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Polynomial identities on superalgebras and exponential growth ☆

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Abstract

Let A be a finitely generated superalgebra over a field F of characteristic 0. To the graded polynomial identities of A one associates a numerical sequence $\{c_n^{\mathrm{sup}}(A)\}_{n\geqslant 1}$ called the sequence of graded codimensions of A. In case A satisfies an ordinary polynomial identity, such sequence is exponentially bounded and we capture its exponential growth by proving that for any such algebra $\lim_{n\to\infty}\sqrt[n]{c_n^{\mathrm{sup}}(A)}$ exists and is a non-negative integer; we denote such integer by $\mathrm{supexp}(A)$ and we give an effective way for computing it. As an application, we construct eight superalgebras A_i, $i=1,\dots,8$, characterizing the identities of any finitely generated superalgebra A with $\mathrm{supexp}(A)>2$ in the following way: $\mathrm{supexp}(A)>2$ if and only if $Id^{\mathrm{sup}}(A)\subseteq Id^{\mathrm{sup}}(A_i)$ for some $i\in\{1,\dots,8\}$, where $Id^{\mathrm{sup}}(B)$ is the ideal of graded identities of the algebra B. We also compare the superexponent and the exponent (see A. Giambruno, M. Zaicev, Adv. Math. 140 (1998) 145–155) of any finitely generated superalgebra.

Keywords: Superalgebras; Polynomial identities; Codimensions; Growth

1. Introduction

Let F be a field of characteristic zero, $X=\{x_1,x_2,\dots\}$ a countable set, and $F\langle X\rangle = F\langle x_1,x_2,\dots\rangle$ the free associative algebra on X over F.

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The connection between T-ideals of $F\langle X\rangle$ (ideals invariant under all endomorphisms of $F\langle X\rangle$) and PI-algebras (algebras satisfying a polynomial identity) is well known: every T-ideal I of $F\langle X\rangle$ is the ideal of identities $I = Id(A)$ of some F-algebra A.

The study of such ideals has been carried out by using methods of representation theory of the symmetric group or of the general linear group. Since in characteristic zero, T-ideals are determined by their multilinear components, to each T-ideal $I = Id(A)$ one attaches a numerical sequence, called the sequence of codimensions $\{c_n(A)\}_{n\geqslant 1}$ of I or of A. $c_n(A)$ is the codimension of the space of multilinear polynomials in the first n variables of I. In [16] Regev proved that for any PI-algebra A, such sequence is exponentially bounded and in [9] and [10] it was shown that $\lim_{n\to\infty}\sqrt[n]{c_n(A)}$ exists and is a non-negative integer. Such integer is called the exponent of the algebras A and it has been a useful tool in PI-theory in the last years.

The main purpose of this paper is to prove an extension of this result in the setting of finitely generated superalgebras satisfying an ordinary identity and to deduce some natural consequences.

Let $A = A^{(0)} \oplus A^{(1)}$ be a superalgebra ($\mathbb{Z}_2$-graded algebra) over F. Write the set $X = Y \cup Z$ as the disjoint union of two sets. Then $F\langle X\rangle = F\langle Y \cup Z\rangle$ has a natural structure of free superalgebra if we require that the variables from Y have degree zero and the variables from Z have degree one.

Recall that an element $f(y_1,\dots,y_n,z_1,\dots,z_m)$ of $F\langle Y\cup Z\rangle$ is a graded identity for A if $f(a_1,\dots,a_n,b_1,\dots,b_m) = 0$, for all $a_1,\dots,a_n \in A^{(0)}$ and $b_1,\dots,b_m \in A^{(1)}$. The set $Id^{\sup}(A)$ of all graded identities of A is a T_2-ideal of $F\langle Y\cup Z\rangle$ i.e., an ideal invariant under all endomorphisms of $F\langle Y\cup Z\rangle$ preserving the grading. Moreover, every T_2-ideal of $F\langle Y\cup Z\rangle$ is the ideal of graded identities of some superalgebra.

As it was shown by Kemer (see [13]), superalgebras and their graded identities play a basic role in the study of the structure of varieties of associative algebras over a field of characteristic zero. More precisely, Kemer showed that any variety is generated by the Grassmann envelope of a suitable finite-dimensional superalgebra.

As in the setting of algebras with involution, in order to study $Id^{\sup}(A)$ in [8] the authors introduced methods pertaining to the representation theory of the hyperoctahedral group.

In case $\operatorname{char} F = 0$, it is well known that $Id^{\sup}(A)$ is completely determined by its multilinear polynomials and we let $V_n^{\sup}$ denote the space of such polynomials in the variables $y_1, z_1,\dots,y_n,z_n$ (i.e., y_i or z_i appears in each monomial at degree 1). Then the sequence of spaces $\{V_n^{\sup}\cap Id^{\sup}(A)\}_{n\geqslant 1}$ determines $Id^{\sup}(A)$ and

$$c_n^{\sup}(A) = \frac{\dim_F V_n^{\sup}}{V_n^{\sup}\cap Id^{\sup}(A)}$$

is called the nth graded codimension of A.

The asymptotic behavior of the graded codimensions plays an important role in the PI-theory of graded algebras. It was shown in [8] that the sequence $\{c_n^{\sup}(A)\}_{n\geqslant 1}$ is exponentially bounded if and only if A satisfies an ordinary polynomial identity. For non-associative algebras, graded identities and codimension growth of graded Lie algebras and Lie superalgebras where studied in [1].

By following the analogous procedure of the ordinary case, we shall study the exponential behavior of such sequence of graded codimensions. We shall prove that if A is a finitely generated superalgebra satisfying a polynomial identity, then $\lim_{n\to\infty} \sqrt[n]{c_n^{\sup}(A)}$ exists and is a non-negative integer. We shall also describe explicitly how to compute such limit. As a consequence, for any such finitely generated superalgebra, we shall define the superexponent of A as

$$\operatorname{supexp}(A) = \lim_{n\to\infty} \sqrt[n]{c_n^{\sup}(A)}.$$

In [7] the authors characterized those T_2-ideals such that the corresponding sequence of graded codimensions is polynomially bounded. For finitely generated superalgebras this means $\operatorname{supexp}(A) \leqslant 1$. Here we shall characterize those T_2-ideals $Id^{\sup}(A)$ such that A is a finitely generated superalgebra and $\operatorname{supexp}(A) > 2$.

More precisely, we shall exhibit eight finite-dimensional superalgebras $A_1, \dots, A_8$ with (distinct) T_2-ideals of graded identities satisfying the following property: if A is any finitely generated superalgebra and $\operatorname{supexp}(A) > 2$, then $Id^{\sup}(A) \subseteq Id^{\sup}(A_i)$ for some $i \in \{1, \dots, 8\}$.

As a consequence of this theorem and the main result of [7], we shall prove that $\operatorname{supexp}(A) = 2$ if and only if $Id^{\sup}(A) \nsubseteq Id^{\sup}(A_i)$ for all $i = 1, \dots, 8$ and $Id^{\sup}(A) \subseteq Id^{\sup}(B_i)$ for some $i = 1, 2, 3$, where $B_1 = UT_2(F)$, the algebra of 2×2 upper triangular matrices with trivial grading, $B_2 = UT_2(F)$ with canonical grading and $B_3 = F \oplus cF$, $c^2 = 1$, are the superalgebras constructed in [7].

Similar results have been found in case of ordinary algebras as well as for algebras with involution (see [12,15]).

2. Preliminaries

Throughout the paper we will denote by F a field of characteristic zero. Recall that an algebra A is a superalgebra (or $\mathbb{Z}_2$-graded algebra) with grading $(A^{(0)}, A^{(1)})$ if $A = A^{(0)} \oplus A^{(1)}$, where $A^{(0)}, A^{(1)}$ are subspaces of A satisfying:

$$A^{(0)}A^{(0)} + A^{(1)}A^{(1)} \subseteq A^{(0)} \quad \text{and} \quad A^{(0)}A^{(1)} + A^{(1)}A^{(0)} \subseteq A^{(1)}.$$

Let $F\langle X\rangle$ be the free associative algebra over F on the countable set X. We represent X in the form $X = Y \cup Z$ where Y and Z are countable disjoint subsets of X. We will denote by $\mathcal{F}^{(0)}$ the subspace of $F\langle X\rangle = F\langle Y \cup Z\rangle$ generated by the monomials of even degree with respect to Z and by $\mathcal{F}^{(1)}$ the subspace generated by the monomials of odd degree with respect to Z. Then clearly, $F\langle X\rangle = F\langle Y \cup Z\rangle = \mathcal{F}^{(0)} \oplus \mathcal{F}^{(1)}$ is a superalgebra with grading $(\mathcal{F}^{(0)}, \mathcal{F}^{(1)})$.

Recall that an ideal I of $F\langle Y \cup Z\rangle$ is a T_2-ideal if it is invariant under all endomorphisms η of $F\langle Y \cup Z\rangle$ such that $\eta(\mathcal{F}^{(0)}) \subseteq \mathcal{F}^{(0)}$ and $\eta(\mathcal{F}^{(1)}) \subseteq \mathcal{F}^{(1)}$. Also, a polynomial $f(y_1, \dots, y_n, z_1, \dots, z_m) \in F\langle Y \cup Z\rangle$ is a graded identity for an F-superalgebra $A = A^{(0)} \oplus A^{(1)}$ if $f(a_1, \dots, a_n, b_1, \dots, b_m) = 0$ for all $a_1, \dots, a_n \in A^{(0)}$ and $b_1, \dots, b_m \in$

$A^{(1)}$. The set $Id^{\sup}(A)$ of all graded identities of A is a T_2-ideal of $F\langle Y \cup Z\rangle$, and every T_2-ideal I of $F\langle Y \cup Z\rangle$ is the ideal of graded identities of some superalgebra A, $I = Id^{\sup}(A)$.

It is well known that any superalgebra A can be viewed as an algebra with ϕ-action where $\phi \in Aut(A)$ is an automorphism of A of order 2. In fact, the homomorphism ϕ of $A = A^{(0)} \oplus A^{(1)}$ defined by $\phi(a_0) = a_0$ and $\phi(a_1) = -a_1$ for any $a_0 \in A^{(0)}$ and $a_1 \in A^{(1)}$ is an automorphism of A of order 2; conversely, if A is an algebra with an automorphism ϕ such that $\phi^2 = 1$, then by setting $A^{(0)} = \{a \in A \mid \phi(a) = a\}$ and $A^{(1)} = \{a \in A \mid \phi(a) = -a\}$ we obtain that $A = A^{(0)} \oplus A^{(1)}$ is a superalgebra with grading $(A^{(0)}, A^{(1)})$.

By applying this remark to the free algebra $F\langle X\rangle = F\langle Y \cup Z\rangle$ we exploit the relation between $\mathbb{Z}_2$-grading and ϕ-action on $F\langle X\rangle$. If we set $x_i = y_i + z_i$, $x_i^{\phi} = y_i - z_i$ for $i = 1, 2, \ldots$, and we require that ϕ acts as an automorphism of order 2 on $F\langle X\rangle$, then $F\langle X\rangle$ becomes the free algebra on X with ϕ-action and we write it as $F\langle X, \phi\rangle$.

The notion of ϕ-identity for an algebra A with automorphism ϕ of order 2 is the obvious one, and we denote by $Id^{\phi}(A)$ the ideal of ϕ-identities of A. It is not difficult to show that $Id^{\phi}(A) = Id^{\sup}(A)$. Motivated by this equality we define the graded codimensions of the superalgebra A as follows (see [7]).

As in the case of ordinary identities, in characteristic zero, the graded identities of a superalgebra are determined by the multilinear ones. We can write the space of all multilinear ϕ-polynomials of degree n as

$$\begin{aligned} V_n^{\phi} &= \operatorname{Span}_F\left\{x_{\sigma(1)}^{\varepsilon_1} \cdots x_{\sigma(n)}^{\varepsilon_n} \,\middle|\, \varepsilon_1, \ldots, \varepsilon_n \in \{1, \phi\}, \sigma \in S_n\right\} \\ &= \operatorname{Span}_F\{w_{\sigma(1)} \cdots w_{\sigma(n)} \mid \sigma \in S_n,\ w_i = y_i \text{ or } w_i = z_i,\ i = 1, \ldots n\} = V_n^{\sup}. \end{aligned}$$

Recall that if $\mathbb{Z}_2$ is the multiplicative cyclic group of order 2, then the hyperoctahedral group of order n is the wreath product $B_n = \mathbb{Z}_2 \wr S_n = \{(a_1, \ldots, a_n; \sigma) \mid a_i \in \mathbb{Z}_2, \sigma \in S_n\}$ with multiplication given by

$$(a_1, \ldots, a_n; \sigma)\,(b_1, \ldots, b_n; \tau) = (a_1 b_{\sigma^{-1}(1)}, \ldots, a_n b_{\sigma^{-1}(n)}; \sigma\tau).$$

The space $V_n^{\sup}$ has a structure of left B_n-module induced by defining for $h = (a_1, \ldots, a_n; \sigma) \in B_n$, $hy_i = y_{\sigma(i)}$, $hz_i = z_{\sigma(i)}^{a_{\sigma(i)}} = \pm z_{\sigma(i)}$. For every algebra A with automorphism ϕ of order 2, the vector space $V_n^{\sup} \cap Id^{\sup}(A)$ is invariant under the above action of $B_n = \mathbb{Z}_2 \wr S_n$. Hence the space $V_n^{\sup}(A) = V_n^{\sup}/(V_n^{\sup} \cap Id^{\sup}(A))$ has a structure of left B_n-module. Its character $\chi_n^{\sup}(A)$ is called the nth graded cocharacter of A and the sequence $\{\chi_n^{\sup}(A)\}_{n \geqslant 1}$ is the graded cocharacter sequence of A. The integer $c_n^{\sup}(A) = \dim_F V_n^{\sup}(A)$ is called the nth graded codimension of A and the sequence $\{c_n^{\sup}(A)\}_{n \geqslant 1}$ is the graded codimension sequence of A.

As in the case of ordinary identities, an approach to the description of the graded identities of A is based on the study of the graded cocharacter sequence of this algebra. While in the theory of associative algebras the space of multilinear identities of degree n is studied through the representation theory of the symmetric group S_n, in this case the space of multilinear graded identities of degree n is explored through the representation theory of B_n (see [8]).

Recall that there is a one-to-one correspondence between irreducible B_n-characters and pairs of partitions (λ, μ), where $\lambda \vdash r$, $\mu \vdash n-r$, for all $r = 0, 1, \ldots, n$. If $\chi_{\lambda,\mu}$ denotes the irreducible B_n-character corresponding to (λ, μ), we write

$$\chi_n^{\mathrm{sup}}(A) = \sum_{r=0}^{n} \sum_{\substack{\lambda \vdash r \\ \mu \vdash n-r}} m_{\lambda,\mu} \chi_{\lambda,\mu}, \tag{1}$$

where $m_{\lambda,\mu} \geqslant 0$ are the corresponding multiplicities.

We now make a reduction following [5]. For fixed r, let $V_{r,n-r}$ be the space of multilinear polynomials in $y_1, \ldots, y_r$, $z_1, \ldots, z_{n-r}$. It is clear that in order to study $V_n^{\mathrm{sup}} \cap Id^{\mathrm{sup}}(A)$ it is enough to study $V_{r,n-r} \cap Id^{\mathrm{sup}}(A)$ for all r. If we let S_r act on the even variables $y_1, \ldots, y_r$ and S_{n-r} on the odd ones $z_1, \ldots, z_{n-r}$, then we obtain an action of $S_r \times S_{n-r}$ on $V_{r,n-r}$. Since T_2-ideals are invariant under this action, we get that

$$V_{r,n-r}(A) = V_{r,n-r} / \big(V_{r,n-r} \cap Id^{\mathrm{sup}}(A)\big)$$

has an induced structure of left $S_r \times S_{n-r}$-module. We denote by $\chi_{r,n-r}(A)$ its character and $c_{r,n-r}(A) = \dim_F V_{r,n-r}(A)$. It is well known that there is a one-to-one correspondence between irreducible $S_r \times S_{n-r}$-characters and pairs of partitions (λ, μ) such that $\lambda \vdash r$, and $\mu \vdash n-r$. Hence, by complete reducibility, we have the following decomposition:

$$\chi_{r,n-r}(A) = \sum_{\substack{\lambda \vdash r \\ \mu \vdash n-r}} \overline{m}_{\lambda,\mu} (\chi_\lambda \otimes \chi_\mu), \tag{2}$$

where χ_λ (respectively χ_μ) denotes the ordinary S_r-character corresponding to $\lambda \vdash n$ (respectively S_{n-r}-character corresponding to $\mu \vdash n$), $\chi_\lambda \otimes \chi_\mu$ is the irreducible $(S_r \times S_{n-r})$-character associated to the pair (λ, μ) and $\overline{m}_{\lambda,\mu} \geqslant 0$ is the corresponding multiplicity.

The relation between B_n-characters and $(S_r \times S_{n-r})$-characters is expressed in the following theorem.

Theorem 1 [5, Theorem 1.3]. *If the nth graded cocharacter of A has the decomposition given in* (1) *and the $(S_r \times S_{n-r})$-character of $V_{r,n-r}(A)$ has the decomposition given in* (2) *then $m_{\lambda,\mu} = \overline{m}_{\lambda,\mu}$ for all λ and μ and*

$$c_n^{\mathrm{sup}}(A) = \sum_{r=0}^{n} \binom{n}{r} c_{r,n-r}(A).$$

3. Determining the superexponent of a finitely generated PI-superalgebra

In this section we shall prove that for any finitely generated superalgebra A, satisfying a polynomial identity, $\operatorname{supexp}(A) = \lim_{n\to\infty} \sqrt[n]{c_n^{\mathrm{sup}}(A)}$ exists. A first reduction is in order.

By a theorem of Kemer (see [13, Theorem 2.2]), if A is a finitely generated superalgebra, then there exists a finite-dimensional superalgebra B such that $Id^{\mathrm{sup}}(A) = Id^{\mathrm{sup}}(B)$. Therefore throughout we may assume that $A = A^{(0)} \oplus A^{(1)}$ is a finite-dimensional superalgebra over F satisfying an ordinary polynomial identity and char $F = 0$.

We write $A = B + J$, where B is a maximal semisimple subalgebra of A and $J = J(A)$ is the Jacobson radical of A. It is well known that $J = J^{(0)} \oplus J^{(1)}$ is a homogeneous ideal. Also, by [13, p. 21] we may assume that B has an induced $\mathbb{Z}_2$-grading, $B = B^{(0)} \oplus B^{(1)}$ and let $B = B_1 \oplus \cdots \oplus B_m$ where $B_1, \ldots, B_m$ are simple superalgebras.

We now define an integer $d = d(A)$ in the following way: we consider all possible nonzero products of the type

$$C_1 J C_2 J \cdots C_{k-1} J C_k \neq 0$$

where $C_1, \ldots, C_k$ are distinct subalgebras from the set $\{B_1, \ldots, B_m\}$ and $k \geqslant 1$. We then define $d = d(A)$ to be the maximal dimension of a subalgebra $C_1 + \cdots + C_k$ satisfying the above inequality. We next prove that in case F is algebraically closed, d coincides with $\mathrm{supexp}(A)$.

The proof of this fact follows closely the proof given in [11] concerning the existence of the $*$-exponent for a finite-dimensional algebra with involution $*$. We start with the following lemma.

Lemma 2. *Let $t \geqslant 0$, $a + b > d$, and let $f(y_1, \ldots, y_a, z_1, \ldots, z_b, x_1, \ldots, x_t)$ be a multilinear polynomial alternating on $\{y_1, \ldots, y_a\}$ and on $\{z_1, \ldots, z_b\}$. Then, for every choice of $\bar{y}_1, \ldots, \bar{y}_a \in B^{(0)}$, $\bar{z}_1, \ldots, \bar{z}_b \in B^{(1)}$, $\bar{x}_1, \ldots, \bar{x}_t \in A$, we have*

$$f(\bar{y}_1, \ldots, \bar{y}_a, \bar{z}_1, \ldots, \bar{z}_b, \bar{x}_1, \ldots, \bar{x}_t) = 0.$$

Proof. For $i = 1, \ldots, m$, let $E_i = E_i^{(0)} \cup E_i^{(1)}$ be a basis of B_i where $E_i^{(0)}$ and $E_i^{(1)}$ are sets of even and odd elements, respectively. Then $E = \bigcup E_i = E^{(0)} \cup E^{(1)}$ is a basis of B. Since f is multilinear, it is enough to evaluate f on a basis of A. The proof now goes on as in [11, Lemma 1], replacing the $*$-simple subalgebras there with the $\mathbb{Z}_2$-graded simple subalgebras. □

Recall that if $\lambda \vdash n$, then $\lambda' = (\lambda'_1, \ldots, \lambda'_r) \vdash n$ denotes the conjugate partition of λ.

Lemma 3. *Let $\lambda \vdash r$, $\mu \vdash n - r$ and let $W_{\lambda,\mu} \subseteq V_{r,n-r}$ be a left irreducible $(S_r \times S_{n-r})$-module. If $W_{\lambda,\mu} \nsubseteq V_{r,n-r} \cap Id^{\mathrm{sup}}(A)$, then $h(\lambda) \leqslant \dim A^{(0)}$, $h(\mu) \leqslant \dim A^{(1)}$, and $\lambda'_{l+1} + \mu'_{l+1} \leqslant d$ where $J^{l+1} = 0$. Moreover, $\dim W_{\lambda,\mu} \leqslant n^a (\lambda'_{l+1})^r (\mu'_{l+1})^{n-r}$ for some $a \geqslant 1$.*

Proof. It is well known (see [11, Remark 1]) that $W_{\lambda,\mu} = F(S_r \times S_{n-r})f$ where f is a multilinear polynomial alternating on disjoint sets of variables $\{y_1^i, \ldots, y_{\lambda'_i}^i\}$, $\{z_1^j, \ldots, z_{\mu'_j}^j\}$, for $1 \leqslant i \leqslant \lambda_1$, $1 \leqslant j \leqslant \mu_1$.

Since f is alternating on $\{y_1^1, \ldots, y_{\lambda'_1}^1\}$, it follows that $\lambda'_1 = h(\lambda) \leqslant \dim A^{(0)}$. Similarly we get that $h(\mu) \leqslant \dim A^{(1)}$. Suppose by contradiction that $\lambda'_{l+1} + \mu'_{l+1} > d$. This implies that $\lambda_i' + \mu_i' > d$ for $i = 1, \ldots, l$. Since by hypothesis f does not vanish on A, then, by the previous lemma, we must have that each set $\{y_1^i, \ldots, y_{\lambda_i'}^i, z_1^i, \ldots, z_{\mu_i'}^i\}$, $1 \leqslant i \leqslant l+1$, must contain at least one variable that evaluates to an element of J. But $J^{l+1} = 0$ says that f vanishes on A, a contradiction. The bound for $\dim W_{\lambda,\mu}$ is obtained by applying the hook formula for the degrees of the irreducible S_n-representations (see [11, Lemma 2]). □

We can now determine an upper bound for $c_n^{\sup}(A)$.

Lemma 4. *There exist constants a, t such that $c_n^{\sup}(A) \leqslant a n^t d^n$.*

Proof. By the previous lemma we can write

$$\chi_{r,n-r}(A) = \sum_{0 \leqslant \lambda'_{l+1} + \mu'_{l+1} \leqslant d} \sum_{\substack{\lambda \vdash r \\ \mu \vdash n-r}} m_{\lambda,\mu} (\chi_\lambda \otimes \chi_\mu).$$

Hence

$$c_{r,n-r}(A) \leqslant \sum_{0 \leqslant \lambda'_{l+1} + \mu'_{l+1} \leqslant d} \sum_{\substack{\lambda \vdash r \\ \mu \vdash n-r}} m_{\lambda,\mu} n^\alpha t(\lambda)^r t(\mu)^{n-r}$$

for some constant α. By [2], the multiplicities $m_{\lambda,\mu}$ are polynomially bounded. Hence, by Theorem 1, we obtain

$$c_n^{\sup}(A) = \sum_{r=0}^{n} \binom{n}{r} c_{r,n-r}(A) \leqslant a n^b \sum_{\substack{t_1+t_2 \leqslant d \\ t_1,t_2 \geqslant 0}} \sum_{r=0}^{n} \binom{n}{r} t_1^r t_2^{n-r} = a n^b \sum_{\substack{t_1+t_2 \leqslant d \\ t_1,t_2 \geqslant 0}} (t_1 + t_2)^n$$
$$\leqslant a n^b d^{n+1}. \qquad \square$$

We next compute a lower bound for $c_n^{\sup}(A)$. We start with the following lemma.

Lemma 5. *Let $C = C^{(0)} \oplus C^{(1)}$ be a finite-dimensional central simple superalgebra over the algebraically closed field F and let $\dim C^{(0)} = p$ and $\dim C^{(1)} = q$. Then, for all $m \geqslant 1$ there exists a multilinear polynomial*

$$f = f\left(y_1^1, \ldots, y_p^1, \ldots, y_1^{2m}, \ldots, y_p^{2m}, z_1^1, \ldots, z_q^1, \ldots, z_1^{2m}, \ldots, z_q^{2m}\right)$$

such that

(1) *f is alternating on each set of variables $\{y_1^i, \ldots, y_p^i\}$, $1 \leqslant i \leqslant 2m$, and $\{z_1^j, \ldots, z_q^j\}$, $1 \leqslant j \leqslant 2m$;*
(2) *there exist $\bar{y}_i^j \in C^{(0)}$, $\bar{z}_i^j \in C^{(1)}$ such that $f(\bar{y}_1^1, \ldots, \bar{y}_p^{2m}, \bar{z}_1^1, \ldots, \bar{z}_q^{2m}) \in F$.*

Proof. By [13, p. 21] we have that C is isomorphic to one of the following superalgebras:

(1) $M_{k,l}(F)$, the algebra of $(k+l)\times(k+l)$ matrices over F with grading $\left(\begin{pmatrix} F_{11} & 0 \\ 0 & F_{22}\end{pmatrix}, \begin{pmatrix} 0 & F_{12} \\ F_{21} & 0\end{pmatrix}\right)$, where F_{11}, F_{12}, F_{21}, F_{22} are $k\times k$, $k\times l$, $l\times k$, and $l\times l$ matrices, respectively, $k\geqslant 1$ and $l\geqslant 0$;

(2) $M_n(F\oplus cF)$, the algebra of $n\times n$ matrices over $F\oplus cF$ with grading $(M_n(F), cM_n(F))$, where $c^2=1$.

Recall that by the result of Formanek in [6], for every $m\geqslant 1$ there exists a multilinear polynomial $f_n(X)=f_n(x_1^1,\dots,x_{n^2}^1,\dots,x_1^{2m},\dots,x_{n^2}^{2m})$ alternating on each set of variables $\{x_1^i,\dots,x_{n^2}^i\}$, $1\leqslant i\leqslant 2m$ and, $f_n(X)$ is a central polynomial for $M_n(F)$. Then $f_{k+l}(X)$ and $f_n(Y)\cdot f_n(Z)$ are the required polynomials for $M_{k,l}(F)$ and $M_n(F\oplus cF)$, respectively. □

In the next step we extend the previous lemma by gluing together the polynomials obtained there.

Lemma 6. *Let A be a finite-dimensional algebra over the algebraically closed filed F. Suppose $C_1JC_2J\cdots C_{k-1}JC_k\neq 0$ where $C_1,\dots,C_k$ are distinct simple subsuperalgebras of A and $C_1+\cdots+C_k=C_1\oplus\cdots\oplus C_k$. If $p=\dim(C_1^{(0)}+\cdots+C_k^{(0)})$, $q=\dim(C_1^{(1)}+\cdots+C_k^{(1)})$, then for all $m\geqslant 1$ there exists a multilinear polynomial*

$$f=f\left(y_1^1,\dots,y_p^1,\dots,y_1^{2m},\dots,y_p^{2m},z_1^1,\dots,z_q^1,\dots,z_1^{2m},\dots,z_q^{2m},y_1,\dots,y_{k_1},z_1,\dots,z_{k_2}\right)$$

where $k_1+k_2=2k-1$, such that

(1) *f is alternating on each set of variables $\{y_1^i,\dots,y_p^i\}$, $\{z_1^j,\dots,z_q^j\}$, $1\leqslant i,j\leqslant 2m$;*

(2) *there exists a valuation of the variables $\bar{y}_i^j\in(C_1^{(0)}+\cdots+C_k^{(0)})$, $\bar{z}_i^j\in(C_1^{(1)}+\cdots+C_k^{(1)})$, $\bar{y}_i\in A^{(0)}$, $\bar{z}_i\in A^{(1)}$ for which f takes a non-zero value.*

Proof. The proof of this lemma is actually given in [11, Lemma 4] if we replace in that lemma symmetric and skew elements with even and odd elements respectively and we use Lemma 5 above instead of [11, Lemma 3]. □

Proposition 7. *Let A be a finite-dimensional superalgebra over the algebraically closed field F and let d be the integer defined at the beginning of the section. Then there exist constants a_1, a_2, b_1, b_2 such that*

$$a_1 n^{b_1} d^n \leqslant c_n^{\sup}(A)\leqslant a_2 n^{b_2} d^n.$$

Hence the superexponent of A exists and $\operatorname{supexp}(A)=\lim_{n\to\infty}\sqrt[n]{c_n^{\sup}(A)}=d$.

Proof. The upper bound for $c_n^{\sup}(A)$ was found in Lemma 4 above. For finding the lower bound, the proof is close to the one given in [11, Theorem 3] if one makes the due changes (symmetric and skew elements become even and odd elements, respectively). We proceed as follows.

Let $C_1, \dots, C_k$ be distinct simple subsuperalgebras of B such that

$$C_1 J C_2 J \cdots C_{k-1} J C_k \neq 0.$$

Let $p = \dim(C_1^{(0)} + \cdots + C_k^{(0)})$, $q = \dim(C_1^{(1)} + \cdots + C_k^{(1)})$ and $d = p + q$.

Take any $n \geqslant 2d + (k_1 + k_2)$, where k_1 and k_2 are as in the previous lemma, and divide $n - (k_1 + k_2)$ by $2d$; then we can write $n = 2m(p+q) + (k_1 + k_2) + t$, for some m, t where $0 \leqslant t < 2d$. Set $s = 2mp + k_1 + t$, $n - s = 2mq + k_2$. Let f be the polynomial constructed in the previous lemma of degree $2mp + 2mq + k_1 + k_2$, and set $g = f y_{k_1+1} \cdots y_{k_1+t} \in V_{s,n-s}$. It is easy to check that g does not vanish on A.

Let the group $G = S_{2mp} \times S_{2mq}$ act on g (S_{2mp} acts on the even variables y_i^l and S_{2mq} acts on the odd variables z_i^l) and let M be the G-submodule of $V_{s,n-s}$ generated by g. By complete reducibility M contains an irreducible G-submodule of the form $W_{\lambda,\mu} = FGe_{T_\lambda} e_{T_\mu}(g)$ for some tableaux T_λ and T_μ of shape $\lambda \vdash 2mp$ and $\mu \vdash 2mq$, respectively.

Let $l(\lambda)$ be the length of the first row of λ. Now, for every $\tau \in S_{2mp}$, $\tau(g)$ is still alternating on $2m$ disjoint sets of even variables and $\sum_{\sigma \in R_{T_\lambda}} \sigma$ acts by symmetrizing $l(\lambda)$ variables. Thus, if $l(\lambda) > 2m$ we would get $e_{T_\lambda}(g) = 0$, a contradiction. Similarly for $l(\mu)$. The outcome of this is that $l(\lambda) \leqslant 2m$ and $l(\mu) \leqslant 2m$.

For a partition $\nu \vdash n$, let D_ν be the corresponding Young tableau and let $h(\nu)$ be the height of D_ν. Notice that from the above, since $l(\lambda) \leqslant 2m$ and $l(\mu) \leqslant 2m$, then $h(\lambda) \geqslant p$ and $h(\mu) \geqslant q$.

Suppose that either $h(\lambda) > p$ or $h(\mu) > q$ and that the total number of boxes out of the first p rows of the diagram D_λ and out of the first q rows of the diagram D_μ is at least $l + 1$ where $J^{l+1} = 0$.

Since $FGe_{T_\lambda} e_{T_\mu}$ is a minimal left ideal of FG, then $FG\overline{C}_{T_\lambda}\overline{C}_{T_\mu} e_{T_\lambda} e_{T_\mu} = FGe_{T_\lambda} e_{T_\mu}$ where $\overline{C}_{T_\lambda} = \sum_{\sigma \in C_{T_\lambda}} (\operatorname{sgn}\sigma)\sigma$ and $\overline{C}_{T_\mu} = \sum_{\sigma \in C_{T_\mu}} (\operatorname{sgn}\sigma)\sigma$. We need to evaluate the polynomial $\bar{g} = \overline{C}_{T_\lambda}\overline{C}_{T_\mu} e_{T_\lambda} e_{T_\mu}(g)$ on A.

Now let $\lambda' = (p + r_1, \dots, p + r_u, \lambda'_{u+1}, \dots, \lambda'_m)$ and let $\mu' = (q + s_1, \dots, q + s_v, \mu'_{v+1}, \dots, \mu'_n)$ be the conjugate partitions of λ and μ respectively. Here $r_1 + \cdots + r_u + s_1 + \cdots + s_v \geqslant l + 1$, $\lambda'_{u+1}, \dots, \lambda'_m \leqslant p$ and $\mu'_{v+1}, \dots, \mu'_n \leqslant q$. Now, the polynomial $\bar{g}$ is alternating on each of u sets of even variables of order $p + r_1, \dots, p + r_u$ and on each of v sets of odd variables of order $q + s_1, \dots, q + s_v$. If we substitute on any of these sets of variables only elements from the semisimple subalgebra $C = C_1 \oplus \cdots \oplus C_k$, we would get zero since $\dim C^{(0)} = p$, $\dim C^{(1)} = q$. It follows that we have to substitute into these sets of variables at least $r_1 + \cdots + r_u + s_1 + \cdots + s_v \geqslant l + 1$ elements form the Jacobson radical J. Since $J^{l+1} = 0$, we get that $\bar{g}$ vanishes in A, a contradiction.

We have proved that D_λ and D_μ must contain at most a total number of l boxes out of the first p and q rows, respectively. Since $l(\lambda) \leqslant 2m$ and $l(\mu) \leqslant 2m$, the outcome of this is that λ must contain the rectangle $\nu = ((2m - l)^p)$ and μ must contain the rectangle $\nu' = ((2m - l)^q)$. Recall that as $m \to \infty$,

$$(\deg \chi_\nu)(\deg \chi_{\nu'}) \simeq a\big((2m - l)p\big)^b \big((2m - l)q\big)^{b'} p^{(2m-l)p} q^{(2m-l)q} \geqslant n^u p^{2mp} q^{2mq},$$

for some constants a, b, b', u. Hence, since

$$\dim W_{\lambda,\mu} = (\deg \chi_\lambda)(\deg \chi_\mu) \geqslant (\deg \chi_\nu)(\deg \chi_{\nu'}),$$

we obtain

$$c_{s,n-s}(A) \geqslant \dim W_{\lambda,\mu} \geqslant n^u p^{2mp} q^{2mq}.$$

Therefore

$$c_n^{\sup}(A) = \sum_{r=0}^{n} \binom{n}{r} c_{r,n-r}(A) \geqslant \binom{n}{s} c_{s,n-s}(A) \geqslant n^u \frac{n!}{s!(n-s)!} p^{2mp} q^{2mq}.$$

If we now recall that $s = 2mp + k_1 + t$ and $n - s = 2mq + k_2$, by making use of the Stirling formula, we get the desired conclusion $c_n^{\sup} \geqslant an^\alpha d^n$ for some a, α. □

Recall that by extending the ground field, the graded codimensions do not change. In fact, let $\overline{F}$ be the algebraic closure of F and $\overline{A} = A \otimes_F \overline{F}$. $\overline{A}$ has an induced grading given by $\overline{A}^{(0)} = A^{(0)} \otimes_F \overline{F}$, $\overline{A}^{(1)} = A^{(1)} \otimes_F \overline{F}$ and it follows that $c_n^{\sup}(\overline{A}) = c_n^{\sup}(A)$. Hence we obtain the following.

Theorem 8. *For any finitely generated superalgebra A satisfying an ordinary polynomial identity*, $\operatorname{supexp}(A) = \lim_{n\to\infty} \sqrt[n]{c_n^{\sup}(A)}$ *exists and is a non-negative integer.*

We record some immediate consequences of Theorem 8. For any superalgebra A over F we let $Z = Z(A)$ be the center of A and $Z^{(0)} = Z(A)^{(0)} = Z \cap A^{(0)}$ the even center of A. Next result gives an exact estimate of $\operatorname{supexp}(A)$ in case of $\mathbb{Z}_2$-simple or $\mathbb{Z}_2$-semisimple superalgebras. For its proof, we refer the reader to [11, Corollaries 3–5].

Corollary 9. *Let A be a finite-dimensional superalgebra over the field F. We have*

(1) *if A is $\mathbb{Z}_2$-simple, then* $\operatorname{supexp}(A) = \dim_{Z^{(0)}} A$;
(2) *if A is $\mathbb{Z}_2$-semisimple, then* $\operatorname{supexp}(A) = \max_i\{\dim_{Z_i^{(0)}} A_i\}$, *where $A = \bigoplus A_i$ and* $Z_i^{(0)} = Z(A_i) \cap A_i^{(0)}$;
(3) $\operatorname{supexp}(A) = \dim_F A$ *if and only if A is $\mathbb{Z}_2$-simple and $F = Z^{(0)}$.*

4. Superalgebras with superexponent equal to two

In this section we shall characterize the T_2-ideals of finitely generated superalgebras having superexponent equal to two. Recall that a characterization of T_2-ideals of superalgebras of superexponent at most one (i.e., with polynomial growth of the graded codimensions) has been given in [7]. We first need to introduce few algebras.

Let us denote by $UT_n(F)$ the algebra of $n \times n$ upper triangular matrices. We define the following eight superalgebras over F that we shall use in the sequel:

$$A_1 = M_2(F), \quad \text{with grading } \big(M_2(F), \{0\}\big);$$

$$A_2 = M_{1,1}(F), \quad \text{with grading } \left(\begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix}, \begin{pmatrix} 0 & F \\ F & 0 \end{pmatrix}\right);$$

$$A_3 = \begin{pmatrix} F \oplus cF & F \oplus cF \\ 0 & F \end{pmatrix}, \quad \text{with grading } \left(\begin{pmatrix} F & F \\ 0 & F \end{pmatrix}, \begin{pmatrix} cF & cF \\ 0 & 0 \end{pmatrix}\right);$$

$$A_4 = \begin{pmatrix} F & F \oplus cF \\ 0 & F \oplus cF \end{pmatrix}, \quad \text{with grading } \left(\begin{pmatrix} F & F \\ 0 & F \end{pmatrix}, \begin{pmatrix} 0 & cF \\ 0 & cF \end{pmatrix}\right);$$

$$A_5 = UT_3(F), \quad \text{with grading } \big(UT_3(F), \{0\}\big);$$

$$A_6 = UT_3(F), \quad \text{with grading } \left(\begin{pmatrix} F & F & 0 \\ 0 & F & 0 \\ 0 & 0 & F \end{pmatrix}, \begin{pmatrix} 0 & 0 & F \\ 0 & 0 & F \\ 0 & 0 & 0 \end{pmatrix}\right);$$

$$A_7 = UT_3(F), \quad \text{with grading } \left(\begin{pmatrix} F & 0 & 0 \\ 0 & F & F \\ 0 & 0 & F \end{pmatrix}, \begin{pmatrix} 0 & F & F \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right);$$

$$A_8 = UT_3(F), \quad \text{with grading } \left(\begin{pmatrix} F & 0 & F \\ 0 & F & 0 \\ 0 & 0 & F \end{pmatrix}, \begin{pmatrix} 0 & F & 0 \\ 0 & 0 & F \\ 0 & 0 & 0 \end{pmatrix}\right).$$

When the field F is algebraically closed and $\operatorname{char} F = 0$, in [17] the authors classified all possible G-gradings on $UT_n(F)$, for G an arbitrary finite abelian group. It turns out that there exist only 4 distinct $\mathbb{Z}_2$-gradings on $UT_3(F)$ and these give rise to the algebras A_5, A_6, A_7, A_8 given above.

We start by examining some special cases in the next two lemmas. If A is a finite-dimensional superalgebra, then we write $A = B + J$ where B is a maximal semisimple subalgebra of A with induced $\mathbb{Z}_2$-grading and $B = B_1 \oplus \cdots \oplus B_m$ where $B_1, \ldots, B_m$ are $\mathbb{Z}_2$-graded simple subalgebras of A.

Lemma 10. *Let A be a finite-dimensional superalgebra over an algebraically closed field F and suppose that* $\operatorname{supexp}(A) > 2$. *If there exist three graded simple components $B_i \cong B_k \cong B_l \cong F$ with grading $(F, \{0\})$ such that $B_i J B_k J B_l \neq 0$, then $Id^{\sup}(A) \subseteq Id^{\sup}(A_i)$ for some $i \in \{5, 6, 7, 8\}$.*

Proof. Let e_1, e_2, e_3 be the unit elements of B_i, B_k, B_l, respectively. It is clear that $e_p^2 = e_p$, $e_p \in B_p^{(0)}$ and $e_r e_s = \delta_{rs} e_r$ for $r, s = 1, 2, 3$ and $p \in \{i, k, l\}$. Since $B_i J B_k J B_l \neq 0$, there exist $j_1 = j_1^{(0)} + j_1^{(1)}$, $j_2 = j_2^{(0)} + j_2^{(1)} \in J$ with $j_1^{(0)}, j_2^{(0)} \in J^{(0)}$, $j_1^{(1)}, j_2^{(1)} \in J^{(1)}$ such that

$$e_1 j_1 e_2 j_2 e_3 = e_1\big(j_1^{(0)} + j_1^{(1)}\big) e_2 \big(j_2^{(0)} + j_2^{(1)}\big) e_3 \neq 0.$$

It follows that one of the four inequalities must hold:

$$e_1 j_1^{(0)} e_2 j_2^{(0)} e_3 \neq 0, \qquad e_1 j_1^{(0)} e_2 j_2^{(1)} e_3 \neq 0,$$
$$e_1 j_1^{(1)} e_2 j_2^{(0)} e_3 \neq 0, \qquad e_1 j_1^{(1)} e_2 j_2^{(1)} e_3 \neq 0.$$

Suppose first that $e_1 j_1^{(0)} e_2 j_2^{(0)} e_3 \neq 0$. Then clearly $e_1 j_1^{(0)} e_2 \neq 0$, $e_2 j_2^{(0)} e_3 \neq 0$ and let D_1 be the subalgebra of A generated by the elements

$$e_1, e_2, \quad e_3, \quad e_1 j_1^{(0)} e_2, \quad e_2 j_2^{(0)} e_3, \quad e_1 j_1^{(0)} e_2 j_2^{(0)} e_3.$$

D_1 is a superalgebra with induced $\mathbb{Z}_2$-grading $(D_1, \{0\})$. Moreover, it is easy to check that D_1 is isomorphic to the superalgebra $UT_3(F)$ with trivial grading $(UT_3(F), \{0\})$. In fact, one can build up the isomorphism of superalgebras $\varphi : D_1 \to UT_3(F)$ induced by setting $\varphi(e_i) = e_{ii}$, $i = 1, 2, 3$, $\varphi(e_1 j_1^{(0)} e_2) = e_{12}$, $\varphi(e_2 j_2^{(0)} e_3) = e_{23}$ and $\varphi(e_1 j_1^{(0)} e_2 j_2^{(0)} e_3) = e_{13}$.

Suppose now that $e_1 j_1^{(0)} e_2 j_2^{(1)} e_3 \neq 0$. Then $e_1 j_1^{(0)} e_2 \neq 0$, $e_2 j_2^{(1)} e_3 \neq 0$ and we construct the subalgebra D_2 of A generated by the elements

$$e_1, \quad e_2, \quad e_3, \quad e_1 j_1^{(0)} e_2, \quad e_2 j_2^{(1)} e_3, \quad e_1 j_1^{(0)} e_2 j_2^{(1)} e_3.$$

D_2 is a superalgebra with induced $\mathbb{Z}_2$-grading $(D_2^{(0)}, D_2^{(1)})$, where $D_2^{(0)} = \mathrm{Span}_F\{e_1, e_2, e_3, e_1 j_1^{(0)} e_2\}$ and $D_2^{(1)} = \mathrm{Span}_F\{e_2 j_2^{(1)} e_3, e_1 j_1^{(0)} e_2 j_2^{(1)} e_3\}$. It is not difficult to show that D_2 is isomorphic as a superalgebra to $UT_3(F)$ with the $\mathbb{Z}_2$-grading

$$\left(\begin{pmatrix} F & F & 0 \\ 0 & F & 0 \\ 0 & 0 & F \end{pmatrix}, \begin{pmatrix} 0 & 0 & F \\ 0 & 0 & F \\ 0 & 0 & 0 \end{pmatrix} \right).$$

Hence $D_2 \cong A_6$.

In the remaining cases, i.e., if either $e_1 j_1^{(1)} e_2 j_2^{(0)} e_3 \neq 0$ or $e_1 j_1^{(1)} e_2 j_2^{(1)} e_3 \neq 0$, we can construct two subsuperalgebras of A isomorphic to either A_7 or A_8.

Finally, it is clear that if C is a subsuperalgebras of A, then $Id^{\sup}(C) \supseteq Id^{\sup}(A)$, and the conclusion of the lemma follows. □

Lemma 11. *Let A be a finite-dimensional superalgebra over an algebraically closed field F and suppose that* $\operatorname{supexp}(A) > 2$. *If there exist two graded simple components $B_i \cong F \oplus cF$, where $c^2 = 1$, with grading (F, cF) and $B_k \cong F$ with grading $(F, \{0\})$ such that either $B_i J B_k \neq 0$ or $B_k J B_i \neq 0$, then $Id^{\sup}(A) \subseteq Id^{\sup}(A_i)$ for some $i \in \{3, 4\}$.*

Proof. Suppose first that $B_i J B_k \neq 0$. Let $e_1 \in B_i$ and $e_2 \in B_k$ be the unit elements of B_i and B_k, respectively. From $B_i J B_k \neq 0$ it follows that there exists some element $j \in J$ such that $e_1 j e_2 \neq 0$. By eventually multiplying by c on the left, we may assume that $e_1 j^{(0)} e_2 \neq 0$ for some $j^{(0)} \in J^{(0)}$.

Let D be the subalgebra of A generated by the elements

$$e_1, \quad ce_1, \quad e_2, \quad e_1 j^{(0)} e_2, \quad ce_1 j^{(0)} e_2.$$

Since all gradings are direct, it is easy to check that these five elements are linearly independent over F. Thus $D = D^{(0)} \oplus D^{(1)}$ is a superalgebra with induced $\mathbb{Z}_2$-grading, where $D^{(0)} = \mathrm{Span}_F\{e_1, e_2, e_1 j^{(0)} e_2\}$ and $D^{(1)} = \mathrm{Span}_F\{ce_1, ce_1 j^{(0)} e_2\}$. We claim that D is isomorphic to the superalgebra A_3 described at the beginning of this section. In fact, it is enough to consider the homomorphism $\varphi : D \to A_3$ induced by setting $\varphi(e_1) = e_{11}$, $\varphi(ce_1) = ce_{11}$, $\varphi(e_2) = e_{22}$, $\varphi(e_1 j^{(0)} e_2) = e_{12}$ and $\varphi(ce_1 j^{(0)} e_2) = ce_{12}$. Thus it follows that if $B_i J B_k \neq 0$, then $Id^{\sup}(A) \subseteq Id^{\sup}(A_3)$.

A similar argument show that in case $B_k J B_i \neq 0$, then $Id^{\sup}(A) \subseteq Id^{\sup}(A_4)$. □

We can now prove the main result of this section.

Theorem 12. *Let A be a finitely generated superalgebra over F satisfying an ordinary polynomial identity. Then* $\operatorname{supexp}(A) > 2$ *if and only if $Id^{\sup}(A) \subseteq Id^{\sup}(A_i)$ for some $i \in \{1, \dots, 8\}$.*

Proof. Recall that by extending the ground field, the graded codimensions do not change. Hence, if $\overline{F}$ denotes the algebraic closure of F, $\operatorname{supexp}(A) = \operatorname{supexp}(A \otimes_F \overline{F})$ and we may assume (as we shall) that F is algebraically closed.

As remarked at the beginning of Section 3, we may also assume that A is a finite-dimensional algebra over F.

Let $A = B_1 \oplus \cdots \oplus B_m + J$ be the Wedderburn–Malcev decomposition of A as above. Recall that a simple finite-dimensional superalgebra B_i over F is isomorphic to one of the following algebras:

(1) $M_{k,l}(F)$ with grading

$$\left(\begin{pmatrix} F_{11} & 0 \\ 0 & F_{22} \end{pmatrix}, \begin{pmatrix} 0 & F_{12} \\ F_{21} & 0 \end{pmatrix} \right),$$

where F_{11}, F_{12}, F_{21}, F_{22} are $k \times k$, $k \times l$, $l \times k$, and $l \times l$ matrices, respectively, $k \geqslant 1$ and $l \geqslant 0$;

(2) $M_n(F \oplus cF)$ with grading $(M_n(F), cM_n(F))$, where $c^2 = 1$.

Now, if one of the simple components B_i is of type 1 with $k + l \geqslant 2$, then $Id^{\sup}(A) \subseteq Id^{\sup}(M_{k,l}(F))$. Hence, either $Id^{\sup}(A) \subseteq Id^{\sup}(M_2(F)) = Id^{\sup}(A_1)$ with $l = 0$ or $Id^{\sup}(A) \subseteq Id^{\sup}(M_{1,1}(F)) = Id^{\sup}(A_2)$ with $l \geqslant 1$, and we are done in this case.

On the other hand, if $B_i \cong M_n(F \oplus cF)$ and $n \geqslant 2$, then $Id^{\sup}(A) \subseteq Id^{\sup}(M_n(F)) \subseteq Id^{\sup}(M_2(F)) = Id^{\sup}(A_1)$, and we are done too.

Recall that by the basic property of the superexponent seen in the previous section, since $\operatorname{supexp}(A) > 2$, it follows that there exist distinct simple components $B_{i_1}, \dots, B_{i_n}$ such that $B_{i_1} J \cdots J B_{i_n} \neq 0$ and $\dim_F(B_{i_1} + \cdots + B_{i_n}) > 2$. Therefore we may assume that one of the following possibilities occurs:

(1) for some $i \neq k$, $B_i J B_k \neq 0$ where $B_i \cong F \oplus cF$ with $c^2 = 1$ and $B_k \cong F$;
(2) for some $i \neq k$, $B_i J B_k \neq 0$ where $B_i \cong F$ and $B_k \cong F \oplus cF$ with $c^2 = 1$;

(3) there exist distinct B_i, B_k, B_l such that $B_i J B_k J B_l \neq 0$ and $B_i \cong B_k \cong B_l \cong F$.

If either (1) or (2) holds, then by Lemma 11, A contains a superalgebra isomorphic to A_3 or A_4, respectively. On the other hand, if case (3) occurs, then by Lemma 10 the superalgebra A contains a superalgebra isomorphic to A_i for some $i \in \{5, 6, 7, 8\}$. In any case the proof is complete. □

We shall prove next that the above list of algebras cannot be reduced. In fact, we have

Proposition 13. *For all $i, j \in \{1, \dots, 8\}$, $i \neq j$, we have that $Id^{\sup}(A_i) \nsubseteq Id^{\sup}(A_j)$.*

Proof. We first notice that $\operatorname{supexp}(A_1) = \operatorname{supexp}(A_2) = 4$ and $\operatorname{supexp}(A_i) = 3$, for $i = 3, \dots, 8$. Hence $Id^{\sup}(A_i) \nsubseteq Id^{\sup}(A_j)$ $i = 3, \dots, 8$, $j = 1, 2$.

Generators for the T_2-ideals of A_1, A_2 and A_5 are well known (see [3,4,14]) and $Id^{\sup}(A_i) \nsubseteq Id^{\sup}(A_j)$ for $i, j \in \{1, 2, 5\}$, $i \neq j$.

Since $z \in Id^{\sup}(A_1)$ but $z \notin Id^{\sup}(A_i)$, $i = 3, 4, 6, 7, 8$ and $[y_1, y_2] \in Id^{\sup}(A_2)$ but $[y_1, y_2] \in Id^{\sup}(A_i)$, $i = 3, 4, 6, 7, 8$ we get that $Id^{\sup}(A_i) \nsubseteq Id^{\sup}(A_j)$ and $Id^{\sup}(A_j) \nsubseteq Id^{\sup}(A_i)$, $j = 1, 2$, $i = 3, 4, 6, 7, 8$.

The following list of polynomial characterize the remaining T_2-ideals:

$$[y_1, y_2][y_3, y_4] \in Id^{\sup}(A_i), \quad i = 3, 4, 6, 7, 8; \qquad [y_1, y_2][y_3, y_4] \notin Id^{\sup}(A_5);$$
$$z \in Id^{\sup}(A_5); \qquad z \notin Id^{\sup}(A_i), \quad i = 3, 4, 6, 7, 8;$$
$$[y_1, y_2]z \in Id^{\sup}(A_i), \quad i = 3, 7, 8; \qquad [y_1, y_2]z \notin Id^{\sup}(A_i), \quad i = 4, 6;$$
$$z[y_1, y_2] \in Id^{\sup}(A_i), \quad i = 4, 6, 8; \qquad z[y_1, y_2] \notin Id^{\sup}(A_i), \quad i = 3, 7;$$
$$z_1 z_2 \in Id^{\sup}(A_i), \quad i = 6, 7; \qquad z_1 z_2 \notin Id^{\sup}(A_i), \quad i = 3, 4, 8;$$
$$[y_1, z][y_2, y_3] \in Id^{\sup}(A_3); \qquad [y_1, z][y_2, y_3] \notin Id^{\sup}(A_7);$$
$$[y, z_1]z_2 \in Id^{\sup}(A_3); \qquad [y, z_1]z_2 \notin Id^{\sup}(A_8);$$
$$[y_1, y_2][y_3, z] \in Id^{\sup}(A_4); \qquad [y_1, y_2][y_3, z] \notin Id^{\sup}(A_6);$$
$$z_1[y, z_2] \in Id^{\sup}(A_4); \qquad z_1[y, z_2] \notin Id^{\sup}(A_8).$$

It follows that $Id^{\sup}(A_i) \nsubseteq Id^{\sup}(A_j)$ for $i, j \in \{1, \dots, 8\}$, $i \neq j$. □

As a consequence of Theorem 12 and the main result of [7], we can now characterize finitely generated superalgebras of superexponent 2. To this end we introduce the following three algebras:

(1) $B_1 = UT_2(F)$ with grading $(UT_2(F), \{0\})$;
(2) $B_2 = UT_2(F)$ with grading $\left(\begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix}, \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}\right)$;
(3) $B_3 = F \oplus cF$ with grading (F, cF), $c^2 = 1$.

We can now prove the following statement.

Corollary 14. *Let A be a finitely generated superalgebra over F satisfying an ordinary polynomial identity. Then* $\mathrm{supexp}(A) = 2$ *if and only if* $Id^{\mathrm{sup}}(A) \not\subseteq Id^{\mathrm{sup}}(A_i)$ *for some* $i \in \{1, \ldots, 8\}$ *and* $Id^{\mathrm{sup}}(A) \subseteq Id^{\mathrm{sup}}(B_i)$ *for some* $i = 1, 2, 3$.

Proof. By [7, Theorem 2] $\mathrm{supexp}(A) > 1$ if and only if either $Id^{\mathrm{sup}}(A) \subseteq Id^{\mathrm{sup}}(B_i)$ for some $i = 1, 2, 3$ or $Id^{\mathrm{sup}}(A) \subseteq Id^{\mathrm{sup}}(G_j)$ for some $j = 1, 2$ where $G_1 = G$ is the Grassmann algebra with trivial grading and $G_2 = G^{(0)} \oplus G^{(1)}$ is the Grassmann algebra with the natural grading.

We next claim that $Id^{\mathrm{sup}}(A) \not\subseteq Id^{\mathrm{sup}}(G_1)$, $Id^{\mathrm{sup}}(G_2)$. In fact, as remarked before, we may assume that A is a finite-dimensional superalgebra. Hence, since $A^{(0)}$ is a finite dimensional, it is well know that $Id(G_1) \not\supseteq Id(A^{(0)})$. We have: $Id^{\mathrm{sup}}(G_1) = Id(G_1) \not\supseteq Id(A^{(0)}) \supseteq Id^{\mathrm{sup}}(A)$.

Let now

$$St_n(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} (\mathrm{sgn}\,\sigma) x_{\sigma(1)} \cdots x_{\sigma(n)}$$

be the standard polynomial of degree n. Notice that for any $n \geqslant 1$, if $e_1, \ldots, e_n \in G^{(1)}$ are generating elements of G, then $St_n(e_1, \ldots, e_n) = n! e_1 \cdots e_n \neq 0$. Hence $St_n(z_1, \ldots, z_n) \notin Id^{\mathrm{sup}}(G_2)$, for any $n \geqslant 1$. On the other hand, since A is finite-dimensional superalgebra, $St_n(z_1, \ldots, z_n) \in Id^{\mathrm{sup}}(A)$ for any $n > \dim_F A^{(1)}$. It follows that $Id^{\mathrm{sup}}(G_1) \not\supseteq Id^{\mathrm{sup}}(A)$. This establishes the claim.

The proof of the corollary now follows from Theorem 12. □

5. Further properties

In this section we collect further properties of finite-dimensional superalgebras and their superexponent.

We start by comparing $\exp(A)$ with $\mathrm{supexp}(A)$. By [8, Lemma 4.7], for any superalgebra A satisfying an ordinary polynomial identity, we have that

$$c_n(A) \leqslant c_n^{\mathrm{sup}}(A) \leqslant 2^n c_n(A).$$

In case A is finitely generated, these inequalities imply the following lemma.

Lemma 15. *If A is a finitely generated superalgebra satisfying an ordinary polynomial identity, then*

$$\exp(A) \leqslant \mathrm{supexp}(A) \leqslant 2\exp(A).$$

Next natural question is:

How large can be the gap between $\exp(A)$ *and* $\mathrm{supexp}(A)$? *In other words, given any positive integers d and $0 \leqslant k \leqslant d$, does there exist a finitely generated superalgebra A such that* $\exp(A) = d$ *and* $\mathrm{supexp}(A) = d + k$?

The answer is yes as the following proposition shows.

Theorem 16. *Given integers $d \geqslant k \geqslant 0$, there exists a finitely generated superalgebra A such that* $\exp(A) = d$ *and* $\operatorname{supexp}(A) = d + k$.

Proof. Let us denote by F_2 the algebra

$$F_2 = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \;\middle|\; a, b \in F \right\}.$$

This algebra is $\mathbb{Z}_2$-graded, $F_2 = F_2^{(0)} \oplus F_2^{(1)}$ where $F_2^{(0)} = F(e_{11} + e_{22})$, $F_2^{(1)} = F(e_{12} + e_{21})$ and clearly $F_2 \cong F \oplus cF$.

We write $d + k = 2k + (d - k)$ and we let $U(k, d - k)$ be the algebra of upper block triangular matrices over F of the type $\left(\begin{smallmatrix} A & B \\ 0 & C \end{smallmatrix}\right)$, where

$$A = \begin{pmatrix} F_2 & & * \\ & \ddots & \\ 0 & & F_2 \end{pmatrix} \subseteq M_{2k}(F)$$

is an algebra of upper block triangular matrices over F, $C = UT_{d-k}(F)$, and B is the space of $2k \times (d - k)$ matrices over F. The Wedderburn–Malcev decomposition of this algebra is

$$U(k, d - k) = \underbrace{F_2 \oplus \cdots \oplus F_2}_{k} \oplus \underbrace{F \oplus \cdots \oplus F}_{d-k} + J$$

where J consists of all strictly upper block triangular matrices.

Recalling the defining properties of $\exp(A)$ and $\operatorname{supexp}(A)$ (see [11] and Section 3 above) it is easy to check that $\exp(U(k, d-k)) = k + d - k = d$ and $\operatorname{supexp}(U(k, d-k)) = 2k + d - k = d + k$. □

We next characterize finite-dimensional superalgebras whose sequence of graded codimensions is polynomially bounded (i.e., $\operatorname{supexp}(A) \leqslant 1$). We have

Theorem 17. *Let A be a finite-dimensional superalgebra over an algebraically closed field F. Then* $\operatorname{supexp}(A) \leqslant 1$ *if and only if*

(1) $\exp(A) \leqslant 1$;
(2) *$A = B + J$ where B is a maximal commutative semisimple subalgebra with trivial induced $\mathbb{Z}_2$-grading.*

Proof. Suppose that $\operatorname{supexp}(A) \leqslant 1$. Then by [8], for all n, $c_n(A) \leqslant c_n^{\mathrm{sup}}(A) \leqslant \alpha n^t$ for some constants α, t, and the sequence of codimensions is polynomially bounded. From Section 3, by the definition of the superexponent it follows that for any maximal semisimple superalgebra $B \subseteq A$, then $B = B_1 \oplus \cdots \oplus B_m$ and, for all i, $B_i \cong F$ has trivial $\mathbb{Z}_2$-grading. This proves the first part of the theorem.

Conversely, suppose that $A = B + J$ where $B = B_1 \oplus \cdots \oplus B_m$ and, for all i, $B_i \cong F$ with trivial $\mathbb{Z}_2$-grading. In this case, if $a \in A$, we write $a = b + j$, $b \in B = B^{(0)}$, $j \in J$. Then $a - a^{(0)} = j - j^{(0)} \in J$ and $A^{(1)} \subseteq J$ follows.

Notice that if we assume that $c_n(A)$ is polynomially bounded, then $c_{r,n-r}(A) \leqslant c_n(A) \leqslant \alpha n^t$, for some α, t, for all $r \geqslant 0$. Let $J^q = 0$. Since $A^{(1)} \subseteq J$, then for all $r \leqslant n - q$, $V_{r,n-r} \cap Id^{\sup}(A) = V_{r,n-r}$ and $c_{r,n-r}(A) = 0$ follows. Hence, for all n, we obtain:

$$c_n^{\sup}(A) = \sum_{r=0}^{n} \binom{n}{r} c_{r,n-r}(A) \leqslant \alpha n^t \sum_{r=n-q+1}^{n} \binom{n}{r} = \alpha n^t \sum_{r=0}^{q-1} \binom{n}{r} \leqslant \alpha n^{t+q},$$

and $c_n^{\sup}(A)$ is polynomially bounded. □

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