Annali di Matematica 185, 105–131 (2006)

Digital Object Identifier (DOI) 10.1007/s10231-004-0130-8

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Flexible Gabor-wavelet atomic decompositions for L^2-Sobolev spaces

Received: June 25, 2003; in final form: March 24, 2004
Published online: April 27, 2005 – © Springer-Verlag 2005

Abstract. In this paper we present a general construction of frames, which allows one to ensure that certain families of functions (atoms) obtained by a suitable combination of translation, modulation, and dilation will form Banach frames for the family of L^2-Sobolev spaces on $\mathbb{R}$ of any order. In this construction a parameter $\alpha \in [0, 1)$ governs the dependence of the dilation factor on the frequency parameter. The well-known Gabor and wavelet frames (also valid for the same scale of Hilbert spaces) using suitable Schwartz functions as building blocks arise as special cases ($\alpha = 0$) and a limiting case ($\alpha \to 1$), respectively. In contrast to those limiting cases, it is no longer possible to use group-theoretical arguments. Nevertheless, we will show how to *constructively* ensure that for Schwartz analyzing atoms and any sufficiently dense but discrete and well-structured family of parameters one can guarantee the frame property. As a consequence of this novel constructive technique, one can generate quasicoherent dual frames by an iterative algorithm. As will be shown in a subsequent paper, the new frames introduced here generate Banach frames for corresponding families of α-modulation spaces.

Mathematics Subject Classification (2000). 42C15, 46S30, 49M27, 65T60

Key words. continuous/discrete frames – Gabor and wavelet frames – non-orthogonal expansions – α-modulation spaces – Sobolev spaces

1. Introduction and motivations

The theory of *frames* or stable redundant nonorthogonal expansions, introduced by Duffin and Schaeffer [12], plays an important role in wavelet theory [9,10] as well as in Gabor analysis [36,25]. Many relevant contributions, among them [1,5,6,10,9,19–21,42–45], describe Gabor and wavelet analysis as two parallel theories with similar but different structures and typically different applications.

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⋆ This work was developed during the Ph.D. program in computational mathematics at the University of Padova, Italy. The author acknowledges the partial support by the EU-network RTN HASSIP (Harmonic Analysis and Statistics for Signal and Image Processing) under Contract HPRN-CT-2002-00285 and the hospitality of NuHAG, Department of Mathematics, University of Vienna, Austria, during the preparation of this work.

We shall propose a general construction of suitably structured frames and its application to present a unified theory for L^2-Sobolev spaces of Gabor and wavelet frames, usually treated separately. We intend our results to be a further answer to the theoretical need of a common interpretation and framework and to define more general intermediate time-frequency tools for applications. To fix notations, consider the following unitary operators on $L^2(\mathbb{R})$ of modulation, translation, and dilation, respectively:

$$M_\omega(f)(t) = e^{2\pi i\omega\cdot t} f(t), \tag{1}$$

$$T_x(f)(t) = f(t-x), \tag{2}$$

$$D_a(f)(t) = |a|^{-1/2} f(t/a). \tag{3}$$

In particular, we shall study families of functions given by:

$$[\pi(x,\omega,\eta_\alpha(\omega))g] = M_\omega T_x D_{\eta_\alpha(\omega)^{-1}} g, \quad \text{for} \quad (x,\omega) \in \mathbb{R}^2, \tag{4}$$

where $\eta_\alpha(\omega) = (1+|\omega|)^\alpha$, for $\alpha \in [0,1]$, and the corresponding *flexible Gabor-wavelet transform*:

$$V_g^\alpha(f)(x,\omega) = \langle f, M_\omega T_x D_{\eta_\alpha(\omega)^{-1}} g\rangle, \quad \forall (x,\omega) \in \mathbb{R}^2. \tag{5}$$

We shall show that for suitable discrete sampling of (4), one can generate Banach frames for L^2-Sobolev spaces $H^s(\mathbb{R})$ once g is sufficiently well localized in the time-frequency plane. One can easily verify that for $\alpha = 0$ the family (4) is just a Gabor family, while for $\alpha \to 1$ the family tends to the situation encountered in the wavelet context, in a very natural (geometric) sense.

Cordoba and Fefferman [7], Folland [28], and Holschneider and Nazaret [42] introduced such families and transforms as new time-frequency tools to study certain pseudodifferential operators. Moreover, L^2-Sobolev spaces are both modulation spaces [14,17,36] whose natural atomic decompositions provide (Banach) Gabor frames as well as Besov spaces [34,47,46], for which wavelet expansions are atomic decompositions. Hence, L^2-Sobolev spaces appear to be natural common spaces for which one can develop an intermediate theory. Furthermore, they are special instances of the more general family of α-modulation spaces [13,35] for each $\alpha \in [0,1]$. Although interesting group-theoretical approaches in generalizing Gabor and wavelet frames are presented in [1,2,41,43,44], as a matter of fact, classical unified (coorbit-space) theories [19–21,37] cannot be applied to generate discrete flexible (parametric) Gabor-wavelet frames. Our approach overcomes these difficulties. We mention, however, that a recent generalization of the coorbit-space theory [8] may open up an alternative approach.

In approximation theory, one might use the parameter α as a *tuning* parameter in analyzing signals. In audio signal encoding, for example, one can use different bases to model different components of signals. The nonlinear approximation of the *transient* component (shock waves) is very effective using wavelet decompositions, but the *tonal* component (harmonic or stationary signal) is much better represented by Gabor type frames (or by local Fourier bases) [11]. Presently it is claimed that possible hybrid versions of Gabor and wavelet expansions can achieve better and more efficient representations of signals. In particular, the almost complementary quality and capability of approximation of Gabor and wavelet expansions has been shown in [11]. This is a further motivation to investigate possible intermediate frames, to introduce more flexible tools of approximation.

The paper is organized as follows. Section 2 lists the elementary properties of frames and a few notational conventions. In Section 3 we briefly recall the coorbit-space theory and the unified approach to Gabor and wavelet Banach frames developed by Feichtinger and Gröchenig [19–21,37]. We explain why it cannot be applied directly to derive any intermediate theory and time-frequency tool. In Section 4 we collect the relevant tools and technical results we need. The concept of admissible coverings of $\mathbb{R}^d$ and BAPUs (bounded admissible partitions of unity), decomposition spaces, in particular Wiener amalgam spaces, and results on series expansions for band-limited functions are presented. In Section 5 we introduce a new constructive technique to generate admissible (discrete) coverings from certain continuous coverings of the real line. In particular, each interval of an admissible covering is identified by a function P, parameterizing the *position* of the interval, and a function S, governing the *size* of the interval. Next we study how to check the Bessel condition for a family of functions of the type (4) associated to such admissible coverings. Finally, we state the definition of α-admissibility for an analyzing function g. In Section 6 we combine the results and tools established in Sections 4 and 5 to give unified sufficient conditions for the existence of *flexible Gabor-wavelet (discrete) frames* for L^2-Sobolev spaces.

The main result can be formulated as follows.

Theorem 1. *Let $\alpha \in [0,1)$, $s \in \mathbb{R}^+$, and $g \in H^s(\mathbb{R})$ be α-admissible. Denote by*

$$P_\alpha(j) = sgn(j)\big((1+(1-\alpha)\cdot b\cdot |j|)^{\frac{1}{1-\alpha}} - 1\big)$$

and

$$S_\alpha(j) = b\cdot(1+(1-\alpha)\cdot b\cdot(|j|+1))^{\frac{\alpha}{1-\alpha}}$$

position and size functions, respectively, where $sgn(\cdot)$ is the sign function. Then, for all $b>0$ and all sufficiently small values $a>0$, the set of functions

$$\big\{g^{j,k}_{\alpha,a,b} = M_{P_\alpha(j)} T_{a\cdot S_\alpha(j)^{-1}\cdot k} D_{S_\alpha(j)^{-1}} g\big\}_{j,k\in\mathbb{Z}} \tag{6}$$

has the property that any $f \in H^s$ can be written as the unconditionally convergent series in $H^s(\mathbb{R})$

$$f = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} c^{j,k}_{\alpha,a,b}(f) g^{j,k}_{\alpha,a,b}, \tag{7}$$

$$\|f\|^2_{H^s} \asymp \sum_{k,j} \left|c^{j,k}_{\alpha,a,b}(f)\right|^2 (1+(1-\alpha)\cdot b\cdot |j|)^{\frac{2s}{1-\alpha}}, \tag{8}$$

for a suitable set of linear functionals $c^{j,k}_{\alpha,a,b}$. In particular, the renormed sequence $\{(1+(1-\alpha)\cdot b\cdot |j|)^{\frac{s}{\alpha-1}} \cdot g^{j,k}_{\alpha,a,b}\}_{j,k\in\mathbb{Z}}$ is a frame, while $\{g^{j,k}_{\alpha,a,b}\}_{j,k\in\mathbb{Z}}$ is a Banach frame for $H^s(\mathbb{R})$. Thus, the space $H^s(\mathbb{R})$ can be interpreted as a generalized coorbit space with respect to (4).

The strategy of the proof has already been introduced in [29] and applied for the Gabor frames case only in [30]. It is essentially inspired by the fundamental works of Frazier and Jawerth [34] and Feichtinger [17], and it works as follows:

- Decompose the space into suitable subspaces of band-limited functions.
- Construct a local family of atoms of translates for each subspace.
- Show that the union of these local families is a Bessel sequence.
- The global system so built is a frame.

Then, one can apply a perturbation result to extend the frame property to larger classes of generating atoms. In fact, under suitable decay-smoothness conditions on g (α-admissibility), one starts to construct a local family of atoms by translates of a band-limited approximation g_ρ of g, which generates global frames. Then, by means of the Neumann series inversion of certain synthesis operators, one proves that the frame property applies also for g.

The α-admissibility condition essentially involves only the upper frame bound condition (Bessel condition) of the system, while the lower bound condition, typically harder to check, is automatically fulfilled by a quite large class of analyzing functions as a consequence of the general and unified construction. This result is an improvement over earlier contributions, for example [5,9,41], where it was necessary to check the lower bound condition, separately, for Gabor or wavelet frames and it is a generalization of the results of [3, Section 3.6]. Moreover, since the developed technique is constructive, we provide a method to calculate suitable coefficients $\{c^{j,k}_{\alpha,a,b}(f)\}_{j,k}$ by means of the choice of (approximated) dual families. A constructive technique for deriving explicit formulas of suitable time-frequency-scale sampling points $(x^\alpha_{j,k}, \omega^\alpha_j)$ for family (4) to define the corresponding discrete frames is also presented. Hence, our approach should be distinguished from those results, see, for example, [41] Theorem 7.1 and [19–21], where the points were required to be just *sufficiently dense*. The construction is also generalizable to describe *irregular* Gabor and wavelet frames [32].

The present paper is the third of a series [29,30] on constructive methods to generate structured Banach frames for function spaces. We have also developed ([31, Chapter 5]) the extension of these expansions as Banach frames for all classes of α-modulation spaces as a generalization of the known coorbit-space theory. These results will be detailed in a subsequent paper [32], and they will make use of the present L^2-Sobolev theory, combined with a novel generalization [33] of the theory of localization of frames recently developed by Gröchenig [38].

2. Preliminaries

Definition 1. *A sequence $\{f_n\}_{n\in\mathbb{N}}$ in $\mathbb{H}$ is a frame for the Hilbert space $\mathbb{H}$ if there exist two positive constants A, $B > 0$ such that*

$$A \cdot \|f\|^2 \leq \sum_{n\in\mathbb{N}} |\langle f, f_n\rangle|^2 \leq B \cdot \|f\|^2, \quad \forall f \in \mathbb{H}. \tag{9}$$

The upper bound in condition (9) is also known as the Bessel condition for the sequence $\{f_n\}_{n\in\mathbb{N}}$, and whenever it holds, the sequence $\{f_n\}_{n\in\mathbb{N}}$ is called a Bessel sequence.

Condition (9) ensures that the *frame operator* $S : \mathbb{H} \to \mathbb{H}$ given by

$$Sf = \sum_{n \in \mathbb{N}} \langle f, f_n \rangle f_n \tag{10}$$

is an invertible operator. This implies that

$$f = SS^{-1} f = \sum_{n \in \mathbb{N}} \langle f, S^{-1} f_n \rangle f_n. \tag{11}$$

The sequence $\{S^{-1} f_n\}_{n \in \mathbb{N}}$ is again a frame and is called the *canonical dual* frame with frame operator S^{-1}. For a frame the coefficient functionals $\{c_n\}_{n \in \mathbb{N}}$ such that

$$f = \sum_{n \in \mathbb{N}} c_n(f) f_n \tag{12}$$

are in general not unique. Typically many dual Bessel sequences $\{\tilde{f}_n\}_{n \in \mathbb{N}}$ in $\mathbb{H}$ exist such that for all $f \in \mathbb{H}$

$$f = \sum_{n \in \mathbb{N}} \langle f, \tilde{f}_n \rangle f_n. \tag{13}$$

Such redundancy of a frame can play an important role in practical problems where robustness and error tolerance are fundamental as, for example, denoising, irregular sampling problems [22–24], pattern matching, data transmission, and communication. We refer the reader to [6] for a recent book on frames and nonorthogonal expansions and to [3,8,37,38] for the more general notion of *Banach frame*.

We write $\mathbb{R}^+$ for the nonnegative reals. For a given function or tempered distribution f, we denote the Fourier transform of f by $\hat{f}$ or $\mathcal{F} f$. Moreover, f is called *band-limited* if $\operatorname{spec}(f) := \operatorname{supp}(\hat{f}) \subset \Omega$, for some Ω compact subset of $\mathbb{R}^d$. We write $\mathcal{S} = \mathcal{S}(\mathbb{R}^d)$ for the Schwartz space. If $B(\mathbb{R}^d)$ is a Banach space of functions on $\mathbb{R}^d$, sometimes we write simply B instead of $B(\mathbb{R}^d)$. For positive quantities F and G, we will write $F \lesssim G$ whenever $F(x) \leq C \cdot G(x)$, for some universal constant $C > 0$ and for all arguments x. If $F \lesssim G$ and $G \lesssim F$, then we will write $F \asymp G$. By convention we write w_s for the weight function $w_s(x) = (1 + |x|^2)^{s/2}$ and $L^p_s := L^p_{w_s}$, the space of functions f such that $f \cdot w_s \in L^p$, and in the following discussion we assume $s \geq 0$. Let us also denote by $\mathcal{A}$ the regular Banach algebra $\mathcal{A} := \mathcal{F} L^1$ and, finally, by M the space of bounded measures, the topological dual space of the continuous function space C^0.

3. The coorbit-space theory: Gabor and wavelet frames

A number of interesting papers in the field suggest a parallel description of Gabor and wavelet decompositions. Let us mention just a few names here, such as Christensen [6,5], Daubechies [9,10], and Triebel [47]. Inspired by the work of Grossmann et al. [39], Feichtinger and Gröchenig have presented a unified group-theoretical approach in [19–21,37]. In those works one can switch between the

Gabor and the wavelet context by exchanging the reduced Heisenberg group for the affine $ax + b$ group.

Let us recall their approach and then introduce a common presentation for these two types of coherent frames. In the sequel G is some (Hausdorff) locally compact group.

Definition 2. *A strongly continuous representation of G on $\mathbb{H}$ is a mapping π from G into the bounded linear operators on $\mathbb{H}$ for which*

(i) $\pi(xy) = \pi(x)\pi(y)$, *for all* $x, y \in G$.
(ii) For all $f \in \mathbb{H}$, the mapping $x \mapsto \pi(x)f$ is continuous from G to $\mathbb{H}$.
We further say that
(iii) π is unitary if all the operators $\{\pi(x)\}_{x\in G}$ are unitary.
(iv) π is irreducible if the only closed subspaces of $\mathbb{H}$ that are invariant under all the operators $\{\pi(x)\}_{x\in G}$ are $\{0\}$ and $\mathbb{H}$.
(v) π is integrable if there exists $g \in \mathbb{H}\backslash\{0\}$ such that

$$\int_G |\langle \pi(x)g, g\rangle| d\mu(x) < \infty,$$

where μ is the Haar measure on G. The set of all $g \in \mathbb{H}$ for which the above integral is finite will be denoted by $\mathbb{A}$ or $\mathbb{A}_\pi$.

Definition 3. *Let X be a locally compact space. A sequence of open, relatively compact subsets $\mathcal{O} = \{\Omega_i\}_{i\in I} \subset X$ is called an admissible covering of X if*

(C1) (Covering) $X = \bigcup_{\in I} \Omega_i$;
(C2) (Uniformly locally finite) $\sup_{i\in I} \#\{j \in I : \Omega_j \cap \Omega_i \neq \emptyset\} \leq N < \infty$.

Theorem 2. *Let π be an irreducible, unitary, and integrable representation of G on $\mathbb{H}$. Then, there exists subspace $\mathbb{B} \subset \mathbb{A}$, which is dense in $\mathbb{H}$, such that for all $g \in \mathbb{B}\backslash\{0\}$ there exists a relatively compact subset Ω of G with nonvoid interior with the following property: for any admissible countable covering $\{x_j\Omega\}_{j\in I}$ of G there exists a bounded operator*

$$\Lambda : \mathbb{H} \to l^2(I), \quad \Lambda f = \{\lambda_j(f)\}_{j\in I}$$

for which

$$f = \sum_{j\in I} \lambda_j(f)\pi(x_j)g, \quad \forall f \in \mathbb{H}. \tag{14}$$

Moreover, $\{\pi(x_j)g\}_{j\in I}$ is a frame for $\mathbb{H}$.

This theorem is just a special case of the general Feichtinger–Gröchenig theory, based on a discretization of convolutions and iterative approximations of a reproducing formula. To a large class of Banach function spaces Y (including weighted L^p-spaces) one can associate a sequence space Y_d and a Banach space $Co_\pi(Y)$ (coorbit space) such that the result holds with $\mathbb{H}$ replaced by $Co_\pi(Y)$ and $l^2(I)$ by Y_d. In particular, Theorem 2 can be applied to obtain Gabor and wavelet frames:

Examples 1. We collect a few relevant examples of atomic decompositions for coorbit spaces:

- (Gabor expansions and modulation spaces) For $\mathbb{H}_{red} = \mathbb{R} \times \mathbb{R} \times \mathbb{T}$, the *(reduced) Heisenberg group* with a suitable group law, and the *Schrödinger representation* on $L^2(\mathbb{R})$ given for $(z, y, x) \in \mathbb{H}_{red}$ by

$$[\pi(z, y, x)g](t) = x \cdot T_z M_y g(t) = x \cdot e^{2\pi i y(t-z)} g(t - z), \quad g \in L^2, \tag{15}$$

  the resulting frames are called *Gabor frames* and the corresponding coorbit spaces are $Co_\pi(L^p_{w_s}) = M^s_{p,p}$, belonging to the family of *modulation spaces* [17,36].
- (Wavelet expansions and Besov–Triebel spaces) For $\mathbb{A}_{ff} = \mathbb{R} \ltimes \mathbb{R}^+$, the *affine* group, one can define the representation on $L^2(\mathbb{R})$ given for $(b, a) \in \mathbb{A}_{ff}$ by

$$[\pi(b, a)f](t) = T_b D_a f(t) = \frac{1}{\sqrt{a}} f\left(\frac{t - b}{a}\right), \quad f \in L^2. \tag{16}$$

  Hence wavelet expansions appear by appropriate sampling of the representation. For the weight function $w(b, a) = |a|^s$, $s \in \mathbb{R}$, the corresponding coorbit spaces are $Co_\pi(L^p_w) = \dot{B}^{s-1/2+1/p}_{p,p}$, the *homogeneous Besov spaces* [34,46,45].
- (L^2-Sobolev spaces) Fractional L^2-Sobolev spaces H^s defined by

$$H^s(\mathbb{R}) = \{f \in \mathcal{S}' : \hat{f}(\omega)(1 + |\omega|^2)^{\frac{s}{2}} \in L^2(\mathbb{R})\}, \tag{17}$$

  endowed with the scalar product $\langle f, g\rangle_{H^s} := \langle \hat{f}, \hat{g}\rangle_{L^2_{w_s}}$, are Hilbert spaces, and $H^s = M^s_{2,2} = \dot{B}^s_{2,2} = Co_\pi(L^2_{w_s})$, for all s, since they belong to both families.

The Feichtinger–Gröchenig approach does *not* suggest any way to move from the Gabor to the wavelet decomposition theory and any possible intermediate time-frequency tools. In fact, Torresani [43,44] investigated the existence of subgroups G of the *affine-Heisenberg group* $G_{aWH} = \mathbb{R} \times \mathbb{R} \ltimes \mathbb{R}^+ \times \mathbb{T}$, such that they admit mixed-square integrable, unitary representations

$$[\pi(b, \omega, \eta_G(\omega))g] = M_\omega T_b D_{\eta_G(\omega)^{-1}} g, \quad \text{for all } (b, \omega) \in \mathbb{R}^2, \tag{18}$$

giving as a unique solution $\eta_G(\omega) = \eta_\lambda(\omega) = (1 + \lambda\omega)$. For $\lambda \neq 0$ it gives a trivial modification of the standard wavelet theory, and for $\lambda = 0$ it gives the standard Gabor theory. Hence *the Feichtinger–Gröchenig theory cannot be used to provide an intermediate theory between Gabor and wavelet decompositions*. However, as we shall show below, an *intermediate theory* is still possible (see also [41]) with a corresponding coorbit-space theory [31,32] by using the *decomposition of Hilbert space* method illustrated in [29] and already used in [30] for generating Gabor frames only. Such an *intermediate* theory provides a new flexible tool that potentially combines the advantages of the two time-frequency techniques to represent signals and operators in a parametric way.

In the construction of intermediate frames we follow the approach of Cordoba–Fefferman [7] and Folland [28] for the representation of functions as continuous superpositions of modulated, translated, and dilated *wave packets*. They introduced such a family of functions to study classes of *pseudodifferential operators*, which are generated by the action of a mixed family of operators given by

$$[\pi(x, \omega, \eta_\alpha(\omega))g] = M_\omega T_x D_{\eta_\alpha(\omega)^{-1}} g, \quad \text{for all } (x, \omega) \in \mathbb{R}^2, \tag{19}$$

where $\eta_\alpha(\omega) = (1 + |\omega|)^\alpha$, for $\alpha \in [0, 1]$, producing a new transform that we wish to call the *flexible Gabor-wavelet transform* or *α-transform*:

$$V_g^\alpha(f)(x, \omega) = \langle f, M_\omega T_x D_{\eta_\alpha(\omega)^{-1}} g\rangle, \quad \text{for all } (x, \omega) \in \mathbb{R}^2. \tag{20}$$

Note that, for $\alpha = 0$, V_g^α is just the short time Fourier transform (STFT) or the windowed Fourier transform and, for $\alpha = 1$, V_g^α is just a slight modification of the continuous wavelet transform (CWT). Further interesting results in this direction were also proposed by Holschneider and Nazaret in [42], and we briefly assemble some of them in what follows.

Theorem 3. *For $g \in \mathcal{S}(\mathbb{R})$ define*

$$\tilde{\sigma}_g^\alpha(\omega) = \int_{\mathbb{R}} |\hat{g}((1 + |\xi|)^{-\alpha}(\omega - \xi))|^2 (1 + |\xi|)^{-\alpha} d\xi. \tag{21}$$

Assume that there exists $A > 0$ such that

$$0 < A^{-1} \le \tilde{\sigma}_g^\alpha(\omega) \le A < \infty, \quad \textit{for a.e} \quad \omega \in \mathbb{R}. \tag{22}$$

Then

$$f \in H^s(\mathbb{R}) \quad \textit{if and only if} \quad V_g^\alpha f \in L^2_{(1+|\omega|^2)^{\frac{s}{2}}}(\mathbb{R}^2). \tag{23}$$

In particular, for $s = 0$, the system $\{M_\omega T_x D_{\eta_\alpha(\omega)^{-1}} g\}_{\omega, x \in \mathbb{R}}$ is a continuous frame [2], and for all $f \in L^2$

$$f = \int_{\mathbb{R}^2} c_{\omega,x}(f) M_\omega T_x D_{\eta_\alpha(\omega)^{-1}} g \quad dx d\omega \quad \textit{and} \quad \|c_{\omega,b}(f)\|_{L^2(\mathbb{R}^2)} \asymp \|f\|_{L^2}. \tag{24}$$

A representative function satisfying (22) is the Gaussian.

It can be interesting to the reader to compare the previous theorem with Theorem 1, which can be considered its discrete version. Similar results are reported in [41] and [1, Proposition 15.2.1].

4. Wiener amalgams, series expansions of band-limited functions

We wish to assemble in this section those tools that we need for a better and self-contained presentation. For more details we refer the reader to [13, 35, 15, 16, 40, 23].

Definition 4. *Given an admissible covering $\mathcal{O} = \{\Omega_i\}_{i\in I}$ of $\mathbb{R}^d$, we call a family $\Phi = \{\varphi_i\}_{i\in I}$ in $\mathcal{A} := \mathcal{F}L^1$ [13, Theorem 4.2] an $\mathcal{O}$-BAPU (bounded admissible partition of unity) subordinate to $\mathcal{O}$, if Φ fulfills the following properties:*

(i) $\sup_{i\in I}\|\varphi_i\|_{\mathcal{A}} = C_0 < \infty$;
(ii) $\mathrm{supp}(\varphi_i) \subseteq \Omega_i$, $\forall i \in I$;
(iii) $\sum_{i\in I}\varphi_i(x) = 1$.

Definition 5. *For a Banach space $(B, \|\cdot\|_B)$ continuously embedded in $\mathcal{S}'$, an $\mathcal{O}$-BAPU $\Phi = \{\varphi_i\}_{i\in I}$ in $\mathcal{A}$ acts boundedly on $(B, \|\cdot\|_B)$ if*

$$\|\varphi_i f\|_B \le \|\varphi_i\|_{\mathcal{A}}\|f\|_B, \forall i \in I,\ f \in B.$$

Definition 6. *Given a Banach space $(B, \|\cdot\|_B)$ continuously embedded in $\mathcal{S}'$, let $\Phi = \{\varphi_i\}_{i\in I} \subset \mathcal{A}$ be an $\mathcal{O}$-BAPU acting boundedly on $(B, \|\cdot\|_B)$ and w a discrete (weight) strictly positive sequence on I. We define the corresponding decomposition space as*

$$D\big(\mathcal{O}, B, l^q_w\big)(\mathbb{R}^d) = \big\{f \in \mathcal{S}' : f\varphi_i \in B \quad \forall i \in I, (\|f\varphi_i\|_B)_{i\in I} \in l^q_w(I)\big\}.$$

Moreover, one can define the natural norm:

$$\|f|D\| := \left(\sum_{i\in I}\|f\varphi_i\|_B^q w(i)^q\right)^{1/q},$$

for $1 \le q < \infty$. The usual modification applies for $q = \infty$.

With such a norm the space $(D(\mathcal{O}, B, l^q_w), \|\cdot|D\|)$ is a Banach space. If, in addition to the already mentioned assumptions, the $\mathcal{O}$-BAPU is a Q-BUPU (bounded uniform partition of unity), i.e., if there exists $(x_i)_{i\in I} \subset \mathbb{R}^d$ such that $\Omega_i = x_i + Q$, with $Q \subset \mathbb{R}^d$ relatively compact, we call $D(\mathcal{O}, B, l^q_w)$ a Wiener amalgam space, and we will write $W(B, l^q_w) := D(\mathcal{O}, B, l^q_w)$. Then the definition does not depend on the particular BUPU Φ taken whenever the weight function w is assumed *submultiplicative*, that is, $w(x+y) \le w(x)w(y)$ for all $x, y \in \mathbb{R}^d$, e.g., w_s for $s \ge 0$. Moreover, there exist characterizations of equivalent general decomposition spaces by means of equivalent coverings, see [13,35]. The properties of Wiener amalgams with respect to inclusions, Fourier transform, and convolutions were described and studied by Feichtinger [15,16]. We refer the reader to [40] for further information.

We must also recall and develop some technical tools on expansions of band-limited functions by translates of a single band-limited function g.

Proposition 1 (Regular case). *Given a band-limited function $g \in L^1(\mathbb{R}^d)$ such that $\hat{g} \neq 0$ on a relatively compact set $\Omega \subset \mathbb{R}^d$, there exists $\delta_0 > 0$ such that for all $0 < \delta \le \delta_0$ $\{g(\cdot - \delta k)\}_{k\in\mathbb{Z}^d}$ is a local family of atoms [27] (see also [29, Definition 3]) for $L^2_\Omega(\mathbb{R}^d) := \{f \in L^2 : \mathrm{supp}(\hat{f}) \subset \Omega\}$. This means that for all $f \in L^2_\Omega(\mathbb{R}^d)$*

$$f = \sum_{k\in\mathbb{Z}^d} c_k(f)T_{\delta k}g \quad \textit{and} \quad \sum_{k\in\mathbb{Z}^d}|\langle f, T_{\delta k}g\rangle|^2 \le B\cdot\|f\|_2^2,$$

for suitable bounded functionals c_k such that $\sum_{k\in\mathbb{Z}^d}|c_k(f)|^2 \le \tilde{B}\cdot\|f\|_2^2$.

Proof. It is well known that $\{\delta^{d/2}e^{2\pi i\delta kx}\}_{k\in\mathbb{Z}^d}$ is an orthonormal basis for $L^2([-1/(2\delta), 1/(2\delta)]^d)$. For $\delta > 0$ small enough $\Omega \subset [-1/(2\delta), 1/(2\delta)]^d$. By Wiener's lemma there exists band-limited function $g_1 \in L^1$ such that $\hat{g}_1 \cdot \hat{g} \equiv 1$ on Ω. Hence, for all $f \in L^2_\Omega(\mathbb{R}^d)$

$$\hat{f}(\omega) = \left[\left(\hat{f}\hat{g}_1\right)\hat{g}\right](\omega) = \sum_{k\in\mathbb{Z}^d} \delta^d \langle \hat{f}\hat{g}_1, e^{2\pi i\delta kx}\rangle e^{2\pi i\delta k\omega}\hat{g}(\omega). \tag{25}$$

By applying the inverse Fourier transform,

$$f = \sum_{k\in\mathbb{Z}^d} \delta^d (f * g_1)(\delta k) T_{\delta k} g, \tag{26}$$

where $*$ is the convolution operator. Observe now that, by Young's inequality,

$$\sum_{k\in\mathbb{Z}^d} \delta^d |(f * g_1)(\delta k)|^2 = \|f * g_1\|_2^2 \le \|f\|_2^2 \|g_1\|_1^2. \tag{27}$$

Let us denote $c_k(f) = \delta^d (f * g_1)(\delta k)$; then one has

$$\sum_{k\in\mathbb{Z}^d} |c_k(f)|^2 \le \left(\delta^d \|g_1\|_1^2\right) \|f\|_2^2. \tag{28}$$

In particular, $\sum_{k\in\mathbb{Z}^d} |\langle f, T_{\delta k} g\rangle|^2 = \sum_{k\in\mathbb{Z}^d} |\langle \hat{f}, e^{2\pi i\delta k\omega}\hat{g}\rangle|^2 \le \delta^{-d}\|f\|_2^2\|\hat{g}\|_\infty^2$. By formula (25) one has that $\{T_{\delta k}\delta^d\overline{g}_1\}_{k\in\mathbb{Z}^d}$ is a dual for the local family of atoms $\{T_{\delta k} g\}_{k\in\mathbb{Z}^d}$ (see [29, Remark]). Moreover, by (28) there exists a universal constant $\tilde{B} > 0$ depending only on the behavior of $\hat{g}$ on Ω such that for all $0 < \delta \le \delta_0$, $\sum_{k\in\mathbb{Z}^d} |c_k(f)|^2 \le \tilde{B}\|f\|^2$.

Similarly, the more general translation irregular sampling case is given by the following theorem.

Theorem 4 (Irregular case). *Let $\Omega \subset \mathbb{R}^d$ be a relatively compact set, and $g \in L^1(\mathbb{R}^d)$ a band-limited function such that $\hat{g}(\omega) \neq 0$ on Ω. Then there exist a relatively compact set $U_0 \subset \mathbb{R}^d$ and a constant $C = C(g, \Omega, U_0) > 0$ such that for any subset $\{y_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}^d$ for which $\{y_k + U_0\}_{k\in\mathbb{Z}}$ is an admissible covering for $\mathbb{R}^d$ there exist linear mappings $f \mapsto c_k(f)$ such that*

$$f = \sum_{k\in\mathbb{Z}} c_k(f) T_{y_k} g, \quad \text{and} \quad \left(\sum_{k\in\mathbb{Z}} |c_k(f)|^p\right)^{1/p} \le C \cdot \|f\|_p, \tag{29}$$

for every $f \in L^p_\Omega(\mathbb{R}^d) = \{f \in L^p(\mathbb{R}^d) : spec(f) \subset \Omega\}$ and $p \in [1, \infty)$.

Proof. [23].

For $p = 2$ the set $\{\psi_k^0 = T_{y_k} g\}_{k\in\mathbb{Z}}$ is a local family of atoms for $\mathbb{W}_0 = L^2_\Omega(\mathbb{R}^d)$ in the sense of [29, Definition 3]. In particular, the projection of this family onto $L^2_\Omega(\mathbb{R}^d)$ is a frame for that subspace. Constant C in (29) depends only on the behavior of $\hat{g}$ on Ω and on the *density* [23, Definition 2.2] of nodes $\{y_k\}_{k\in\mathbb{Z}}$. In fact, there exists $\delta = \delta(U_0) > 0$ such that

$$C = \frac{1}{1 - c(\delta, \Omega_1)\|h\|_1} \cdot \|g_1\|_1, \quad c(\delta, \Omega_1)\|h\|_1 < 1, \tag{30}$$

where

(1) g_1 is a band-limited function $g_1 \in L^1$, with $\mathrm{supp}(\hat{g}_1) \subset \mathrm{supp}(\hat{g})$ and $\hat{g}_1 \cdot \hat{g} \equiv 1$ on Ω;
(2) $h \in L^1$ is a band-limited function such that $\hat{h} \equiv 1$ on $\mathrm{supp}(\hat{g})$ and $\Omega_1 := \mathrm{supp}(\hat{h})$;
(3) $c(\delta, \Omega_1) := \inf \|\mathrm{osc}_\delta p\|_1$, $\mathrm{osc}_\delta p(x) = \big(\sup_{z\in B_\delta(x)} |p(z) - p(x)|\big)$, where the infimum ranges over all functions $p \in L^1$, with $\hat{p} \equiv 1$ on Ω_1.

Remark 1. Observe that $\|h\|_1$ in (30) is independent of the size $|\mathrm{supp}(\hat{g})|$. In fact, for any band-limited function $h \in L^1$ such that $\hat{h} \equiv 1$ on Ω and for all band-limited $g \in L^1$ there exists $\rho > 0$ such that $h_\rho(t) = \rho^d \cdot h(\rho \cdot t) \in L^1$ is a band-limited function, $\hat{h}_\rho \equiv 1$ on $\mathrm{supp}(\hat{g})$, and $\|h_\rho\|_1 = \|h\|_1$.

Lemma 1. *Let U be a relatively compact subset of $\mathbb{R}^d$. Then there exists an increasing function $C_U(\xi)$ on $\mathbb{R}^+$, $C_U(\xi) \to 0$ for $\xi \to 0$ such that, for all constants $\delta > 0$ and $\rho > 0$,*

$$c(\delta, \rho \cdot U) \le C_U(\delta \cdot \rho). \tag{31}$$

Proof. Consider some band-limited function $p \in L^1(\mathbb{R}^d)$ with $\hat{p} \equiv 1$ on U. Hence $p_\rho(x) = \rho^d \cdot p(\rho \cdot x)$ is a band-limited function with $\hat{p}_\rho \equiv 1$ on $\rho \cdot U$ and $c(\delta, \rho \cdot U) \le \|\mathrm{osc}_\delta p_\rho\|_1$. By the mean-value theorem, one has

$$\mathrm{osc}_\delta p_\rho(x) \le \delta \cdot |\nabla p_\rho|^\#_\delta(x), \tag{32}$$

where $f^\#_\delta(x) := \sup_{z\in B_\delta(x)} |f(z)|$ is the *local maximal function* of f. Observe now that $\nabla p_\rho(x) = \rho^{d+1} \cdot \nabla p(\rho \cdot x)$ and

$$|\nabla p_\rho|^\#_\delta(x) = \rho^{d+1} \cdot \sup_{z\in B_\delta(x)} |\nabla p(\rho \cdot z)| = \rho^{d+1} \cdot \sup_{\rho z\in B_{\rho\cdot\delta}(\rho x)} |\nabla p(\rho \cdot z)|.$$

Thus one has

$$\|\mathrm{osc}_\delta p_\rho\|_1 \le \delta \cdot \rho \cdot \big\| |\nabla p|^\#_{\delta\rho} \big\|_1. \tag{33}$$

Let us choose $C_U(\xi) = \xi \cdot \||\nabla p|^\#_\xi\|_1$.

Remark 2. By the previous lemma, for any $\varepsilon > 0$ and $\rho > 0$, for the dilation of a fundamental domain Ω by ρ there exists $\delta > 0$, $\delta \asymp \rho^{-1}$ such that $c(\delta, \rho\cdot\Omega) < \varepsilon$. Hence, a fixed constant C in (30) can be used for all pairs of functions g and sampling sets $\{y_k\}_k$, as long as the product of maximal gap size of $\{y_k\}_k$ and the size of $\mathrm{supp}(\hat{g})$ is uniformly bounded.

In order to develop a constructive approach, we present a method to calculate a possible dual $\{\tilde{\psi}_k^0\}_k$ for $\{\psi_k^0\}_k$ (also valid for the irregular case).

Remark 3. ([LDA], **L**ocal **D**ual **A**lgorithm). Under the assumptions and notations of Theorem 4, let $\Phi = \{\varphi_k\}_{k\in\mathbb{Z}}$ be a U_0-BUPU associated to $\{y_k + U_0\}_{k\in\mathbb{Z}}$. One can define the discrete measure $D_\Phi f$ in $W(M, l^2)$ by the operator

$$D_\Phi : L^2 \to W(M, l^2), \quad D_\Phi f = \sum_{k\in\mathbb{Z}} \langle f, \varphi_k\rangle_{L^2} \delta_{y_k}. \tag{34}$$

Observe that for all $g \in W(C^0, l^2)$

$$\langle g, D_\Phi f\rangle = \langle g, \sum_{k\in\mathbb{Z}} \langle f, \varphi_k\rangle_{L^2} \delta_{y_k}\rangle = \sum_{k\in\mathbb{Z}} \langle f, \varphi_k\rangle_{L^2} g(y_k) = \langle f, \overline{\sum_{k\in\mathbb{Z}} g(y_k)\varphi_k}\rangle_{L^2}. \tag{35}$$

Hence, one can also define

$$S_\Phi : W(C^0, l^2) \to L^2, \quad S_\Phi g = \sum_{k\in\mathbb{Z}} g(y_k)\varphi_k, \tag{36}$$

and $\langle g, D_\Phi f\rangle$[1] $= \langle f, \overline{S_\Phi g}\rangle_{L^2}$. Let us consider the convolution operator:

$$C_h f = f * h. \tag{37}$$

Denote $C_h^* g = h^\nabla * g = C_g(h^\nabla)$, where $h^\nabla = h(-x)$. By the proof of [23, Theorem 2, formula (64)], one has that the coefficients $\{c_k(f)\}_{k\in\mathbb{Z}}$ can be calculated by means of the Neumann series:

$$c_k(f) = \langle \varphi_k, \sum_{n=0}^{\infty} (C_h(\mathcal{I} - D_\Phi))^n \, f_0\rangle, \tag{38}$$

where $f_0 = C_{g_1} f$. Hence, for $p = 2$, a dual sequence $\{\tilde{\psi}_k^0\}_{k\in\mathbb{Z}}$ for the local family of atoms $\{T_{y_k} g\}_{k\in\mathbb{Z}}$ satisfying $c_k(f) = \langle f, \tilde{\psi}_k^0\rangle_{L^2}$ can be constructed by

$$\tilde{\psi}_k^0 = \overline{C_{g_1}^* \left(\sum_{n=0}^{\infty} \left((\mathcal{I} - S_\Phi) C_h^* \right)^n \right) \varphi_k}. \tag{39}$$

This construction can be implemented by the following algorithm:

$$\Psi_k^0 = \varphi_k, \quad \Psi_k^{n+1} = \left(h^\nabla * \Psi_k^n\right) - \sum_{j\in\mathbb{Z}} \left(h^\nabla * \Psi_k^n\right)(y_j)\varphi_j, \quad \forall n \in \mathbb{N}, \quad \forall k \in \mathbb{Z}, \tag{40}$$

[1] Here the symbol $\sigma(g) = \langle g, \sigma\rangle$ is the action of the distribution σ on the function g as an extension of the bilinear form $\langle g, \overline{\sigma}\rangle_{L^2} = \int_{\mathbb{R}^d} g(x)\sigma(x)dx$ when g and σ are both functions. This notation is in fact the one used in [22,23].

and

$$\tilde{\psi}_k^0 = \overline{g_1^{\nabla} * \left(\sum_{n=0}^{\infty} \Psi_k^n \right)}. \tag{41}$$

For the regular case, $y_k = \delta \cdot k$ and $\varphi_k = T_{\delta \cdot k}\varphi_0$ imply

$$[S_\Phi, T_{\delta \cdot k}] = [C_{g_1}^*, T_{\delta \cdot k}] = [C_h^*, T_{\delta \cdot k}] \equiv 0, \tag{42}$$

where $[A, B] = A \circ B - B \circ A$ is the commutator operator of A and B. These commutator equations, together with (39), imply that

$$\tilde{\psi}_k^0 = T_{\delta \cdot k}\tilde{\psi}_0^0, \tag{43}$$

where

$$\tilde{\psi}_0^0 = \overline{C_{g_1}^* \left(\sum_{n=0}^{\infty} \left((\mathcal{I} - S_\Phi) C_h^* \right)^n \right) \varphi_0}. \tag{44}$$

In this case we will say that the dual is (locally) *coherent* because it preserves the structure of the original local family of atoms.

5. Admissible coverings and α-admissibility

Let us assume in what follows $d = 1$. The multidimensional theory will be the subject of subsequent contributions. We want to show how to define suitable decompositions ([29, Definition 2], [30]) of H^s as a Hilbert space in order to construct frames derived for sampling of (19) in the time-frequency-scale domain. This will be done by suitable partitioning of the frequency line and corresponding decomposition of H^s into suitable subspaces of band-limited functions. For that, the following technical results are useful.

Lemma 2. *Let $P, S : \mathbb{R}^+ \to \mathbb{R}^+$ be two positive nondecreasing (unbounded) functions, $P \in C^1(\mathbb{R}^+)$, such that*

$$\frac{dP}{dw}(w) = S(w), \quad P(0) = 0. \tag{45}$$

For all $b > 0$ and $j \in \mathbb{N}$, denote by $\Omega_j^{P,S}$ the interval $P(b \cdot j) + [0, b \cdot S(b(j+1))]$. Then $\mathcal{O} = \{\Omega_j^{P,S}\}_{j \in \mathbb{N}}$ is an admissible covering for $\mathbb{R}^+$. We call P the position function and S the size function of the covering.

Proof. By the mean-value theorem, one has

$$P(b(j+1)) - P(b \cdot j) = b \cdot S(b \cdot \xi_j),$$

for some $\xi_j \in (j, j+1)$. Since S is nondecreasing, one has also

$$P(b(j+1)) \leq P(b \cdot j) + b \cdot S(b(j+1)).$$

Moreover, P is also nondecreasing, and then

$$\begin{aligned} P(b(j+1)) &\leq P(b \cdot j) + b \cdot S(b(j+1)) \\ &\leq P(b(j+1)) + b \cdot S(b(j+1)) \leq P(b(j+2)). \end{aligned}$$

Hence, for all j, $\Omega_j^{P,S}$ intersects $\Omega_{j+1}^{P,S}$, but no further successive elements of the covering and

$$\sup_{j \in \mathbb{N}} \#\left\{ i \in \mathbb{N} : \Omega_i^{P,S} \bigcap \Omega_j^{P,S} \neq \emptyset \right\} \leq 3. \tag{46}$$

Lemma 3. *For all* $\alpha \in [0, 1)$*, the functions*

$$P_\alpha(\omega) = (1 + (1-\alpha)\omega)^{\frac{1}{1-\alpha}} - 1, \quad S_\alpha(\omega) = (1 + (1-\alpha)\omega)^{\frac{\alpha}{1-\alpha}} \tag{47}$$

are position and size functions. Then, for each fixed constant $b > 0$*, denote*

$$\Omega_j^\alpha = \left((1 + (1-\alpha) \cdot b \cdot j)^{\frac{1}{1-\alpha}} - 1\right) + \left[0, b \cdot ((1 + (1-\alpha) \cdot b \cdot (j+1))^{\frac{\alpha}{1-\alpha}}\right]. \tag{48}$$

Hence $\mathcal{O}^\alpha = \{\Omega_j^\alpha\}_{j \in \mathbb{N}}$ *is an admissible covering of* $\mathbb{R}^+$*.*

Proof. Consider the functions

$$\tilde{P}(\omega) = \omega, \quad \tilde{S}(\omega) = \eta_\alpha(\omega) = (1+\omega)^\alpha. \tag{49}$$

They are position and size functions only for $\alpha = 0$. In order to generate some position and size functions, it suffices to show that there exists a positive smooth parameterizing function $h(\omega)$ such that

$$P_\alpha(\omega) = \tilde{P}(h(\omega)), \quad S_\alpha(\omega) = \tilde{S}(h(\omega)) \tag{50}$$

satisfy (45). This means that one has to solve the following differential equation:

$$\tilde{P}'(h(\omega))h'(\omega) = \tilde{S}(h(\omega)). \tag{51}$$

In our particular case one has to find h such that

$$h'(\omega) = (1 + h(\omega))^\alpha. \tag{52}$$

The general solution of (52) is given by

$$h_C(\omega) = (C + (1-\alpha)\omega)^{\frac{1}{1-\alpha}} - 1. \tag{53}$$

We should choose C such that $P_\alpha(0) = 0$ and hence $C = 1$. One concludes by using Lemma 2.

Remark 4. Observe that, for $\alpha = 0$, $\mathcal{O}^0$ is a regular covering of positions $b \cdot j$ and fixed size b. But for $\alpha \to 1$, $\mathcal{O}^\alpha$ tends to an *exponential* covering $\mathcal{O}^1$, where position and size are of the type e^{bj}. In particular, for $b = \ln(2)$, $\mathcal{O}^1$ is a dyadic covering of $\mathbb{R}^+$. Moreover, up to mirroring, $\mathcal{O}^\alpha$ defines an admissible covering for all $\mathbb{R}$.

Remark 5. We will say that a couple of functions $(\tilde{P},\tilde{S})$ are an *admissible parameterization* of the frequency axis if there exists a smooth, increasing, and nonnegative function h such that $\lim_{x\to\infty} h(x) = \infty$ and

$$P(\omega) = \tilde{P}(h(\omega)), \quad S(\omega) = \tilde{S}(h(\omega))$$

are position and size functions. In particular, let us observe that, for $\alpha > 1$, there exists *no* h satisfying the requirements.

The following results, up to suitable modifications, will hold also for more general admissible parameterizations of the frequency axis, and equivalent admissible coverings are defined as in [18, Definition 3.3].

In [6,5,3] sufficient conditions to have Gabor or wavelet L^2-frames are presented separately. In what follows we generalize those sufficient conditions (see in particular [5, Theorem 5.1]) to check the frame upper bound (Bessel) condition for a more general class of sequences in L^2-Sobolev spaces [3] related to suitable positions and size functions.

Lemma 4. *For $\omega \in \mathbb{R}$ and $j \in \mathbb{Z}$, write $\xi_{\omega,j} = \frac{(\omega - P(j))}{S(j)}$. If for some $c > 0$ and $s \in \mathbb{R}^+$, a function $g \in H^s(\mathbb{R})$ has the property that, for almost all $\omega \in \mathbb{R}$,*

$$(B) \quad \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \left| \hat{g}(\xi_{\omega,j}) \hat{g}\left(\xi_{\omega,j} - \frac{k}{c}\right)\right| \left(\frac{1+|\omega|^2}{1+|P(j)|^2}\right)^s \le B_c < \infty; \tag{54}$$

then the system

$$\left\{g^{j,k}_{P,S,c} = (1+|P(j)|^2)^{-\frac{s}{2}} \cdot M_{P(j)} D_{S(j)^{-1}} T_{c\cdot k}\, g\right\}_{j,k\in\mathbb{Z}} \tag{55}$$

is a Bessel family for H^s with Bessel bound B_c/c.

Proof. We first assume that $f \in \mathcal{S}$ is band-limited. The general case follows later by a standard density argument.

[Step 1] For a fixed $j \in \mathbb{Z}$ one has

$$\begin{aligned}
\int_0^{S(j)/c} \sum_{k\in\mathbb{Z}} &\left| \hat{f}\left(\omega - \frac{S(j)k}{c}\right) \hat{g}\left(\xi_{\omega,j} - \frac{k}{c}\right)\right| \left(1 + \left|\omega - \frac{S(j)k}{c}\right|^2\right)^s d\omega \\
&= \int_{\mathbb{R}} \left|\hat{f}(\omega)\hat{g}(\xi_{\omega,j})\right| (1+|\omega|^2)^s d\omega \\
&\le \|f\|_{H^s} \cdot \left(\int_{\mathbb{R}} \left|\hat{g}(\xi_{\omega,j})\right|^2 (1+|\omega|^2)^s d\omega\right)^{1/2} \\
&\lesssim S(j)(1+S(j))^s (1+P(j))^s \cdot \|f\|_{H^s} \cdot \|g\|_{H^s}.
\end{aligned}$$

Therefore, the periodic function

$$F_j(\omega) = \sum_{k\in\mathbb{Z}} \hat{f}\left(\omega - \frac{S(j)k}{c}\right) \overline{\hat{g}\left(\xi_{\omega,j} - \frac{k}{c}\right)} \left(1 + \left|\omega - \frac{S(j)k}{c}\right|^2\right)^s$$

is in $L^1[0, S(j)/c]$, and one can show that it is also in $L^2[0, S(j)/c]$. Moreover:

$$\int_{\mathbb{R}} \hat{f}(\omega)\overline{\hat{g}(\xi_{\omega,j})}e^{2\pi i m S(j)^{-1}c\omega} \cdot (1+|\omega|^2)^s d\omega = \int_0^{\frac{S(j)}{c}} F_j(\omega)e^{2\pi i m S(j)^{-1}c\omega} d\omega. \quad (56)$$

By Fourier series, one has also

$$\sum_{m\in\mathbb{Z}} \left| \int_0^{\frac{S(j)}{c}} F_j(\omega)e^{2\pi i m S(j)^{-1}c\omega} d\omega \right|^2 = \frac{S(j)}{c} \int_0^{\frac{S(j)}{c}} |F_j(\omega)|^2 d\omega. \quad (57)$$

[Step 2]

$$\sum_{j,k\in\mathbb{Z}} \left|\left\langle f, g^{j,k}_{P,S,c}\right\rangle_{H^s}\right|^2 = \sum_{j,m\in\mathbb{Z}} (1+|P(j)|^2)^{-s} \cdot \left|\langle \hat{f}, M_{cmS(j)^{-1}} T_{P(j)} D_{S(j)} \hat{g}\rangle_{L^2_{w_s}}\right|^2$$

$$= \sum_{j,m\in\mathbb{Z}} S(j)^{-1} \cdot (1+|P(j)|^2)^{-s} \cdot \left| \int_{\mathbb{R}} \hat{f}(\omega)\overline{\hat{g}(\xi_{\omega,j})e^{2\pi i m S(j)^{-1}c\omega}} \cdot (1+|\omega|^2)^s d\omega \right|^2.$$

By (56) and (57) one has

$$\sum_{j,k\in\mathbb{Z}} \left|\left\langle f, g^{j,k}_{P,S,c}\right\rangle_{H^s}\right|^2$$

$$= \sum_{j\in\mathbb{Z}} c^{-1} \cdot (1+|P(j)|^2)^{-s} \cdot$$

$$\cdot \int_0^{\frac{S(j)}{c}} \left| \sum_{k\in\mathbb{Z}} \hat{f}\left(\omega - \frac{S(j)k}{c}\right) \hat{g}\left(\xi_{\omega,j} - \frac{k}{c}\right) \left(1 + \left|\omega - \frac{S(j)k}{c}\right|^2\right)^s \right|^2 d\omega$$

$$\leq c^{-1} \sum_{j\in\mathbb{Z}} (1+|P(j)|^2)^{-s} \cdot \int_0^{\frac{S(j)}{c}} \sum_{l\in\mathbb{Z}} \left| \hat{f}\left(\omega - \frac{S(j)l}{c}\right) \hat{g}\left(\xi_{\omega,j} - \frac{l}{c}\right) \right| \times$$

$$\times \left(1 + \left|\omega - \frac{S(j)l}{c}\right|^2\right)^s \sum_{k\in\mathbb{Z}} \left| \hat{f}\left(\omega - \frac{S(j)k}{c}\right) \hat{g}\left(\xi_{\omega,j} - \frac{k}{c}\right) \right| \times$$

$$\times \left(1 + \left|\omega - \frac{S(j)k}{c}\right|^2\right)^s d\omega$$

$$= c^{-1} \sum_{j\in\mathbb{Z}} (1+|P(j)|^2)^{-s} \cdot \sum_{l\in\mathbb{Z}} \int_0^{\frac{S(j)}{c}} \left| \hat{f}\left(\omega - \frac{S(j)l}{c}\right) \hat{g}\left(\xi_{\omega,j} - \frac{l}{c}\right) \right| \times$$

$$\times \left(1 + \left|\omega - \frac{S(j)l}{c}\right|^2\right)^s \sum_{k\in\mathbb{Z}} \left| \hat{f}\left(\omega - \frac{S(j)k}{c}\right) \hat{g}\left(\xi_{\omega,j} - \frac{k}{c}\right) \right| \left(1 + \left|\omega - \frac{S(j)k}{c}\right|^2\right)^s d\omega$$

$$= c^{-1} \sum_{j\in\mathbb{Z}} (1+|P(j)|^2)^{-s} \cdot \int_{\mathbb{R}} \left|\hat{f}(\omega)\hat{g}(\xi_{\omega,j})\right| (1+|\omega|^2)^s \times$$

$$\times \sum_{k\in\mathbb{Z}} \left| \hat{f}\left(\omega - \frac{S(j)k}{c}\right)\hat{g}\left(\xi_{\omega,j} - \frac{k}{c}\right)\right| \left(1 + \left|\omega - \frac{S(j)k}{c}\right|^2\right)^s d\omega$$

$$= c^{-1}\int_{\mathbb{R}} |\hat{f}(\omega)|^2(1+|\omega|^2)^s \sum_{j\in\mathbb{Z}}(1+|P(j)|^2)^{-s}\cdot \left|\hat{g}(\xi_{\omega,j})\right|^2 (1+|\omega|^2)^s d\omega$$

$$+ c^{-1}\sum_{k\neq 0}\sum_{j\in\mathbb{Z}}(1+|P(j)|^2)^{-s}\cdot \int_{\mathbb{R}} \left|\hat{f}(\omega)\hat{f}\left(\omega - \frac{S(j)k}{c}\right)\right|(1+|\omega|^2)^s \times$$

$$\times \left|\hat{g}(\xi_{\omega,j})\hat{g}\left(\xi_{\omega,j} - \frac{k}{c}\right)\right| \left(1 + \left|\omega - \frac{S(j)k}{c}\right|^2\right)^s d\omega$$

$$= c^{-1}\int_{\mathbb{R}} |\hat{f}(\omega)|^2(1+|\omega|^2)^s \sum_{j\in\mathbb{Z}}(1+|P(j)|^2)^{-s}\cdot \left|\hat{g}(\xi_{\omega,j})\right|^2 (1+|\omega|^2)^s d\omega + c^{-1}R.$$

We now estimate the second term R. Using Cauchy–Schwarz first on the integral and then on the sum over k as in the proof of [5, Theorem 5.1], we obtain

$$R \leq \int_{\mathbb{R}} |\hat{f}(\omega)|^2(1+|\omega|^2)^s \sum_{k\neq 0}\sum_{j\in\mathbb{Z}}(1+|P(j)|^2)^{-s}\cdot \left|\hat{g}(\xi_{\omega,j})\hat{g}\left(\xi_{\omega,j} - \frac{k}{c}\right)\right| \times$$
$$\times (1+|\omega|^2)^s d\omega.$$

Hence, using property (B) one finally has

$$\sum_{j,k\mathbb{Z}} \left|\langle f, g^{j,k}_{P,S,c}\rangle_{H^s}\right|^2 \leq \frac{B_c}{c}\cdot \|f\|_{H^s}. \tag{58}$$

This concludes the proof.

On the basis of the previous lemma we give the following definition.

Definition 7. *Let $P_\alpha(j) = sgn(j)\left((1+(1-\alpha)\cdot b\cdot |j|)^{\frac{1}{1-\alpha}} - 1\right)$ and $S_\alpha(j) = b\cdot(1+(1-\alpha)\cdot b\cdot(|j|+1))^{\frac{\alpha}{1-\alpha}}$, where $\alpha \in [0,1)$, and $s \in \mathbb{R}^+$. A function $g \in H^s(\mathbb{R})$ is called α-admissible with respect to (P_α, S_α) if*

(AD1) $g \in L^1$ and $\hat{g} \neq 0$ on $\Omega = [-1,1]$ or g is band-limited, $\hat{g} \in L^\infty$ and $\hat{g} \neq 0$ on $spec(g)$, $\Omega \subset spec(g)$;

(AD2) g satisfies condition (B) of Lemma 4 for all $a \in (0,1]$:

$$\sigma^\alpha_{g,a}(\omega) = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \left|\hat{g}\left(\frac{\omega - P_\alpha(j)}{S_\alpha(j)}\right)\hat{g}\left(\frac{\omega - P_\alpha(j)}{S_\alpha(j)} - \frac{k}{a}\right)\right| \left(\frac{1+|\omega|^2}{1+|P_\alpha(j)|^2}\right)^s$$
$$\leq B_{g,a} < \infty, \tag{59}$$

for a.e. $\omega \in \mathbb{R}$;

(AD3) for all $\rho, a > 0$ and all g_ρ band-limited approximations of g, i.e., $g_\rho = g * \gamma_\rho$, where $\gamma \in \mathcal{S}$ is a band-limited function, with $\hat{\gamma} \equiv 1$ on Ω and $\gamma_\rho(t) = \rho^d\gamma(\rho\cdot t)$, there exist $B_{g_\rho,a} > 0$ and $B_{g-g_\rho,a} > 0$ such that $\sigma^\alpha_{g_\rho,a}(\omega) \leq B_{g_\rho,a} < \infty$ and $\sigma^\alpha_{g-g_\rho,a}(\omega) \leq B_{g-g_\rho,a} < \infty$ for a.e. $\omega \in \mathbb{R}$;

(AD4) If $a = a(\rho) = c_0^{-1}\rho^{-1}$, then $a^{-1}\cdot B_{g-g_\rho,a} \to 0$, for $\rho \to +\infty$.

Lemma 5. *For $s \geq 0$ and $\alpha \in [0,1)$, write $(s;\alpha) = \frac{2s}{1-\alpha}$. Then every function $g \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$ such that $|\widehat{g}| \lesssim h_t$, with $h_t(\omega) = (1+|\omega|^2)^{-\frac{t}{2}}$, $t \geq 2+(s;\alpha)$, and $\hat{g}(\omega) \neq 0$ on Ω, is α-admissible with respect to (P_α, S_α).*

Proof. Assume for simplicity $b=1$. By assumption $\widehat{g}(\omega) \neq 0$ for $\omega \in \Omega$ (AD1). Let $\xi_{\omega,j} = \frac{\omega - P_\alpha(j)}{S_\alpha(j)}$ and, since $|\widehat{g}| \lesssim h_t$ and h_t is a symmetric decreasing function, one has

$$\sum_{k\in\mathbb{Z}} \left|\widehat{g}\left(\xi_{\omega,j} - \frac{k}{a}\right)\right| \leq K\|h_t\|_1 := b_{g,a} < \infty, \tag{60}$$

where K is uniform with respect to $a \in (0,1]$.

Hence, one also has

$$\sigma^\alpha_{g,a}(\omega) \leq b_{g,a} \cdot \sum_{j\in\mathbb{Z}} |\widehat{g}(\xi_{\omega,j})| \left(\frac{1+|\omega|^2}{1+|P_\alpha(j)|^2}\right)^s. \tag{61}$$

By standard analysis and calculations one can check that $|\widehat{g}(\xi_{\omega,j})| \lesssim h_t((|\omega|+1)^{1-\alpha} - 1 - \operatorname{sgn}(j)(1-\alpha)j)$. This implies that

$$\begin{aligned}
\sup_{\omega\in\mathbb{R}} \sigma^\alpha_{g,a}(\omega) &\leq K_1 b_{g,a} \cdot \sup_{\omega\in\mathbb{R}} \sum_{j\in\mathbb{Z}} h_t\left((|\omega|+1)^{1-\alpha} - 1 - \operatorname{sgn}(j)(1-\alpha)j\right) \times \\
&\quad \times \left(\frac{1+|\omega|^2}{1+|P_\alpha(j)|^2}\right)^s \\
&= K_1 \cdot b_{g,a} \cdot \sup_{\xi\in\mathbb{R}} \sum_{j\in\mathbb{Z}} h_t\left(|\xi| - \operatorname{sgn}(j)(1-\alpha)j\right) \left(\frac{1+|P_\alpha(\frac{\xi}{1-\alpha})|^2}{1+|P_\alpha(j)|^2}\right)^s \\
&\leq K_2 \cdot b_{g,a} \cdot \sup_{\xi\in\mathbb{R}} \sum_{j\in\mathbb{Z}} h_t\left(\xi - (1-\alpha)j\right) \left(\frac{1+|P_\alpha(\frac{\xi}{1-\alpha})|^2}{1+|P_\alpha(j)|^2}\right)^s (*).
\end{aligned}$$

Observe that $h_t \in W(C^0, l^1_{(s;\alpha)})$; hence $\sum_{j\in\mathbb{Z}} (1+|P_\alpha(j)|^2)^{-s} \delta_{(1-\alpha)j} \in W(M, l^\infty_{(s;\alpha)})$ and

$$\sum_{j\in\mathbb{Z}} \frac{h_t\left(\xi - (1-\alpha)j\right)}{(1+|P_\alpha(j)|^2)^s} = \left(\sum_{j\in\mathbb{Z}} (1+|P_\alpha(j)|^2)^{-s} \delta_{(1-\alpha)j}\right) * h_t.$$

Applying standard results concerning the behavior of Wiener amalgams under convolution and pointwise multiplication [15,16,40] may recall

$$W\big(M, l^\infty_{(s;\alpha)}\big) * W\big(C^0, l^1_{(s;\alpha)}\big) \subset W\big(C^0, l^\infty_{(s;\alpha)}\big)$$

and

$$W\big(C^0, l^\infty_{-(s;\alpha)}\big) \cdot W\big(C^0, l^\infty_{(s;\alpha)}\big) \subset W(C^0, l^\infty).$$

In particular, $(1+|P_\alpha(\frac{\xi}{1-\alpha})|^2)^s \in W(C^0, l^\infty_{-(s;\alpha)})$, which implies (AD2):

$$(*) \leq K_2 b_{g,a} \cdot \|h_t\|_{W(C^0, l^1_{(s;\alpha)})} = K_3 \cdot \|h_t\|_1 \cdot \|h_t\|_{W(C^0, l^1_{(s;\alpha)})} := B_{g,a}. \tag{62}$$

In the same way, one shows that, for g_ρ, and denoting $g^\rho := g - g_\rho$

$$\sup_{\omega\in\mathbb{R}} \quad \sigma^{\alpha}_{g_\rho,a}(\omega) \le O\left(\|(h_t)_\rho\|_1 \cdot \|(h_t)_\rho\|_{W(C^0,l^1_{(s;\alpha)})}\right) := B_{g_\rho,a}$$

$$\sup_{\omega\in\mathbb{R}} \quad \sigma^{\alpha}_{g^\rho,a}(\omega) \le O\left(\|h^\rho_t\|_1 \cdot \|h^\rho_t\|_{W(C^0,l^1_{(s;\alpha)})}\right) := B_{g^\rho,a} \tag{63}$$

i.e., (AD3) is valid. Moreover, for $a = a(\rho) = c_0^{-1}\rho^{-1}$, one achieves (AD4) as follows:

$$a^{-1} \cdot B_{g^\rho,a} = O\left(\rho \cdot \|h^\rho_t\|_1 \cdot \|h^\rho_t\|_{W(C^0,l^1_{(s;\alpha)})}\right) = O(\rho^{-1}) \to 0 \ \text{ for } \ \rho \to \infty.$$

Remark 6. If $\widehat{g} \in W(C^0, l^\infty_t)$, then $|\widehat{g}(\omega)| \lesssim h_t(\omega)$. Moreover, $\mathcal{F}\left(W(\mathcal{F}L^2_t, l^1)\right) \subset W(\mathcal{F}L^1, l^2_t) \subset W(\mathcal{F}L^1, l^\infty_t) \cap L^2_t$. Hence, if $t \ge 2 + (s;\alpha)$, $g \in W(\mathcal{F}L^2_t, l^1)$ and $\widehat{g}(\omega) \ne 0$ on Ω, then g is α-admissible with respect to (P_α, S_α).

Remark 7. Assume $t \ge 2 + (s;\alpha)$, $g \in W(\mathcal{F}L^2_t, l^1)$, and $\widehat{g}(\omega) \ne 0$ on Ω. Then g is α-admissible with respect to (P'_α, S'_α) (in the sense that (AD1–4) are valid substituting (P'_α, S'_α) for (P_α, S_α)) whenever $P'_\alpha \asymp P_\alpha$ and $S'_\alpha \asymp S_\alpha$. In fact, following the proof of the previous lemma, it will be sufficient to check that $h_t(\frac{\omega - P_\alpha(j)}{S_\alpha(j)}) \asymp h_t(\frac{\omega - P'_\alpha(j)}{S'_\alpha(j)})$ uniformly with respect to w and j. Hence, in this case, we will say that "g is α-admissible" instead of "g is α-admissible with respect to (P_α, S_α)."

Example 1. By the previous remark it is not hard to show that all $g \in \mathcal{S}(\mathbb{R})$ such that $\hat{g}(\omega) \ne 0$ on Ω are α-admissible for any $\alpha \in [0, 1)$ and $s \ge 0$. Obviously the Gaussian function $g(x) = e^{-\pi x^2}$ is an ideal candidate for an α-admissible function for all $s \ge 0$ and $\alpha \in [0, 1)$.

6. Proof of Theorem 1

Denote $\Omega_0 = [0, 1]$. Let $\mathbb{V}_0 = H^s_{\Omega_0}$ be the closed subspace of $H^s(\mathbb{R})$ of the band-limited functions f such that $\mathrm{spec}(f) \subseteq \Omega_0$ and $b > 0$ are a fixed positive constant. Assume, for example, for $j \ge 0$ (for $j < 0$ is analogous, up to mirroring),

$$D^\alpha_j = M_{P_\alpha(j)} D_{S_\alpha(j)^{-1}}, \quad \mathbb{W}_j = D^\alpha_j(\mathbb{V}_0) = H^s_{\Omega^\alpha_j}.$$

Since $\mathcal{O}^\alpha$ is an admissible covering of the frequency domain, by application of the Fourier transform identification of H^s with L^2_s, one easily has that $(\mathbb{V}_0, \{D^\alpha_j\}_{j\in\mathbb{Z}})$ is a decomposition of H^s in the sense of [29, Definition 2]. Moreover, for any BAPU $\{\varphi^\alpha_j\}_{j\in\mathbb{Z}} \subset \mathcal{S}$ associated to $\mathcal{O}^\alpha$, one can consider the *system of bounded quasiprojectors* $\mathcal{P}^\alpha = \{\mathcal{P}^\alpha_j\}_{j\in\mathbb{Z}}$, in the sense of [29, Definition 4], $\mathcal{P}^\alpha_j$ mapping H^s into $\mathbb{W}_j$, given by

$$\mathcal{P}^\alpha_j(f) = f * \mathcal{F}^{-1}\varphi^\alpha_j, \quad \text{for all } f \in H^s. \tag{64}$$

In fact, $\langle \mathcal{P}_j f, g\rangle = \langle \varphi_j^\alpha \cdot \hat{f}, \hat{g}\rangle = \langle \hat{f}, \varphi_j^\alpha \cdot \hat{g}\rangle = \langle f, \mathcal{P}_j g\rangle$. Hence $\mathcal{P}_j^\alpha = (\mathcal{P}_j^\alpha)^*$. $\hat{f} = 1 \cdot \hat{f} = (\sum_j \varphi_j^\alpha)\hat{f} = \sum_j \mathcal{F}(\mathcal{P}_j f)$ implies $\sum_j \mathcal{P}_j = I$.

$$\begin{aligned}\sum_j \left\|\mathcal{P}_j^\alpha f\right\|^2 &= \sum_j \int_{\mathbb{R}} \varphi_j^\alpha(\omega)^2 |\hat{f}(\omega)|^2 (1+|\omega|)^s d\omega = \\ &= \sum_j \int_{\mathbb{R}} \varphi_j^\alpha(\omega)^2 \widehat{D_j^\alpha} \chi_{\Omega_0}(\omega) |\hat{f}(\omega)|^2 (1+|\omega|)^s d\omega \le \\ &\le \int_{\mathbb{R}} K(\omega) \cdot |\hat{f}(\omega)|^2 (1+|\omega|)^s d\omega \le \left(\max_{\omega\in\mathbb{R}} K(\omega)\right) \|f\|_{H^s}^2, \end{aligned} \tag{65}$$

where $\widehat{D_j^\alpha} = T_{\operatorname{sgn}(j)((1+(1-\alpha)\cdot b\cdot|j|)^{\frac{1}{1-\alpha}}-1)} D_{b\cdot(1+(1-\alpha)\cdot b\cdot(|j|+1))^{\frac{\alpha}{1-\alpha}}} = \mathcal{F} D_j^\alpha$ and $K(\omega) = \sum_j \widehat{D_j^\alpha} \chi_{\Omega_0}(\omega)$.

Finally, $\mathcal{F}\mathcal{P}_j^\alpha(\pi_{W_j}(f)) = \varphi_j^\alpha \cdot \widehat{D_j^\alpha}\chi_{\Omega_0} \cdot \hat{f} = \varphi_j^\alpha \cdot \hat{f} = \mathcal{F}\mathcal{P}_j^\alpha(f)$. By (AD1) and Proposition 1 (or Theorem 4), one has that, for all $\rho > 1$ and $0 < a = a(\rho) = c_0^{-1}\rho^{-1}$ small enough, $\{\psi_k^0 = T_{ak} g_\rho\}_{k\in\mathbb{Z}}$ is a local family of atoms for $\mathbb{V}_0$, where g_ρ is a band-limited approximation of g in the sense of Definition 7 (AD3). Observe now that the assumptions on D_j^α of [29, Theorem 1] hold if one takes as lower constant $\alpha_j = \frac{1}{2^s} \min_{\omega\in\Omega_j^\alpha}(1+|\omega|^2)^s$ and upper constant $\beta_j = \max_{\omega\in\Omega_j^\alpha}(1+|\omega|^2)^s$. In fact, for all $f \in \mathbb{V}_0$

$$\begin{aligned}\left\|D_j^\alpha(f)\right\|_{H^s}^2 &= \int_{\Omega_j^\alpha} \left|\widehat{D_j^\alpha(f)}(\omega)\right|^2 (1+|\omega|^2)^s d\omega \\ &\le \left(\max_{\xi\in\Omega_j^\alpha}(1+|\xi|^2)^s\right) \cdot \int_{\Omega_j^\alpha} \left|\widehat{D_j^\alpha(f)}(\omega)\right|^2 d\omega \\ &= \left(\max_{\xi\in\Omega_j^\alpha}(1+|\xi|^2)^s\right) \int_{\Omega_0} |\hat{f}(\omega)|^2 d\omega \\ &\le \left(\max_{\omega\in\Omega_j^\alpha}(1+|\omega|^2)^s\right) \cdot \|f\|_{H^s}^2. \end{aligned}$$

In the same way:

$$\begin{aligned}\frac{1}{2^s} \cdot \left(\min_{\xi\in\Omega_j^\alpha}(1+|\xi|^2)^s\right) \int_{\Omega_0} |\hat{f}(\omega)|^2 (1+|\omega|^2)^s d\omega & \\ &\le \left(\min_{\xi\in\Omega_j^\alpha}(1+|\xi|^2)^s\right) \int_{\Omega_0} |\hat{f}(\omega)|^2 d\omega \\ &\le \int_{\Omega_j^\alpha} \left|\widehat{D_j^\alpha(f)}(\omega)\right|^2 (1+|\omega|^2)^s \\ &= \left\|D_j^\alpha(f)\right\|_{H^s}^2. \end{aligned}$$

Moreover, making use of the fact that

$$\max_{\omega\in\Omega_j^\alpha}(1+|\omega|^2)^s \asymp \min_{\omega\in\Omega_j^\alpha}(1+|\omega|^2)^s \asymp (1+(1-\alpha)b\cdot|j|)^{\frac{2s}{1-\alpha}},$$

one achieves by application of [29, Theorem 1] that for all $f \in H^s$

$$f = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} c^{j,k}_{\alpha,a,b,\rho}(f)(g_\rho)^{j,k}_{\alpha,a,b}, \tag{66}$$

$$\|f\|^2_{H^s} \asymp \sum_{k,j} \left|c^{j,k}_{\alpha,a,b,\rho}(f)\right|^2 (1+(1-\alpha)\cdot b\cdot |j|)^{\frac{2s}{1-\alpha}}\,. \tag{67}$$

Due to condition (AD2), consider the well-defined linear bounded operator

$$S_\rho f = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} c^{j,k}_{\alpha,a,b,\rho}(f) g^{j,k}_{\alpha,a,b}. \tag{68}$$

By Lemma 4 and (AD3)

$$\begin{aligned}\|(I-S_\rho)f\|^2_{H^s} &= \Big\| \sum_{k\in\mathbb{Z}} c^{j,k}_{\alpha,a,b,\rho}(f)(g^\rho)^{j,k}_{\alpha,a,b}\Big\|^2_{H^s}\\ &\le a^{-1}\cdot B_{g^\rho,a}\cdot \sum_{k,j} \left|c^{j,k}_{\alpha,a,b,\rho}(f)\right|^2 (1+(1-\alpha)\cdot b\cdot |j|)^{\frac{2s}{1-\alpha}}\\ &\le a^{-1}\cdot B_{g^\rho,a}\cdot C\cdot \|f\|^2_{H^s}.\end{aligned}$$

By the proof of [29, Theorem 1] and of Proposition 1 and by Remark 2, constant C can be chosen uniformly with respect to ρ (perhaps at the cost of increasing the density of the translation sampling points). Moreover, by (AD4) $a^{-1}\cdot B_{g^\rho,a} \to 0$, and therefore one has

$$\||I-S_\rho|\|_{H^s\to H^s} \le \eta < 1, \text{ for } \rho > \rho_0 > 0. \tag{69}$$

Hence S_ρ is invertible by using the Neumann series expansion $S_\rho^{-1} = \sum_{n=0}^{\infty}(I - S_\rho)^n$, and

$$f = S_\rho S_\rho^{-1} f = \sum_{k\in\mathbb{Z}} c^{j,k}_{\alpha,a,b,\rho}\big(S_\rho^{-1} f\big) g^{j,k}_{\alpha,a,b}. \tag{70}$$

Furthermore, $\||S_\rho^{-1}|\|_{H^s\to H^s} \le \frac{1}{1-\eta}$, and one has

$$\sum_{k,j} \left|c^{j,k}_{\alpha,a,b,\rho}\big(S_\rho^{-1} f\big)\right|^2 (1+(1-\alpha)\cdot b\cdot |j|)^{\frac{2s}{1-\alpha}} \asymp \|f\|^2_{H^s}. \tag{71}$$

This concludes the proof.

Remark 8. Theorem 1 holds for $\alpha \to 1$ (and for exponential coverings), but, in this case, one should modify the admissibility condition, which becomes a pure wavelet-type condition, and the proof also should be adapted. Since Theorem 4 works for irregular nodes $\{y_k\}_{k\in\mathbb{Z}}$ and Theorem 1 does not depend on the particular choice of equivalent covering $\mathcal{O}^\alpha$, up to suitable modifications of (56–57), Theorem 1 can be extended to *irregular* (Gabor and wavelet) cases where the frame is of the type $\{M_{P'_\alpha(j)} T_{y_k S'_\alpha(j)^{-1}} D_{S'_\alpha(j)^{-1}} g\}_{j,k\in\mathbb{Z}}$ [32]. In fact, the α-admissibility does not depend on (small) perturbations of $P_\alpha(j)$ and $S_\alpha(j)$ in the sense of Remark 7, nor does Theorem 1 depend on the particular choice of equivalent position and size functions. The multidimensional case can be treated in an analogous way by suitable geometrical decomposition into coronas of the frequency space centered in $P_\alpha(j)$ and with width $S_\alpha(j)$ or by suitable *stretching* of multidimensional regular coverings by means of the function $P_\alpha(\omega)$, see for example [18,13,35,47,46,34].

6.1. Computing and approximating duals

If $g \in H^s$ is an α-admissible band-limited function, then $\{T_{y_k} g\}_{k\in\mathbb{Z}}$ is a local family of atoms for $\mathbb{V}_0$ for any sufficiently small $\delta > 0$, and one considers $\{\tilde{\psi}_k^0\}_{k\in\mathbb{Z}}$ as its (local) dual. Then, by [29, Theorem 1] (see also [30]), for any BAPU $\{\varphi_j^\alpha\}_{j\in\mathbb{Z}}$ associated to $\mathcal{O}^\alpha$ and for all $f \in H^s$

$$f = \sum_{j,k\in\mathbb{Z}} \langle f, \left(\left((D_j^\alpha)^{-1}\right)^* \tilde{\psi}_k^0\right) * \mathcal{F}^{-1}\varphi_j^\alpha \rangle_{L^2} D_j^\alpha T_{y_k} g. \tag{72}$$

Hence, calculating the local dual, perhaps using the [LDA]-algorithm (Remark 3), one can completely analyze and recover any H^s function by means of flexible Gabor-wavelet frames (for all $\alpha \in [0, 1)$!). One example of a band-limited analyzing function g is given by

$$\hat{g}(x) = \begin{cases} e^{\frac{1}{1-|\frac{x}{1+\varepsilon}|^2}}, & |x| < 1+\varepsilon\,, \\ 0, & \text{otherwise} \end{cases} \tag{73}$$

which is an α-admissible function (Fig. 1) for every choice of $\alpha \in [0, 1)$.

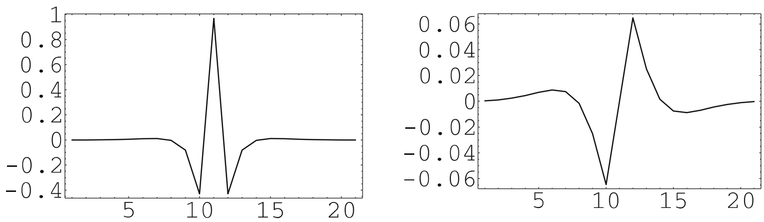

Fig. 1. Real and imaginary part of a band-limited *analyzing function*

Moreover, a dual frame for $\{g_{\alpha,a,b}^{j,k}\}_{j,k\in\mathbb{Z}}$ can be provided by

$$\tilde{g}_{\alpha,a,b}^{j,k} = \left(D_j^\alpha \tilde{\psi}_k^0\right) * \mathcal{F}^{-1}\varphi_j^\alpha. \tag{74}$$

Hence (74) expresses a dual in a *quasicoherent* form (with respect to the same family of operators $\{D_j^\alpha\}_{j\in\mathbb{Z}}$). If there exists Φ_0 such that $S_\alpha(j)^{-1/2} \cdot D_j^\alpha \Phi_0 = \mathcal{F}^{-1}\varphi_j^\alpha$, then the dual frame is coherent with respect to the local construction $(\mathbb{V}_0, \{D_j^\alpha\}_j)$ and can be written as

$$\tilde{g}_{\alpha,a,b}^{j,k} = D_j^\alpha \tilde{\psi}_k^*, \tag{75}$$

where $\tilde{\psi}_k^* = \tilde{\psi}_k^0 * \Phi_0$. Moreover, in the regular case (Proposition 1), for all $f \in H^s$, one has, in the sense of an unconditional convergent expansion,

$$f = \sum_{j,k\in\mathbb{Z}} \langle f, D_j^\alpha T_{\delta\cdot k} \tilde{\psi}_0^* \rangle D_j^\alpha T_{\delta\cdot k} g. \tag{76}$$

When $\alpha = 0$ or $\alpha = 1$, there exist always *coherent* BAPUs (regular and e^b-adic resp.) such that $S_\alpha(j)^{-1/2} \cdot D_j^\alpha \Phi_0 = \mathcal{F}^{-1}\varphi_j^\alpha$, but, in general, one cannot expect that this will be true for arbitrary $\alpha \in (0, 1)$ because no group structure is available anymore.

If the function g is not band-limited, one can consider for ρ large enough the dual given by

$$\tilde{g}_{\alpha,a,b,\rho}^{j,k} = \left(S_\rho^{-1}\right)^* \left(\left(D_j^\alpha \tilde{\psi}_k^0\right) * \mathcal{F}^{-1}\varphi_j^\alpha\right), \tag{77}$$

where $\tilde{\psi}_k^0$ is a dual of the local family of atoms generated by a band-limited approximation g_ρ of g and S_ρ is the operator defined in (68). Since

$$\left(S_\rho^{-1}\right)^* = \sum_{n=0}^{\infty} \left(I - S_\rho^*\right)^n, \tag{78}$$

one has that

$$\left\| \left| I - \left(S_\rho^{-1}\right)^* \right\| \right|_{H^s \to H^s} \le \frac{\eta}{1-\eta}, \tag{79}$$

where η is as in (69). Moreover, because of Definition 4(i),

$$\begin{aligned} \left\| (\tilde{g}_\rho)_{\alpha,a,b}^{j,k} - \tilde{g}_{\alpha,a,b,\rho}^{j,k} \right\|_{H^s} &= \left\| \left(I - \left(S_\rho^{-1}\right)^*\right) \left(\left(D_j^\alpha \tilde{\psi}_k^0\right) * \mathcal{F}^{-1}\varphi_j^\alpha\right) \right\|_{H^s} \\ &\le \frac{\eta}{1-\eta} \left\| \left(D_j^\alpha \tilde{\psi}_k^0\right) * \mathcal{F}^{-1}\varphi_j^\alpha \right\|_{H^s} \\ &\le \frac{\eta}{1-\eta} \left\| \varphi_j^\alpha \right\|_{\mathcal{F}L_s^1} \left\| D_j^\alpha \tilde{\psi}_k^0 \right\|_{H^s} \\ &\le \frac{C_0\eta}{1-\eta} (1 + (1-\alpha)b \cdot |j|)^{\frac{s}{1-\alpha}} \left\| \tilde{\psi}_k^0 \right\|_{H^s}. \end{aligned} \tag{80}$$

Hence, up to choosing $\eta > 0$ small enough and, as a consequence, ρ large enough, a dual for the L^2-frame (i.e., for s=0) $\{g_{\alpha,a,b}^{j,k}\}_{j,k}$ can be approximated by a dual of the type (74) of a band-limited approximation g_ρ of g.

Corollary 1. *Under the assumptions of Theorem 1 and using the notations of its proof, for $f \in L^2$ the following conditions are equivalent:*

(i) $f \in H^s$;
(ii) $\sum_{j,k} |\langle f, \left((D_j^\alpha \tilde{\psi}_k^0) * \mathcal{F}^{-1}\varphi_j^\alpha\right)\rangle_{L^2}|^2 \, (1 + (1-\alpha) \cdot b \cdot |j|)^{\frac{2s}{1-\alpha}} < \infty$;
(iii) $\sum_{j,k} |\langle f, \left((S_\rho^{-1})^* (D_j^\alpha \tilde{\psi}_k^0) * \mathcal{F}^{-1}\varphi_j^\alpha\right)\rangle_{L^2}|^2 \, (1 + (1-\alpha) \cdot b \cdot |j|)^{\frac{2s}{1-\alpha}} < \infty$.

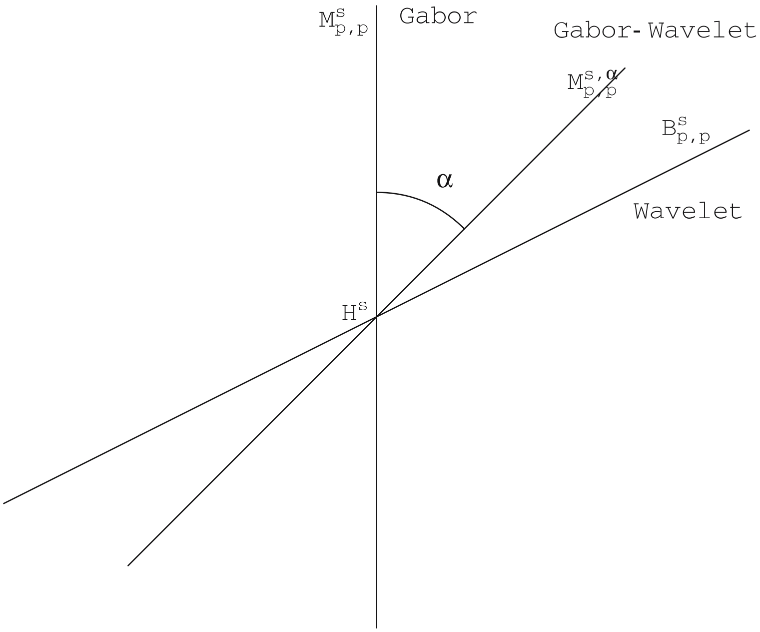

Fig. 2. α-modulation spaces

6.2. L^2-Sobolev spaces as α-modulation spaces

Theorem 3 by Holschneider and Nazaret [42] completes the analysis of the characterization of H^s by means of the (continuous) flexible Gabor-wavelet transform V_g^α, and it represents the continuous version of Theorem 1. In particular, condition (22) corresponds to condition (AD2) and the reconstruction formula (24) to formulas (7–8).

Fractional L^2-Sobolev spaces are just particular instances of a more general class of spaces.

Definition 8. *Related to the covering $\mathcal{O}^\alpha$ one can fix an $\mathcal{O}^\alpha$-BAPU $\{\varphi_j^\alpha\}_{j\in\mathbb{Z}}$ in $\mathcal{S}(\mathbb{R})$, and, for $1 \le p, q < \infty$, $s \in \mathbb{R}$ and $\alpha \in [0,1)$ ($\alpha = 1$ as limit case), one can define the spaces*

$$M^{s,\alpha}_{p,q}(\mathbb{R}) = \left\{ f \in S' : \left(\sum_{j\in\mathbb{Z}} \left\| \mathcal{P}_j^\alpha f \right\|_p^q (1 + (1-\alpha)b \cdot |j|)^{\frac{s\cdot q}{1-\alpha}} \right)^{1/q} < \infty \right\}, \quad (81)$$

where $\mathcal{P}_j^\alpha$ is defined as in (64). Endowed with the norm

$$\left\| f \middle| M^{s,\alpha}_{p,q} \right\| := \left(\sum_{j\in\mathbb{Z}} \left\| \mathcal{P}_j^\alpha f \right\|_p^q (1 + (1-\alpha)b \cdot |j|)^{\frac{s\cdot q}{1-\alpha}} \right)^{1/q}, \quad (82)$$

they are Banach spaces. The usual modifications apply for $p \cdot q = \infty$.

One can prove that $M^{s,\alpha}_{p,q}(\mathbb{R})$ does not depend on the particular value of $b > 0$, nor on the particular $\mathcal{O}^\alpha$-BAPU chosen. These spaces, introduced by Gröbner and Feichtinger in [18,13,35], are called α-modulation spaces, and they appear to be the right spaces in which to study flexible Gabor-wavelet analysis as generalized *coorbit spaces* related to the *Stone–Von Neumann representation* (19) (restricted on a suitable *homogeneous space*, see [1,43,44,8]) as it has been discussed in [31, Chapter 5] and is detailed in the later paper [32]. In particular, note that

$$M^{s,\alpha}_{2,2} = H^s, \quad \text{for all } s \in \mathbb{R} \text{ and } \alpha \in [0,1]. \tag{83}$$

Hence L^2-Sobolev spaces are α-modulation spaces for any exponent $s \in \mathbb{R}$ and any $\alpha \in [0,1]$. Moreover, for $\alpha = 0$, these spaces are just modulation spaces $M^s_{p,q}$ [17,36] (Gabor analysis) and for $\alpha \to 1$ are just Besov–Triebel spaces $B^s_{p,q}$ [47, 46,34] (wavelet analysis).

7. Conclusion

Theorem 1 can be seen as a *unified* approach to Gabor and wavelet frames and introduces new classes of intermediate frames. It gives unified sufficient conditions for the existence of such frames, which are satisfied by a quite large class of interesting functions. The construction improves the flexibility in the possible choice of coefficients for the decompositions obtained by duals that depend essentially on the choice of suitable BAPUs. The duals built in this way are typically nice smooth functions that are well localized in time and in frequency and show a "quasicoherent" behavior. The calculation of this kind of dual is also possible in the "irregular case" and can be done iteratively by means of the proposed algorithm.

Acknowledgement. The authors want to thank Karlheinz Gröchenig for his fundamental and deep hints and Norbert Kaiblinger for the fruitful discussions during the preparation of this work.

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