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On double coverings of hyperelliptic curves

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Abstract

We prove various properties of varieties of special linear systems on double coverings of hyperelliptic curves. We show and determine the irreducibility, generically reducedness and singular loci of the variety W^r_d for bi-elliptic curves and double coverings of genus two curves. Similar results for double coverings of hyperelliptic curves of genus $h \geq 3$ are also presented.

MSC: 14H50

1. Introduction

In the articles [2] and [3], the variety of pencils on a double covering X of a **general** curve C (in the sense of Brill–Noether) has been studied extensively. In this paper, we treat double covering of a very **special** curve, i.e. when the base curve C of a double covering $\pi : X \to C$ is a hyperelliptic curve. Denoting by $W^r_d(X)$ the subscheme of the Picard variety $\mathrm{Pic}^d(X)$ consisting of line bundles of degree d of dimension at least $r+1$, we study and determine the dimension, number of irreducible components and their properties (e.g. generically reducedness) and possibly the singular locus of the irreducible

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components of the variety of special pencils $W^r_d(X)$ almost completely when the genus of the base curve C is low. We also present similar results for double coverings of hyperelliptic curves of genus $h \geq 3$. Specifically, the following is a partial list of results which will be proved in subsequent sections.

(i) If X is hyperelliptic, $W^r_d(X)$ is irreducible and singular exactly along $W^{r+1}_d(X)$; see Proposition 2.4.

(i) If X is bi-elliptic, $W^r_d(X)$ is irreducible and singular exactly along $W^{r+1}_d(X)$ for every $d \neq g-1$ and $r \geq 0$; see Proposition 2.6.

(iii) A double covering X of a curve of genus two does not carry a base-point-free and complete g^2_{g-1}. Furthermore $W^2_{g-1}(X)$ is irreducible; see Theorem 2.10.

(iv) A double covering of a hyperelliptic curve of genus $h \geq 3$ does not carry a birationally very ample g^r_{g-2h+r}; see Proposition 3.4.

The organization of this paper is as follows. In Section 2, after mentioning a few basic facts such as the Castelnuovo–Severi inequality, we treat hyperelliptic curves and bi-elliptic curves in some detail. We then proceed further to consider a double covering of a curve of genus two. After proving the non-existence of a base-point-free and complete net of degree $g-1$, we determine the minimal degree of a plane model as well as the primitive length of the double covering of genus two. In the last section, we treat double covering of a hyperelliptic curve of genus $h \geq 3$.

For all the notations and conventions used but not explained, we refer the reader to [1]. Unless otherwise stated, every curve considered in this paper is smooth irreducible and projective defined over the field of complex numbers. Throughout X is always a smooth projective curve of genus g.

2. Double coverings of low genus curves

We first recall the so-called Castelnuovo–Severi inequality and its immediate consequence which we shall make use of quite often.

Proposition 2.1 (*Castelnuovo–Severi, [1, p. 336]*). *Let X be a curve of genus g which admits a morphism $\pi_i : X \longrightarrow C_i$ of degree k_i where C_i is a curve of genus h_i for $i = 1, 2$. If $(\pi_1, \pi_2) : X \longrightarrow C_1 \times C_2$ is birational onto its image then*

$$g \leq (k_1 - 1)(k_2 - 1) + k_1h_1 + k_2h_2.$$

As a simple application of the Castelnuovo–Severi inequality, we make a note of the following regarding certain base-point-free pencils on a double covering.

Corollary 2.2. *Let $\pi : X \to C$ be a double covering of a curve C of genus h. Then any base-point-free pencil of degree $d \leq g - 2h$ is a pull-back from C via π.*

We will also use the following simple fact [4, Lemma 2.2.1] several times.

Lemma 2.3. *Let $f : X \to C$ be a non-trivial covering of smooth curves and g^r_d $(r \geq 1)$ a base-point-free linear series on X which is not induced by C. Let $p_1, \ldots, p_{r-1}$ be $r-1$ general points of X. Then the base-point-free part of the pencil $g^r_d(-p_1 - \cdots - p_{r-1})$ on X is not induced by C.*

We now start with the case when the genus of the base curve C is zero, i.e. X itself is hyperelliptic.

Proposition 2.4. *Let X be a hyperelliptic curve of genus $g \geq 2$. Then $W^r_d(X)$ is irreducible and generically reduced of dimension $d - 2r$ for $d \leq g$ and $r \geq 1$. Furthermore,*

$$\operatorname{Sing} W^r_d(X) = W^{r+1}_d(X).$$

Proof. Since any complete g^r_d with $d \leq g$ is of the form $|rg^1_2 + p_1 + \cdots + p_{d-2r}|$ where no two of the p_i's are conjugate under the hyperelliptic involution, we have

$$W^r_d(X) = \{rg^1_2\} + W_{d-2r}(X)$$

which is irreducible of dimension $d - 2r$.

We now show that $\operatorname{Sing} W^r_d(X) = W^{r+1}_d(X)$, from which the generically reducedness of $W^r_d(X)$ follows easily; note that no irreducible component of $W^r_d(X)$ is entirely contained in $W^{r+1}_d(X)$. Take $L \in \operatorname{Sing} W^r_d(X) \setminus W^{r+1}_d(X)$. We may write L as $rg^1_2 \otimes \mathcal{O}(p_1 + \cdots + p_{d-2r})$ where no two of the p_i's are conjugate under the hyperelliptic involution. We recall the following well-known description of the Zariski tangent space to $W^r_d(X)$ at $L \in W^r_d(X) \setminus W^{r+1}_d(X)$:

$$T_L W^r_d(X) \cong \operatorname{coker} \mu_0, \tag{2.4.1}$$

where

$$\mu_0 : H^0(X, L) \otimes H^0(X, KL^{-1}) \to H^0(X, K)$$

is the natural cup product map. Let $l = r + 1 = h^0(X, L)$ and $i = g - d + r$. Since $L \in \operatorname{Sing} W^r_d(X)$, we have $\dim T_L W^r_d(X) \gneq \dim W^r_d(X) = d - 2r$. By (2.4.1), we have

$$\dim T_L W^r_d(X) = g - li + \dim \ker \mu_0 \geq d - 2r + 1, \tag{2.4.2}$$

and hence

$$\dim \ker \mu_0 \geq li - l - i + 2. \tag{2.4.3}$$

Fix a basis $\{s_1, \ldots, s_l\}$ of $H^0(X, L)$, set $W_h := \operatorname{span}\{s_1, \ldots, s_h\}$ for $h = 1, \ldots, l$ and let

$$\mu_{0,W_h} : W_h \otimes H^0(X, KL^{-1}) \to H^0(X, K)$$

be the restriction of μ_0 to the subspace $W_h \otimes H^0(X, KL^{-1})$. By noting that

$$\dim \ker \mu_{0,W_h} - \dim \ker \mu_{0,W_{h-1}} \leq i$$

for $h = 3, \ldots, l$, we have

$$\dim \ker \mu_{0,W_l} - \dim \ker \mu_{0,W_2} \leq i(l - 2)$$

and hence

$$\dim \ker \mu_{0,W_2} \geq \dim \ker \mu_0 - i(l - 2) \geq li - l - i + 2 - i(l - 2) \tag{2.4.4}$$

by (2.4.3). On the other hand, by the base-point-free pencil trick

$$\begin{aligned}\dim\ker\mu_{0,W_2} &= h^0(X, K\otimes\mathcal{O}(-2rg^1_2 - p_1 - \cdots - p_{d-2r}))\\ &= g-2r-(d-2r) = g-d\end{aligned}$$

since $p_1, \ldots, p_{d-2r}$ were chosen so that no two of the p_i's are conjugate under the hyperelliptic involution. Therefore (2.4.4) becomes a numerical absurdity, showing that $\mathrm{Sing}W^r_d(X) \setminus W^{r+1}_d(X) = \emptyset$ and hence $\mathrm{Sing}W^r_d(X) \subset W^{r+1}_d(X)$. By a theorem of Mayer [8, Lemma 6], one knows $W^{r+1}_d(X) \subset \mathrm{Sing}W^r_d(X)$; see also [1, Proposition 4.2(ii), p. 189]. □

Remark 2.5. The fact that $\mathrm{Sing}W^r_d(X)$ coincides with $W^{r+1}_d(X)$ for a hyperelliptic curve was stated without proof in [8, bottom line p. 164]. We presented an easy (and perhaps standard) proof just for the convenience of the reader.

We next turn to the case of a bi-elliptic curve, i.e. when the base curve C of the double covering is an elliptic curve.

Proposition 2.6. *Let $\pi : X \to C$ be a bi-elliptic curve of genus $g \geq 6$. Then the following holds.*

(i) *For any $d \leq g-2$,*

$$W^1_d(X) = \pi^* W^1_2(C) + W_{d-4}(X),$$

which is irreducible of dimension $d-3$. Furthermore, $W^1_d(X)$ is generically reduced and

$$\mathrm{Sing}W^1_d(X) = W^2_d(X).$$

(ii) *For $d = g-1$, $W^1_{g-1}(X)$ is reducible and every irreducible component is generically reduced.*

(iii) *For $d = g$, $W^1_d(X)$ is irreducible of dimension $g-2$ with the singular locus $W^2_g(X)$ which is isomorphic to $W^1_{g-2}(X)$.*

(iv) *For $r \geq 2$ and $g-d+r \geq 2$,*

$$W^r_d(X) = \pi^* W^r_{r+1}(C) + W_{d-2r-2}(X),$$

which is irreducible and generically reduced. Moreover, if $d \leq g-2$, then

$$\mathrm{Sing}W^r_d(X) = W^{r+1}_d(X).$$

Proof. The only statements which may possibly be non-trivial (or not previously known) are the statement (i) about the singular locus of $W^1_d(X)$ for $d \leq g-2$ and the statement (iv). Indeed, the first statement of (i) is an immediate consequence of the Castelnuovo–Severi inequality. The fact that $W^1_{g-1}(X)$ is reducible with at least two irreducible components, a component, say B, containing a base-point-free pencil and the other one consisting of pencils with non-empty base locus, is due to J. Harris [1, Exercise F, p. 372]. For a base-point-free and complete $L \in B$, $\dim T_L W^1_{g-1}(X) > \dim W^1_{g-1}(X)$ if and only if $\ker\mu_0 \cong H^0(X, KL^{-2}) \neq 0$, i.e. L is a theta characteristic. Since there are only finitely many theta characteristics, B is generically reduced. For the generically reducedness of the component $\Sigma := \pi^* W_2^1(C) + W_{g-5}(X)$, see Remark 2.7(ii). The statement (iii) follows from Riemann–Roch formula and the Riemann–Kempf singularity theorem; cf. [1, p. 241].

(i) We suppose that there exists $L \in [\mathrm{Sing}\, W_d^1(X)] \setminus W_d^2(X)$, which should be of the form $L = \pi^* g_2^1 \otimes \mathcal{O}(p_1 + \cdots + p_{d-4})$ where no two p_i's are conjugate under the bi-elliptic involution; cf. Corollary 2.2. In the same way as in the proof of Proposition 2.4,

$$\dim T_L W_d^1(X) = \rho(d, g, 1) + \dim \ker \mu_0 \gneq \dim W_d^1(X) = d - 3,$$

and hence

$$\begin{aligned} \dim \ker \mu_0 &= h^0(X, KL^{-2}(\Delta)) = h^0(X, K - 2\pi^* g_2^1 - \Delta) \\ &= h^0(X, K - \pi^* g_4^3 - \Delta) \gneq g - d - 1, \end{aligned}$$

where $\Delta = p_1 + \cdots + p_{d-4}$. Therefore it follows that

$$\begin{aligned} &h^0(X, 2\pi^* g_2^1 \otimes \mathcal{O}(p_1 + \cdots + p_{d-4})) \\ &\quad = h^0(X, \pi^* g_4^3 \otimes \mathcal{O}(p_1 + \cdots + p_{d-4})) \gneq 4. \end{aligned} \tag{2.6.1}$$

Let $\mathcal{E}$ the base-point-free part of the complete linear series

$$\mathcal{D} := |\pi^* g_4^3 \otimes \mathcal{O}(p_1 + \cdots + p_{d-4})| = g_{d+4}^\alpha, \quad (\alpha \geq 4)$$

and let $\phi_\mathcal{E}$ be the morphism induced by $\mathcal{E}$. In the case $d = 4$, $\mathcal{D} = g_8^\alpha (\alpha \geq 4)$ and X is hyperelliptic by Clifford's theorem. But this is a contradiction since X cannot be both hyperelliptic and bi-elliptic by the Castelnuovo–Severi inequality. We now assume $d \geq 5$. Since $|\mathcal{E} - \pi^* g_4^3| \neq \emptyset$, the morphism induced by $\pi^* g_4^3$ (which is just π) factors through $\phi_\mathcal{E}$ and hence $\deg \phi_\mathcal{E} | \deg \pi$. If $\deg \phi_\mathcal{E} = 2$ then $\phi_\mathcal{E}(X) \overset{\mathrm{bir}}{\cong} C$ whence $\mathcal{E}$ is induced by π. Therefore there exists a pair among $\{p_1, \ldots, p_{d-4}\}$ which is in the same fiber of π, a contradiction. Hence $\phi_\mathcal{E} = 1$ and $\mathcal{E}$ is birationally very ample. By Lemma 2.3, for general $q_1, \ldots, q_{\alpha-1} \in X$, the complete series $\mathcal{E}(-q_1 - \cdots - q_{\alpha-1})$ is a pencil which is not induced by π. If $\mathcal{D}$ has a base point, or $\alpha \geq 5$, or $d \leq g - 3$,

$$\deg |\mathcal{E}(-q_1 - \cdots - q_{\alpha-1})| \leq g - 2$$

which should be induced by π by the Castelnuovo–Severi inequality, which is a contradiction. Therefore, we are left with the case $\mathcal{D}$ being base-point-free, $d = g - 2$ and $\alpha = 4$. In this case, we have

$$|K - \mathcal{D}| = g_{g-4}^1 = |\pi^* g_2^1 + r_1 + \cdots + r_{g-8}|$$

for some $r_1, \ldots, r_{g-8} \in X$ by the Castelnuovo–Severi inequality. Hence $\mathcal{D} = |K - \pi^* g_2^1 - r_1 - \cdots - r_{g-8}|$ and therefore for a general fiber $p + p' = \pi^{-1}(s)$, we have

$$\begin{aligned} h^0(X, \mathcal{D} - p - p') &= h^0(X, K - \pi^* g_2^1 - r_1 - \cdots - r_{g-8} - p - p') \\ &= h^0(X, K - \pi^* g_3^2 - r_1 - \cdots - r_{g-8}) \\ &\geq h^0(X, K - \pi^* g_3^2) - (g - 8) = (g - 4) - (g - 8) = 4, \end{aligned}$$

which in turn implies that $\mathcal{D} = \mathcal{E}$ induces a morphism of degree $t \geq 2$, a contradiction to $\mathcal{E}$ being birationally very ample. What we have shown so far is that $[\mathrm{Sing}\, W_d^1(X)] \setminus W_d^2(X) = \emptyset$ and hence $\mathrm{Sing}\, W_d^1(X) = W_d^2(X)$.

(iv) We treat the cases $g-d+r=2$ and $g-d+r\geq 3$ separately.

If $g-d+r=2$ and $r\geq 2$, we have $d\geq g$. By (i), we have

$$K-W^r_d(X)=W^1_{2g-2-d}(X)=\pi^*W^1_2(C)+W^1_{2g-d-6}(X),$$

hence $W^r_d(X)$ is irreducible of dimension $2g-d-5=d-2r-1$. The closed sublocus $\pi^*W^r_{r+1}(C)+W_{d-2r-2}(X)\subset W^r_d(X)$ is also of dimension $d-2r-1$, and it follows that $W^r_d(X)=\pi^*W^r_{r+1}(C)+W_{d-2r-2}(X)$.

If $g-d+r\geq 3$ and $r\geq 2$, we may argue as follows. Given $L\in W^r_d(X)$ and general $p_1,\ldots,p_{r-1}\in X$, $L(-p_1-\cdots-p_{r-1})\in W^1_{d-r+1}(X)$, where $d-r+1\leq g-2$, and hence $L(-p_1-\cdots-p_{r-1})$ is induced by π by the Castelnuovo–Severi inequality. Therefore L is also induced by π by Lemma 2.3 and we have

$$W^r_d(X)=\pi^*W^r_{r+1}(C)+W_{d-2r-2}(X),$$

which is irreducible of dimension $d-2r-1$.

To determine the singular locus of $W^r_d(X)$, we take $L\in[\mathrm{Sing}W^r_d(X)]\setminus W^{r+1}_d(X)$. We may write $L=\pi^*g^r_{r+1}\otimes\mathcal{O}(p_1+\cdots+p_{d-2r-2})$ where no two of the p_i's are conjugate under the bi-elliptic involution. Let $l=r+1=h^0(X,L)$ and $i=g-d+r$. By (2.4.1), we have

$$\dim T_LW^r_d(X)=g-li+\dim\ker\mu_0\geq d-2r, \tag{2.6.2}$$

and hence

$$\dim\ker\mu_0\geq li-l-i+1. \tag{2.6.3}$$

Fix a basis $\{s_1,\ldots,s_l\}$ of $H^0(X,L)$ such that s_1,s_2 has no common zeros other than $p_1,\ldots,p_{d-2r-2}$. Set $W_h:=\mathrm{span}\{s_1,\ldots,s_h\}$ for $h=1,\ldots,l$, and let

$$\mu_{0,W_h}:W_h\otimes H^0(X,KL^{-1})\to H^0(X,K)$$

be the restriction of μ_0 to the subspace $W_h\otimes H^0(X,KL^{-1})$. By noting that

$$\dim\ker\mu_{0,W_h}-\dim\ker\mu_{0,W_{h-1}}\leq i$$

for $h=3,\ldots,l$, we have

$$\dim\ker\mu_{0,W_l}-\dim\ker\mu_{0,W_2}\leq i(l-2)$$

and hence

$$\dim\ker\mu_{0,W_2}\geq\dim\ker\mu_0-i(l-2)\geq li-l-i+1-i(l-2)=g-d \tag{2.6.4}$$

by (2.6.3). Note that $\dim\ker\mu_{0,W_2}\neq 0$ by the assumption $d\leq g-1$. By the projection formula and the Riemann–Hurwitz relation for double coverings, unless $d=2r+2=g-1$, one has

$$\begin{aligned}h^0(X,2\pi^*g^r_{r+1})&=h^0(C,\pi_*(\pi^*(2g^r_{r+1})\otimes\mathcal{O}_X))=h^0(C,2g^r_{r+1}\otimes\pi_*\mathcal{O}_X)\\&=h^0(C,2g^r_{r+1}\otimes(\mathcal{O}_C\oplus\mathcal{O}_C(T)))\\&=h^0(C,2g^r_{r+1})+h^0(C,2g^r_{r+1}\otimes T)=2r+2,\end{aligned}$$

where T is a divisor such that $2T\cong -R$ with R the branch divisor of π.

Let $\mathcal{F}$ the base-point-free part of the complete linear series

$$\mathcal{G} := |2\pi^* g^r_{r+1} \otimes \mathcal{O}(p_1 + \cdots + p_{d-2r-2})| = g^{\beta}_{d+2r+2}.$$

Claim: $\beta = 2r+1$. Suppose $\beta \geq 2r+2$ and let $\phi_{\mathcal{F}}$ be the morphism induced by $\mathcal{F}$. Since $|\mathcal{F} - 2\pi^* g^r_{r+1}| = |\mathcal{F} - \pi^* g^{2r+1}_{2r+2}| \neq \emptyset$, the morphism induced by $\pi^* g^{2r+1}_{2r+2}$ (which is equal to π) factors through $\phi_{\mathcal{F}}$ and hence $\deg \phi_{\mathcal{F}} \mid \deg \pi$. If $\deg \phi_{\mathcal{F}} = 2$ then $\phi_{\mathcal{F}}(X) \overset{\text{bir}}{\cong} C$ whence $\mathcal{F}$ is induced by π. Therefore there exists a pair among $\{p_1, \ldots, p_{d-2r-2}\}$ which is in the same fiber of π, a contradiction. Hence $\phi_{\mathcal{F}} = 1$ and $\mathcal{F}$ is birationally very ample. By Lemma 2.3, for general $q_1, \ldots, q_{\beta-1} \in X$, the complete series $\mathcal{F}(-q_1 - \cdots - q_{\beta-1})$ is a pencil which is not induced by π. If $\mathcal{G}$ has a base point, or $\beta \geq 2r+3$, or $d \leq g-3$,

$$\deg |\mathcal{F}(-q_1 - \cdots - q_{\beta-1})| \leq g-2$$

which should be induced by π by the Castelnuovo–Severi inequality, a contradiction. Therefore, we are left with the case $\mathcal{G}$ being base-point-free, $d = g-2$ and $\beta = 2r+2$. In this case, we have

$$|K - \mathcal{G}| = g^1_{g-2r-2} = |\pi^* g^1_2 + r_1 + \cdots + r_{g-2r-6}|$$

for some $r_1, \ldots, r_{g-2r-6} \in X$ by the Castelnuovo–Severi inequality. Hence $\mathcal{G} = |K - \pi^* g^1_2 - r_1 - \cdots - r_{g-2r-6}|$ and therefore for a general fiber $p + p' = \pi^{-1}(s)$, we have

$$\begin{aligned} h^0(X, \mathcal{G} - p - p') &= h^0(X, K - \pi^* g^1_2 - r_1 - \cdots - r_{g-2r-6} - p - p') \\ &= h^0(X, K - \pi^* g^2_3 - r_1 - \cdots - r_{g-2r-6}) \\ &\geq h^0(X, K - \pi^* g^2_3) - (g - 2r - 6) = 2r+2, \end{aligned}$$

which in turn implies that $\mathcal{G} = \mathcal{F}$ induces a morphism of degree $t \geq 2$, a contradiction to $\mathcal{F}$ being birationally very ample, thereby finishing the proof of the Claim.

On the other hand, by the base-point-free pencil trick and the Claim,

$$\dim \ker \mu_{0,W_2} = h^0(X, K \otimes \mathcal{O}(-2\pi^* g^r_{r+1} - p_1 - \cdots - p_{d-2r-2})) = g - d - 1.$$

Therefore (2.6.4) becomes a numerical absurdity, showing that $\operatorname{Sing} W^r_d(X) \setminus W^{r+1}_d(X) = \emptyset$ and hence $\operatorname{Sing} W^r_d(X) \subset W^{r+1}_d(X)$. By a theorem of Mayer [8, Lemma 6], one knows $W^{r+1}_d(X) \subset \operatorname{Sing} W^r_d(X)$; see also [1, Proposition 4.2(ii), p. 189]. □

Remark 2.7. (i) In general, $\operatorname{Sing} W^r_d(X)$ may not be equal to $W^{r+1}_d(X)$ for a given curve X. It has been known that $\operatorname{Sing} W^r_d(X) = W^{r+1}_d(X)$ for $r = 0$, for an hyperelliptic curve X or for a general curve X of genus g. Proposition 2.6 adds bi-elliptic curves to the list of such curves (at least for $d \neq g-1$).

(ii) It seems worthwhile to remark that on a bi-elliptic curve X, $W^1_{g-1}(X)$ has the singular locus which does not coincide with $W^2_{g-1}(X)$. Note that there is a unique irreducible component of $W^1_{g-1}(X)$ whose general member has a base point, namely $\Sigma := \pi^* W^1_2(C) + W_{g-5}(X)$, and we may describe the singular points of $W^1_{g-1}(X)$ in Σ as follows. By the identification (2.4.1) for the Zariski tangent spaces to $W^r_d(X)$,

$$L := \pi^* g^1_2 \otimes \mathcal{O}(p_1 + \cdots + p_{g-5}) = \pi^* g^1_2 \otimes \mathcal{O}(\Delta) \in \operatorname{Sing}\Sigma \setminus W^2_{g-1}(X)$$

if and only if

$$h^0(X, KL^{-2}(\Delta)) = h^0(X, K - 2\pi^*(g_2^1) - \Delta) > 0.$$

Therefore, for a given $g_2^1 \in W_2^1(C) = \mathrm{Jac}(C)$,

$$\pi^* g_2^1 \otimes \mathcal{O}(p_1 + \cdots + p_{g-5}) = \pi^* g_2^1 \otimes \mathcal{O}(\Delta)$$

is singular if and only if $\Delta \in X_{g-5}$ fails to impose independent conditions on the complete series $|K - 2\pi^*(g_2^1)|$, and the collection of such divisors Δ becomes a degeneracy locus of certain bundle maps; cf. [1, Chapter VIII]. Therefore we deduce that

$$[\mathrm{Sing} W_{g-1}^1(X)] \cap \Sigma \setminus W_{g-1}^2(X) =$$
$$\{|\pi^* E + \Delta' + p|;\, E \in W_2^1(C),\, \Delta' \in X_{g-6} \setminus X_{g-6}^1 \text{ and } |K - 2\pi^* E - \Delta'| \geq p\}.$$

By noting that

$$W_{g-1}^2(X) = \pi^* W_4^2(C) + W_{g-9}(X)$$
$$\subset \{|\pi^* E + \Delta' + p|;\, E \in W_2^1(C),\, \Delta' \in X_{g-6} \text{ and } |K - 2\pi^* E - \Delta'| \geq p\}$$

we finally conclude

$$[\mathrm{Sing} W_{g-1}^1(X)] \cap \Sigma =$$
$$\{|\pi^* E + \Delta' + p|;\, E \in W_2^1(C),\, \Delta' \in X_{g-6} \text{ and } |K - 2\pi^* E - \Delta'| \geq p\}$$

which is easily seen to be irreducible of dimension $g - 5 = \dim W_2^1(C) + \dim X_{g-6}$.

(iii) For $r \geq 2$ and $d = g-1 = 2r+2$, which somehow occurred in the course of the proof of the last statement of Proposition 2.6, $\mathrm{Sing} W_d^r(X) \neq W_d^{r+1}(X)$. Indeed, $W_{2r+2}^{r+1}(X) = \emptyset$ since X is not hyperelliptic. Singular points of $W_{2r+2}^r(X)$ are just theta characteristics.

We next consider double coverings $\pi : X \to C$ where $g(C) = 2$. As for bi-elliptic curves, many things are known and we collect some of them as follows; [4, Remark (2.4.2) and Claim in p. 251].

Remark 2.8. Let $\pi : X \to C$ be a double covering of a genus two curve C and assume $g \geq 11$. Then

(1)	$W_d^1(X) = W_6^1(X) + W_{d-6}(X)$	for $6 \leq d \leq g-4$
(2)	$W_d^2(X) = W_8^2(X) + W_{d-8}(X)$	for $8 \leq d \leq g-2$
(3)	$W_d^r(X) = W_{2r+4}^r(X) + W_{d-(2r+4)}(X)$	for $2r+4 \leq d \leq g-1$ and $r \geq 3$.

For the rest of this section, we will try to make Remark 2.8 a little bit more precise and look for some possible extensions. For $r = 1$, one can make the following statement which is partially based on the above Remark 2.8(1).

Proposition 2.9. *Let $\pi : X \to C$ be a double covering of a curve of genus two where $g \geq 13$. Then the following hold.*

(i) *$W_{g-1}^1(X)$ is generically reduced and irreducible. Furthermore, a general element in $W_{g-1}^1(X)$ is base-point-free.*

(ii) $W^1_{g-2}(X)$ *is reducible with at least three irreducible components, and every component is generically reduced.*

(iii) $W^1_{g-3}(X)$ *is reducible with at least three irreducible components, and every component is generically reduced.*

(iv) *For* $6 \le d \le g-4$, $W^1_d(X)$ *is reducible with only two irreducible components which are generically reduced.*

Proof. (i) By a result of Teixidor [9], for a curve X of genus $g \ge 5$, $W^1_{g-1}(X)$ is reducible if and only if X is either (a) trigonal, (b) bi-elliptic or (c) an unramified double covering of a genus three non-hyperelliptic curve. By the Castelnuovo–Severi inequality a double covering of a curve of genus two can be neither trigonal (if $g \ge 7$), bi-elliptic (if $g \ge 9$) nor an unramified double cover of non-hyperelliptic curves of genus three. Hence $W^1_{g-1}(X)$ is irreducible. By using a theorem of Mumford [1, p. 193], it is not hard to see that a general element of a component of $W^1_{g-1}(X)$ has no base point unless X is bi-elliptic, trigonal or a smooth plane quintic; cf. [1, Exercise F-1, p. 373]. Since a double cover of a genus two curve cannot be such types of curves by the Castelnuovo–Severi inequality as long as $g \ge 8$, we deduce that a general member in $W^1_{g-1}(X)$ is base-point-free (and complete). Since a complete and base-point-free pencil $L \in W^1_{g-1}(X)$ is singular if and only if L is a theta characteristic, $W^1_{g-1}(X)$ is smooth at general points.

(ii) Note that every irreducible component of $W^1_{g-2}(X)$ has dimension $\rho(g-2, g, 1) = g-6$ by a theorem of Mumford. We first identify components of $W^1_{g-1}(X)$ whose general element has a base point. Consider two irreducible closed loci

$$\Sigma_1 := \pi^* W^1_2(C) + W_{g-6}(X) \qquad \text{and} \qquad \Sigma_2 := \pi^* W^1_3(C) + W_{g-8}(X).$$

Since $\dim \Sigma_1 = \dim \Sigma_2 = g-6$, Σ_i's are components of $W^1_{g-2}(X)$. A priori there may exist further components of $W^1_{g-2}(X)$ whose general element has a base point. Let Σ such a component. A general element $L \in \Sigma$ has only one base point, otherwise L is either in Σ_1 or Σ_2. Therefore Σ is of the form $\Sigma = A + W_1(X)$, where $A \subset W^1_{g-3}$ is an irreducible closed locus which is generically base-point-free. By Proposition 3.1, which we shall prove in the next section, a generically base-point-free component of W^1_{g-3} has dimension $\rho(g-3, g, 1) = g-8$, hence

$$\dim \Sigma \le \dim A + 1 \le g-7 \lneq \rho(g-2, g, 1),$$

which is a contradiction. Therefore we conclude that

$$W^1_{g-2}(X) = \bigcup_{j=1}^{n} B_j \cup \Sigma_1 \cup \Sigma_2,$$

where the B_j's are irreducible components which are generically base-point-free. The B_j's surely exist by [7] and are generically reduced by Proposition 3.1.

(iii) Like the case for $d = g-2$, the Castelnuovo–Severi inequality implies $W^1_{g-3}(X)$ has exactly two components whose general element has a base point, namely

$$\tilde{\Sigma}_1 := \pi^* W^1_2(C) + W_{g-7}(X) \qquad \text{and} \qquad \tilde{\Sigma}_2 := \pi^* W^1_3(C) + W_{g-9}(X)$$

and both of them has dimension $g-7$; note that the irreducible loci $\tilde{\Sigma}_i$ cannot be contained in a bigger irreducible component by a theorem of Mumford [1, p. 193]. By the main result in [7], there exists a base-point-free and complete g^1_{g-3} on X. Therefore we also have

$$W^1_{g-2}(X) = \bigcup_{j=1}^{n} B_j \cup \tilde{\Sigma}_1 \cup \tilde{\Sigma}_2,$$

where the B_j's are irreducible components which are generically base-point-free and complete. Again the B_j's are generically reduced by Proposition 3.1.

(iv) By the Castelnuovo–Severi inequality, we have

$$W^1_d(X) = \pi^* W^1_2(C) + W_{d-4}(X) \cup \pi^* W^1_3(C) + W_{d-6}(X).$$

The proof for the generically reducedness of the other components of $W^1_d(X)$ (Σ_1 & Σ_2 in (ii), $\tilde{\Sigma}_1$ & $\tilde{\Sigma}_2$ in (iii) and those in (iv)) shall be provided in a more general setting in the next section; cf. Proposition 3.7. □

To improve Remark 2.8 one step further, we will prove a theorem addressing the non-existence of a base-point-free and complete net of degree $g-1$ on a double covering of a curve of genus two.

Theorem 2.10. *A double covering $\pi : X \to C$ of a curve of genus two with $g \geq 11$ does not carry a base-point-free and complete g^2_{g-1}. Furthermore $W^2_{g-1}(X)$ is irreducible.*

Proof. Note that for a double covering X of a curve of genus two, $\dim W^2_{g-1}(X) = g-7$ and

$$\Sigma_0 := \pi^* W^2_4(C) + W_{g-9}(X) = K - \pi^* W^2_4(C) - W_{g-9}(X) \tag{2.10.1}$$

is an irreducible component of maximal dimension; cf. [5, Corollary 2.3] and [4, Claim (ii) in p. 251]. By Remark 2.8(2), we also note that Σ_0 is the only component whose general element has a non-empty base locus. Assume the existence of a component of $W^2_{g-1}(X)$, say Σ, whose general element is base-point-free. Let $g^1_4 := \pi^* g^1_2(C)$ and we choose two sections $s_0, s_1 \in H^0(X, g^1_4)$ without common zeros. For a general $L \in \Sigma$, we consider the natural map

$$H^0(X, L) \oplus H^0(X, L) \xrightarrow{\mu} H^0(X, L \otimes g^1_4),$$

defined by $\mu(t_0, t_1) := s_0 \cdot t_0 + s_1 \cdot t_1$; $t_i \in H^0(X, L)$.

Claim: $h^0(X, L \otimes g^1_4) \geq 5$.

Proof of the Claim. If $h^0(X, L \otimes g^1_4) \leq 4$, then $h^0(X, L(-g^1_4)) = \dim \ker \mu = 2$ by the base-point-free pencil trick. On the other hand, since $\deg L(-g^1_4) = g-5 \leq g-4$, $L(-g^1_4)$ is induced by π by the Castelnuovo–Severi inequality, and hence $L(-g^1_4) = |g^1_6| + \Delta$ for some effective divisor Δ of degree $g-11$. Then we would have $L = |g^1_4 + g^1_6| + \Delta$, contradicting L being base-point-free.

Now we consider the following two possibilities separately.
(i) $h^0(X, L\otimes g_4^1) \geq 6$: In this case, we have $h^0(X, KL^{-1}(-g_4^1)) \geq 2$ by the Riemann–Roch formula, whereas $\deg KL^{-1}(-g_4^1) = g-5 \leq g-4$. Hence by the Castelnuovo–Severi inequality and Proposition 2.9(iv), we have

$$KL^{-1}(-g_4^1) \in W_{g-5}^1(X) = \pi^* W_2^1(C) + W_{g-9}(X) \cup \pi^* W_3^1(C) + W_{g-11}(X),$$

implying

$$K - \Sigma - \{g_4^1\} \subset \pi^* W_2^1(C) + W_{g-9}(X) \cup \pi^* W_3^1(C) + W_{g-11}(X).$$

Since the locus $K - \Sigma - \{g_4^1\}$ is irreducible, we have either

$$K_X - \Sigma - \{g_4^1\} \subset \pi^* W_2^1(C) + W_{g-9}(X)$$

or

$$K - \Sigma - \{g_4^1\} \subset \pi^*(W_3^1(C)) + W_{g-11}(X).$$

If $K_X - \Sigma - g_4^1 \subset \pi^*(W_3^1(C)) + W_{g-11}(X)$, then

$$\begin{aligned} K - \Sigma &\subset \pi^*(W_2^1(C)) + \pi^*(W_3^1(C)) + W_{g-11}(X) \\ &\subset \pi^*(W_5^3(C)) + W_{g-11}(X) \subset W_{g-1}^3(X), \end{aligned}$$

which is impossible; cf. [1, Lemma 3.5, p. 182]. Therefore we must have

$$K - \Sigma - \{g_4^1\} \subset \pi^*(W_2^1(C)) + W_{g-9}(X),$$

implying $K - \Sigma \subset \Sigma_0 = K - \Sigma_0$, which is again a contradiction.
(ii) $h^0(X, L \otimes g_4^1) = 5$: In this case, we have $h^0(X, L - g_4^1) = \dim\ker\mu > 0$ and hence $L = g_4^1 \otimes \mathcal{O}(p_1 + \cdots + p_{g-5})$ and we may assume that $h^0(X, \mathcal{O}(p_1 + \cdots + p_{g-5})) = 1$; otherwise L would not be base-point-free by the Castelnuovo–Severi inequality. We also note that among the points $p_1, \ldots, p_{g-5}$, at most one pair of points, say $\{p_{g-6}, p_{g-5}\}$, is in the same fiber of π; otherwise L is not base-point-free either.

(ii-a) $\{p_{g-6}, p_{g-5}\}$ is in the same fiber of π: In this case, $L = |\pi^*(g_3^1)+p_1+\cdots+p_{g-7}|$. For $j = 1, \ldots, g-7$, we consider the complete linear series

$$|\pi^*(g_2^1) + p_{g-6} + p_{g-5} + p_1 + \cdots + p_j + \bar{p}_1 + \cdots + \bar{p}_j| = |\pi^*(g_{3+j}^{1+j})|,$$

which is base-point-free. Since no two p_j's (for $j = 1, \ldots, g-7$) are in the same fiber of π, $\bar{p}_j$ is not a base point of the linear series

$$|L + \bar{p}_1 + \cdots + \bar{p}_j| = |\pi^*(g_{3+j}^{1+j}) + p_{j+1} + \cdots + p_{g-7}|$$

for each $j = 1, \ldots, g-7$. Hence we have

$$\dim|L + \bar{p}_1 + \cdots + \bar{p}_j| > \dim|L + \bar{p}_1 + \cdots + \bar{p}_{j-1}|,$$

implying $|L + \bar{p}_1 + \cdots + \bar{p}_{g-7}| = g_{2g-8}^{g-5}$ whose dual is a g_6^2. Since X is neither trigonal nor bi-elliptic by the Castelnuovo–Severi inequality, we are done with this case.

(ii-b) No two among $\{p_1, \ldots, p_{g-5}\}$ are in the same fiber of π: We may use the same argument as (ii-a) to deduce $|L + \bar{p}_1 + \cdots + \bar{p}_{g-5}| = g_{2g-6}^{g-3}$ whose dual is a g_4^2, a contradiction. □

For the variety of linear series of dimension more than one, one may refine Remark 2.8 as follows.

Theorem 2.11. *Let $\pi : X \to C$ be a double covering of a curve of genus two and $g \geq 11$. Then*

(i) $W^2_d(X) = \pi^* W^2_4(C) + W_{d-8}(X)$ *for* $8 \leq d \leq g-1$
(ii) $W^r_d(X) = \pi^* W^r_{r+2}(C) + W_{d-(2r+4)}(X)$ *for* $2r+4 \leq d \leq g$ *and* $r \geq 3$,

which are irreducible.

Furthermore, $W^2_d(X)$ is generically reduced for $8 \leq d \leq g-1$ and $W^r_d(X)$ is also generically reduced for $2r+4 \leq d \leq g-4$ and $r \geq 3$, under the assumption $g \geq 12$.

Proof. (i) For $d \leq g-2$, it is clear by Remark 2.8(2). By Theorem 2.10 and the fact that $W^2_{g-2}(X) = \pi^* W^2_4(C) + W_{g-10}(X)$, we have

$$W^2_{g-1}(X) = W^2_{g-2}(X) + W_1(X) = \pi^* W^2_4(C) + W_{g-9}(X).$$

The irreducibility of those $W^2_d(X)$'s and $W^r_d(X)$'s is also clear. (ii) is also clear. Again, the proof for the generically reducedness of $W^r_d(X)$ shall be provided in a more general setting in Proposition 3.7. □

Remark 2.12. Following the notation in [6, Section 2], we set $s(r) = s(r, X)$ the minimal degree of a complete, base-point-free and simple linear series of dimension $r \geq 2$ on a given curve X.

(i) By Theorem 2.10 and [6, Corollary 2.5], we have $s(2, X) = g$ for the double coverings of genus two curves; cf. [6, Remark 2.6]. In fact, the linear series $|K - \pi^* g^1_2 - p_1 - \cdots - p_{g-6}| = g^2_g$ for generically chosen $p_1, \ldots, p_{g-6} \in X$ gives a plane model of X.

(ii) A double covering X of a curve of genus $h \leq 2$ has $s(2, X) \geq g$. It would be nice if one can identify all the curves with $s(2, X) \geq g$.

Recall that a complete and base-point-free linear series g^r_d on a given algebraic curve is called primitive if its residual series is also base-point-free. For a curve of genus $g \geq 4$, there always exists a primitive linear series other than the trivial (zero and canonical) linear series. Following [4], let the primitive length $l(X)$ of X be the cardinality of the finite set of integers consisting of Clifford indices of all non-trivial primitive linear series on X. It has been shown in [4] that the primitive length is an invariant detecting double coverings; cf. [4, Theorem 3.4.1]. The results we obtained so far in this section determine the primitive length of a double covering of a curve of genus two.

Corollary 2.13. *Let X be a smooth double covering of a genus two curve C where $g \geq 11$. Then*

(i) *there is no primitive net of degree $g-1$;*
(ii) *X has primitive length* 5.

Proof. (i) By Theorem 2.11, $W^2_{g-1}(X) = \pi^* W^2_4(C) + W_{g-9}(X) = \pi^* J(C) + W_{g-9}(X)$ and hence every complete net g^2_{g-1} has non-empty base locus, and hence is not primitive.

(ii) Remark that for a base-point-free, complete and non-primitive linear series $|D|$, there exists $p \in X$ such that $h^0(X, D + p) = h^0(X, D) + 1$ and $|D + p|$ is birationally very ample. By a result of [7], there always exists a base-point-free and complete g^1_{g-3} on X. On the other hand, since there does not exist a birationally very ample and complete g^2_{g-2} on X, any base-point-free and complete g^1_{g-3} is primitive. Likewise, any base-point-free and complete g^1_{g-2} (of which we know its existence by [7]) is primitive by Theorem 2.10. Therefore the primitive linear series g^r_d on X are complete pencils $\pi^* g^1_2, \pi^* g^1_3, g^1_{g-3}, g^1_{g-2}, g^1_{g-1}$ which have different Clifford indices. □

3. Double coverings over a curve of high genus

In this section we deal with double coverings $\pi : X \to C$ when the genus h of the base curve C is given arbitrarily. Recall that by the Castelnuovo–Severi inequality for double coverings (Corollary 2.2), any base-point-free pencil of degree $d \le g - 2h$ is a pull-back of the one on the base curve C. Therefore it is quite natural to ask if there exists a base-point-free pencil of degree $d > g - 2h$ which is not a pull-back from C. This question has been answered in the affirmative recently in [7] where the dimensions estimate of an irreducible component containing such pencils has been left open. In the next proposition we show that any component consisting of base-point-free pencils not a pull-back from C has the expected dimension. We also determine the components of $W^r_d(X)$ and their generically reducedness. As a by-product, we derive a lower bound of the minimal degree $s(2)$ of a plane model of the double covering X of a hyperelliptic curve of genus h.

Proposition 3.1. *Let $\pi : X \to C$ be a double covering of a curve C which is not necessarily hyperelliptic and $g \ge 8h - 3$. For $d \ge g - 2h + 1$, let B be an irreducible component of $W^1_d(X)$ whose general element is a base-point-free complete pencil not composed with the involution induced by π. Then B is generically reduced and*

$$\dim B = \rho(d, g, 1) = 2d - g - 2.$$

Proof. The existence of such component B is the main result of [7]. We only give a proof for the border line case $d = g - 2h + 1$; the case $d > g - 2h + 1$ is similar. Using the same notations and conventions as in the proof of Proposition 2.4, we take a general element $L \in B \subset W^1_{g-2h+1}(X)$ and consider the cup product map

$$\mu_0 : H^0(X, L) \otimes H^0(X, KL^{-1}) \to H^0(X, K).$$

Since L is a base-point-free and complete pencil, we have $\ker \mu_0 = H^0(X, KL^{-2})$ by the base-point-free pencil trick. If $\ker \mu_0 = \{0\}$, then

$$\dim T_L W^1_{g-2h+1}(X) = \rho(g - 2h + 1, g, 1) = \dim W^1_{g-2h+1}(X)$$

by (2.4.1).

Assume that $h^0(X, KL^{-2}) > 0$. Then we have a finite map

$$\begin{array}{rcl} B & \to & W_{4h-4}(X) \\ L & \mapsto & KL^{-2} \end{array}$$

and hence $\rho(g-2h+1,g,1) = g-4h \le \dim B \le \dim W_{4h-4h}(X) = 4h-4$, contrary to the assumption $g \ge 8h-3$. The generically reducedness is clear. □

Proposition 3.2. *Let $\pi : X \to C$ be a double covering of a hyperelliptic curve of genus $h \ge 3$ and $g \ge \max\{6h, 8h-3\}$. Set*

$$\Sigma_1 := \pi^* W_2^1(C) + W_{d-4}(X) \ \&\ \Sigma_2 := \pi^* W_{h+1}^1(C) + W_{d-2h-2}(X).$$

Then the following holds.

(i) *For any $d \le g-2h$,*

$$W_d^1(X) = \Sigma_1 \cup \Sigma_2$$

and both Σ_1 & Σ_2 are generically reduced irreducible components of $W_d^1(X)$.

(ii) *In the range $g-2h+1 \le d \le g-h$,*

$$W_d^1(X) = \bigcup_{i=1}^{2} \Sigma_i \cup \bigcup_{j=1}^{\alpha} B_j,$$

where the B_j's are possibly several irreducible components whose general element is base-point-free. In this case also, all the Σ_i's and the B_j's are generically reduced irreducible components of $W_d^1(X)$.

(iii) *For $d \gneq g-h$,*

$$W_d^1(X) = \Sigma_1 \cup \bigcup_{j=1}^{\alpha} B_j,$$

and every irreducible component Σ_1 and the B_j's of $W_d^1(X)$ are generically reduced.

Proof. (i) Recall again that no component of $W_d^r(C)$ is entirely contained in $W_d^{r+1}(X)$ if $g-d+r \ge 0$; [1, Lemma 3.5, p. 182]. Hence a general element L in any irreducible component of $W_d^1(X)$ is a complete pencil whose base-point-free part is a pull-back of a base-point-free and complete pencil M on C by the Castelnuovo–Severi inequality (Corollary 2.2), i.e. $L = \pi^* M + \Delta$ where Δ is the base locus of L. Therefore $L \in \Sigma_1$ if M is special or $L \in \Sigma_2$ if M is non-special. Note that $\dim \Sigma_1 = d-4 \ge \dim \Sigma_2 = d-h-2$ and hence the closed loci Σ_i are indeed the two distinct irreducible components of $W_d^1(X)$ by semi-continuity.

(ii) In the range $g-2h+1 \le d \le g-h$, by Proposition 3.1

$$\dim B_j = \rho(d,g,1) = 2d-g-2 \le \dim \Sigma_2 = d-h-2 \le \dim \Sigma_1 = d-4.$$

Hence by semi-continuity, Σ_1, Σ_2 and the B_j's are indeed different irreducible components of $W_d^1(X)$.

(iii) For $d \gneq g-h$, the closed locus Σ_2 cannot be an irreducible component since $\dim \Sigma_2 < \rho(d,g,1)$, hence $W_d^1(X) = \Sigma_1 \cup \bigcup_{j=1}^{\alpha} B_j$. In this case the closed locus Σ_2 is contained in one of the B_j's also by semi-continuity.

The generically reducedness of the components of $W_d^1(X)$ follows from Propositions 3.1 and 3.7. □

Remark 3.3. For a double covering $\pi : X \to C$ when the base curve C is a general curve of genus h, it has been shown in [3, Theorem 1.1] that $W^1_d(X)$, $d \geq g-h+1$ is irreducible if the genus g of X is sufficiently high. Proposition 3.2(iii) tells us that we cannot expect such a phenomenon any more if the base curve C is a special curve.

Proposition 3.4. *Let $\pi : X \to C$ be a double covering of a hyperelliptic curve C of genus $h \geq 3$, with $g \geq 6h-1$. Then there does not exist a birationally very ample g^r_{g-2h+r} for any $r \geq 2$.*

Proof. By successive projections, we only need to prove the statement for $r = 2$, $d = g-2h+2$ and for the complete linear series. Suppose that there is a base-point-free and complete $g^2_{g-2h+2} = |D|$ on X. $|D|$ cannot be very ample, otherwise X is a smooth plane curve of degree $g-2h+2$ of gonality $g-2h+1 = 4$, a contradiction. Therefore there exists an effective divisor $p_1+p_2 \in X_2$ such that

$$|D-p_1-p_2| = g^1_{g-2h} \in [\pi^*W^1_{h+1}(C) + W_{g-4h-2}(X)] \\ \cup[\pi^*W^1_2(C) + W_{g-2h-4}(X)]$$

by Proposition 3.2.

(i) First we assume that $|D-p_1-p_2| \in [\pi^*W^1_{h+1}(C) + W_{g-4h-2}(X)]$. Then

$$|D| = |\pi^*g^1_{h+1} + p_1 + p_2 + p_3 + \cdots + p_{g-4h}|,$$

where no two among $\{p_1, \ldots, p_{g-4h}\}$ are conjugate; otherwise, $|D| = |\pi^*g^1_{h+1} + p_1 + p_2 + p_3 + \cdots + p_{g-4h}| = |\pi^*g^2_{h+2} + \Delta|$ has the non-empty base locus Δ. For each j such that $1 \leq j \leq g-4h$, we consider

$$|D + p'_1 + \cdots + p'_j| = |\pi^*g^1_{h+1} + p_1 + \cdots + p_{g-4h} + p'_1 + \cdots + p'_j| \\ = |\pi^*g^{j+1}_{h+j+1} + p_{j+1} + \cdots + p_{g-4h}|,$$

where p and p' are in the same fiber of π. Note that p'_j is not a base point of the series $|D + p'_1 + \cdots + p'_j|$; if so, we have

$$j+1 \leq r(\pi^*(g^1_{h+1} + \pi(p_1) + \cdots + \pi(p_j))) \leq r(D + p'_1 + \cdots + p'_{j-1} + p'_j) \\ = r(\pi^*g^{j+1}_{h+j+1} + p_{j+1} + \cdots + p_{g-4h}) = r(D + p'_1 + \cdots + p'_{j-1}) \\ \leq r(D) + (j-1) = j+1$$

and hence $p'_j \in \{p_{j+1}, \ldots, p_{g-4h}\}$ contrary to the fact that $\{p_1, \ldots, p_{g-4h}\}$ contains no conjugate pairs. From this, we get

$$\dim|D + p'_1 + \cdots + p'_{g-4h}| = \dim|D| + g - 4h = g - 4h + 2$$

and hence there exits a base-point-free and complete $g^{g-4h+2}_{2g-6h+2}$ on X whose residual series is $|K - g^{g-4h+2}_{2g-6h+2}| = g^{2h-1}_{6h-4}$. If g^{2h-1}_{6h-4} is birationally very ample then by the Castelnuovo genus bound, $g \leq 6h-3$. Suppose g^{2h-1}_{6h-4} induces a multiple sheeted map $\alpha : X \to \alpha(X) \subset \mathbb{P}^{2h-1}$. Then

$$\deg\alpha \cdot (2h-1) \leq \deg\alpha \cdot \deg\alpha(X) \leq 6h-4$$

implies $\deg \alpha = 2$ and $\deg \alpha(X) \le 3h - 2$. Therefore, by the Castelnuovo genus bound again, $g(\alpha(X)) \le h - 1$. On the other hand, by the Castelnuovo–Severi inequality,

$$g \le (\deg \pi - 1) \cdot (\deg \alpha - 1) + \deg \pi \cdot g(C) + \deg \alpha \cdot g(\alpha(X)) \le 4h - 1,$$

which takes care of case (i).

(ii) We are now left with the case

$$\begin{aligned} |D - p_1 - p_2| = g^1_{g-2h} &= |\pi^* g^1_2 + p_3 + \cdots + p_{g-2h-2}| \\ &\in \pi^* W^1_2(C) + W_{g-2h-4}(X) \end{aligned}$$

and we set

$$|D| = |\pi^* g^1_2 + p_1 + p_2 + \cdots + p_{g-2h-2}|.$$

Note that among $\{p_1, \ldots, p_{g-2h-2}\}$, there exist at most $h - 1$ pairs of conjugate points; otherwise, $|D|$ would have non-empty base locus.

If there are exactly $(h - 1)$ pairs of conjugate points among $\{p_1, \ldots, p_{g-2h-2}\}$, then

$$\begin{aligned} |D| = |\pi^* g^1_2 &+ (p_1 + p'_1) + (p_2 + p'_2) + \cdots + (p_{h-1} + p'_{h-1}) \\ &+ p_{2h-1} + \cdots + p_{g-2h-2}| = |\pi^* g^1_{h+1} + p_{2h-1} + \cdots + p_{g-2h-2}|, \end{aligned}$$

which we treated already in (i). Therefore we may assume that there exist fewer than $(h-1)$ pairs of conjugate points and we write

$$|D| = |\pi^* g^1_2 + \pi^*(r_1) + \cdots + \pi^*(r_t) + p_{2t+1} + \cdots + p_{g-2h-2}|, \quad t \le h - 2$$

where no two among $\{p_{2t+1}, \ldots, p_{g-2h-2}\}$ are conjugate. We now claim that

$$|D + p'_{2t+1} + \cdots + p'_{g-2h-2}| \ge g - 3h - t + 1. \tag{3.4.1}$$

Suppose $|D + p'_{2t+1} + \cdots + p'_{g-2h-2}| \le g - 3h - t$. Then we have

$$\begin{aligned} g - 3h - t &\le \dim |\pi^*(g^{g-3h-t}_{g-2h-t})| \\ &= \dim |\pi^*(g^1_2 + r_1 + \cdots + r_t + \pi(p_{2t+1}) + \cdots + \pi(p_{g-2h-2}))| \\ &= \dim |\pi^* g^1_2 + \pi^*(r_1) + \cdots + \pi^*(r_t) + p_{2t+1} + \cdots + p_{g-2h-2} \\ &\quad + p'_{2t+1} + \cdots + p'_{g-2h-2}| \\ &= \dim |D + p'_{2t+1} + \cdots + p'_{g-2h-2}| \le g - 3h - t, \end{aligned}$$

hence

$$\dim |\pi^*(g^{g-3h-t}_{g-2h-t})| = \dim |D + p'_{2t+1} + \cdots + p'_{g-2h-2}| = g - 3h - t.$$

If this is the case, the morphism defined by $|\pi^*(g^{g-3h-t}_{g-2h-t})|$ is nothing but the double covering π. On the other hand, since $|\pi^*(g^{g-3h-t}_{g-2h-t}) - D| \ne \emptyset$, the birational morphism defined by $|D|$ factors through the morphism π, which is an absurdity finishing the proof of (3.4.1). Now we have

$$|E| := |D + p'_{2t+1} + \cdots + p'_{g-2h-2}| = g^{\alpha}_{2g-4h-2t} \qquad \text{with } \alpha \ge g - 3h - t + 1,$$

which is clearly base-point-free and birationally very ample. Taking the residual series, we have $|F| := |K - E| = g^{\beta}_{4h+2t-2}$, with $\beta \geq h + t$. Let $\eta : X \to \mathbb{P}^{\beta}$ be the morphism defined by the series $|F|$ (which may have non-empty base locus) and let $f \leq 4h + 2t - 2$ be the degree of the moving part of $|F|$.

(i) If η is a birationally morphism onto its image $\eta(X)$, then we may conclude $g \leq 6h - 2$ by using the Castelnuovo genus bound. In fact by an elementary calculation, one ends up with the following three possibilities where $m = [(f - 1)/(\beta - 1)]$.

(a) $m = 2 \Rightarrow g(X) \leq 6h - 5$
(b) $m = 3 \Rightarrow g(X) \leq 6h - 3$
(c) $m = 4$ (in which case $t = 0$ and $\beta = h$) $\Rightarrow g(X) \leq 6h - 2$.

(ii) Assume that η defines a k-sheeted morphism onto the image curve $\eta(X)$ of degree $e := \frac{f}{k}$ in $\mathbb{P}^{\beta}$. Since $\eta(X)$ is non-degenerate,

$$k \cdot (h + t) \leq k \cdot \deg \eta(X) \leq 4h + 2t - 2,$$

and hence $k \leq 3$. Let $\mathcal{D} := g^{\beta}_{e}$ be the complete linear series on $\eta(X)$ (or on its normalization). Suppose $\mathcal{D}$ is special. Then by Clifford's theorem we have

$$h + t \leq \beta \leq \deg \mathcal{D}/2 = \frac{f}{2k} \leq \frac{4h + 2t - 2}{2k},$$

which is a contradiction. Therefore, $\mathcal{D}$ is non-special and we have the upper bound for the genus j of $\eta(X)$:

$$j \leq \begin{cases} h - 1 & \text{if } k = 2 \\ \dfrac{h - 2}{3} & \text{if } k = 3. \end{cases}$$

Obviously, the morphism η does not factor through π. Therefore, by the Castelnuovo–Severi inequality we have $g \leq 4h - 1$. □

Corollary 3.5. *Let $\pi : X \to C$ be a double covering of a hyperelliptic curve C of genus $h \geq 3$ with $g \geq 6h - 1$. Set*

$$\Sigma_1 := \pi^* W^r_{2r}(C) + W_{d-4r}(X) \quad \textit{and} \quad \Sigma_2 := \pi^* W^r_{r+h}(C) + W_{d-2r-2h}(X).$$

Then for any $d \leq g - 2h + r$ and $r \geq 2$ we have

(i) $W^r_d(X) = \Sigma_1 \cup \Sigma_2$. *Furthermore, Σ_1 and Σ_2 are irreducible components of $W^r_d(X)$ if $r \lneq h$. If $r \geq h$, $\Sigma_1 \subset \Sigma_2 = W^r_d(X)$.*

(ii) $s(r, X) \geq g - 2h + r + 1$.

Proof. For $d \leq g - 2h + (r - 1)$, by taking off $r - 1$ general points, the fact that $W^r_d(X) = \Sigma_1 \cup \Sigma_2$ (set theoretically) follows from Proposition 3.2 and Lemma 2.3. For $d = g - 2h + r$, $W^r_d(X) = \Sigma_1 \cup \Sigma_2$ is also obvious by Proposition 3.4. We need to determine the irreducible components of $W^r_d(X)$. Let f_i be the degree of the base locus of a general element of the closed loci Σ_i. We have $f_1 = d - 4r$, $f_2 = d - 2r - 2h$ and $\dim \Sigma_1 = d - 4r$, $\dim \Sigma_2 = d - 2r - h$.

(1) If $\dim \Sigma_1 \geq \dim \Sigma_2$ (i.e. in the range $h \geq 2r$) then $f_1 > f_2$. Hence by semi-continuity we see that both Σ_1 and Σ_2 are irreducible components of $W^r_d(X)$.

(2) If $\dim \Sigma_1 \lneq \dim \Sigma_2$, i.e. in the range $h \lneq 2r$ there are three possibilities:

(2-a) $f_1 \lneq f_2 \Leftrightarrow h \lneq r$: since $\Sigma_1 \subset W^{r+1}_d(X)$, Σ_1 is not a component.

(2-b) $f_1 = f_2 \Leftrightarrow h = r$: in this case we clearly have $\Sigma_1 = \Sigma_2$.

(2-c) $f_1 \gneq f_2 \Leftrightarrow h \gneq r$: in this case we cannot use semi-continuity. Instead, we may argue as follows. Suppose $\Sigma_1 \subset \Sigma_2$. Then for a general choice $p_1 + \cdots + p_{d-4r} \in X_{d-4r}$,

$$|\pi^* r g^1_2 + p_1 + \cdots + p_{d-4r}| = |\pi^* g^r_{2r} + p_1 + \cdots + p_{d-4r}| = |\pi^* g^r_{r+h}| + E$$

for some $g^r_{r+h} \in W^r_{r+h}(C)$ and $E \in X_{d-2r-2h}$. Therefore it follows that $g^r_{r+h} = g^r_{2r} + q_1 + \cdots + q_{h-r}$ and hence $X_{d-4r} \subset \pi^* C_{h-r} + X_{d-2r-2h}$, whereas $\dim X_{d-4r} > \dim[\pi^* C_{h-r} + X_{d-2r-2h}]$, a contradiction.

(ii) follows immediately from Proposition 3.4. □

Remark 3.6. Compared with bi-elliptic curves or double coverings of genus two curves, Corollary 3.5 do not seem to be optimal. We expect that the same statement would hold in a wider range of the degree d of the linear series.

Finally we prove the generically reducedness of various components of $W^r_d(X)$ for a double covering of a curve of genus $h \geq 2$ which we have been putting off.

Proposition 3.7. *Let X be double covering of a hyperelliptic curve C of genus $h \geq 2$ with $g \geq 6h$. Set*

$$\Sigma_1 := \pi^* W^r_{2r}(C) + W_{d-4r}(X) \quad \textit{and} \quad \Sigma_2 := \pi^* W^r_{r+h}(C) + W_{d-2r-2h}(X).$$

Then Σ_1 and Σ_2 are generically reduced for the following cases:

(i) *$r \leq h$ and $d \leq g + r - 2$.*

(ii) *$r > h$ and $d \leq g - 2h$.*

Proof. We only give a proof for Σ_2 because the proof for Σ_1 is similar. We take a general $L \in \Sigma_2$ which may be written as $L = \pi^* g^r_{r+h} \otimes \mathcal{O}(p_1 + \cdots + p_{d-2r-2h})$, where g^r_{r+h} is non-special and $p_1 + \cdots + p_{d-2r-2h}$ is a general degree $d - 2r - 2h$ effective divisor on X. Assume that L is singular and let $l = r + 1 = h^0(X, L)$ and $i = g - d + r$. By (2.4.1), we have

$$\dim T_L W^r_d(X) = g - li + \dim\ker \mu_0 \geq d - 2r - h + 1, \tag{3.7.1}$$

and hence

$$\dim\ker \mu_0 \geq li - l - i - h + 2. \tag{3.7.2}$$

Fix a basis $\{s_1, \ldots, s_l\}$ of $H^0(X, L)$ such that s_1, s_2 has no common zeros other than $p_1, \ldots, p_{d-2r-2h}$. Set $W_k := \mathrm{span}\{s_1, \ldots, s_k\}$ for $k = 1, \ldots, l$ and let

$$\mu_{0,W_k} : W_k \otimes H^0(X, KL^{-1}) \to H^0(X, K)$$

be the restriction of μ_0 to the subspace $W_k \otimes H^0(X, KL^{-1})$. By noting that

$$\dim\ker \mu_{0,W_k} - \dim\ker \mu_{0,W_{k-1}} \leq i$$

for $k = 3, \ldots, l$ we have

$$\dim \ker \mu_{0,W_l} - \dim \ker \mu_{0,W_2} \leq i(l-2)$$

and hence

$$\begin{aligned} \dim \ker \mu_{0,W_2} &\geq \dim \ker \mu_0 - i(l-2) \geq li - l - i - h + 2 - i(l-2) \\ &= g - d + 1 - h \end{aligned} \tag{3.7.3}$$

by (3.7.2). By using the projection formula and the Riemann–Hurwitz relation for double coverings, one has

$$\begin{aligned} h^0(X, 2\pi^* g^r_{r+h}) &= h^0(C, \pi_*(\pi^*(2g^r_{r+h}) \otimes \mathcal{O}_X)) = h^0(C, 2g^r_{r+h} \otimes \pi_* \mathcal{O}_X) \\ &= h^0(C, 2g^r_{r+h} \otimes (\mathcal{O}_C \oplus \mathcal{O}_C(T))) \\ &= h^0(C, 2g^r_{r+h}) + h^0(C, 2g^r_{r+h} \otimes T) = 2r + h + 1, \end{aligned}$$

where T is a divisor such that $2T \cong -R$ with R the branch divisor of π; note that $\deg 2g^r_{r+h} \otimes T = 2r + 4h - g - 1 < 0$ by our numerical data $g \geq 6h$, $d - 2r - 2h \geq 0$ and $d \leq g - 2h$ (if $r > h$).

On the other hand, by the base-point-free pencil trick

$$\begin{aligned} \dim \ker \mu_{0,W_2} &= h^0(X, K \otimes \mathcal{O}(-2\pi^* g^r_{r+h} - p_1 - \cdots - p_{d-2r-2h})) \\ &= h^0(X, K \otimes \mathcal{O}(-2\pi^* g^r_{r+h})) - (d - 2r - 2h) \\ &= g - 4(r+h) + (2r+h) - (d - 2r - 2h) = g - d - h \end{aligned}$$

since $p_1, \ldots, p_{d-2r-2h}$ were chosen generically. Therefore (3.7.3) becomes a numerical absurdity. □

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