\xi)B_M Q_N^{-1}(\xi)A_{=}^{-1}(\xi)\widetilde{(\ell v)}(\xi) + \\ + A_{\neq}^{-1}(\xi)V_\varphi^{-1}\left(\sum_{l_1=1}^{n_1}\sum_{l_2=1}^{n_2}\cdots\sum_{l_n=1}^{n_n} \tilde{c}_L(\xi'_K)\xi_{k_1}^{l_1-1}\xi_{k_2}^{l_2-1}\cdots\xi_{k_n}^{l_n-1}\right).\end{aligned} \tag{9}$$

To obtain a priori estimates let us note that all summands in the formula (8) should belong to the space $\widetilde{H}^{S-æ}(\mathbf{R}^M)$. We take one of summands and estimate corresponding integral.

$$||\tilde{c}_L(\xi'_K)\xi_{k_1}^{l_1-1}\xi_{k_2}^{l_2-1}\cdots\xi_{k_n}^{l_n-1}||_{S-æ}^2 \le$$

$$\le \int_{\mathbf{R}^M} |\tilde{c}_L(\xi'_K)|^2 \prod_{j=1}^{n}(1 + |\xi_{K_j})^{2(s_j-æ_j)} \prod_{j=1}^{n} |\xi_{k_j}|^{2(l_j-1)} d\xi'_K \prod_{j=1}^{n} d\xi_{k_j} \le$$

$$\le \int_{\mathbf{R}^M} |\tilde{c}_L(\xi'_K)|^2 \prod_{j=1}^{n}(1 + |\xi_{K_j}|^{2(s_j-æ_j+l_j-1)} d\xi'_K \prod_{j=1}^{n} d\xi_{k_j},$$

and for existence of each integral of the type

$$\int\limits_{-\infty}^{+\infty}(1+|\xi'_{K_j}|+|\xi_{k_j}|)^{2(s_j-æ_j+l_j-1)}d\xi_{k_j}$$

the condition

$$2(s_j-æ_j+l_j-1)<-1 \tag{10}$$

is necessary. It is equivalent to the following condition

$$s_j-æ_j+l_j<1.$$

Since we have $s_j-æ_j+l_j=-n_j-\delta_j+l_j$ then we see that the condition (10) is satisfied for all

$$l_j=1,2,\dots,n_j,$$

but it is not satisfied for $l_j=n_j+1$. After integration on all ξ_{k_j} we will find that $\tilde{c}_L(\xi'_K)\in\widetilde{H}^{S_L}(\mathbf{R}^{M-n})$, where $S_L=(s_1-æ_1+l_1-1/2,\dots,s_n-æ_n+l_n-1/2)$, and $l_j=1,2,\dots,n_j$, $j=1,2.\dots,n$.

For a priori estimates we have

$$||A_{\neq}^{-1}(\xi)Q_N(\xi)B_MQ_N^{-1}(\xi)A_{=}^{-1}(\xi)\widetilde{(\ell v)}(\xi)||_S\le$$

$$\le const\ ||B_MQ_N^{-1}(\xi)A_{=}^{-1}(\xi)\widetilde{(\ell v)}(\xi)||_{S-æ+N}\le$$

$$\le const\ ||Q_N^{-1}(\xi)A_{=}^{-1}(\xi)\widetilde{(\ell v)}(\xi)||_{S-æ+N}\le$$

$$\le const\ ||\widetilde{(\ell v)}(\xi)||_{S-æ+N-N+æ-\alpha}=const\ ||\widetilde{(\ell v)}(\xi)||_{S-\alpha}\le const\ ||v||^{+}_{S-\alpha}$$

according to Lemma 1 and the fact that $S-æ+N=\delta.|\delta_j|<1/2$, $j=1,\dots,n$.

To estimate other summands in the formula (9) we use above considerations. Really, if $\tilde{c}_L(\xi'_K)\in\widetilde{H}^{S_L}(\mathbf{R}^{M-n})$ then each summand $\tilde{c}_L(\xi'_K)\xi_{k_1}^{l_1-1}\xi_{k_2}^{l_2-1}\cdots\xi_{k_n}^{l_n-1}$ in the formula (9) belongs to the space $\widetilde{H}^{S-æ}(\mathbf{R}^M)$. Thus, we have

$$||A_{\neq}^{-1}(\xi)V_{\varphi}^{-1}\tilde{c}_L(\xi'_K)\xi_{k_1}^{l_1-1}\xi_{k_2}^{l_2-1}\cdots\xi_{k_n}^{l_n-1}||_S\le$$

$$\le const\ ||V_{\varphi}^{-1}\tilde{c}_L(\xi'_K)\xi_{k_1}^{l_1-1}\xi_{k_2}^{l_2-1}\cdots\xi_{k_n}^{l_n-1}||_{S-æ}\le$$

$$\le const\ ||\tilde{c}_L(\xi'_K)\xi_{k_1}^{l_1-1}\xi_{k_2}^{l_2-1}\cdots\xi_{k_n}^{l_n-1}||_{S-æ}\le const\ ||\tilde{c}_L||_{S_L}.$$

The latter estimate was obtained above. The Theorem 2 is proved. □

Remark 1 This formula includes the operator V_φ. Examples 2 and 3 give exact representation for this operator for certain concrete cones.

Conclusion

These studies led to different boundary value problems for such elliptic pseudo-differential equations in cones similar to [14, 17, 18]. Particularly, for the case of Theorem 2 a general solution of the Eq. (3) includes a lot of arbitrary functions from corresponding Sobolev–Slobodetskii spaces. To determine these functions uniquely one needs some additional conditions (not necessary boundary conditions). We will try to describe certain statements of boundary value problems in forthcoming papers.

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