JOURNAL OF
FUNCTION SPACES AND APPLICATIONS
Volume 4, Number 1 (2006), 1-6

Characterizations of inner product spaces by orthogonal vectors

Marco Baronti and **Emanuele Casini**

(*Communicated by Jürgen Appell*)

2000 Mathematics Subject Classification. 46C15, 46B99, 41A65.

Keywords and phrases. Orthogonality, Characterizations of inner product spaces.

Abstract. Let X be a real normed space with unit closed ball B. We prove that X is an inner product space if and only if it is true that whenever x, y are points in ∂B such that the line through x and y supports $\frac{\sqrt{2}}{2}B$ then $x \perp y$ in the sense of Birkhoff.

1. Introduction

Several known characterizations of inner product spaces are available in the literature [1]. This paper concerns a new characterization by means of orthogonal vectors. Let X be a real normed space (of dimension ≥ 2) and let S_X denote its unit sphere. If $x, y \in S_X$, we set:

$$k(x,y) = \inf\{\|tx + (1-t)y\| : t \in [0,1]\}$$

moreover if $x, y \in S_X$ then $x \perp y$ denotes the orthogonality in the sense of Birkhoff [1] i.e. $\|x\| \leq \|x + \lambda y\|$ $\forall \lambda \in \mathbb{R}$. In this paper we consider the following properties

$$(P_1): \quad x, y \in S_X \quad x \perp y \Rightarrow k(x,y) = \frac{\|x+y\|}{2}$$

$$(P_2): \quad x, y \in S_X \quad k(x,y) = \frac{\sqrt{2}}{2} \Rightarrow x \perp y$$

We prove that these properties characterize inner product spaces. The definition of the constant $k(x,y)$ is suggested by the following result by Benitez and Yanez [4]:

$$\left\{ x, y \in S_X \quad k(x,y) = \frac{1}{2} \Rightarrow x + y \in S_X \right\} \Leftrightarrow X \text{ is an inner product space.}$$

It is easy to prove that every inner product spaces X satisfies the properties (P_1) and (P_2). Moreover from the particular structure of (P_1) and (P_2) we can suppose that X is to be a 2-dimensional real normed space.

2. Some preliminary results

We start with some preliminary observations on the constant $k(x,y)$.
If $x, y \in S_X$ and $x \perp y$ we set:

$$\mu(x,y) = \sup_{\lambda \geq 0} \frac{1+\lambda}{\|x+\lambda y\|}$$

and

$$\mu(X) = \sup\{\mu(x,y);\ x, y \in S_X\ x \perp y\}.$$

These constants were introduced in [6] and in [6] and [3] the following results were proved:

$$\mu(X) \geq \sqrt{2} \text{ for any space } X$$

and

$$\mu(X) = \sqrt{2} \Leftrightarrow X \text{ is an inner product space}$$

Let $x, y \in S_X$, $x \perp y$, we have

$$\begin{aligned} \mu(x,y) &= \sup_{\lambda \geq 0} \frac{1+\lambda}{\|x+\lambda y\|} = \sup_{0 \leq t < 1} \frac{1+\frac{t}{1-t}}{\|x + \frac{t}{1-t} y\|} \\ &= \sup_{0 \leq t \leq 1} \frac{1}{\|tx + (1-t)y\|} \\ &= \frac{1}{k(x,y)} \end{aligned}$$

and so

$$\mu(X)=\frac{1}{\inf\{k(x,y):x,y\in S_X\quad x\perp y\}} \tag{1}$$

From known results on the constant μ [3], [4] we obtain the following results:

$$\inf\{k(x,y):x,y\in S_X\quad x\perp y\}\le\frac{\sqrt{2}}{2} \tag{2}$$

$$\inf\{k(x,y):x,y\in S_X,\ x\perp y\}=\frac{\sqrt{2}}{2}\Leftrightarrow X\ \text{ is inner product space.} \tag{3}$$

Theorem 1. *If X fulfills (P_1) then X is an inner product space.*

Proof. Let $x,y\in S_X$ with $x\perp y$. From (P_1) we have $k(x,y)=\frac{\|x+y\|}{2}$. We define $g:\mathbb{R}\to\mathbb{R}$ by $g(t)=\|tx+(1-t)y\|$. g is a convex function and $g(\frac{1}{2})\le g(t)\quad\forall\ t\in[0,1]$. So we have $g(\frac{1}{2})\le g(t)\quad\forall\ t\in\mathbb{R}$. Hence

$$\frac{\|x+y\|}{2}\le\left\|y+t(x-y)+\frac{x+y}{2}-\frac{x+y}{2}\right\|=\left\|\frac{x+y}{2}+\frac{2t-1}{2}(x-y)\right\|.$$

So

$$\frac{||x+y||}{2}\le\left\|\frac{x+y}{2}+\lambda(x-y)\right\|\qquad\forall\lambda\in\mathbb{R}$$

that is equivalent to

$$x+y\perp x-y.$$

Therefore $\forall\ x,y\in S_X\quad x\perp y\Rightarrow x+y\perp x-y$ that is a characteristic property of inner product spaces [2]. □

3. Main result

To prove our main result we start with the following Lemma

Lemma 2. *Let X be a 2-dimensional real normed space and let x be in S_X. Let γ be one of the two oriented arcs of S_X joining x to $-x$. Then the function $y\in\gamma\mapsto k(x,y)$ is nonincreasing.*

Proof. Let $\bar{y}$ and $y\in\gamma$ and we suppose that $\bar{y}$ belongs to the part of the arc γ joining $-x$ to y. We will prove that

$$k(x,\bar{y})\le k(x,y).$$

We can suppose $\bar{y} \neq \pm x$ so there exist $a, b \in \mathbb{R}$ such that $y = ax + b\bar{y}$. From the assumption about $\bar{y}$ we have $a \geq 0$ and $b \geq 0$ and so

$$\|y\| = 1 = \|ax + b\bar{y}\| \leq a + b\,.$$

If $a + b = 1$, then $y = ax + (1-a)\bar{y}$ hence x, y and $\bar{y}$ are collinear and this implies:

$$1 = k(x, \bar{y}) = k(x, y)\,.$$

Now we suppose $a + b > 1$ and $k(x, \bar{y}) > k(x, y)$. Then there exists $t_1 \in (0, 1)$ such that $||t_1 x + (1 - t_1)y|| < k(x, \bar{y}) \leq 1$. Let

$$\lambda = \frac{1}{t_1(1 - a - b) + a + b}\,.$$

It is easy to verify that $\lambda \in (0, 1)$. Let

$$\begin{aligned} q_\lambda &= \lambda(t_1 x + (1 - t_1)y) \\ &= \frac{t_1}{t_1(1 - a - b) + a + b} x + \frac{(1 - t_1)(ax + b\bar{y})}{t_1(1 - a - b) + a + b} \\ &= \frac{(t_1 + a - at_1)}{t_1(1 - a - b) + a + b} x + \frac{(1 - t_1)b}{t_1(1 - a - b) + a + b} \bar{y}\,. \end{aligned}$$

So q_λ belongs to the segment joining x to $\bar{y}$ and

$$\|q_\lambda\| = \lambda\|t_1 x + (1 - t_1)y\| \geq k(x, \bar{y}) > \|t_1 x + (1 - t_1)y\|\,.$$

So $\lambda > 1$. This contradiction implies the thesis. □

Theorem 3. *If X fulfills (P_2) then X is an inner product space.*

Proof. We suppose that X is not an inner product space. So by using results (2) and (3) there exist $x, y \in S_X$, $x \perp y$ such that $k(x, y) < \frac{\sqrt{2}}{2}$. Let γ be one of the two oriented arcs of S_X joining x to $-x$. Let z_1 and z_2 be in γ such that $x \perp z_1$, $x \perp z_2$ and the part of γ joining z_1 to z_2 is the arc containing all points z such that $x \perp z$ and moreover we suppose that z_1 belongs to the part of γ joining x to z_2; we will write $z_1 \in \widehat{xz_2}$. Note that $y \in \widehat{z_1 z_2}$. Then using a result by Precupanu [7] we can suppose that there exists a sequence (x_n) such that:

1. $x_n \in \widehat{xz_1}$
2. $x_n \neq x$
3. $x_n \to x$
4. x_n are smooth.

Let y_n be in γ such that $x_n \perp y_n$. From the monotony of the Birkhoff orthogonality it follows that $y_n \in \widehat{-xz_2}$ Let ϵ be a positive number such that

$$k(x,y) + \epsilon < \frac{\sqrt{2}}{2}\,.$$

From Lemma 2 and the continuity of the function $k(\cdot\,, y)$ there exists $\bar{n} \in \mathbb{N}$ such that for $n > \bar{n}$

$$k(x_n, y_n) \leq k(x_n, y) \leq k(x,y) + \epsilon < \frac{\sqrt{2}}{2}\,.$$

Now if we fix $n_0 > \bar{n}$, we have

$$k(x_{n_0}, y_{n_0}) < \frac{\sqrt{2}}{2}; \quad x_{n_0} \perp y_{n_0}; \quad x_{n_0} \text{ is smooth}\,.$$

The continuous function $k(x_{n_0}, \cdot)$ assumes values between $k(x_{n_0}, y_{n_0}) < \frac{\sqrt{2}}{2}$ and $k(x_{n_0}, x_{n_0}) = 1$. So there exists $y^* \neq y_{n_0}$, $y^* \in \widehat{x_{n_0} y_{n_0}}$, $k(x_{n_0}, y^*) = \frac{\sqrt{2}}{2}$. From (P_2) we have $x_{n_0} \perp y^*$. But x_{n_0} is a smooth point and so $y^* = y_{n_0}$. This contradiction completes the proof. □

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Dipartimento di Ingegneria della Produzione e Metodi e Modelli Matematici
Università degli studi di Genova
Piazzale Kennedy, Pad. D
16129 Genova
Italy
(*E-mail : baronti@dimet.unige.it*)

Dipartimento di Fisica e Matematica
Università dell'Insubria
Via Valleggio 11
22100 Como
Italy
(*E-mail : emanuele.casini@uninsubria.it*)

(*Received : July 2004*)