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On Solvability of Certain Non-Standard Elliptic Problems

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Abstract. We consider a model elliptic pseudo-differential equation in a 4-wedge conical canonical 3D singular domain with two parameters. It is shown that the solution of a special boundary value problem for this equation can have a limit with respect to endpoint values of the parameters in appropriate Sobolev – Slobodetskii space if the boundary function is a solution of a special functional singular integral equation.

INTRODUCTION

A lot of papers (see, for example, [1, 2, 3]) are devoted to constructing and developing the theory of elliptic pseudo-differential operators and equations on non-smooth manifolds or manifolds with non-smooth boundaries. The term "theory" means an existence in these stidies Fredholmness and index theorems.

Using the local principle and studying invertibility properties for model operators in so called canonical domains we at the same time investigate Fredholm properties for general elliptic pseudo-differential operators [4, 5, 6, 7]. Let us note that distinct model operators and canonical domains generate distinct Fredholm theories.

By canonical domain as usual we mean a certain cone in m-dimensional space. It may be a whole space $\mathbb{R}^m$, a half-space $\mathbb{R}^m_+ = \{x \in \mathbb{R}^m : x = (x', x_m), x_m > 0\}$ or a certain typical cone in $\mathbb{R}^m$.

Let C be a convex cone in $\mathbb{R}^m$ non including a straight line. Let us consider a pseudo-differential operator of the following type

$$(Au)(x) = \int\limits_{C}\int\limits_{\mathbb{R}^m} A(\xi)e^{i(x-y)\cdot\xi}u(y)d\xi dy, \quad x \in C,$$

and a model equation

$$(Au)(x) = v(x), \quad x \in C, \tag{1}$$

assuming that the symbol $A(\xi)$ of the operator A satisfies the condition

$$c_1(1+|\xi|)^{\alpha} \leq |A(\xi)| \leq c_2(1+|\xi|)^{\alpha}, \alpha \in \mathbb{R}. \tag{2}$$

We would like to remind that new approach to studying pseudo-differential equations on manifolds with a non-smooth boundary was developed in [8]. It is based on studying invertibility conditions for a model pseudo-differential operator A or conditions of unique solvability for the model equation (1) in appropriate functional spaces. To describe these conditions the concept of wave factorization for an elliptic symbol was used [8].

ELLIPTIC EQUATION IN A MULTIDIMENSIONAL CONE

We will start from three-dimensional case. Let C^{ab}_+ be a conical canonical domain of the following type

$$C^{ab}_+ = \{x \in \mathbb{R}^3 : x = (x_1, x_2, x_3), x_3 > a|x_1| + b|x_2|, a, b > 0\}.$$

As usual $\tilde{A}(\xi)$ denotes symbol of the pseudo-differential operator A, it does not depend on a spatial variable x and satisfies the condition (2). Here we remind some definitions from [9].

A radial tube domain over the cone C^{ab}_+ is called a domain in 3-dimensional complex space $\mathbb{C}^3$ of the following type

$$T(C^{ab}_+) \equiv \{z \in \mathbb{C}^3 : z = x + iy, x \in \mathbb{R}^3, y \in C^{ab}_+\}.$$

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A conjugate cone $\overset{*}{C}{}_{+}^{ab}$ is called such a cone in which for all points the condition

$$x \cdot y > 0, \quad \forall y \in C_{+}^{ab},$$

holds; $x \cdot y$ means inner product for x and y.

According to [10] we introduce the wave factorization for a symbol.

Definition 1. *Wave factorization of elliptic symbol $\tilde{A}(\xi)$ with respect to the cone C_{+}^{ab} is called its representation in the form*

$$\tilde{A}(\xi) = \tilde{A}_{\neq}(\xi)\tilde{A}_{=}(\xi),$$

where the factors $\tilde{A}_{\neq}(\xi), \tilde{A}_{=}(\xi)$ satisfy the following conditions:

1) $\tilde{A}_{\neq}(\xi), \tilde{A}_{=}(\xi)$ are defined everywhere excluding may be the points $\{\xi \in \mathbb{R}^3 : \xi_3 = \frac{1}{2a}|\xi_1| + \frac{1}{2b}|\xi_2|\}$;

2) $\tilde{A}_{\neq}(\xi), \tilde{A}_{=}(\xi)$ admit analytic continuation into radial tube domains $T(\overset{*}{C}{}_{+}^{ab}), T(-\overset{*}{C}{}_{+}^{ab})$ respectively with estimates

$$|A_{\neq}^{\pm 1}(\xi + i\tau)| \leq c_1(1 + |\xi| + |\tau|)^{\pm},$$

$$|A_{=}^{\pm 1}(\xi - i\tau)| \leq c_2(1 + |\xi| + |\tau|)^{\pm(\alpha -)}, \ \forall \tau \in \overset{*}{C}{}_{+}^{ab}.$$

The real number is called an index of the wave factorization.

Remark 1. *Everywhere below we assume that such a factorization exists.*

We consider multidimensional equation (1) in analogous Sobolev–Slobodetskii space $H^s(C_{+}^{ab})$ [10]. For simplicity we study the homogeneous equation

$$(Au)(x) = 0, x \in C_{+}^{ab}. \tag{3}$$

To describe a general solution of the equation (1) for the case $-s = 1 + \delta, |\delta| < 1/2$ we use some results from [11]. Further, we introduce the following singular integral operators [12, 13]

$$(S_1 u)(\xi_1, \xi_2, \xi_3) = v.p \frac{i}{2\pi} \int\limits_{-\infty}^{+\infty} \frac{u(\tau, \xi_2, \xi_3) d\tau}{\xi_1 - \tau},$$

$$(S_2 u)(\xi_1, \xi_2, \xi_3) = v.p \frac{i}{2\pi} \int\limits_{-\infty}^{+\infty} \frac{u(\xi_1, \eta, \xi_3) d\eta}{\xi_2 - \eta}$$

and write

$$A_{\neq}(\xi)\tilde{u}(\xi) = \tilde{C}_1(\xi_1 - a\xi_3, \xi_2 - b\xi_3) + \tilde{C}_2(\xi_1 - a\xi_3, \xi_2 + b\xi_3)$$

$$+\tilde{C}_3(\xi_1 + a\xi_3, \xi_2 - b\xi_3) + \tilde{C}_4(\xi_1 + a\xi_3, \xi_2 + b\xi_3), \tag{4}$$

where

$$\tilde{C}_1(\xi_1 - a\xi_3, \xi_2 - b\xi_3) = \frac{1}{4}\tilde{c}_0(\xi_1 - a\xi_3, \xi_2 - b\xi_3) - \frac{1}{2}(S_1\tilde{c}_0)(\xi_1 - a\xi_3, \xi_2 - b\xi_3)$$

$$-\frac{1}{2}(S_2\tilde{c}_0)(\xi_1 - a\xi_3, \xi_2 - b\xi_3) + (S_1 S_2 \tilde{c}_0)(\xi_1 - a\xi_3, \xi_2 - b\xi_3);$$

$$\tilde{C}_2(\xi_1-a\xi_3,\xi_2+b\xi_3)=\frac{1}{4}\tilde{c}_0(\xi_1-a\xi_3,\xi_2+b\xi_3)-\frac{1}{2}(S_1\tilde{c}_0)(\xi_1-a\xi_3,\xi_2+b\xi_3)$$

$$+\frac{1}{2}(S_2\tilde{c}_0)(\xi_1-a\xi_3,\xi_2+b\xi_3)-(S_1S_2\tilde{c}_0)(\xi_1-a\xi_3,\xi_2+b\xi_3);$$

$$\tilde{C}_3(\xi_1+a\xi_3,\xi_2-b\xi_3)=\frac{1}{4}\tilde{c}_0(\xi_1+a\xi_3,\xi_2-b\xi_3)+\frac{1}{2}(S_1\tilde{c}_0)(\xi_1+a\xi_3,\xi_2-b\xi_3)$$

$$-\frac{1}{2}(S_2\tilde{c}_0)(\xi_1+a\xi_3,\xi_2-b\xi_3)-(S_1S_2\tilde{c}_0)(\xi_1+a\xi_3,\xi_2-b\xi_3);$$

$$\tilde{C}_4(\xi_1+a\xi_3,\xi_2+b\xi_3)=\frac{1}{4}\tilde{c}_0(\xi_1+a\xi_3,\xi_2+b\xi_3)+\frac{1}{2}(S_1\tilde{c}_0)(\xi_1+a\xi_3,\xi_2+b\xi_3)$$

$$+\frac{1}{2}(S_2\tilde{c}_0)(\xi_1+a\xi_3,\xi_2+b\xi_3)+(S_1S_2\tilde{c}_0)(\xi_1+a\xi_3,\xi_2+b\xi_3).$$

To determine uniquely the arbitrary function $c_0(\xi_1,\xi_2)$ we require certain additional condition. for example, we assume that the restriction $\tilde{u}(\xi_1,\xi_2,0)$ is given, i.e. the following integral

$$\int_{-\infty}^{+\infty} u(x_1,x_2,x_3)dx_3 \equiv g(x_1,x_2), \tag{5}$$

it gives the equality

$$\tilde{u}(\xi_1,\xi_2,0)=\tilde{g}(\xi_1,\xi_2). \tag{6}$$

Substituting (6) into (4) we obtain all summands. Indeed,

$$\tilde{A}_{\neq}(\xi)\tilde{u}(\xi)=\sum_{k=1}^{4}\tilde{C}_k(\xi_1,\xi_2)$$

$$=\frac{1}{4}\tilde{c}_0(\xi_1,\xi_2)-\frac{1}{2}(S_1\tilde{c}_0)(\xi_1,\xi_2)-\frac{1}{2}(S_2\tilde{c}_0)(\xi_1,\xi_2)+(S_1S_2\tilde{c}_0)(\xi_1,\xi_2)$$

$$+\frac{1}{4}\tilde{c}_0(\xi_1,\xi_2)-\frac{1}{2}(S_1\tilde{c}_0)(\xi_1,\xi_2)+\frac{1}{2}(S_2\tilde{c}_0)(\xi_1,\xi_2)-(S_1S_2\tilde{c}_0)(\xi_1,\xi_2)$$

$$+\frac{1}{4}\tilde{c}_0(\xi_1,\xi_2)+\frac{1}{2}(S_1\tilde{c}_0)(\xi_1,\xi_2)-\frac{1}{2}(S_2\tilde{c}_0)(\xi_1,\xi_2)-(S_1S_2\tilde{c}_0)(\xi_1,\xi_2)$$

$$+\frac{1}{4}\tilde{c}_0(\xi_1,\xi_2)+\frac{1}{2}(S_1\tilde{c}_0)(\xi_1,\xi_2)+\frac{1}{2}(S_2\tilde{c}_0)(\xi_1,\xi_2)+(S_1S_2\tilde{c}_0)(\xi_1,\xi_2)=\tilde{c}_0(\xi_1,\xi_2).$$

Taking into account the condition (6) we find

$$\tilde{c}_0(\xi')=\tilde{A}_{\neq}(\xi',0)\tilde{g}(\xi'). \tag{7}$$

Theorem 1. *Let $-s=1+\delta, |\delta|<1/2, g\in H^{s+1/2}(\mathbb{R}^2)$. Then the unique solution of the problem (3),(5) is given by the formula (4), and $c_0(x_1,x_2)$ is determined by the formula (7).*

DEGENERATED CASE

In the above section we have two parameters of the cone a and b. Degenerated case corresponds to that case when one of the parameters (or both) tends to 0 or $+\infty$. Let us note that case $a,b \to 0$ was studied in [4], the cases $a \to 0, b = const$ and $a = const, b \to 0$ were studied in [10].Here we consider limit case $a \to +\infty, b = const$, the case $a = const, b \to +\infty$ looks almost the same..

Starting point for this consideration will be the equality (4). We use the change of variables $\xi_1 - a\xi_3 = t_1, \xi_1 + a\xi_3 = t_3$ from which we have $\xi_1 = \frac{t_3+t_1}{2}, \xi_3 = \frac{t_3-t_1}{2a}$. So, we have new variables t_1, ξ_2, t_3. If we use the condition (5) we can find the unknown function $\tilde{c}_0$ by the formula (7). Now let us write the formula (4) for the new variables t_1, ξ_2, t_3. Then we obtain

$$A_{\neq}\left(\frac{t_2+t_1}{2}, \xi_2, \frac{t_3-t_1}{2a}\right)\tilde{u}\left(\frac{t_2+t_1}{2}, \xi_2, \frac{t_3-t_1}{2a}\right) = \tilde{C}_1\left(t_1, \xi_2 - b\frac{t_3-t_1}{2a}\right) + \tilde{C}_2\left(t_1, \xi_2 + b\frac{t_3-t_1}{2a}\right)$$

$$+\tilde{C}_3\left(t_3, \xi_2 - b\frac{t_3-t_1}{2a}\right) + \tilde{C}_4\left(t_3, \xi_2 + b\frac{t_3-t_1}{2a}\right). \tag{8}$$

Tending a to $+\infty$ we obtain the following relation

$$A_{\neq}\left(\frac{t_2+t_1}{2}, \xi_2, 0\right)\tilde{u}\left(\frac{t_2+t_1}{2}, \xi_2, 0\right) = \tilde{C}_1(t_1, \xi_2) + \tilde{C}_2(t_1, \xi_2) + \tilde{C}_3(t_3, \xi_2) + \tilde{C}_4(t_3, \xi_2).$$

After accurate calculations we find

$$\tilde{C}_1(t_1, \xi_2) + \tilde{C}_2(t_1, \xi_2) + \tilde{C}_3(t_3, \xi_2) + \tilde{C}_4(t_3, \xi_2) = \frac{\tilde{c}_0(t_1, \xi_2) + \tilde{c}_0(t_3, \xi_2)}{2} - (S_1\tilde{c}_0)(t_1, \xi_2) + (S_1\tilde{c}_0)(t_3, \xi_2).$$

Taking into account the condition (6), the formula (7) and new notation

$$\tilde{A}_{\neq}(\xi_1, \xi_2, 0)\tilde{g}(\xi_1, \xi_2) \equiv h(\xi_1, \xi_2),$$

we obtain the following equation with parameter ξ_2

$$h\left(\frac{t_2+t_1}{2}, \xi_2\right) = \frac{h(t_1, \xi_2) + h(t_3, \xi_2)}{2} - (S_1 h)(t_1, \xi_2) + (S_1 h)(t_3, \xi_2). \tag{9}$$

Thus, we have the following property.

Theorem 2. *If the symbol $A(\xi)$ admits the wave factorization with respect to C_+^{ab} with the index such that $-s = 1+\delta, |\delta| < 1/2$ for enough large a then the unique solution of the boundary value problem (3),(5) has a limit under $a \to +\infty$ if and only if the boundary function $g \in H^{s+1/2}(\mathbb{R}^2)$ satisfies the equation (9).*

CONCLUSION

We have considered here a homogeneous equation only, it was done for simplicity. More general and more interesting situation is considering non-homogeneous equations. Some steps in this direction were suggested, but this approach should be verified.

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