

## A NOTE ON MINIMUM DEGREE, BIPARTITE HOLES, AND HAMILTONIAN PROPERTIES\*

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### Abstract

We adopt the recently introduced concept of the bipartite-hole-number due to McDiarmid and YOLOV, and extend their result on Hamiltonicity to other Hamiltonian properties of graphs with a large minimum degree in terms of this concept. An  $(s, t)$ -bipartite-hole in a graph  $G$  consists of two disjoint sets of vertices  $S$  and  $T$  with  $|S| = s$  and  $|T| = t$  such that  $E(S, T) = \emptyset$ . The bipartite-hole-number  $\tilde{\alpha}(G)$  is the maximum integer  $r$  such that  $G$  contains an  $(s, t)$ -bipartite-hole for every pair of nonnegative integers  $s$  and  $t$  with  $s + t = r$ . Our main results are that a graph  $G$  is traceable if  $\delta(G) \geq \tilde{\alpha}(G) - 1$ , and Hamilton-connected if  $\delta(G) \geq \tilde{\alpha}(G) + 1$ , both improving the analogues of Dirac's Theorem for traceable and Hamilton-connected graphs.

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## 1. INTRODUCTION

Our motivation for the presented results is a recent generalization of a classic result of Dirac [3] on Hamiltonicity (Theorem 1 below) due to McDiarmid and YOLOV [8] (Theorem 2 below). We answer the natural question whether similar extensions can be established for analogues of Dirac's Theorem for traceability and Hamilton-connectivity. Throughout this note, we use Bondy and Murty [1] for terminology and notation not defined here and only consider finite simple graphs.

For a graph  $G$ , we use  $V(G)$  and  $E(G)$  to denote the vertex set and the edge set of  $G$ , respectively. For  $v \in V(G)$ , we use  $N_G(v)$  to denote the set of neighbors of  $v$  in  $G$ , and we let  $d(v) = d_G(v) = |N_G(v)|$  denote the degree of  $v$  in  $G$ . Moreover, we use  $N_G[v] = N_G(v) \cup \{v\}$ . If the graph  $G$  is clear from the context, we will usually drop the subscript  $G$ . Let  $\delta(G)$  denote the minimum degree of (the vertices of)  $G$ . An independent set of  $G$  is a set of vertices no two of which are adjacent. The cardinality of a maximum independent set in  $G$  is called the independence number of  $G$ , and denoted by  $\alpha(G)$ . A spanning subgraph of a graph  $G$  is a subgraph obtained by edge deletions only. If  $H$  is a spanning subgraph of  $G$ , we use  $G - H$  to denote the graph with vertex set  $V(G)$  and edge set  $E(G) \setminus E(H)$ . For two disjoint nonempty subsets  $S$  and  $T$  of  $V(G)$ ,  $E[S, T]$  denotes the set of edges with one end in  $S$  and one end in  $T$ . The disjoint union of  $G$  and  $H$ , denoted by  $G + H$ , is the graph with vertex set  $V(G) \cup V(H)$  and edge set  $E(G) \cup E(H)$ . The join of  $G$  and  $H$ , denoted by  $G \vee H$ , is the graph obtained from the disjoint union of  $G$  and  $H$  by adding edges joining every vertex of  $G$  to every vertex of  $H$ . The complement  $\bar{G}$  of  $G$  is the graph with vertex set  $V(G)$  and the property that  $uv \in E(\bar{G})$  if and only if  $uv \notin E(G)$ .

A connected graph  $G$  is said to be  $k$ -connected if it has more than  $k$  vertices and remains connected whenever fewer than  $k$  vertices are removed. The connectivity  $\kappa(G)$  of  $G$  is the maximum value of  $k$  for which  $G$  is  $k$ -connected.

If  $C$  is a cycle in  $G$ , we let  $\vec{C}$  denote the cycle  $C$  with a clockwise or anticlockwise orientation. For  $u, v \in V(C)$  with a fixed chosen orientation for  $C$ , we let  $\vec{C}[u, v]$  denote the consecutive vertices on  $C$  from  $u$  to  $v$  in the direction specified by  $\vec{C}$ . The same vertices, in reverse order, are given by  $\overleftarrow{C}[v, u]$ . Both  $\vec{C}[u, v]$  and  $\overleftarrow{C}[v, u]$  are considered as paths and as vertex sets in the sequel. Note that we do not exclude the possibility that  $u = v$ ; in this case both  $\vec{C}[u, v]$  and  $\overleftarrow{C}[v, u]$  reduce to one vertex.

A cycle passing through all the vertices of a graph is called a Hamilton cycle. Similarly, a path passing through all the vertices of a graph is called a Hamilton path. A graph  $G$  is said to be Hamiltonian if  $G$  has a Hamilton cycle, traceable if  $G$  has a Hamilton path, and Hamilton-connected if every two vertices of  $G$  are connected by a Hamilton path.

Already back in 1952, Dirac [3] gave the following minimum degree condition for a graph to be Hamiltonian.

**Theorem 1** [3]. *A graph  $G$  with  $n \geq 3$  vertices is Hamiltonian if  $\delta(G) \geq n/2$ .*

There exist many generalizations of Dirac's Theorem. In this note we refrain from providing more details. For more information on some of these generalizations, we refer the reader to [2, 4–7, 9, 10].

Motivated by Dirac's Theorem, in a paper of 2017 McDiarmid and YOLOV [8] introduced a new graph parameter which they named the bipartite-hole-number.

**Definition** [8]. An  $(s, t)$ -bipartite-hole in a graph  $G$  consists of two disjoint sets of vertices  $S$  and  $T$  with  $|S| = s$  and  $|T| = t$  such that  $E(S, T) = \emptyset$ . The bipartite-hole-number  $\tilde{\alpha}(G)$  is the least integer  $r$  that can be written as  $r = s + t - 1$  for some positive integers  $s$  and  $t$  such that  $G$  does not contain an  $(s, t)$ -bipartite-hole.

As stated in [8], an equivalent definition of  $\tilde{\alpha}(G)$  is the maximum integer  $r$  such that  $G$  contains an  $(s, t)$ -bipartite-hole for every pair of nonnegative integers  $s$  and  $t$  with  $s + t = r$ .

In [8], the authors presented the following tight sufficient condition for Hamiltonicity in terms of the minimum degree and the bipartite hole number, improving Theorem 1.

**Theorem 2** [8]. *A graph  $G$  with at least three vertices is Hamiltonian if  $\delta(G) \geq \tilde{\alpha}(G)$ .*

As noted in [8], it is easy to check that a graph  $G$  with  $\delta(G) \geq n/2$  has no  $(1, \lfloor \frac{n}{2} \rfloor)$ -bipartite-hole, so for such a graph  $\delta(G) \geq n/2 \geq \tilde{\alpha}(G)$ . Motivated by this result, it is natural to consider possible counterparts of this result for other Hamiltonian properties.

In [8], the authors also presented the following result.

**Theorem 3** [8]. *Let  $r \geq 0$  be an integer and let  $G$  be a graph with at least three vertices such that  $\delta(G) \geq (r + 1)\tilde{\alpha}(G) + 3r$ . Then  $G$  contains  $r + 1$  edge-disjoint Hamilton cycles.*

The rest of this note is organized as follows. In Section 2, we will present our results, including the natural counterparts of Theorem 2 for traceable graphs and for Hamilton-connected graphs. In Section 3, we will present the proofs of our results.

## 2. MAIN RESULTS

We start with the following counterpart of Theorem 2 for traceable graphs.

**Theorem 4.** *A graph  $G$  on at least three vertices is traceable if  $\delta(G) \geq \tilde{\alpha}(G) - 1$ .*

It is easy to come up with examples showing that the result is sharp. Consider, e.g., the nontraceable graph  $G = K_{r,r+2}$ , for which clearly  $\delta(G) = r$  and  $\tilde{\alpha}(G) = r+2$ . Our next result is a counterpart of Theorem 3, providing a sufficient condition for the existence of many edge-disjoint Hamilton paths.

**Theorem 5.** *Let  $r \geq 0$  be an integer, and let  $G$  be a graph on at least three vertices with  $\delta(G) \geq (r+1)\tilde{\alpha}(G) + 3r - 1$ . Then  $G$  contains  $r+1$  edge-disjoint Hamilton paths, which have  $2(r+1)$  distinct end vertices.*

We believe that the above result is only sharp for  $r = 0$ , but we were not able to relax the condition either, and leave it as an open problem. Next, we present the analogue of Theorem 4 for Hamilton-connected graphs.

**Theorem 6.** *A graph  $G$  on at least three vertices is Hamilton-connected if  $\delta(G) \geq \tilde{\alpha}(G) + 1$ .*

This result is also sharp, in the sense that there exist non-Hamilton-connected graphs  $G$  with  $\delta(G) = \tilde{\alpha}(G) = r$  for any positive integer  $r$ . An obvious example is the graph  $G = K_{r,r}$ , satisfying  $\delta(G) = \tilde{\alpha}(G) = r$ . An analogue of Dirac's Theorem for Hamilton-connected graphs states that a graph  $G$  of order  $n$  is Hamilton-connected if  $\delta(G) \geq \frac{n+1}{2}$ . It is not difficult to show that Theorem 6 improves this result. For a graph satisfying  $\delta(G) \geq \frac{n+1}{2}$ , it is easy to check that there is no  $(1, \lfloor \frac{n-1}{2} \rfloor)$ -bipartite-hole. Hence, for such a graph  $\delta(G) \geq \frac{n+1}{2} \geq \tilde{\alpha}(G) + 1$ .

The following generalization of Dirac's Theorem for Hamilton-connected graphs is due to Chvátal and Erdős.

**Theorem 7** [2]. *A graph  $G$  with at least three vertices is Hamilton-connected if  $\kappa(G) \geq \alpha(G) + 1$ .*

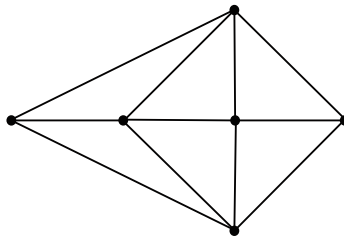


Figure 1. Graph  $G$ .

We observe that the condition of Theorem 6 is very similar to that of Theorem 7. But comparing these two theorems, neither condition implies the other. We first show an example of a graph  $G$  meeting the condition of Theorem 6 but

not of Theorem 7. Let  $G$  be the graph on vertex set  $V(G) = V(A) \cup V(B) \cup V(C)$ , where  $A = K_\ell$ ,  $B = \overline{K}_k$ ,  $C = K_k$ ,  $\ell \leq k$ , and all these subgraphs are mutually vertex-disjoint. Let the edge set of  $G$  be defined as  $E(G) = E(A) \cup E(C) \cup \{ab \mid a \in V(A), b \in V(B)\} \cup \{bc \mid b \in V(B), c \in V(C)\}$ . Obviously, we have  $\kappa(G) = k = \alpha(G)$ , and if we take  $\ell \geq 3$  and  $k \geq \ell + 3$ , we get  $\delta(G) = \ell - 1 + k \geq \min\{2\ell + 1, k + 1\} + 1 = \tilde{\alpha}(G) + 1$ . In the other direction, the graph  $G$  that is depicted in Figure 1 satisfies  $\kappa(G) = 3$ ,  $\alpha(G) = 2$  but  $\delta(G) = \tilde{\alpha}(G) = 3$ .

In the next section, we will present the details of our proofs of the above theorems.

### 3. THE PROOFS

Our proof of Theorem 4 is an easy consequence of Theorem 2 and the following observation.

**Lemma 8** (Exercise 18.1.6 on Page 474 of [1]). *Let  $G$  be a graph on at least two vertices. Then  $G$  is traceable if and only if  $G \vee K_1$  is Hamiltonian.*

**Proof of Theorem 4.** Suppose  $H = G \vee K_1$  with vertex set  $V(H) = V(G) \cup \{v\}$  and edge set  $E(H) = E(G) \cup \{vx \mid x \in V(G)\}$ . By the definition of the bipartite-hole-number, we know that  $\tilde{\alpha}(H) = \tilde{\alpha}(G)$ . Then  $\delta(H) = \delta(G) + 1 \geq \tilde{\alpha}(G) - 1 + 1 = \tilde{\alpha}(G) = \tilde{\alpha}(H)$ . Using Theorem 2, we obtain that  $H$  is Hamiltonian. Then by Lemma 8,  $G$  is traceable. ■

Our proof of Theorem 5 is also based on Lemma 8, and makes use of Theorem 3.

**Proof of Theorem 5.** Let  $H = G \vee K_1$  be defined as above. Similarly as in the above proof, we get  $\delta(H) \geq (r + 1)\tilde{\alpha}(G) + 3r - 1 + 1 = (r + 1)\tilde{\alpha}(H) + 3r$ . By Theorem 3,  $H$  has  $r + 1$  edge-disjoint Hamilton cycles. Using Lemma 8, we conclude that  $G$  has  $r + 1$  edge-disjoint Hamilton paths, and that these paths have  $2(r + 1)$  distinct end vertices. ■

At the end of this note, we present our proof of Theorem 6.

**Proof of Theorem 6.** If  $\tilde{\alpha}(G) = 1$ , then  $G$  is complete, and so  $G$  is Hamilton-connected. Hence we may suppose that  $\tilde{\alpha}(G) \geq 2$  and  $G$  is not Hamilton-connected. Then there exist two vertices  $u$  and  $v$  such that there is no Hamilton path connecting them. By Theorem 2, we know  $G$  is Hamiltonian. Let  $C$  be a Hamilton cycle in  $G$ , and let  $|V(C)| = n$ . Label the vertices in  $V(C)$  with  $[n] = \{1, 2, \dots, n\}$  in order according to the clockwise direction, where  $u = n$  and  $v = k$  for some  $k \notin \{1, n - 1, n\}$ . For a set  $S \subseteq V(C)$ , denote by  $S^+$  the set of

immediate successors  $x^+$  on  $C$  of elements  $x$  in  $S$ , and denote by  $S^-$  the set of immediate predecessors  $x^-$ .

Let  $1 \leq s \leq t$  be such that  $\tilde{\alpha}(G) + 1 = s + t$  and  $G$  has no  $(s, t)$ -bipartite-hole. Since  $\tilde{\alpha}(G) \geq 2$ , we have  $1 \leq s \leq \frac{\tilde{\alpha}(G)+1}{2} < \tilde{\alpha}(G)$ , and hence

$$|N(1) \cap \{1, 2\}| = 1 \leq s \leq \delta(G) - 2 \leq |N(1) \cap (2, n)| = d(1) - 2.$$

Therefore we can choose  $\ell \in (1, n)$  such that  $|N(1) \cap (1, \ell)| = s$ . We choose the smallest  $\ell$  with this property and note that this choice implies  $1\ell \in E(G)$ .

We know that 1 is not adjacent to  $k + 1$  since there is no Hamiltonian path from  $n$  to  $k$ . Hence, we have  $\ell \in (1, k]$  or  $\ell \in (k + 1, n)$ . Next, we consider these two cases.

*Case 1.*  $\ell \in (1, k]$ . We describe five situations (referring to Figure 2) in which there is a Hamiltonian path connecting  $n$  and  $k$ , denoted as an  $(n, k)$ -H-path in the remainder of the proof.

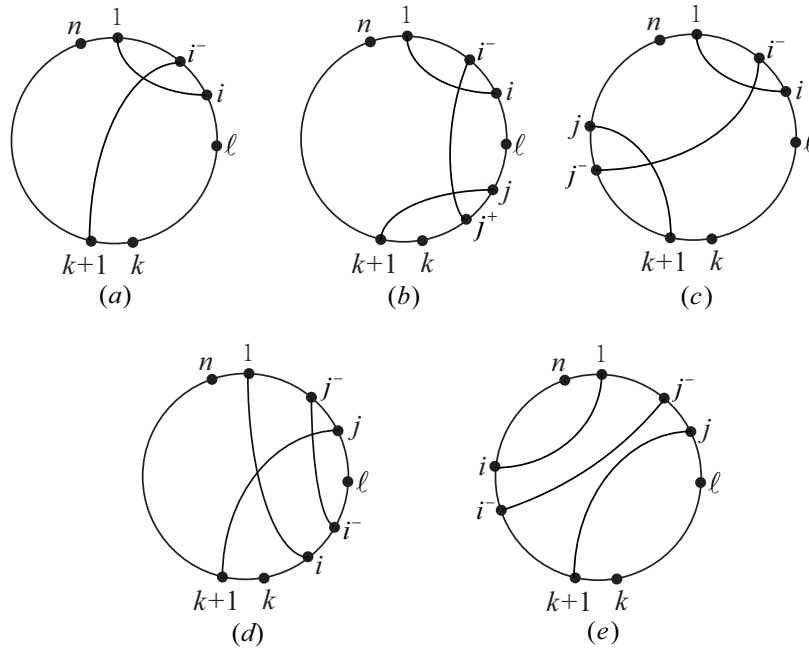


Figure 2. Situations (a)–(e).

(a) If for some  $i \in (1, \ell]$  we have  $i \in N(1)$  and  $i^- \in N(k + 1)$ , then  $\overleftarrow{C}[n, k + 1] \overleftarrow{C}[i^-, 1] \overrightarrow{C}[i, k]$  is an  $(n, k)$ -H-path.

(b) If for some  $i \in (1, \ell]$  and  $j \in (\ell, k]$  we have  $i \in N(1)$ ,  $j \in N(k + 1)$  and  $i^-j^+ \in E(G)$ , then  $\overleftarrow{C}[n, k + 1] \overleftarrow{C}[j, i] \overrightarrow{C}[1, i^-] \overrightarrow{C}[j^+, k]$  is an  $(n, k)$ -H-path. In the particular case that  $j = k$ , then  $\overleftarrow{C}[n, k + 1] \overleftarrow{C}[i^-, 1] \overrightarrow{C}[i, k]$  is an  $(n, k)$ -H-path.

(c) If for some  $i \in (1, \ell]$  and  $j \in (k + 1, n]$  we have  $i \in N(1)$ ,  $j \in N(k + 1)$  and  $i^-j^- \in E(G)$ , then  $\overleftarrow{C}[n, j]\overrightarrow{C}[k + 1, j^-]\overleftarrow{C}[i^-, 1]\overrightarrow{C}[i, k]$  is an  $(n, k)$ -H-path.

(d) If for some  $i \in (\ell, k]$  and  $j \in (1, \ell]$  we have  $i \in N(1)$ ,  $j \in N(k + 1)$  and  $i^-j^- \in E(G)$ , then  $\overleftarrow{C}[n, k + 1]\overrightarrow{C}[j, i^-]\overleftarrow{C}[j^-, 1]\overrightarrow{C}[i, k]$  is an  $(n, k)$ -H-path.

(e) If for some  $i \in (k + 1, n]$  and  $j \in (1, \ell]$  we have  $i \in N(1)$ ,  $j \in N(k + 1)$  and  $i^-j^- \in E(G)$ , then  $\overleftarrow{C}[n, i]\overrightarrow{C}[1, j^-]\overleftarrow{C}[i^-, k + 1]\overrightarrow{C}[j, k]$  is an  $(n, k)$ -H-path.

We shall show that at least one of these situations must occur.

Suppose for a contradiction that this is not the case. Then for every  $\ell \in (1, k]$

$$(1) \quad E[(N(1) \cap (1, \ell])^-, (N(k + 1) \cap (\ell, k])^+ \cup (N(k + 1) \cap (k + 1, n])^-] = \emptyset,$$

since (a), (b) and (c) do not occur; and

$$(2) \quad E[(N(1) \cap (\ell, k])^- \cup (N(1) \cap (k + 1, n])^-, (N(k + 1) \cap (1, \ell])^-] = \emptyset,$$

since (d) and (e) do not occur.

Then equation (1) implies  $|(N(k + 1) \cap (\ell, k])^+ \cup (N(k + 1) \cap (k + 1, n])^-| < t$ .

Since the two sets  $(N(k + 1) \cap (\ell, k])^+$  and  $(N(k + 1) \cap (k + 1, n])^-$  both contain the vertex  $k + 1$ , we have

$$|(N(k + 1) \cap (\ell, k])^+ \cap (N(k + 1) \cap (k + 1, n])^-| = |\{k + 1\}| = 1$$

when  $\ell < k$ , and

$$|(N(k + 1) \cap (\ell, k])^+ \cap (N(k + 1) \cap (k + 1, n])^-| = |\emptyset| = 0$$

when  $\ell = k$ . Then

$$\begin{aligned} d(k + 1) &= |N(k + 1) \cap (1, \ell]| + |N(k + 1) \cap (\ell, k]| + |N(k + 1) \cap (k + 1, n]| \\ &= |N(k + 1) \cap (1, \ell]| + |(N(k + 1) \cap (\ell, k])^+| + |(N(k + 1) \cap (k + 1, n])^-| \\ &= |N(k + 1) \cap (1, \ell]| + |(N(k + 1) \cap (\ell, k])^+ \cup (N(k + 1) \cap (k + 1, n])^-| \\ &\quad + |(N(k + 1) \cap (\ell, k])^+ \cap (N(k + 1) \cap (k + 1, n])^-| \\ &\geq \delta(G). \end{aligned}$$

Now we have  $|N(k + 1) \cap (1, \ell]| > \delta(G) - t - 1 \geq \tilde{\alpha}(G) + 1 - t - 1 = s - 1$ , i.e.,  $|N(k + 1) \cap (1, \ell]| \geq s$  when  $\ell < k$ , and  $|N(k + 1) \cap (1, \ell]| > \delta(G) - t \geq \tilde{\alpha}(G) + 1 - t = s$ , i.e.,  $|N(k + 1) \cap (1, \ell]| \geq s + 1$  when  $\ell = k$ . No matter whether  $\ell < k$  or  $\ell = k$ , equation (2) implies  $|(N(1) \cap (\ell, k])^- \cup (N(1) \cap (k + 1, n])^-| < t$ . It is obvious that  $(N(1) \cap (\ell, k])^-$  and  $(N(1) \cap (k + 1, n])^-$  are disjoint. Hence

$$\begin{aligned} \delta(G) \leq d(1) &= |N(1) \cap (1, \ell]| + |N(1) \cap (\ell, k]| + |N(1) \cap (k + 1, n]| \\ &= |N(1) \cap [1, \ell]| + |(N(1) \cap (\ell, k])^-| + |(N(1) \cap (k + 1, n])^-| \\ &= |N(1) \cap [1, \ell]| + |(N(1) \cap (\ell, k])^- \cup (N(1) \cap (k + 1, n])^-| \\ &< s + t = \tilde{\alpha}(G) + 1 \leq \delta(G), \end{aligned}$$

a contradiction.

*Case 2.*  $\ell \in (k+1, n)$ . Here, we describe four situations (referring to Figure 3) in which there is an  $(n, k)$ -H-path. Recall that  $1\ell \in E(G)$ .

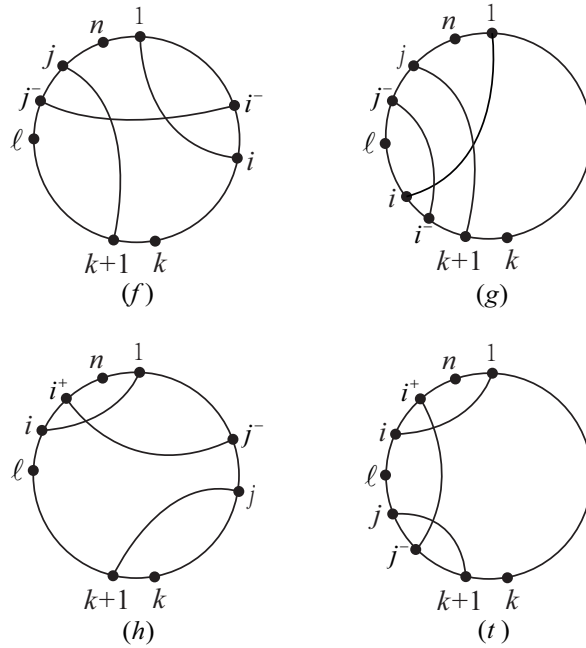


Figure 3. Situations (f)–(t).

(f) If for some  $i \in (1, k]$  and  $j \in (\ell, n]$  we have  $i \in N(1)$ ,  $j \in N(k+1)$  and  $i^-j^- \in E(G)$ , then  $\overleftarrow{C}[n, j]\overrightarrow{C}[k+1, j^-]\overleftarrow{C}[i^-, 1]\overrightarrow{C}[i, k]$  is an  $(n, k)$ -H-path.

(g) If for some  $i \in (k+1, \ell]$  and  $j \in (\ell, n]$  we have  $i \in N(1)$ ,  $j \in N(k+1)$  and  $i^-j^- \in E(G)$ , then  $\overleftarrow{C}[n, j]\overrightarrow{C}[k+1, i^-]\overleftarrow{C}[j^-, i]\overrightarrow{C}[1, k]$  is an  $(n, k)$ -H-path.

(h) If for some  $i \in [\ell, n)$  and  $j \in [1, k]$  we have  $i \in N(1)$ ,  $j \in N(k+1)$  and  $i^+j^- \in E(G)$ , then  $\overleftarrow{C}[n, i^+]\overleftarrow{C}[j^-, 1]\overleftarrow{C}[i, k+1]\overrightarrow{C}[j, k]$  is an  $(n, k)$ -H-path.

(t) If for some  $i \in [\ell, n)$  and  $j \in (k+1, \ell]$  we have  $i \in N(1)$ ,  $j \in N(k+1)$  and  $i^+j^- \in E(G)$ , then  $\overleftarrow{C}[n, i^+]\overleftarrow{C}[j^-, k+1]\overrightarrow{C}[j, i]\overrightarrow{C}[1, k]$  is an  $(n, k)$ -H-path.

We shall show that at least one of these situations must occur. Suppose for a contradiction that this is not the case. Then for every  $\ell \in (1, k)$

$$(3) \quad E[(N(1) \cap (1, \ell])^-, (N(k+1) \cap (\ell, n])^-] = \emptyset,$$

since (f) and (g) do not occur; and

$$(4) \quad E[(N(1) \cap [\ell, n))^+, (N(k+1) \cap [1, \ell])^-] = \emptyset,$$



since (h) and (t) do not occur.

Then equation (3) implies  $|N(k+1) \cap (\ell, n]| = |(N(k+1) \cap (\ell, n))^-| < t$ .  
Then

$$\begin{aligned} |(N(k+1) \cap [1, \ell])^-| &= |(N(k+1) \cap [1, \ell])| \geq \delta(G) - |(N(k+1) \cap (\ell, n])| \\ &> \delta(G) - t \geq \tilde{\alpha}(G) + 1 - t = s + t - t = s. \end{aligned}$$

Now we have  $|(N(k+1) \cap [1, \ell])^-| \geq s + 1$ . Then equation (4) implies that  $|N(1) \cap [\ell, n]| = |(N(1) \cap [\ell, n])^+| < t$ . Therefore

$$\begin{aligned} \delta(G) \leq d(1) &= |N(1) \cap (1, \ell]| + |N(1) \cap [\ell, n]| + |\{n\}| - |\{\ell\}| \\ &\leq s + t - 1 = \tilde{\alpha}(G) + 1 - 1 \leq \delta(G) - 1, \end{aligned}$$

a contradiction.

This completes the proof of Theorem 6. ■

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