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Cauchy Problem for Evolution Equations of Schrödinger Type

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In (R. Agliardi, 1995, *Internat. J. Math.* **6**, 791–804) we proved the well-posedness of the Cauchy problem in H^∞ for some p-evolution equations ($p \geqslant 1$) with real characteristic roots. For this purpose some assumptions on the lower order terms are needed, which, in the special case $p=1$, recapture well-known results for hyperbolic operators. In (R. Agliardi, 1995, *Internat. J. Math.* **6**, 791–804) the leading coefficients are assumed to be constant. In this paper we allow them to be variable. Our result is applicable to 2-evolution differential operators with real characteristics, i.e., to Schrödinger type operators. This class of operators comprehends, for example, Schrödinger operator $D_t - \Delta_x$ or the plate operator $D_t^2 - \Delta_x^2$. The Cauchy problem in H^∞ for such evolution operators has been studied extensively by Takeuchi when the coefficients in the principal part are constant and the characteristic roots are distinct.

1. NOTATION AND MAIN ASSUMPTIONS

Let $\mathscr{C}^j(\Omega)$ denote the set of all functions with continuous jth-order partial derivatives on Ω and let $\mathscr{B}^j(\Omega)$ denote the set of all functions whose derivatives of order $\leqslant j$ are continuous and bounded in Ω. Let S^m denote the class of the pseudo-differential operators $p(x, D_x)$ whose symbol $p(x, \xi)$ satisfies the following condition:

$$\sup_{\alpha, \beta \in \mathbb{N}^n} \sup_{\substack{x, \xi \in \mathbb{R}^n \\ |\xi| \geqslant B}} |\partial_\xi^\alpha D_x^\beta p(x, \xi)| \cdot \langle \xi \rangle^{|\alpha| - m} < \infty$$

for some $B \geqslant 0$. (Here $\langle \xi \rangle = \sqrt{1 + |\xi|^2}$). For any real s we denote the Sobolev space $\{u \in \mathscr{S}'(\mathbb{R}^n);\ \langle D_x \rangle^s u \in L^2(\mathbb{R}^n)\}$ by $H^s(\mathbb{R}^n)$ and set $H^\infty = \bigcap_s H^s$.

Consider a linear operator of the form

$$P = \Pi_{2m}(t, x, D_t, D_x) + \sum_{j=r}^{m} a_j(t, x, D_x)\, D_t^{m-j}, \tag{1.1}$$

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where

$$\Pi_{2m}(t, x, D_t, D_x) = \prod_{j=1}^{r} (D_t - \lambda_j^1(t, x, D_x)) \cdots (D_t - \lambda_j^{s_j}(t, x, D_x)), \quad (1.2)$$

where $\sum_{j=1}^{r} s_j = m$, $s_r \geqslant s_{r-1} \geqslant \cdots \geqslant s_1$. We assume that the $\lambda(t, x, \xi)$'s satisfy the following properties:

$$\begin{aligned} &\text{(i)} && \partial_t^k \lambda_j^i \in S^2, \quad k = 0, ..., m-1, \\ &\text{(ii)} && \lambda_j^i \text{ are real-valued and } \lambda_j^i - (\lambda_j^i)^* \in S^0, \\ &\text{(iii)} && \lambda_j^i(t, x, \xi) \neq \lambda_k^h(t, x, \xi) \quad \text{if} \quad i \neq h \quad \text{and} \quad \xi \neq 0. \end{aligned} \quad (1.3)$$

Moreover,

$$a_j(t, x, D_x) = \sum_{|\alpha| \leqslant 2(j-r)} a_{\alpha j}(t, x)\, D_x^\alpha, \quad (1.4)$$

where $a_{\alpha j} \in \mathscr{B}([-T, T]\,;\, \mathscr{B}^\infty(\mathbb{R}^n))$.

Let $\bar{s}_j = \sum_{h=1}^{j} s_h$. Denote λ_j^i by λ_i if $j = 1$ and by $\lambda_{\bar{s}_{j-1}+1}$ if $j > 1$. Let ∂_i denote $D_t - \lambda_i(t, x, D_x)$. If $J = (j_1, ..., j_k)$ set $\{J\} = \{j_1, ..., j_k\}$, $|J| = k$, $\partial_J = \partial_{j_1} \cdots \partial_{j_k}$. Let $\mathscr{I}_h^{(1)} = \{J = (j_1, ..., j_h);\ j_1 < \cdots < j_h,\ \{J\} \subset \{1, ..., s_1\}\}$ and, for $k = 2, ..., r$, $\mathscr{I}_h^{(k)} = \{J = (j_1, ..., j_h); j_1 < \cdots < j_h, \{J\} \subset \{\bar{s}_{k-1}, ..., \bar{s}_k\}\}$. Thus Π_{2m} can be written in the form $\partial_{J_1} \cdots \partial_{J_r}$, with $J_k \in \mathscr{I}_{s_k}^{(k)}$.

EXAMPLES. Assume that $P = \sum_{h=0}^{2m} P_{2m-h}$ where $P_{2m-h}(t, x, D_t, D_x) = \sum_{j=[(h+1)/2]}^{m} \sum_{|\alpha| = 2j-h} a_{\alpha j}(t, x)\, D_x^\alpha D_t^{m-j}$ and $a_{\alpha j} \in \mathscr{B}([-T, T];\ \mathscr{B}^\infty(\mathbb{R}^n))$. Suppose that the characteristic roots are real with constant multiplicity; that is, P_{2m} can be written in the form

$$P_{2m}(t, x; \tau, \xi) = \prod_{i=1}^{k} (\tau - \lambda^i(t, x, \xi))^{r_i},$$

with $\sum_{i=1}^{k} r_i = m$, $r = r_1 \geqslant \cdots \geqslant r_k$, $\lambda^i(t, x, \xi) \neq \lambda^h(t, x, \xi)$ if $i \neq h$ and $\xi \neq 0$.

(a) If the characteristic roots do not depend on t and x, then $P_{2m} = \Pi_{2m}$. We assume that P_{2m-h} vanishes for $h = 1, ..., 2r-1$.

(b) Let $\{.,.\}$ denote the Poisson bracket with respect to the space variables. Assume that

$$\{\lambda_i, \lambda_j\} \in \mathscr{B}^\infty([-T, T]; S^2) \quad (1.5)$$

so that Lemma 2.2 holds with $s_J \in S^0$. Therefore, in this case, if $r=1$, in order to obtain the main result (Theorem 2.1), it suffices to assume that P_{2m-1} vanishes.

Note that (1.5) holds if the characteristic roots do not depend on x.

(c) Suppose that P_{2m} does not depend on x and has a double characteristic root $\lambda(t, \xi)$, while the other roots are simple. Then a simple calculation shows that our result holds if we assume the following conditions: P_{2m-1} and P_{2m-3} vanish, and

$$L_{2m-2}(t, x, \xi) = (P_{2m-2}(t, x; \tau, \xi) + \tfrac{1}{2}\partial_\tau^2 P_{2m}(t, x; \tau, \xi)\, D_t\lambda(t, \xi))_{\tau=\lambda(t, \xi)}$$

vanishes.

2. CAUCHY PROBLEM IN H^∞

If $u \in H^s(\mathbb{R}^n)$, its Sobolev norm $\|\langle D_x\rangle^s u\|_{L^2}$ will be denoted by $\|u\|_s$. Let $M, s \in \mathbb{N}$ be fixed. If $u(t, x) \in \bigcap_{j=0}^M \mathscr{C}^j([-T, T]; H^{s+2(M-j)}(\mathbb{R}^n))$, we shall use the following notation:

$$\| \|u(t)\| \|_{s,M}^2 = \sum_{j=0}^{M} \|\partial_t^j u(t, .)\|_{s+2(M-j)}^2. \tag{2.1}$$

The result we prove in this section is the following:

THEOREM 2.1. *If P satisfies the assumptions* (1.1), (1.2), (1.3), (1.4), *the initial data g_h are in H^∞ and $f \in \mathscr{C}([-T, T]; H^\infty)$, then the Cauchy problem*

$$\begin{aligned} Pu(t) &= f(t) \\ D_t^h u(0) &= g_h \qquad h = 0, ..., m-1 \end{aligned} \tag{2.2}$$

has a solution $u(t, \cdot) \in H^\infty$, $\forall t \in [-T, T]$. Moreover, the following energy inequality holds for every $s \in \mathbb{N}$:

$$\| \|u(t)\| \|_{s,m-r} \leqslant M(T)\left\{ \| \|u(0)\|_{s,m-1} + \left| \int_0^t \|f(\tau, \cdot)\|_s \, d\tau \right| \right\}. \tag{2.3}$$

This theorem is proved at the end of this section. We premise a few lemmas. Throughout all the following lemmas we assume that the λ_j's satisfy (1.3) (i). In writing the identities involving pseudo-differential operators we shall omit the regularizers. Moreover, for the sake of brevity, we shall only point out the properties of the pseudo-differential operators with respect to the space variables, disregarding their behavior with respect to time.

LEMMA 2.1. *Assume that λ_i and λ_j are distinct. Then, for any positive integer N, we can write the identity in the following way:*

$$\mathrm{Id.} = d_{ij}^{(N)}(t, x, D_x)\, \partial_j + d_{ji}^{(N)}(t, x, D_x)\, \partial_i + r^{(N)}(t, x, D_x), \tag{2.4}$$

where $d_{ij}^{(N)}, d_{ji}^{(N)} \in S^{-2}$ and $r^{(N)} \in S^{-N}$.

Proof. Let us define $d_{ij}(t, x, \xi) = (\lambda_i(t, x, \xi) - \lambda_j(t, x, \xi))^{-1}$. (We modify the λ's for small $|\xi|$, if it is the case.) Then we can write: $\mathrm{Id.} = d_{ij}(t, x, D_x)\, \partial_j + d_{ji}(t, x, D_x)\, \partial_i + r^{(1)}(t, x, D_x)$, where $r^{(1)} \in S^{-1}$. Reiteration of this procedure on the last addendum yields the desired identity.

LEMMA 2.2. *Assume that the λ_j's are distinct for $j \in \{1, ..., s\}$ and let λ_m be distinct from every λ_j with $j \in \{1, ..., s\}$. Denote $\mathscr{I}_h = \{J = (j_1, ..., j_h);\ j_1 < \cdots < j_h,\ \{J\} \subset \{1, ..., s\}\}$ for $h = 1, ..., s$. (We mean $\mathscr{I}_0 = \varnothing$ and $\partial_\varnothing =$ Identity). Let $\mathscr{I}'_h$ denote $\mathscr{I}_h \cup \{J = (j_1, ..., j_{h-1}, m);\ (j_1, ..., j_{h-1}) \in \mathscr{I}_{h-1}\}$. Then, for every $k = 1, ..., s$ and for every $\tilde{J} \in \mathscr{I}_k$, we have, for any arbitrary integer $N > 0$,*

$$[\partial_m, \partial_{\tilde{J}}] = \sum_{J \in \mathscr{I}'_k} s_J(t, x, D_x)\, \partial_J + \sum_{h=0}^{k-1} \sum_{J \in \mathscr{I}'_h} r_J^{(N)}(t, x, D_x)\, \partial_J, \tag{2.5}$$

where $s_J \in S^1$ and $r_J^{(N)} \in S^{-N}$. (Here $[A, B]$ denotes AB–BA.)

Proof. Let us start with $k = 1$. Since $[\partial_m, \partial_i] \in S^3$, in view of Lemma 2.1 we have

$$[\partial_m, \partial_i] = [\partial_m, \partial_i](d_{im}^{(N+3)}\, \partial_m + d_{mi}^{(N+3)}\, \partial_i + r^{(N+3)})$$

with $d_{im}^{(N+3)}, d_{mi}^{(N+3)} \in S^{-2}$ and $r^{(N+3)} \in S^{-N-3}$. Thus we have established (2.5) in this case. More generally, for $\tilde{J} = (j_1, ..., j_k) \in \mathscr{I}_k$, we have

$$\partial_m \partial_{\tilde{J}} = (\partial_{j_1} \partial_m + s_m^{(1)}\, \partial_m + s_{j_1}^{(1)}\, \partial_{j_1} + r_{-N}^{(1)})\, \partial_{j_2} \cdots \partial_{j_k}$$

for some $s_m^{(1)}, s_{j_1}^{(1)} \in S^1$ and $r_{-N}^{(1)} \in S^{-N}$. By induction we obtain:

$$\begin{aligned}\partial_m \partial_{\tilde{J}} = (\partial_{j_1} + s_m^{(1)}) \Big\{ \partial_{j_2} \cdots \partial_{j_k} \partial_m + \sum_{J \in \mathscr{I}''_k} \tilde{s}_J \partial_J + \sum_{h=0}^{k-2} \sum_{J \in \mathscr{I}''_h} \tilde{r}_J^{(N+1)}\, \partial_J \Big\} \\ + s_{j_1}^{(1)}\, \partial_{\tilde{J}} + r_{-N}^{(1)}\, \partial_{j_2} \cdots \partial_{j_k},\end{aligned}$$

where $\tilde{s}_J \in S^1$, $\tilde{r}_J^{(N+1)} \in S^{-N-1}$, and $\mathscr{I}'_h = \{J;\ (j_1, J) \in \mathscr{I}'_h\}$.

Note that, for $J \in \mathscr{I}''_k$, $(\partial_{j_1} + s_m^{(1)})\, \tilde{s}_J = \tilde{s}_J \partial_{j_1} + w_J^{(2)}$ with $w_J^{(2)} \in S^2$. Therefore, inspecting our expression, we see that, in order to put it in the form (2.5), we only have to examine the terms containing $w_J^{(2)}$. For every $J \in \mathscr{I}''_k$ there exists $\nu \in \{2, ..., k+1\}$ such that J lacks j_ν. (We have set $j_{k+1} = m$.) Let us denote such a J by J_ν. Then, in view of Lemma 2.1, we have $w_{J_\nu}^{(2)} = \mathring{s}_\nu \partial_{j_1} + \mathring{w}_\nu \partial_{j_\nu} + r_\nu$ where $\mathring{s}_\nu$, $\mathring{w}_\nu \in S^0$ and $r_\nu \in S^{-N}$. Therefore, in the sum $\sum_{J \in \mathscr{I}''_k} w_J^{(2)} \partial_J$, the only term that requires further handling is:

$$\sum_{\nu=2}^{k+1} \mathring{w}_\nu \partial_{j_\nu} \partial_{J_\nu}.$$

Again induction enables us to write $\partial_{j_\nu} \partial_{J_\nu}$ in the desired ordering, so that each term is included in the formula (2.5).

LEMMA 2.3. *Assume that the λ_j's are distinct for $j \in \{1, ..., s\}$. We define $\mathscr{I}_h$, $h = 1, ..., s$, as in Lemma 2.2. For $h = 1, ..., s-1$, let Σ_h be a subset of $\{1, ..., s\}$ with $h+1$ elements. Then, for any positive integer N, we can write the identity as the following sum:*

$$\sum_{\substack{J \in \mathscr{I}_h \\ \{J\} \subset \Sigma_h}} d_J^{(N)}(t, x, D_x)\, \partial_J + \sum_{k=0}^{h-1} \sum_{\substack{J \in \mathscr{I}_k \\ \{J\} \subset \Sigma_h}} r_J^{(N)}(t, x, D_x)\, \partial_J, \tag{2.6}$$

where $d_J^{(N)} \in S^{-2h}$ and $r_J^{(N)} \in S^{-N}$.

Proof. First note that the case $h = 1$ follows from Lemma 2.1. More generally, if $1 < h \leqslant s-1$ and Σ_h is given, then (2.6) can be proved inductively. Let $\ell = \min\{i; i \in \Sigma_h\}$ and define $\Sigma_{h-1} = \Sigma_h - \{\ell\}$. Suppose that

$$\mathrm{Id.} = \sum_{\substack{J \in \mathscr{I}_{h-1} \\ \{J\} \subset \Sigma_{h-1}}} \tilde{d}_J^{(N)}(t, x, D_x)\, \partial_J + \sum_{k=0}^{h-2} \sum_{\substack{J \in \mathscr{I}_k \\ \{J\} \subset \Sigma_{h-1}}} \tilde{r}_J^{(N)}(t, x, D_x)\, \partial_J,$$

where $\tilde{d}_J^{(N)} \in S^{-2h+2}$ and $\tilde{r}_J^{(N)} \in S^{-N}$. For every J in the first sum let $i_j \in \Sigma_{h-1} - \{J\}$. Thus, inserting $\mathrm{Id} = d_{i_j\ell} \partial_\ell + d_{\ell i_j} \partial_{i_j} + \tilde{\tilde{r}}_J^{(N)}$ (with $d_{i_j\ell}$, $d_{\ell i_j} \in S^{-2}$ and $\tilde{\tilde{r}}_J^{(N)} \in S^{-N}$, as in Lemma 2.1) between $\tilde{d}_J^{(N)}$ and ∂_J, we obtain

$$\sum_{\substack{J \in \mathscr{I}_{h-1} \\ \{J\} \subset \Sigma_{h-1}}} d_J'^{(N)} \partial_\ell \partial_J + \sum_{\substack{J \in \mathscr{I}_{h-1} \\ \{J\} \subset \Sigma_{h-1}}} d_J''^{(N)} \partial_{i_j} \partial_J + \sum_{k=0}^{h-1} \sum_{\substack{J \in \mathscr{I}_k \\ \{J\} \subset \Sigma_{h-1}}} r_J'^{(N)}(t, x, D_x)\, \partial_J,$$

where $d_J'^{(N)}$, $d_J''^{(N)} \in S^{-2h}$ and $r_J'^{(N)} \in S^{-N}$. We just have to put the second term in the desired form. This can be accomplished by using Lemma 2.2 to set $\partial_{i_j} \partial_J$ in the right ordering. Since some pseudo-differential operators of order 1-2h appear, we insert a suitable term of the form (2.4) to lower their order. A final application of Lemma 2.2 gives the form (2.6).

LEMMA 2.4. *Let the λ_j's be as in Lemma 2.3. Then for all $k = 0, ..., s-1$ and for any positive integer N we can write*

$$D_t^{s-1-k} = \sum_{J \in \mathscr{I}_{s-1}} c_J^{(k)}(t, x, D_x)\, \partial_J + \sum_{h=0}^{s-2} \sum_{J \in \mathscr{I}_h} r_J(t, x, D_x)\, \partial_J \tag{2.7}$$

for some $c_J^{(k)}$ and r_J depending on N and belonging to S^{-2k} and S^{-N}, respectively.

Proof. For $h = 1, ..., s-1$, let Σ_h be a subset of $\{1, ..., s\}$ with $h+1$ elements. First we prove that, for any Σ_h and for any positive integer N,

$$D_t^h = \sum_{\substack{J \in \mathscr{I}_h \\ \{J\} \subset \Sigma_h}} b_J^{(N)}(t, x, D_x)\, \partial_J + \sum_{m=0}^{h-1} \sum_{\substack{J \in \mathscr{I}_m \\ \{J\} \subset \Sigma_h}} r_J^{(N)}(t, x, D_x)\, \partial_J \tag{2.8}$$

holds with $b_J^{(N)} \in S^0$ and $r_J^{(N)} \in S^{-N}$.

If $i \neq j$, let $c_{ij}(t, x, \xi) = \lambda_j(t, x, \xi)/(\lambda_j(t, x, \xi) - \lambda_i(t, x, \xi))$. Then, if $\Sigma_1 = \{i, j\}$, we can write $D_t = c_{ij}(t, x, D_x)\, \partial_i + c_{ji}(t, x, D_x)\, \partial_j + \tilde{r}_1$ where $\tilde{r}_1 \in S^{-1}$. Applying Lemma 2.1 to the last addendum we obtain (2.8) for arbitrary N. The general case $h > 1$ can be proved by induction. Let N and Σ_h be given. For the following notation we refer to the proof of Lemma 2.3. Then

$$D_t^{h-1} = \sum_{\substack{J \in \mathscr{I}_{h-1} \\ \{J\} \subset \Sigma_{h-1}}} \tilde{b}_J^{(N+1)}\, \partial_J + \sum_{m=0}^{h-2} \sum_{\substack{J \in \mathscr{I}_m \\ \{J\} \subset \Sigma_{h-1}}} \tilde{r}_J^{(N+1)}\, \partial_J$$

for some $\tilde{b}_J^{(N+1)} \in S^0$ and $\tilde{r}_J^{(N+1)} \in S^{-N-1}$.

For every J in this sum there exists $i_j \in \Sigma_{h-1} - \{J\}$ and then we can write

$$\begin{aligned} D_t^h = & \sum_{\substack{J \in \mathscr{I}_{h-1} \\ \{J\} \subset \Sigma_{h-1}}} (c_{\ell i_j}\, \partial_\ell + c_{i_j \ell}\, \partial_{i_j} + \mathring{r}_J^{(N)})\, \tilde{b}_J^{(N+1)}\, \partial_J \\ & + \sum_{m=0}^{h-2} \sum_{\substack{J \in \mathscr{I}_m \\ \{J\} \subset \Sigma_{h-1}}} (c_{\ell i_j}\, \partial_\ell + c_{i_j \ell}\, \partial_{i_j} + \mathring{r}_J^{(N)})\, \tilde{r}_J^{(N+1)}\, \partial_J \end{aligned}$$

where $c_{\ell i_j}, c_{i_j \ell} \in S^0$ and $\mathring{r}_J^{(N)} \in S^{-N}$. The above expression is of the form

$$\begin{aligned} & \sum_{\substack{J \in \mathscr{I}_{h-1} \\ \{J\} \subset \Sigma_{h-1}}} \{\hat{\beta}_J\, \partial_\ell + \beta_J\, \partial_{i_j} + \gamma_J + \hat{r}_J^{(N)}\}\, \partial_J \\ & + \sum_{m=0}^{h-2} \sum_{\substack{J \in \mathscr{I}_m \\ \{J\} \subset \Sigma_{h-1}}} \{\hat{\rho}_J^{(N+1)}\, \partial_\ell + \rho_J^{(N+1)}\, \partial_{i_j} + \hat{\hat{r}}_J^{(N)}\}\, \partial_J, \end{aligned}$$

where the β's belong to S^0, γ to S^1, the ρ's to S^{-N-1}, and the r's to S^{-N}. By applying Lemma 2.1 we insert a term of the form $d_{\ell i_j}\partial_\ell + d_{i_j\ell}\partial_{i_j} + \mathring{r}_J^{(N+1)}$ between γ_J and ∂_J, thus replacing $\gamma_J\partial_J$ with $(\hat{\delta}_J\partial_\ell + \delta_J\partial_{i_j} + r_J'^{(N)})\,\partial_J$, where the $\delta's$ belong to S^{-1}. Now we put the terms in $\partial_{i_j}\partial_J$ in order, by using Lemma 2.2. The only terms which are not yet in the desired form are some terms arising from $\beta_J\partial_{i_j}\partial_J$: they include pseudo-differential operators of order 1. However they eventually reduce to the desired form, if we apply Lemma 2.1 and Lemma 2.2.

Now we prove (2.7). If $k=0$, then setting $h=s-1$ in (2.8), we obtain (2.7). If $k>0$, let $\Sigma_{s-1-k}=\{j_1, ..., j_k\}$. In view of (2.7) we have

$$D_t^{s-1-k} = \sum_{\substack{J''\in\mathscr{I}_{s-1-k}\\ \{J''\}\subset\Sigma_{s-1-k}}} b_{J''}\,\partial_{J''} + \sum_{m=0}^{s-2-k}\ \sum_{\substack{J\in\mathscr{I}_m\\ \{J\}\subset\Sigma_{s-1-k}}} r_J^{(N)}\,\partial_J \tag{2.9}$$

with $b_{J''}\in S^0$ and $r_J^{(N)}\in S^{-N}$. Denoting $\{1, ..., s\}-\{J''\}$ by $\Sigma^*_{J''}$ and applying Lemma 2.3 to each term of the former sum in the right-hand side of (2.9), we get

$$\sum_{\substack{J''\in\mathscr{I}_{s-1-k}\\ \{J''\}\subset\Sigma_{s-1-k}}} b_{J''}\left(\sum_{\substack{J'\in\mathscr{I}_m\\ \{J'\}\subset\Sigma^*_{J''}}} d_{J'}\,\partial_{J'} + \sum_{m=0}^{k-1}\ \sum_{\substack{J'\in\mathscr{I}_m\\ \{J'\}\subset\Sigma^*_{J''}}} r_{J'}^{(N+1)}\,\partial_{J'}\right)\partial_{J''}$$

with $d_J\in S^{-2k}$ and $r_{J'}^{(N+1)}\in S^{-N-1}$. Now (2.7) is nearly established, with only the order of some terms in $\partial_{J'}\partial_{J''}$ reversed. Therefore, as above, we use Lemma 2.2 and, if it is the case, Lemma 2.1 to obtain the desired form (2.7).

PROPOSITION 2.1. *If the operator P satisfies* (1.1), (1.2), (1.3), (1.4), *then, for any positive integer N, P can be written in the following form (modulo regularizers),*

$$\Pi_{2m} + \sum_{J_1\in\mathscr{I}^{(1)}_{s_1-1}, ..., J_r\in\mathscr{I}^{(r)}_{s_r-1}} \tilde{a}_{J_1, ..., J_r}(t, x, D_x)\,\partial_{J_1}\cdots\partial_{J_r}$$
$$+ \sum_{\substack{h_i=0, ..., s_i-1\\ i=1, ..., r}}\ \sum_{J_i\in\mathscr{I}^{(i)}_{h_i}} \rho_{J_1, ..., J_r}\,\partial_{J_1}\cdots\partial_{J_r}, \tag{2.10}$$

where $\tilde{a}_{J_1, ..., J_r}\in\mathscr{B}([-T, T]; S^0)$ and $\rho_{J_1, ..., J_r}\in\mathscr{B}([-T, T]; S^{-N})$.

Proof. If $s_r=1$ then $r=m$ and $P=\Pi_{2m}+a_r(t, x, D_x)$, which is in the form (2.10). If $s_r>1$ we can write P in the form $\Pi_{2m}+Q_r(t, x, D_t, D_x)\,D_t^{s_r-1}+\sum_{j=1}^r K_j(t, x, D_t, D_x)$, where $Q_r(t, x, D_t, D_x)=a_r(t, x, D_x)\,D_t^{\bar{s}_{r-1}+1-r}$, $K_r(t, x, D_t, D_x)=\sum_{h=1}^{s_r-1} a_{r+h}(t, x, D_x)\,D_t^{m-r-h}$, and for $j=1, ..., r-1$, $K_j(t, x, D_t, D_x)=\sum_h a_{r+h}(t, x, D_x)\,D_t^{m-r-h}$, where the sum $\sum_h$ runs over all $h=\sum_{k=j+1}^r(s_k-1)+1, ..., \sum_{k=j}^r(s_k-1)$. Of course K_j is found in the sum only if $s_j > 1$. If $s_{r-1} = 1$, since $m - r = s_r - 1$, in view of Lemma 2.4, we write P in the form:

$$\Pi_{2m} + Q_r \left(\sum_{J_r \in \mathscr{I}^{(r)}_{s_r - 1}} c^{(o)}_{J_r} \partial_{J_r} + \sum_{i=0}^{s_r - 2} \sum_{J_r \in \mathscr{I}^{(r)}_1} \rho^{(N)}_{J_r} \partial_{J_r} \right)$$
$$+ \sum_{h=1}^{s_r - 1} a_{r+h}(t, x, D_x) \left(\sum_{J_r \in \mathscr{I}^{(r)}_{s_r - 1}} c^{(h)}_{J_r} \partial_{J_r} + \sum_{i=0}^{s_r - 2} \sum_{J_r \in \mathscr{I}^{(r)}_1} \rho^{(N+2h)}_{J_r} \partial_{J_r} \right),$$

with $c^{(h)}_{J_r} \in S^{-2h}$ and $\rho^{(M)}_{J_r} \in S^{-M}$.

Thus P is in the form (2.10). If $s_{r-1} > 1$, but $s_{r-2} = 1$, by applying Lemma 2.4 to Q_r, K_r, and K_{r-1}, and a combination of Lemmas 2.1 and 2.2, if it is necessary, we can write P in the form (2.10). More generally, if $s_{r-k} > 1$, but $s_{r-k-1} = 1$ for some k, $1 \leqslant k \leqslant r - 1$, then we can apply Lemma 2.4 to each K_{r-j}, $j = 0, ..., k$, which is written in the form

$$K_{r-j} = \sum_{h=1}^{s_{r-j} - 1} a_{r-j+h+s_r+ \cdots +s_{r-j+1}}(t, x, D_x)\, D_t^{s_{r-k}-1}$$
$$\cdots D_t^{s_{r-j-1}-1} D_t^{s_{r-j}-1-h} \qquad \text{for} \quad j \leqslant k - 1,$$

and for $j = k$,

$$K_{r-k} = \sum_{h=1}^{S_{r-k} - 1} a_{r-k+h+s_r+ \cdots +s_{r-k+1}}(t, x, D_x)\, D_t^{s_{r-k}-1-h},$$

and to:

$$Q_r(t, x, D_t, D_x)\, D_t^{s_r - 1} = a_r(t, x, D_x)\, D_t^{s_{r-k}-1} \cdots D_t^{s_{r-1}-1} D_t^{s_r - 1}.$$

Applying a combination of Lemmas 2.1 and 2.2 when necessary, we obtain (2.10).

Now we are going to prove Theorem 2.1. In what follows, let $\mathscr{F}$ denote $\{J = (J_1, ..., J_r);\ J_i \in \mathscr{I}^{(i)}_{h_i}$ for some $h_i \in \{0, ..., s_i\}$ $i = 1, ..., r\}$ and let ∂_J denote $\partial_{J_1} \cdots \partial_{J_r}$ if $J \in \mathscr{F}$.

Proof of Theorem 2.1. We reduce our Cauchy problem to a Cauchy problem for a first-order system with diagonal principal part. The $2^m - 1$ entries of the unknown vector valued function $\mathscr{U} = (U_J)_{J \in \mathscr{F}, |J| \leqslant m-1}$ are defined as $U_0 = u$ and $U_J = \partial_J u$ if $0 < |J| \leqslant m-1$. For every $J = (j_1, ..., j_h) \in \mathcal{F}$, $1 \leqslant |J| \leqslant m-1$, we set $U_J = \partial_{j_1} U_{(j_2, ..., j_h)}$ if $h > 1$ and $U_J = \partial_{j_1} U_0$ if $h = 1$. In view of Proposition 2.1, $Pu = f$ can be written as

$$\partial_1 U_{(2, ..., m)} + \sum_{J \in \mathcal{F}, |J| \leqslant m-r} a_J^*(t, x, D_x) U_J = f,$$

with $a_J^* \in \mathcal{B}([-T, T]; S^0)$. Thus we are led to consider a system of the form

$$D_t \mathcal{U} - \mathcal{D}(t, x, D_x) \mathcal{U} - \mathcal{A}(t, x, D_x) \mathcal{U} = \mathcal{F}(t, x)$$
$$\mathcal{U}(t = 0) = \Psi,$$

where the entries of the diagonal matrix $\mathcal{D}$ are the λ's, the entries of $\mathcal{A}$ belong to $\mathcal{B}([-T, T]; S^0)$, and the initial values Ψ of $\mathcal{U}$ are determined as follows:

$$U_0(t = 0) = g_0,$$

$$U_j(t = 0) = \sum_{\substack{l \leqslant |J| \\ j_1, ..., j_k \in \{J\} \\ j_1 < \cdots < j_k}} \delta_k^{(J)}(0, x, D_x)\, g_{|J|-k}, \qquad \text{if} \quad 0 < |J| \leqslant m-1,$$

for some $\delta_k^{(J)}(t, x, D_x) \in \mathcal{B}([-T, T]; S^{2k})$. Then we have the energy estimate $\|\mathcal{U}(t)\|_s \leqslant C(T)(\|\Psi\|_s + |\int_0^t \|\mathcal{F}(\tau, \cdot)\|_s \, d\tau|)$, which yields:

$$\sum_{|J| \leqslant m-1} \|U_J(t)\|_s \leqslant C'(T) \left\{ \sum_{j=0}^{m-1} \|g_j\|_{s+2(m-1-j)} + \left| \int_0^t \|f(\tau, \cdot)\|_s \, d\tau \right| \right\}. \quad (2.11)$$

Now note that we can write

$$\|\partial_t^j u(t, \cdot)\|_{s+2(m-r-j)} \leqslant \sum_{|J| \leqslant m-1} c_J \|U_J(t)\|_s \quad (2.12)$$

for some positive constants c_J. Indeed, we can write $\|\partial_t^j u(t, \cdot)\|_{s+2(m-r-j)} = \|\partial_t^{\sum_{k=1}^r (s_k - 1 - h_k)} u(t, \cdot)\|_{s+2h}$, for some h_k such $\sum_{k=1}^r h_k = m - r - j$ and $s_k - 1 - h_k \geqslant 0$. Applying Lemma 2.4 with $N \geqslant 2h$ to each $\partial_t^{s_k - 1 - h_k}$, we obtain $\partial_t^j u(t, \cdot) = \sum_{J \in \mathcal{F}, |J| \leqslant m-r} \tilde{c}_J^{(h)}(t, x, D_x) \partial_J$, for some $\tilde{c}_J^{(h)} \in \mathcal{B}([-T, T]; S^{-2h})$, which yields (2.12). Finally, combining (2.11) with (2.12), we get the energy estimate (2.3).

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