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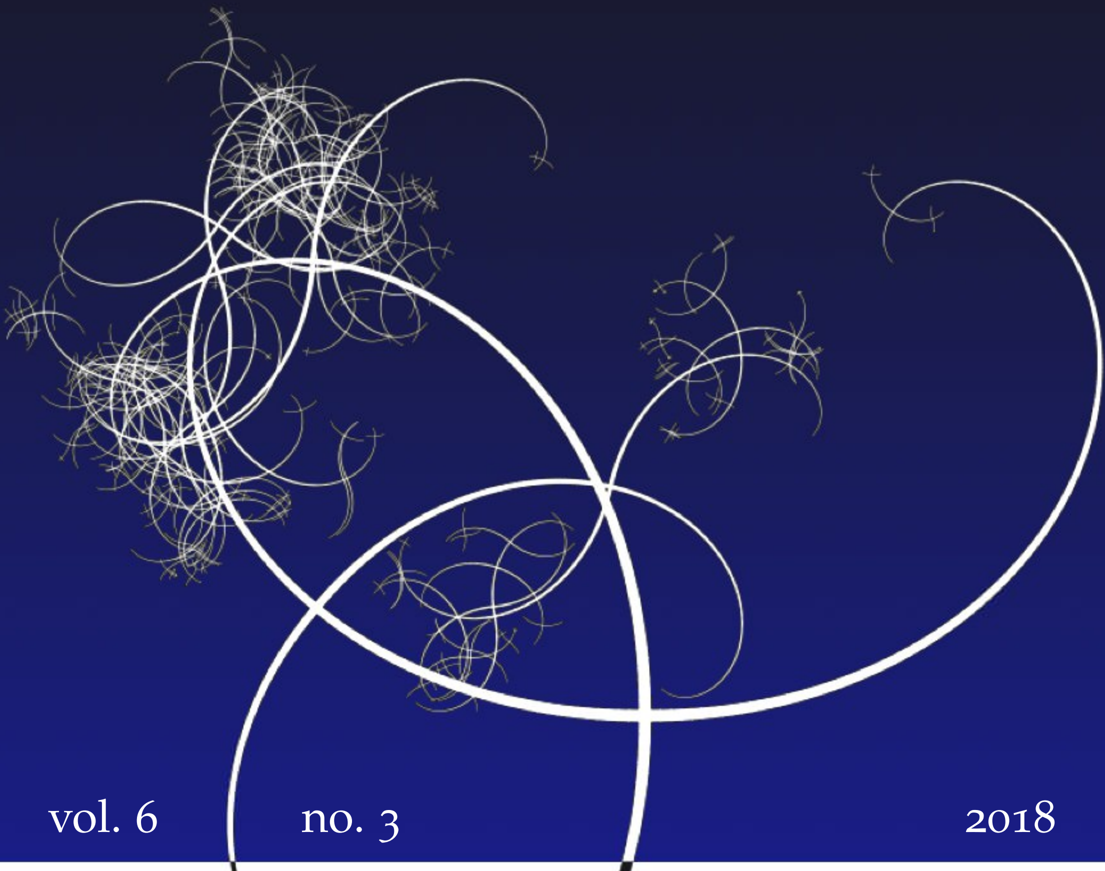

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## THE VARIATIONAL STRUCTURE OF CLASSICAL PLASTICITY

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# THE VARIATIONAL STRUCTURE OF CLASSICAL PLASTICITY

GIANPIETRO DEL PIERO

A unified approach to classical plasticity, including metal plasticity, geomaterials, and crystal plasticity, is presented. A distinctive feature of this approach is that the basic constitutive elements (yield criterion, flow rule, consistency condition, and hardening rule), instead of being assumed on a phenomenological basis or deduced from ad hoc principles, are obtained directly from the stationarity of the energy. The plastic continuum is regarded as a particular micromorphic continuum, and its energy has the form resulting from a homogenization procedure introduced in the theory of structured deformations. This form of the energy requires an additive decomposition of the deformation gradient, in place of the multiplicative decomposition usually adopted in finite plasticity. It is shown by examples that many of the models adopted in classical plasticity can be obtained from ad hoc specifications of the energy.

## 1. Introduction

Plasticity is a branch of continuum mechanics characterized by the presence of a state variable, the plastic strain, which describes rearrangements of the material structure at the microscopic level. With the progress of microstructural multiscale theories, it became important to specify the nature of the continuum in which a plasticity model is embedded. For *classical plasticity* the underlying continuum is the *classical continuum*, that is, a continuum whose external power is produced by body forces and surface tractions alone. This excludes nonlocal models such as *gradient plasticity*, in which the plastic strain is supposed to produce an extra power when multiplied by microscopic external forces, and the latter produce an extra stress measure plus a hyperstress represented by a third-order tensor.[1] Rate-dependent theories, and in particular viscoplasticity, are also excluded from the present treatment.

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[1] For this and other models of classical and nonclassical plasticity, see the book by Gurtin, Fried, and Anand [Gurtin et al. 2010, Part XV] and the references therein.

The bases of classical plasticity were laid down in the 1940s. Of the basic constitutive elements of the theory,

- *the yield condition*,
- *the flow rule*,
- *the consistency condition*, and
- *the hardening rule*,

the first three had been fixed by the end of that decade, and the fourth followed a few years later.[2] Though these elements take origin from experimental observation, several efforts were made to relate them to general principles, in order to show that plastic response is not a caprice of nature, but obeys a precise mathematical structure. In this spirit, Prager [1949] proved that the flow rule, in the form of a *normality law*, is a consequence of the uniqueness of the solution of the incremental equilibrium problem. In a paper published just before, Hill [1948] had proved that uniqueness is, in turn, a consequence of a *principle of maximum plastic work*.

Soon afterwards, Drucker [1952] showed that normality is also a consequence of a *quasithermodynamic postulate* of material stability. Actually this was not progress, since Drucker's postulate came out to be more restrictive than Hill's principle. Progress was also not brought by the postulate of Il'yushin [1961] which, though less restrictive in general, in the case of classical plasticity is equivalent to Drucker's [Lucchesi and Podio-Guidugli 1990]. On the contrary, some progress came with the introduction of supplementary state variables.[3] This opened the way to the study of *nonassociated plasticity*, which is in contradiction with Drucker's postulate, since by its own definition nonassociated plasticity does not obey the normality law.[4]

Several variational principles were formulated at the earlier stages of the theory.[5] In the development of such principles, a turning point was marked by the introduction of the concept of plastic potential.[6] In the broader context of classical continuum mechanics, related concepts of dissipation function and dissipation potential were introduced by Ziegler [1963] and Moreau [1970; 1974]. Incremental minimum principles involving a strain energy made of the sum of an elastic energy and a dissipation potential were formulated by Fedelich and Ehrlacher [1989]

[2]See Prager's overview [1949]. Prager's kinematical hardening model [1955] was introduced in the mid 1950s.

[3]See [Lemaitre and Chaboche 1990, p. 193] or [Ziegler 1983, §14].

[4]The interest in nonassociated plasticity was stimulated by the study of geomaterials, such as soils, concrete, and stones; see, e.g., [Vermeer and de Borst 1984].

[5]For the variational principles formulated before the 1950s, see Hill's book [1950].

[6]See [Hill 1950] for plasticity, and [Rice 1971] for viscoplasticity. Later, the existence of a special type of potential, called *maximal responsive*, was proved to be equivalent to Hill's principle; see the article by Eve, Reddy, and Rockafellar [Eve et al. 1990, Theorem 4.1].

and by Petryk [2003]. Subsequent contributions by Dal Maso et al. [2006; 2008], Mielke [Carstensen et al. 2002; Mielke 2003], and their schools marked substantial progress in this direction.

The variational approach adopted in the present paper, formally similar to those in [Fedelich and Ehrlacher 1989; Petryk 2003], has the peculiarity of deducing all constitutive elements from the stationarity condition on the energy functional. Indeed, this condition determines the incremental response law without any supplementary assumption, such as the existence or convexity of the elastic range, or the form of the flow rule.[7]

After fixing the incremental response law, the incremental equilibrium problem can be formulated. This is the problem of determining the small deformations from a given equilibrium placement, due to a conveniently small load increment. Though the deformations to be determined are small, it is sometimes convenient to formulate the problem in large deformations.[8] This is the case, for example, when one has in mind to approximate a problem with large load increments by a sequence of problems with small load increments.

The present study is restricted to stationarity, that is, to the condition of nonnegativeness of the first variation of the energy. In this way only equilibrium conditions are obtained, without any information about stability. This is a serious limitation. Indeed, a stability analysis would show that in classical plasticity a softening response is unstable, because the plastic strain localizes on arbitrarily small regions of the body. Initially, this led to considering softening materials as inadmissible. But this viewpoint, consecrated by Drucker's postulate, conflicted with the evidence of the softening response exhibited by many real materials. Later, it was realized that softening can be described by adding to the energy a nonlocal stabilizing term, depending on the gradient of the plastic strain rate [Aifantis 1984; Bažant et al. 1984].[9]

In this paper some preliminary definitions, including a new ad hoc notation for homogeneous maps, form Section 2, and the transitions from the equilibrium problem in finite deformation to the evolution problem and from this one to the incremental equilibrium problem are briefly illustrated in Section 3. This section deals with *two-scale*, or *micromorphic*, continua, of which the plastic continua

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[7]The idea of deducing the constitutive properties from two scalar potentials, an elastic energy and a dissipation function, had already been exploited by Collins, Houlsby, and coworkers. They initially applied it to geomaterials [Collins and Houlsby 1997], and then to general rate-independent dissipative materials [Houlsby and Purzrin 2000; Collins 2003].

[8]The linearized equations for large deformations differ from those of the linear theory; see Section 3.2 below.

[9]More recent one-dimensional analyses of the softening response in the proximity of fracture can be found in [Pham et al. 2011a; 2011b] for damage and in [Del Piero 2013; Del Piero et al. 2013] for plasticity.

are a subclass. For them, in Section 3.4 it is shown that the stationarity of the energy determines the response law relating the Piola stress to the elastic part of the deformation, and produces a local stationarity condition which will be shown to fully characterize the constitutive response.

In Section 4 the incremental problem is reformulated as a minimum problem for the energy. The selected form of the energy is based on the additive decomposition of the deformation gradient into an elastic and a plastic part, a rather unusual choice in the context of large deformations. As shown in Section 4.1, this choice leads to a particular form of the plastic strain rate, in which a symmetric *plastic stretching* is followed by a rotation to be determined by a constitutive assumption. In most models this rotation is taken equal to the identity. An exception is the *crystal plasticity* model discussed in Section 7.[10] An indifference argument developed in Section 4.2 shows that the plastic part of the energy is independent of this rotation.

In the crucial Section 5, the constitutive elements of the theory are deduced from the local stationarity condition, now called *plastic stationarity condition*. A major result in this paper is that this condition determines a *bounding map* in the stress space, related to the directional derivatives of a dissipation potential. The values taken by this map in different directions impose directional limit values for the stress. This leads to the definition of an *elastic range*, a region in the stress space which by its own construction turns out to be closed and convex. From the plastic stationarity condition it also follows that a plastic strain rate can only occur if the stress is a boundary point of the elastic range, and that its direction belongs to the normal cone at that boundary point. This is the *normality law* which determines the *associated flow rule*. Thus, *nonassociated flow rules are not provided by the variational procedure*.

In Section 6 some well known plasticity models are reobtained assuming particular forms of the plastic energy. Section 6.1 deals with the three basic models of perfect, kinematic, and dilatational plasticity, which include kinematic and isotropic hardening as special cases. The assumption of isochoricity of the plastic strain rate is studied in Section 6.2. It is well known that this assumption gives the opportunity of taking the hydrostatic pressure as a supplementary state variable.[11] As a consequence of stationarity, this extra variable generates a pressure-dependent family of elastic ranges, such that normality holds for each member of the family. This makes possible to include in the present scheme some plasticity models usually described by nonassociated flow rules.

[10] In the models based on the multiplicative decomposition, an equivalent constitutive assumption is the assumption of *plastic irrotationality*, by which the *plastic spin*, which is the skew-symmetric part of the plastic strain rate, is set to zero. Here, too, crystal plasticity is an exception.

[11] See, e.g., [Srinivasa 2010; Vermeer and de Borst 1984, Ziegler 1983, §17.6; Ziegler and Wehrli 1987, §VII.A].

The isotropic case, in which the plastic energy is independent of the direction of the plastic strain rate, is investigated in Section 6.3. This case includes the energies of Drucker and Prager and of Mises. These energies are used as paradigms to compare the plastic behaviors of metals and geomaterials. The fact that isochoric plasticity cannot describe some plastic volume changes observed in geomaterials, such as the dilatancy of soils, motivates the *Cam-clay model* summarized in Section 6.4, specifically conceived to describe such phenomena [Roscoe and Poorooshasb 1963; Roscoe et al. 1958].

Finally, Section 7 deals with the plasticity of crystals. This is a special case of isochoric plasticity, in which the plastic strain has the form of slips occurring on predetermined slip planes. As said above, this is the only case considered in this paper in which the form of the plastic strain is given a priori, so that there is no need of specifying constitutively any rotation. The single-slip and the multislip models are illustrated in Sections 7.1 and 7.2, respectively. In Section 7.3, the periodic energies used to study the two-level shear of single crystals are considered. Within the exception constituted by crystal plasticity, this model exhibits the further exception that the plastic potential which governs the evolution of the plastic strain need not be nonsmooth. Instead of being diffused along the whole process, the plastic dissipation concentrates on singular instability events of catastrophic nature. This opens perspectives of revision of the bases of classical plasticity, including the revisitation of the concept of elastic range and of the other constitutive elements. For reasons of brevity, only a mention of such perspectives can be made here.

It is the present author's opinion that the possibility of treating in a unified way many models reproducing the behavior of materials of different natures, just acting on the shape of the plastic energy and then looking at the consequences of the plastic stationarity condition, is a paramount advantage of the variational approach.

## 2. Notation and preliminaries

**2.1.** ***Linear spaces and linear maps.*** By *linear space* we mean a finite-dimensional vector space endowed with an inner product. A *linear map* from a linear space $\mathscr{A}$ to a linear space $\mathscr{B}$ is a map $\ell : \mathscr{A} \to \mathscr{B}$ such that

$$\ell(\alpha H + \beta K) = \alpha\ell(H) + \beta\ell(K) \quad \text{for all } H, K \in \mathscr{A} \text{ and } \alpha, \beta \in \mathbb{R}. \tag{2-1}$$

The set of all linear maps of $\mathscr{A}$ into $\mathscr{B}$ is a linear space which will be denoted by $\mathscr{L}(\mathscr{A}, \mathscr{B})$.

If $\mathscr{B}$ is the real line $\mathbb{R}$, the elements of $\mathscr{L}(\mathscr{A}, \mathbb{R})$ are the *linear functionals on* $\mathscr{A}$. By the representation theorem of linear functionals, $\mathscr{L}(\mathscr{A}, \mathbb{R})$ is isomorphic to $\mathscr{A}$. That is, with every $\ell \in \mathscr{L}(\mathscr{A}, \mathbb{R})$ one can associate a unique element $H$ of $\mathscr{A}$ such that

$$\ell(K) = H \cdot K \quad \text{for all } K \in \mathscr{A}, \tag{2-2}$$

with $(\cdot)$ the inner product of $\mathscr{A}$.[12]

For $\mathscr{A}$ fixed and $\mathscr{B} = \mathscr{A}$, we write $\mathscr{L}$ in place of $\mathscr{L}(\mathscr{A}, \mathscr{A})$ and we call its elements *second-order tensors*. The linear maps of $\mathscr{L}$ into itself are called *fourth-order tensors*.

A *subspace* of a linear space is a subset which is itself a linear space. Proper subspaces of $\mathscr{L}$ are the set $\mathscr{S}$ of all symmetric tensors and the set $\mathscr{W}$ of all skew-symmetric tensors.

**2.2.** ***Homogeneous and bihomogeneous maps.*** In plasticity, an important role is played by homogeneous and bihomogeneous maps. Therefore, it does not seem inappropriate to recall some basic definitions and to introduce some ad hoc notation. A map $\mathfrak{h}$ from a linear space $\mathscr{A}$ to a linear space $\mathscr{B}$ is *homogeneous of order one*, in short, *homogeneous*, if

$$\mathfrak{h}(tH) = t\mathfrak{h}(H) \quad \text{for all } H \in \mathscr{A} \text{ and } t \geq 0. \tag{2-3}$$

The set $\mathscr{H}(\mathscr{A}, \mathscr{B})$ of all homogeneous maps from $\mathscr{A}$ to $\mathscr{B}$ is a vector space, with obvious definitions of the sum and multiplication by a scalar:

$$(\mathfrak{h} + \mathfrak{l})(H) = \mathfrak{h}(H) + \mathfrak{l}(H), \quad (\alpha\mathfrak{h})(H) = \alpha\mathfrak{h}(H) \quad \text{for all } \alpha \in \mathbb{R} \text{ and } H \in \mathscr{A}. \tag{2-4}$$

If a homogeneous map is additive,

$$\mathfrak{h}(H + K) = \mathfrak{h}(H) + \mathfrak{h}(K) \quad \text{for all } H, K \in \mathscr{A}, \tag{2-5}$$

it is linear. Therefore, $\mathscr{L}(\mathscr{A}, \mathscr{B})$ is a proper subspace of $\mathscr{H}(\mathscr{A}, \mathscr{B})$.

If $\mathscr{B}$ is the real line $\mathbb{R}$, the elements of $\mathscr{H}(\mathscr{A}, \mathbb{R})$ are the *homogeneous functionals on $\mathscr{A}$*. In this case we write

$$\mathfrak{h} \triangleright K \quad \text{in place of} \quad \mathfrak{h}(K). \tag{2-6}$$

The operator "$\triangleright$" maps the elements of $(\mathscr{H}(\mathscr{A}, \mathbb{R}) \times \mathscr{A})$ into the real numbers. In particular, by (2-2), a linear $\mathfrak{h}$ can be identified with an element $H$ of $\mathscr{A}$. In this case, the pairing reduces to the inner product of $\mathscr{A}$:

$$\mathfrak{h} \in \mathscr{L}(\mathscr{A}, \mathbb{R}) \quad \Longrightarrow \quad \mathfrak{h} \triangleright K = H \cdot K \quad \text{for all } K \in \mathscr{A}. \tag{2-7}$$

A *bihomogeneous map* on $\mathscr{A}$ is a homogeneous map $\mathfrak{K}$ from $\mathscr{A}$ to $\mathscr{H}(\mathscr{A}, \mathbb{R})$. For any such map, the homogeneous functional on $\mathscr{A}$ obtained applying $\mathfrak{K}$ to $K \in \mathscr{A}$

[12] See, e.g., [Halmos 1942, §67].

and the real number obtained by applying this homogeneous functional to $H \in \mathscr{A}$ are denoted by

$$\mathfrak{K}\{K\}, \qquad \mathfrak{K}\{K\} \triangleright H, \tag{2-8}$$

respectively. For every bihomogeneous map we have

$$\mathfrak{K}\{\lambda K\} \triangleright \mu H = \lambda\mu\mathfrak{K}\{K\} \triangleright H \quad \text{for all } \lambda, \mu \geq 0. \tag{2-9}$$

If $\mathfrak{K}$ is linear, we write $\mathfrak{K}[K]$ in place of $\mathfrak{K}\{K\}$. If $\mathfrak{K}$ is linear and $\mathfrak{K}[K]$ is linear for all $K$, then $\mathfrak{K}$ is a linear map from $\mathscr{A}$ to itself, that is, a fourth-order tensor $\mathbb{K}$. In this case, we have

$$\mathfrak{K}\{K\} \triangleright H = \mathbb{K}[K] \cdot H. \tag{2-10}$$

**2.3.** ***Directional derivatives.*** Let $\mathscr{A}$ be a linear space, and let $\phi$ be a map of $\mathscr{L} = \mathscr{L}(\mathscr{A}, \mathscr{A})$ into the real line $\mathbb{R}$. For $A \in \mathscr{L}$ and $H \in \mathscr{L} \setminus \{0\}$, the limit

$$\breve{\nabla}\phi(A) \triangleright H = \lim_{\varepsilon \to 0^+} \frac{\phi(A + \varepsilon H) - \phi(A)}{\varepsilon} \tag{2-11}$$

is the *directional derivative* at $A$ in the direction $H$. If this limit exists for all $H$, we say that $\phi$ is *Gâteaux differentiable* at $A$. From its very definition it is clear that $\breve{\nabla}\phi(A)$ is a homogeneous functional, that is, an element of $\mathscr{H}(\mathscr{L}, \mathbb{R})$. If $\breve{\nabla}\phi(A)$ is additive, it reduces to the ordinary derivative $\nabla\phi(A)$, and $\phi$ is said to be Fréchet differentiable, in short, *differentiable*, at $A$.

The map $\phi$ is *twice Gâteaux differentiable* at $A$ if $\breve{\nabla}\phi$ is Gâteaux differentiable in a neighborhood of $A$ and the limit

$$\breve{\nabla}^2\phi(A)\{H\} \triangleright K = \lim_{\varepsilon \to 0^+} \frac{\breve{\nabla}\phi(A + \varepsilon H) \triangleright K - \breve{\nabla}\phi(A) \triangleright K}{\varepsilon} \tag{2-12}$$

exists for all $H \in \mathscr{L} \setminus \{0\}$. In this case $\breve{\nabla}^2\phi(A)$ is said to be the *second directional derivative at* $A$ in the direction $H$. It is clear that $\breve{\nabla}^2\phi(A)\{H\}$ is an element of $\mathscr{H}(\mathscr{L}, \mathbb{R})$, and that $\breve{\nabla}^2\phi(A)$ is a homogeneous map from $\mathscr{L}$ to $\mathscr{H}(\mathscr{L}, \mathbb{R})$, that is, a bihomogeneous map on $\mathscr{L}$. If $\breve{\nabla}^2\phi(A)$ is linear in both $H$ and $K$, it reduces to the second Fréchet derivative $\nabla^2\phi(A)$, which is identified with a fourth-order tensor. Here we shall be interested in the case $H = K$, in which the expansions

$$\begin{aligned} \phi(A + \varepsilon H) &= \phi(A) + \varepsilon\breve{\nabla}\phi(A) \triangleright H + \tfrac{1}{2}\varepsilon^2\breve{\nabla}^2\phi(A)\{H\} \triangleright H + o(\varepsilon^2), \\ \breve{\nabla}\phi(A + \varepsilon H) &= \breve{\nabla}\phi(A) + \varepsilon\breve{\nabla}^2\phi(A)\{H\} + o(\varepsilon) \end{aligned} \tag{2-13}$$

are direct consequences of the definitions.

**2.4.** ***Nonsmooth potentials and dissipation potentials.*** A function $\phi$ from $\mathscr{L}$ to the real line

(i) *with* $\phi(A) = 0$,

(ii) *twice differentiable at* $\mathscr{L} \setminus \{A\}$,

(iii) *twice Gâteaux differentiable at* $A$,

will be called a *nonsmooth potential from* $A$. A *smooth potential* is the special case in which $\phi$ is also twice differentiable at $A$. A nonsmooth potential

(iv) *strictly increasing along every direction* $H$,

$$\lambda > \mu \geq 0 \quad \Longrightarrow \quad \phi(A+\lambda H) > \phi(A+\mu H) \quad \text{for all } H \in \mathscr{L} \setminus \{0\}, \tag{2-14}$$

will be called a *dissipation potential*. We point out that (iv) implies

$$\breve{\nabla}\phi(A) \rhd H > 0 \quad \text{for all } H \in \mathscr{L} \setminus \{0\}. \tag{2-15}$$

A dissipation potential is *homogeneous* from $A \in \mathscr{L}$ if[13]

$$\phi(A+\varepsilon H) = \varepsilon\phi(A+H) \quad \text{for all } H \in \mathscr{L} \text{ and } \varepsilon \geq 0. \tag{2-16}$$

For a homogeneous potential from $A$, from $(2\text{-}13)_1$ we have

$$\phi(A+H) = \breve{\nabla}\phi(A) \rhd H, \qquad \breve{\nabla}^2\phi(A)\{H\} \rhd H = 0 \quad \text{for all } H \in \mathscr{L}. \tag{2-17}$$

## 3. The incremental problem

**3.1.** ***From the equilibrium problem to the evolution problem.*** Let $\Omega_R$ be the region occupied by a continuous body in the reference placement. Suppose that on $\Omega_R$ is prescribed a system of external loads, consisting of a body force field $b_R$ at the interior of $\Omega_R$ and of a surface traction field $s_R$ on a given portion $\partial_s\Omega_R$ of the boundary $\partial\Omega_R$.

A stress field $T_R$ on $\Omega_R$ is said to be *equilibrated* with the given loads if it satisfies the *virtual work equation*[14]

$$\int_{\Omega_R} T_R \cdot \nabla v \, dV_R = \int_{\Omega_R} b_R \cdot v \, dV_R + \int_{\partial_s\Omega_R} s_R \cdot v \, dA_R \tag{3-1}$$

for all vector fields $v$ on $\Omega_R$ which vanish on $\partial\Omega_R \setminus \partial_s\Omega_R$. The *Piola stress tensor* $T_R$ is related to the deformation gradient $\nabla f$ by a *response law*. The *equilibrium problem* consists of finding the deformation $f$ from $\Omega_R$ produced by the given loads. The *weak formulation* of the problem is obtained substituting the response law into the virtual work equation and imposing the boundary condition of place

$$f(x) = \hat{f}(x), \quad x \in \partial\Omega_R \setminus \partial_s\Omega_R. \tag{3-2}$$

[13] See, e.g., [Eve et al. 1990; Martin and Reddy 1993]. Here, nonhomogeneous potentials appear in the dilatational plasticity models in Section 6. A dissipation potential is also frequently assumed to be convex, but this assumption is not essential; see, e.g., [Petryk 2003, Remark 1]. In fact, nonconvexity is required to describe the softening response; see Section 6.1 below.

[14] See, e.g., [Ciarlet 1988, §2.6]. For simplicity, the variable of integration $x$ is omitted.

For an elastic body, the response law is a functional relation between the punctual values of $T_R$ and $\nabla f$:

$$T_R(x) = \mathcal{G}(\nabla f(x)), \quad x \in \Omega_R. \tag{3-3}$$

For nonelastic bodies, the response law is less simple. In the class of *simple materials* it is assumed that, at each point $x$, the stress $T_R(x)$ is determined by the past history of the gradient of $f$ at $x$ [Truesdell and Noll 1965, §28; Noll 1972]. By consequence, the final deformation does not only depend on the final load $(b_R, s_R)$, but also on the loading path along which this load has been reached.

To follow the evolution of the load, the equilibrium problem is replaced by an *evolution problem*, which can be formulated as follows. Let an initial placement $f_0$ of the body, an initial stress $T_{R0}$, a loading path $t \mapsto (b_{Rt}, s_{Rt})$, and a family $t \mapsto \hat{f}_t$ of constraints be given.[15] The initial placement is required to satisfy the constraint $\hat{f}_0$ on the constrained part of the boundary, and the initial stress is required to be equilibrated with the initial load $(b_{R0}, s_{R0})$. The problem consists of determining a deformation process $t \mapsto f_t$ such that, for each $t > 0$, $f_t$ satisfies the constraint $\hat{f}_t$ and the stress $T_{Rt}$ is equilibrated with the loads $(b_{Rt}, s_{Rt})$.

**3.2.** ***From the evolution problem to the incremental problem.*** In the evolution problem, $t$ varies over a finite interval $[0, t^\dagger]$. In the *incremental equilibrium problem*, this interval is restricted by taking the limit of $t^\dagger$ to $0^+$. Then, from the expansions

$$b_{Rt} = b_{R0} + t\delta b_R + o(t), \quad s_{Rt} = s_{R0} + t\delta s_R + o(t), \quad T_{Rt} = T_{R0} + t\delta T_R + o(t), \tag{3-4}$$

the incremental version of the virtual work equation (3-1)

$$\int_{\Omega_R} \delta T_R \cdot \nabla v \, dV_R = \int_{\Omega_R} \delta b_R \cdot v \, dV_R + \int_{\partial_s \Omega_R} \delta s_R \cdot v \, dA_R \tag{3-5}$$

follows. The weak formulation of the problem is completed by the prescription of an incremental constraint $\hat{u}$ on $\partial\Omega_R \setminus \partial_s\Omega_R$ and of an incremental response law of the form

$$\delta T_R = \mathfrak{h}(\nabla u), \tag{3-6}$$

where $u(x) = f(x) - x$ is the displacement vector and $\mathfrak{h}$ is a homogeneous map depending on a set of variables which define the current *state* of each specific continuum.[16]

[15] The parameter $t$ is an *internal time* which describes the deformation path. It need not be related to the physical time.

[16] This map is an *evolution function* in the sense of W. Noll's *new theory of simple materials* [Noll 1972].

For the solution of the evolution problem, a natural strategy is to subdivide the interval $[0, t^\dagger]$ into subintervals, and to solve the incremental problem on each subinterval, using the solution of each problem as the input for the next one. It is expected that, in the presence of sufficient regularity, the solution to the evolution problem would be achieved in the limit, when the lengths of all subintervals tend uniformly to zero. There is, however, a difficulty in passing from a subinterval to the next. Indeed, in this passage the initial deformation $f_0$, the loads $b_R$, $s_R$, and the constraint $\hat{u}$ change. By consequence, the response function $\mathfrak{h}$ and the Piola stress $T_R$ must be updated. This is perhaps the most difficult part of the solution procedure. It will be not considered in the present paper, which is restricted to the formulation of a single incremental problem.

The assumption of infinitesimal deformations, by which these changes are neglected, drastically simplifies the problem. However, since the increments obey different laws in small and in large deformations, the incremental problem is different in the two cases. Here, only the formulation in the context of large deformations will be considered.

**3.3.** ***The energetic formulation.*** An alternative formulation of the equilibrium problem consists of transforming it into a minimum problem for the total energy of the body. In this approach the equilibrium placements of the body are characterized by the stationarity of the energy, that is, by the nonnegativeness of the first variation. As already said in the Introduction, the stability of the equilibrium, which is decided by the sign of the second variation, will not be considered here.

The energy of a body is assumed to be the sum of an internal strain energy plus the energy communicated by the external loads:

$$\mathcal{E}_{\text{tot}} = \mathcal{E}_{\text{int}} + \mathcal{E}_{\text{ext}}. \tag{3-7}$$

A *classical continuum* is a continuum with an external energy of the form[17]

$$\mathcal{E}_{\text{ext}}(b_R, s_R, u) = -\int_{\Omega_R} b_R \cdot u \, dV_R - \int_{\partial\Omega_R} s_R \cdot u \, dA_R. \tag{3-8}$$

The internal energy has the form of a volume integral, with an energy density depending on the strain measures which characterize each specific continuum. In particular, the class of *two-scale*, or *micromorphic*, continua is characterized by two strain measures, the macroscopic deformation $f$ and the local microscopic deformation $F$.[18] For a two-scale deformation $(f, F)$ we assume an internal energy

[17] In a nonclassical, or *generalized* continuum, the external energy has supplementary terms involving external actions associated with the state variables.

[18] Here *local* means that the function $F$ is not in general the gradient of a function defined over the whole body.

of the form

$$\mathcal{E}_{\text{int}}(f, F) = \int_{\Omega_R} (\varphi(F) + \phi(F^d))\, dV_R, \tag{3-9}$$

where

$$F^d = \nabla f - F \tag{3-10}$$

is the deformation due to the *disarrangements* [Owen 1995] which occur at the microscopic scale, and the energy densities $\varphi$ and $\phi$ are a smooth and a nonsmooth potential, respectively.

This energy is the relaxed limit of the energies associated with the discontinuities in a sequence of piecewise continuous macroscopic deformations approximating $(f, F)$.[19] The existence of such approximating sequences for any two-scale deformation is ensured by the approximation theorem of the theory of *structured deformations* [Del Piero and Owen 1993].[20]

**3.4.** ***Incremental minimization.*** Let $t \mapsto (f_t, F_t)$ be a solution of the incremental problem from an initial placement which, for convenience, we take as the reference placement. Then $f_0$ and $F_0$ are the identity map $\imath$ and the identity tensor $I$, respectively, and $f_t$ and $F_t$ have the expansions

$$f_t = \imath + t\nabla u + o(t), \qquad F_t = I + tL + o(t). \tag{3-11}$$

The region $\Omega_R$ is now the region $\Omega_0$ occupied by the body at $t = 0$, and the loads and stresses (3-4) are now

$$b_t = b_0 + t\delta b_R + o(t), \quad s_t = s_0 + t\delta s_R + o(t), \quad T_t = T_0 + t\delta T_R + o(t). \tag{3-12}$$

For $F_t^d = \nabla f_t - F_t$ we have

$$F_0^d = 0, \qquad F_t^d = tL^d + o(t), \qquad L^d = \nabla u - L, \tag{3-13}$$

[19]The dependence of the relaxed energy on the pair $(F, F^d)$ was proved by Choksi and Fonseca [1997] for a model of defective crystals. They also found that the sum decomposition (3-9) holds in some special cases, and conjectured that it may hold in general. In [Del Piero 2001; Owen 2000] their conjecture was proved to be true in one dimension. Later this result was extended to three dimensions, under different regularity assumptions. In the papers [Baía et al. 2012; Owen and Paroni 2015; Šilhavý 2017], $\phi$ was found to be a homogeneous potential. For plastic materials, we shall see later that the homogeneous potentials can only describe a perfectly plastic response. To describe work-hardening, more general potentials have been obtained either assuming special forms of $\phi$ [Deseri and Owen 2002] or taking the relaxed limit of approximating sequences for *second-order structured deformations* [Barroso et al. 2017].

[20]That $F^d$ is an independent energetic variable is due to the fact that, in the limit, the discontinuities in the piecewise continuous approximating deformations have a volume density, which is exactly $F^d$; see [Del Piero and Owen 1995].

and, for the energy densities $\varphi$ and $\phi$,

$$\begin{aligned}\varphi(F_t) &= t\nabla\varphi(I)\cdot L + \tfrac{1}{2}t^2\nabla^2\varphi(I)[L]\cdot L + o(t^2),\\ \phi(F_t^d) &= t\breve{\nabla}\phi(0)\triangleright L^d + \tfrac{1}{2}t^2\breve{\nabla}^2\phi(0)[L^d]\triangleright L^d + o(t^2).\end{aligned} \tag{3-14}$$

For fixed $t > 0$, the solution $(f_t, F_t)$ is the minimum of the total energy in the class of all deformations $(f_\varepsilon, F_\varepsilon)$ such that

$$f_\varepsilon = f_t + \varepsilon v, \qquad F_\varepsilon = F_t + \varepsilon\mathcal{L}, \qquad F_\varepsilon^d = F_t^d + \varepsilon\mathcal{L}^d, \tag{3-15}$$

with $(v, \mathcal{L})$ a field of virtual velocities on $\Omega_0$ and $\varepsilon > 0$ a scalar parameter. The total energy of $(f_\varepsilon, F_\varepsilon)$ is

$$\begin{aligned}\mathcal{E}_{\text{tot}}(f_\varepsilon, F_\varepsilon) = \mathcal{E}_{\text{tot}}(f_t, F_t) + \varepsilon\int_{\Omega_0}&(\nabla\varphi(F_t)\cdot\mathcal{L} + \breve{\nabla}\phi(F_t^d)\triangleright\mathcal{L}^d)\,dV_0\\ &-\varepsilon\Big(\int_{\Omega_0} b_t\cdot v\,dV_0 + \int_{\partial\Omega_0} s_t\cdot v\,dA_0\Big) + o(\varepsilon).\end{aligned} \tag{3-16}$$

The energy is said to be *stationary* at $(f, F)$ if

$$\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}(\mathcal{E}_{\text{tot}}(f_\varepsilon, F_\varepsilon) - \mathcal{E}_{\text{tot}}(f_t, F_t)) \geq 0 \tag{3-17}$$

for all perturbations $(v, \mathcal{L})$. In particular, for $\mathcal{L} = \nabla v$ we have $\mathcal{L}^d = 0$, and for $v = 0$ on $\partial\Omega_0 \setminus \partial_s\Omega_0$ we get

$$\int_{\Omega_0}\nabla\varphi(F_t)\cdot\nabla v\,dV_0 = \int_{\Omega_0} b_t\cdot v\,dV_0 + \int_{\partial_s\Omega_0} s_t\cdot v\,dA_0, \tag{3-18}$$

with the equality sign because the inequality holds for both $v$ and $-v$. A comparison with the virtual work equation (3-1) then leads to the identification

$$T_t = \nabla\varphi(F_t) \tag{3-19}$$

of $\nabla\varphi$ with a stress field $T_t$ equilibrated with the loads $(b_t, s_t)$. Therefore, *at every stationarity point of the energy there is a stress field $T_t$ equilibrated with the given loads*. In particular, the same relation at $t = 0$

$$T_0 = \nabla\varphi(I) \tag{3-20}$$

provides the stress $T_0$ associated with the initial deformation $F_0 = I$.

As a consequence of (3-18), all terms in $v$ cancel from (3-16) with $\mathcal{L}$ replaced by $\nabla v - \mathcal{L}^d$. Then (3-17) reduces to the inequality

$$\int_{\Omega_0}\breve{\nabla}\phi(F_t^d)\triangleright\mathcal{L}^d\,dV_0 - \int_{\Omega_0} T_t\cdot\mathcal{L}^d\,dV_0 \geq 0. \tag{3-21}$$

The punctual inequality

$$T_t \cdot \mathcal{L}^d \leq \check{\nabla}\phi(F_t^d) \triangleright \mathcal{L}^d \quad \text{for all } \mathcal{L}^d \in \mathscr{L} \tag{3-22}$$

follows almost everywhere in $\Omega_R$.[21] This is a necessary condition for the stationarity of the energy.

Since the left-hand side of (3-21) is the first-order approximation of the difference between the energy of $(f_\varepsilon, F_\varepsilon)$ and the energy of the minimizer $(f_t, F_t)$, the minimum of this difference is zero and is attained for $\mathcal{L}^d = L^d$. Then for $\mathcal{L}^d = L^d$ inequality (3-22) is satisfied as an equality

$$T_t \cdot L^d = \check{\nabla}\phi(F_t^d) \triangleright L^d. \tag{3-23}$$

## 4. Incremental minimization in plasticity

Classical plasticity deals with continua which are both classical and micromorphic. That is, the deformations are two-scale deformations $(f, F)$, and the total energy is the sum of the energies (3-8) and (3-9). The difference $\nabla f - F$ is the *plastic strain* $F^d$, and inequality (3-22) is the *plastic stationarity condition*. In what follows, this condition is used to characterize the incremental response law. This requires a preliminary definition of the set of the admissible plastic strain rates and the determination of the restrictions imposed on the energy densities $\varphi$ and $\phi$ by material indifference.

**4.1.** ***Admissible plastic strain rates.*** The definition $F^d = \nabla f - F$ of the plastic strain can be read as the *additive decomposition*

$$\nabla f = F + F^d \tag{4-1}$$

of the macroscopic deformation gradient into the sum of an elastic and a plastic part.[22] The choice of the additive decomposition has notable consequences on the structure of the plastic strain rate and, consequently, on the restrictions on the plastic potential $\phi$ imposed by material indifference.

Consider a deformation process $t \mapsto (f_t, F_t)$ with the initial placement as reference placement. In the polar decomposition $F_t^d = R_t^d U_t^d$ of the plastic strain,

[21] Indeed, if for some $\mathcal{L}^d$ the punctual inequality were violated on a region of positive volume, the integral inequality would be violated by any perturbation with support in that region and with $\mathcal{L}^d(x)$ parallel to $\mathcal{L}^d$.

[22] This decomposition is standard in infinitesimal plasticity, while in finite plasticity the Kröner–Lee multiplicative decomposition $\nabla f = F^e F^p$ is largely preferred. The reason for the present choice of the additive decomposition is the form (3-9) assumed for the energy of a micromorphic continuum. For a detailed discussion on additive and multiplicative decompositions in plasticity, see the present author's paper [Del Piero 2018].

at $t = 0$ we have $F_0^d = 0$. Then $U_0^d$ is zero, $R_0^d$ is indeterminate, and $R_t^d$ and $U_t^d$ have the expansions[23]

$$R_t^d = R_0^d(I + tW^d) + o(t), \qquad U_t^d = tD^d + o(t). \tag{4-2}$$

Then, by consequence,

$$L^d = R_0^d D^d. \tag{4-3}$$

Thus, in the additive decomposition the *plastic strain rate* $L^d$ and the *plastic stretching* $D^d$ differ by the rotation $R_0^d$. This rotation is determined by a constitutive assumption.[24] Then either a set $\mathscr{S}_0^d$ of *admissible plastic stretchings* for the initial deformation is selected a priori and the set of the *admissible plastic strain rates* $\mathscr{L}_0^d = R_0^d \mathscr{S}_0^d$ is deduced, or vice versa.

The most common constitutive choice is $R_0^d = I$, which corresponds to assuming the symmetry of the plastic strain rate.[25] In the following we keep this choice. The important exception of crystal plasticity is treated separately in Section 7.

**4.2.** ***Indifference requirements.*** Like every scalar-valued function, the potentials $\varphi, \phi$ must be invariant under distance-preserving changes of placement.[26] This results in the indifference conditions

$$\varphi(F) = \varphi(QF), \qquad \phi(F^d) = \phi(QF^d) \tag{4-4}$$

to be satisfied by all $F$, $F^d$ in $\mathscr{L}$ and by all proper orthogonal tensors $Q$.[27] For the function $\varphi$, from the polar decomposition $F = RU$ of $F$, taking $Q = R^T$ we get the condition

$$\varphi(F) = \varphi(U), \tag{4-5}$$

[23]In the multiplicative decomposition, $F_t^p$ is defined in an *intermediate placement* at which both $F_0^p$ and $R_0^p$ are equal to $I$ and therefore are nonsingular. In the additive decomposition there is no such placement, since both $f$ and $F$ are defined on the reference placement.

[24]In the multiplicative decomposition, $F_t^p = I + tL^p$ implies $R_0^p = I$. Nevertheless, the decomposition is determined only to within an indeterminate rotation; see [Dashner 1986; Green and Naghdi 1971; Lubarda and Lee 1981] and the paper [Del Piero 2018] by the present author. Therefore, a constitutive assumption on a rotation has to be made anyway. Note also that, if $R_t^p = I + tW^p$ and $U_t^p = I + tD^p$, the *plastic stretching* $D^p$ and the *plastic spin* $W^p$ are the symmetric and skew-symmetric parts of $L^p$, respectively. This property is not preserved in the additive decomposition.

[25]In the multiplicative decomposition, the corresponding constitutive choice is the *irrotationality assumption* $W^p = 0$ [Gurtin and Anand 2005; Reddy 2011; Reddy et al. 2008]. This assumption has been made, tacitly or explicitly, in classical textbooks [Hill 1950; Prager 1949; 1955], as well as in some more recent models of gradient plasticity [Fleck and Hutchinson 2001; Gurtin and Anand 2005; Gudmundson 2004].

[26]See, e.g., [Truesdell and Noll 1965, §17].

[27]The transformation law for $F^d$ is a consequence of the laws $\nabla f \to Q\nabla f$ and $F \to QF$. In the multiplicative decomposition, while the transformation law $F^e \to QF^e$ was universally accepted, the question of the transformation law to be adopted for $F^p$ has a long and controversial history [Dashner 1986; Green and Naghdi 1971; Lubarda and Lee 1981; Gurtin et al. 2010, §95.2].

according to which $\varphi$ is insensitive to the rotation $R$ which follows the pure stretch $U$ in the microscopic deformation. This condition can be used to determine an indifference restriction on $\varphi$ at $F = I$. Indeed, for $R$ and $U$ consider the expansions

$$R = I + \varepsilon W + \tfrac{1}{2}\varepsilon^2(W^2 + \widetilde{W}) + o(\varepsilon^2), \qquad U = I + \varepsilon D + \tfrac{1}{2}\varepsilon^2 \widetilde{D} + o(\varepsilon^2), \quad (4\text{-}6)$$

with $D$, $\widetilde{D}$ symmetric and $W$, $\widetilde{W}$ skew-symmetric. Then,

$$F = RU = I + \varepsilon(W + D) + \tfrac{1}{2}\varepsilon^2(W^2 + \widetilde{W} + \widetilde{D} + 2WD) + o(\varepsilon^2), \quad (4\text{-}7)$$

and therefore,

$$\begin{aligned}\varphi(F) = \nabla\varphi(I)\cdot(\varepsilon(W + D) + \tfrac{1}{2}\varepsilon^2(W^2 + \widetilde{W} + \widetilde{D} + 2WD))\\ + \tfrac{1}{2}\varepsilon^2\mathbb{C}[W + D]\cdot(W + D) + o(\varepsilon^2),\end{aligned} \quad (4\text{-}8)$$

where $\mathbb{C}$ is the fourth-order tensor

$$\mathbb{C} = \nabla^2\varphi(I), \quad (4\text{-}9)$$

which, by the interchangeability of the order of differentiation, has the symmetry property

$$\mathbb{C}[H]\cdot K = \mathbb{C}[K]\cdot H. \quad (4\text{-}10)$$

The expansion of $\varphi(U)$ is the same as (4-8), with $W = 0$. Subtracting the two expansions, from (4-5) we get

$$\nabla\varphi(I)\cdot(\varepsilon W + \varepsilon^2\widetilde{W}) + \varepsilon^2(\mathbb{C}[W] - W\nabla\varphi(I))\cdot(\tfrac{1}{2}W + D) = 0, \quad (4\text{-}11)$$

and due to the arbitrariness of $\varepsilon$, $W$, and $D$, the separate conditions

$$\begin{aligned}\nabla\varphi(I)\cdot W &= 0,\\ (\mathbb{C}[W] - W\nabla\varphi(I))\cdot W &= 0, \quad \text{for all } W \in \mathscr{W} \text{ and } D \in \mathscr{S},\\ (\mathbb{C}[W] - W\nabla\varphi(I))\cdot D &= 0,\end{aligned} \quad (4\text{-}12)$$

follow. The last two conditions merge in the single condition

$$\mathbb{C}[W] = W\nabla\varphi(I) \quad \text{for all } W \in \mathscr{W}, \quad (4\text{-}13)$$

according to which the restriction of $\mathbb{C}$ to $\mathscr{W}$ is determined by $\nabla\varphi(I)$.

For the nonsmooth potential $\phi$, for $f_t$ and $F_t$ as in (3-11) and from $(3\text{-}13)_2$ and (4-3) we have

$$\phi(F_t^d) = \phi(tL^d) + o(t) = \phi(tR_0^d D^d) + o(t), \quad (4\text{-}14)$$

and from the indifference condition $(4\text{-}4)_2$ the condition

$$\phi(F_t^d) = \phi(tD^d) + o(t) \quad (4\text{-}15)$$

follows. Then from the representations

$$\begin{aligned} \phi(F_t^d) &= t\breve{\nabla}\phi(0) \triangleright L^d + \tfrac{1}{2}t^2\breve{\nabla}^2\phi(0)\{L^d\} \triangleright L^d, \\ \phi(F_t^d) &= t\breve{\nabla}\phi(0) \triangleright D^d + \tfrac{1}{2}t^2\breve{\nabla}^2\phi(0)\{D^d\} \triangleright D^d, \end{aligned} \tag{4-16}$$

we get

$$\breve{\nabla}\phi(0) \triangleright L^d = \breve{\nabla}\phi(0) \triangleright D^d, \qquad \breve{\nabla}^2\phi(0)\{L^d\} \triangleright L^d = \breve{\nabla}^2\phi(0)\{D^d\} \triangleright D^d. \tag{4-17}$$

Since $R_0^d = I$ implies $L^d = D^d$, the indifference conditions on $\phi$ are trivially satisfied assuming $R_0^d = I$. Effective restrictions on $\phi$ follow from any other constitutive choice of $R_0^d$.

## 5. Construction of the incremental response law

The purpose of this section is to deduce the incremental response law from the plastic stationarity condition (3-22). From (3-19), (3-12)$_3$, and (3-14)$_1$ we have

$$T_0 + t\delta T = \nabla\varphi(I) + t\nabla^2\varphi(I)[L] + o(t), \tag{5-1}$$

and from (4-9), (3-13)$_3$, and (4-3) with $R_0^d = I$,

$$\delta T = \mathbb{C}[L] = \mathbb{C}[\nabla u - L^d] = \mathbb{C}[\nabla u - D^d]. \tag{5-2}$$

By comparison with the incremental stress-strain relation (3-6), we get

$$\mathbb{C}[\nabla u - D^d] = \mathfrak{h}(\nabla u). \tag{5-3}$$

This is a homogeneous map $\mathfrak{g}$ from $\nabla u$ to $D^d$

$$D^d = \mathfrak{g}(\nabla u). \tag{5-4}$$

Therefore, the determination of the incremental stress-strain relation (3-6) is reduced to the determination of the map $\mathfrak{g}$. With this goal in mind, we proceed to the characterization of the basic constitutive elements of the theory, in the order in which they are listed in the Introduction.

**5.1.** ***The yield condition.*** With the identification $\mathcal{L}^d = \mathcal{D}^d$ due to the assumption $R_0^d = I$, the plastic stationarity condition (3-22) at $t = 0$ becomes

$$T_0 \cdot \mathcal{D}^d \le \breve{\Phi}_0 \triangleright \mathcal{D}^d, \quad \breve{\Phi}_0 = \breve{\nabla}\phi(0), \tag{5-5}$$

for all $\mathcal{D}^d$ in a given set $\mathscr{S}_0^d$ of admissible plastic stretchings.[28] This inequality imposes the upper bound $\breve{\Phi}_0 \triangleright \mathcal{D}^d$ on the projection of $T_0$ in the direction of $\mathcal{D}^d$.

[28]Usually, $\mathscr{S}_0^d$ is assumed to be a cone in $\mathscr{S}$, that is, a subset of $\mathscr{S}$ such that $\mathcal{D}^d \in \mathscr{S}_0^d \implies \lambda\mathcal{D}^d \in \mathscr{S}_0^d$ for all $\lambda > 0$.

Then, $\breve{\Phi}_0$ is a *bounding map* for $T_0$. Indeed, inequality (5-5) says that for every direction $N = \mathcal{D}^d/|\mathcal{D}^d|$ in $\mathscr{S}_0^d$ the stress $T_0$ must belong to the set [29]

$$\mathcal{H}_0^N = \{T \in \mathscr{S} \mid T \cdot N \le \breve{\Phi}_0 \triangleright N\}, \tag{5-6}$$

which is the closed half-space of $\mathscr{S}$ bounded by the hyperplane with exterior unit normal $N$, placed at the distance $\breve{\Phi}_0 \triangleright N$ from the origin. Since this holds for all directions in $\mathscr{S}_0^d$, $T_0$ must belong to the intersection

$$\mathcal{C}_0 = \bigcap_{N \in \mathscr{S}_0^d,\ |N|=1} \mathcal{H}_0^N \tag{5-7}$$

of all $\mathcal{H}_0^N$. Thus, a first consequence of (3-22) is the *yield condition*[30]

$$T_0 \in \mathcal{C}_0. \tag{5-8}$$

Since all $\mathcal{H}_0^N$ are closed convex sets, their intersection $\mathcal{C}_0$ is also a closed convex set. Moreover, if $\phi$ is a dissipation potential, then $\breve{\Phi}_0 \triangleright N > 0$ by (2-15), and the inequality in (5-6) is strict at $T = 0$. That is, the null tensor is an interior point of $\mathcal{H}_0^N$. Since this holds for all $N$, the point $T = 0$ also belongs to the interior of $\mathcal{C}_0$.

The boundary of $\mathcal{C}_0$ consists of all $T_0 \in \mathscr{S}$ at which the inequality in (5-6) is satisfied as an equality:

$$T_0 \in \partial\mathcal{C}_0 \quad \Longleftrightarrow \quad \text{there is } N_0 \in \mathscr{S}_0^d \text{ such that } T_0 \cdot N_0 = \breve{\Phi}_0 \triangleright N_0. \tag{5-9}$$

By (5-6), $T_0$ also belongs to the boundary of the half-space $\mathcal{H}_0^{N_0}$. Since $\mathcal{C}_0$ is included in $\mathcal{H}_0^{N_0}$, it follows that $T \cdot N_0 \le \breve{\Phi}_0 \triangleright N_0$ for all $T \in \mathcal{C}_0$. Then subtracting from the previous equation we get

$$(T_0 - T) \cdot N_0 \ge 0 \quad \text{for all } T \in \mathcal{C}_0; \tag{5-10}$$

that is, $N_0$ belongs to the normal cone to $\mathcal{C}_0$ at $T_0$.

From (5-6) and (5-7) we see that $\mathcal{C}_0$ is determined by $\phi$. Conversely, for a given $\mathcal{C}_0$ the relation (5-10) associates with every direction $N_0$ a (possibly nonunique)

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[29] We recall that $T_0 = \nabla\varphi(I)$ is a symmetric tensor by the indifference condition $(4\text{-}12)_1$.

[30] In the terminology of convex analysis, inequality (5-2) says that the map $\breve{\Phi}_0$ is subdifferentiable at $\mathcal{D}^d = 0$ and $T_0$ belongs to the subdifferential of $\breve{\Phi}_0$ at 0. Moreover, if $\mathscr{S}_0^d$ is a proper subset of $\mathscr{S}$, then $\mathcal{C}_0$ is unbounded, and $\mathscr{S}_0^d$ and the recession cone of $\mathcal{C}_0$ are polar to each other [Rockafellar 1970, Theorem 14.6]. In particular, if $\mathscr{S}_0^d$ is a subspace, then the recession cone of $\mathcal{C}_0$ is its orthogonal complement. Equation (5-11) below also says that $\breve{\Phi}_0$ is the support function, that is, the conjugate of the indicator function, of $\mathcal{C}_0$; see [Eve et al. 1990] or [Rockafellar 1970, Theorem 13.2]. Though the formalism of convex analysis fully captures the mathematical structure of classical plasticity, I prefer to keep the present exposition at a more elementary level.

boundary point $T_0$ of $\mathcal{C}_0$ such that

$$T_0 \cdot N_0 = \sup_{T \in \mathcal{C}_0} T \cdot N_0. \tag{5-11}$$

Then $\breve{\Phi}_0 \triangleright N_0$ is specified by (5-9). Since this can be repeated for all directions $N$, the whole restriction of $\breve{\Phi}_0$ to $\mathscr{S}_0^d$ is specified in this way. Thus, there is a one-to-one correspondence between the closed convex sets $\mathcal{C}_0$ of $\mathscr{S}$ and the homogeneous maps $\breve{\Phi}_0$ on $\mathscr{S}_0^d$.[31] On the contrary, the correspondence between directions $N$ in $\mathscr{S}_0^d$ and boundary points of $\mathcal{C}_0$ need not be one-to-one.[32]

**5.2. *The flow rule.*** At $t = 0$ and for $R_0^d = I$, $(4\text{-}17)_1$ holds and (3-23) takes the form

$$T_0 \cdot D^d = \breve{\Phi}_0 \triangleright D^d. \tag{5-12}$$

By consequence, a nonnull plastic stretching may occur only if $T_0$ is a boundary point of $\mathcal{C}_0$. The *flow rule* is a law prescribing the direction of $D^d$ at each boundary point. For example, the *associated flow rule* states that $D^d$ belongs to the normal cone of $\mathcal{C}_0$ at $T_0$. This is indeed the rule provided by the variational approach, since (5-12) says precisely that $D^d$ belongs to the normal cone at $T_0$. Calling $N_0$ the unit tensor $D^d/|D^d|$, the *normality rule*

$$D^d = \begin{cases} \lambda^d N_0,\ \lambda^d \geq 0 & \text{if } T_0 \in \partial\mathcal{C}_0, \\ 0 & \text{if } T_0 \in \mathcal{C}_0 \setminus \partial\mathcal{C}_0 \end{cases} \tag{5-13}$$

is obtained. The fact that $D^d$ is zero at the interior point justifies the name of *elastic range* given to $\mathcal{C}_0$.

The product $T_0 \cdot D^d$ is the *plastic power*. According to (5-11), we have

$$(T_0 - T) \cdot N_0 \geq 0 \quad \text{for all } T \in \mathcal{C}_0. \tag{5-14}$$

This is Hill's *principle of maximum plastic work*.[33] The inequality is strict for $T = 0$, because the origin of $\mathscr{S}$ is an interior point of $\mathcal{C}_0$. Then the strict inequality

$$T_0 \cdot D^d > 0 \tag{5-15}$$

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[31] If $\phi$ is a homogeneous map, then $\breve{\Phi}_0 \triangleright N$ is equal to $\phi(N)$ by $(2\text{-}17)_1$. Therefore, there is a one-to-one correspondence between closed convex sets and homogeneous dissipation potentials. Eve et al. [1990] called *canonical yield function* the homogeneous dissipation potential associated with $\mathcal{C}_0$.

[32] For example, if $\mathcal{C}_0$ is polyhedral, the normal to a face is a normal at the infinitely many points which belong to that face, and at each point on an edge or vertex of $\mathcal{C}_0$ there are the infinitely many normals which form the normal cone at that point.

[33] According to Srinivasa [2010], the existence and convexity of the elastic range is a consequence of Ziegler's *principle of maximum rate of dissipation* [1963], of which Hill's principle is a special case. That the elastic range is a derived concept was also pointed out by Martin and Reddy [1993]. In the present context, elastic range and Hill's principle are both derived concepts, since both are consequences of the plastic stationarity condition.

holds. This is the *dissipation inequality* which establishes the dissipative character of the plastic power $T_0 \cdot D^d$.

**5.3.** ***The consistency condition.*** Equation (3-23) states that, in the limit $t \to 0^+$, the stress $T_t$ and the bounding map $\breve{\nabla}\phi(F_t^d)$ have the same projection in the direction of the plastic stretching $D^d$. This is the *consistency condition*, which says that for $t \to 0^+$ and $D^d \neq 0$ the stress $T_t$ is a boundary point of the elastic range $\mathcal{C}_t$ at $t$ [Prager 1949].[34]

With the identification $L^d = D^d$ due to the assumption $R_0^d = I$, for $\mathcal{L}^d = L^d$ from (3-23) and (5-12) we have

$$\delta T \cdot D^d = \breve{\mathbb{K}}\{D^d\} \triangleright D^d, \quad \breve{\mathbb{K}} = \breve{\nabla}^2\phi(0), \tag{5-16}$$

and since $D^d = \lambda^d N_0$ we get

$$\lambda^d(\delta T \cdot N_0 - \lambda^d \breve{\mathbb{K}}\{N_0\} \triangleright N_0) = 0. \tag{5-17}$$

This *complementarity condition*, which states that either $\lambda^d$ or the term within parentheses is zero, is the mathematical form taken by the consistency condition.

**5.4.** ***The incremental response law.*** Due to the normality rule (5-13), the incremental relation (5-2) takes the form

$$\delta T = \mathbb{C}[\nabla u] - \lambda^d \mathbb{C}[N_0]. \tag{5-18}$$

On the right-hand side, only the *plastic multiplier* $\lambda^d$ is unknown. For its determination we have at our disposal the inequality $\lambda^d \geq 0$ and the complementarity condition (5-17). Together with (5-18) multiplied by $N_0$,

$$\delta T \cdot N_0 = \mathbb{C}[\nabla u] \cdot N_0 - \lambda^d \mathbb{C}[N_0] \cdot N_0, \tag{5-19}$$

they form a system in the unknowns $\delta T \cdot N_0$ and $\lambda^d$. If $\mathbb{C}[\nabla u] \cdot N_0 < 0$, the last equation implies $\delta T \cdot N_0 < 0$, and then $\lambda^d = 0$ by the complementarity condition. If $\lambda^d > 0$, from (5-17) and (5-19) by elimination of $\delta T \cdot N_0$ we get

$$\lambda^d(\mathbb{C}[N_0] \cdot N_0 + \breve{\mathbb{K}}\{N_0\} \triangleright N_0) = \mathbb{C}[\nabla u] \cdot N_0. \tag{5-20}$$

Then, assuming

$$\mathbb{C}[N_0] \cdot N_0 + \breve{\mathbb{K}}\{N_0\} \triangleright N_0 > 0, \tag{5-21}$$

it follows that $\mathbb{C}[\nabla u] \cdot N_0 \geq 0$ for $\lambda^d \geq 0$. Then the problem has the unique solution

$$\lambda^d(N_0) = \frac{\langle \mathbb{C}[N_0] \cdot \nabla u \rangle}{\mathbb{C}[N_0] \cdot N_0 + \breve{\mathbb{K}}\{N_0\} \triangleright N_0}, \tag{5-22}$$

[34] Since $T_t$ is not symmetric in general, $\mathcal{C}_t$ is not, in general, a subset of $\mathscr{S}$. The symmetry of $T_t$ and the inclusion of $\mathcal{C}_t$ in $\mathscr{S}$ can be recovered by a change of reference placement, taking at each $t$ the current placement as reference placement.

where $\langle\alpha\rangle = \max\{\alpha, 0\}$ denotes the positive part of a real number $\alpha$, and $\mathbb{C}[\nabla u]\cdot N_0 = \mathbb{C}[N_0]\cdot\nabla u$ by the symmetry property (4-10). This is a homogeneous relation between $\lambda^d(N_0)$ and $\nabla u$, and its substitution into (5-18) provides the desired incremental law (3-6).

**5.5.** ***Elastic unloading.*** A strain rate $\nabla u$ is said to determine a regime of

$$\begin{cases} \textit{loading} & \text{if } \mathbb{C}[N_0]\cdot\nabla u > 0, \\ \textit{unloading} & \text{if } \mathbb{C}[N_0]\cdot\nabla u < 0, \\ \textit{neutral loading} & \text{if } \mathbb{C}[N_0]\cdot\nabla u = 0. \end{cases} \tag{5-23}$$

By (5-22), the response law can be split into two linear relations, one for loading and one for unloading. At loading, from (5-22) we have

$$\delta T = \mathbb{L}[\nabla u], \tag{5-24}$$

with

$$\mathbb{L} = \mathbb{C} - \frac{\mathbb{C}[N_0]\otimes\mathbb{C}[N_0]}{\mathbb{C}[N_0]\cdot N_0 + \breve{\mathbb{K}}\{N_0\}\triangleright N_0}, \tag{5-25}$$

and at unloading we have

$$\delta T = \mathbb{C}[\nabla u]. \tag{5-26}$$

The latter is an *elastic* law, corresponding to $\lambda^d = 0$. It describes the phenomenon of *elastic unloading*, typical of plasticity.

**5.6.** ***The hardening rule.*** In the loading regime, for $\lambda^d > 0$ and for sufficiently small $t > 0$, the stress $T_t = T_0 + t\delta T + o(t)$ is placed outside, inside, or on the boundary of the elastic range $\mathcal{C}_0$, depending on the sign of the product $\delta T\cdot N_0$. Since by the consistency condition $T_t$ is a boundary point of the elastic range $\mathcal{C}_t$, the three possible locations of $T_t$ correspond to an enlargement, to a contraction, or to invariance of the elastic range at $T_0$. In the three cases we say that the response at loading is *hardening, softening*, and *perfectly plastic*, respectively. From (5-24) and (5-25) we get the *hardening rule*[35]

$$\delta T\cdot N_0 = \mathbb{L}[\nabla u]\cdot N_0 = h(N_0)\mathbb{C}[\nabla u]\cdot N_0, \tag{5-27}$$

where

$$h(N_0) = \frac{\breve{\mathbb{K}}\{N_0\}\triangleright N_0}{\mathbb{C}[N_0]\cdot N_0 + \breve{\mathbb{K}}\{N_0\}\triangleright N_0} \tag{5-28}$$

is the *hardening modulus*. Note that $h(N_0) < 1$ if $\mathbb{C}$ restricted to $\mathscr{S}$ is positive definite and that, the denominator being positive by assumption (5-21), $h(N_0)$ has the same sign of $\breve{\mathbb{K}}\{N_0\}\triangleright N_0$. Then in the loading regime, in which $\mathbb{C}[\nabla u]\cdot N_0$

[35] See, e.g., [Lemaitre and Chaboche 1990; Lubliner 1990].

is positive, the response in the direction $N_0$ is hardening, softening, or perfectly plastic if $h(N_0)$ is positive, negative, or zero, respectively.

## 6. Some models of classical plasticity

After establishing the incremental response law, we now show how the models of classical plasticity can be obtained taking particular forms of the plastic energy density $\phi$. Throughout this section we assume $R_0^d = I$, that is, $L^d = D^d$.

**6.1.** ***The three basic models.*** We assume that the elastic energy density $\varphi$ is a smooth potential and that the plastic energy density $\phi$ is a nonsmooth potential from $F^d = 0$. We consider three basic models of classical plasticity: perfect, kinematic, and dilatational plasticity, which correspond to the special forms of $\phi$

$$\begin{cases} \textit{perfect plasticity} & \phi = \phi^h, \\ \textit{kinematic plasticity} & \phi = \psi + \phi^h, \\ \textit{dilatational plasticity} & \phi = \phi^{nh}, \end{cases} \tag{6-1}$$

with $\phi^h$ a homogeneous dissipation potential, $\phi^{nh}$ a nonhomogeneous dissipation potential, and $\psi$ a smooth potential.

**6.1.1.** *Perfect plasticity.* If $\phi^h$ is a homogeneous potential from $A = 0$, from (2-17)$_1$ we have $\phi^h(\mathcal{D}^d) = \breve{\nabla}\phi^h(0) \triangleright \mathcal{D}^d$, and inequality (5-5) reduces to

$$T_0 \cdot \mathcal{D}^d \leq \phi^h(\mathcal{D}^d). \tag{6-2}$$

Then the boundary of each generating half-space $\mathcal{H}_0^N$ is placed at the distance $\phi^h(N)$ from the origin. Moreover, from (2-17)$_2$ we have

$$\breve{\mathbb{K}}\{N\} \triangleright N = 0; \tag{6-3}$$

that is, the hardening modulus (5-28) is zero. The hardening rule (5-27) then reduces to

$$\delta T \cdot N_0 = 0, \tag{6-4}$$

in agreement with the definition of a perfectly plastic response given in the preceding section. Therefore,

$$T_t \cdot N_0 = (T_0 + t\delta T) \cdot N_0 = T_0 \cdot N_0. \tag{6-5}$$

Since $T_0$ is a boundary point of $\mathcal{C}_0$ and $T_t$ is a boundary point of $\mathcal{C}_t$ by the consistency condition, this equation shows that for every direction $N_0$ in $\mathscr{S}_0^d$ the generating half-space $\mathcal{H}^{N_0}$ for $\mathcal{C}_0$ is also a generating half-space for the projection $\mathcal{C}_t^S$ of $\mathcal{C}_t$ on $\mathscr{S}$.[36] Then $\mathcal{C}_0$ and $\mathcal{C}_t^S$ have the same generating half-spaces; that is, $\mathcal{C}_t^S = \mathcal{C}_0$.

**6.1.2.** *Kinematic plasticity.* If $\phi$ is the sum of a smooth potential $\psi$ and a homogeneous dissipation potential $\phi^h$, from (4-15) we have

$$\phi(F_t^d) = \psi(tD^d) + \phi^h(tD^d) + o(t). \tag{6-6}$$

Let us use the differentiability of $\psi$ to introduce the tensor

$$T_{Bt} = \nabla\psi(tD^d), \tag{6-7}$$

for which, from the power expansions

$$T_{Bt} = T_{B0} + t\delta T_B + o(t), \qquad \nabla\psi(tD^d) = \nabla\psi(0) + t\nabla^2\psi(0)[D^d] + o(t), \tag{6-8}$$

we have

$$T_{B0} = \nabla\psi(0), \qquad \delta T_B = \mathbb{D}[D^d], \quad \mathbb{D} = \nabla^2\psi(0). \tag{6-9}$$

Inequality (5-5) then takes the form

$$(T_0 - T_{B0}) \cdot N \le \phi^h(N) \quad \text{for all } N \in \mathscr{S}_0^d,\ |N| = 1. \tag{6-10}$$

This tells us that the boundary of each generating half-space $\mathcal{H}_0^N$ of $\mathcal{C}_0$ is placed at the distance $\phi^h(N)$ from the projection $T_{B0}^S$ of $T_{B0}$ on $\mathscr{S}$. This distance is the same as in perfect plasticity, but now is measured from the point $T_{B0}^S$ and not from the origin. Moreover, since in the identities

$$\breve{\mathbb{K}}\{N_0\} \triangleright N_0 = \breve{\nabla}^2\phi(0)\{N_0\} \triangleright N_0 = \mathbb{D}[N_0] \cdot N_0 + \breve{\nabla}^2\phi^h(0)\{N_0\} \triangleright N_0, \tag{6-11}$$

the last term is zero by $(2\text{-}17)_2$, the plastic multiplier (5-22) and the hardening modulus (5-28) take the form

$$\lambda^d(N_0) = \frac{\mathbb{C}[\nabla u] \cdot N_0}{(\mathbb{C} + \mathbb{D})[N_0] \cdot N_0}, \qquad h(N_0) = \frac{\mathbb{D}[N_0] \cdot N_0}{(\mathbb{C} + \mathbb{D})[N_0] \cdot N_0}. \tag{6-12}$$

Then, from (5-27) and $(6\text{-}9)_2$,

$$(\delta T - \delta T_B) \cdot N_0 = h(N_0)\, \mathbb{C}[\nabla u] \cdot N_0 - \lambda^d(N_0)\, \mathbb{D}[N_0] \cdot N_0 = 0. \tag{6-13}$$

This is the same relation (6-4) of perfect plasticity with $T_t$ replaced by $(T_t - T_{Bt})$. It says that the generating half-spaces for $\mathcal{C}_t^S$ are the generating half-spaces $\mathcal{H}^{N_0}$ for $\mathcal{C}_0$ translated of $\delta T_B^S$, that is, that $\mathcal{C}_t^S = \mathcal{C}_0 + \{\delta T_B^S\}$. The fact that the translation of the

[36] We recall that, in general, $\mathcal{C}_t$ is not included in $\mathscr{S}$; see Footnote 34. Here and in the following, the superscript $S$ denotes the projection on $\mathscr{S}$ symmetric part of a tensor. In particular, $A^S$ denotes the symmetric part of a tensor $A$.

elastic range is controlled by the translation of $T_B$ motivates the name *backstress tensor* attributed to $T_B$.

In kinematic plasticity, the response in a direction $N_0$ is hardening, softening, or perfectly plastic depending on the sign of $h(N_0)$. For $h(N_0)$ positive, we have Prager's *kinematic hardening* model [1955].[37]

**6.1.3.** *Dilatational plasticity.* As discussed in Section 5.6, to a positive hardening modulus corresponds an enlargement of the elastic range

$$h(N_0) > 0 \quad \text{for all } N_0 \in \mathscr{S}^d \quad \Longrightarrow \quad \mathcal{C}_0 \subset \mathcal{C}_t^S. \tag{6-14}$$

By (5-21), under assumption (5-28), $h(N_0)$ is positive if $\breve{\mathbb{K}}\{N_0\} \triangleright N_0$ is positive. If $\phi$ is a nonhomogeneous dissipation potential $\phi^{nh}$, then $\breve{\mathbb{K}} = \breve{\nabla}^2\phi^{nh}(0)$. Therefore, a hardening response occurs in all directions $N_0$ if $\phi^{nh}$ is strictly convex. *Isotropic hardening* is the special case of $h(N_0)$ independent of $N_0$.

If $h(N_0)$ is positive at $t = 0$, it remains positive for sufficiently small $t$. For all such $t$ there is no way of producing a contraction, that is, the expansion of the elastic range is irreversible.

In conclusion, comparing $\mathcal{C}_t^S$ and $\mathcal{C}_0$ we see that a homogeneous dissipation potential leaves $\mathcal{C}_0$ unchanged, a smooth potential produces a rigid translation, and a nonhomogeneous dissipation potential produces a dilatation or a contraction, depending on the sign of the hardening modulus. More complex evolutions of the elastic range can be described by combinations of the potentials considered here.[38]

**6.2.** ***Isochoric plasticity.*** The experiments show that for many materials the plastic strain rate is practically isochoric. The presence of this internal constraint modifies the form of the energy, and this requires a nontrivial reformulation of the theory.

**6.2.1.** *The isochoricity constraint.* We still consider two-scale deformations $(f, F)$ and take the difference $(\nabla f - F)$ as the plastic deformation $F^d$. *Isochoricity* is the assumption that the volume changes in the macroscopic and microscopic deformations are the same:

$$\det F = \det \nabla f. \tag{6-15}$$

In a deformation process $t \mapsto (f_t, F_t)$ from $(\imath, I)$, from the expansions (3-11),

$$\det \nabla f_t = 1 + tI \cdot \nabla u + o(t), \qquad \det F_t = 1 + tI \cdot L + o(t), \tag{6-16}$$

[37]The dependence of the backstress tensor on the differentiable part of the plastic strain energy was pointed out by Aifantis [1987]. See also [Anand and Gurtin 2003].

[38]In [Gurtin et al. 2010, p. 421], it has been pointed out that "for many metals, the actual strain-hardening behavior...may be approximated by a combination of nonlinear isotropic hardening and nonlinear kinematic hardening." In fact, the possibility of more general combinations of kinematic and dilatational plasticity emerges from the present analysis.

and since $\nabla u - L = L^d$, isochoricity results in the constraint

$$I \cdot L^d = 0. \tag{6-17}$$

Consider the decomposition of a second-order tensor $A$ into the sum of a hydrostatic and a deviatoric part:

$$A = A^H + A^D, \quad A^H = \tfrac{1}{3}(I \cdot A)I, \quad I \cdot A^D = 0. \tag{6-18}$$

They are the perpendicular projections of $A$ on the *hydrostatic axis* and on the *deviatoric hyperplane*,

$$\mathscr{L}^H = \{A \in \mathscr{L} \mid A = \alpha I,\ \alpha \in \mathbb{R}\}, \qquad \mathscr{L}^D = \{A \in \mathscr{L} \mid I \cdot A = 0\}, \tag{6-19}$$

respectively. Then the constraint (6-17) can be written in any of the equivalent forms

$$L^d \in \mathcal{L}^D, \qquad L^d = L^{dD}, \qquad L^{dH} = 0. \tag{6-20}$$

**6.2.2.** *The plastic strain energy.* The elastic energy density $\varphi$ is a function of the variable $F$, which is not restricted by the constraint (6-17). Then $\varphi$ is not constrained as well. On the contrary, the plastic energy density $\tilde{\phi}$ depends on the plastic strain rate $L^d$, which is now assumed to be deviatoric. It is then appropriate to assume a dependence on the part of the stress which does no work in any deformation process which satisfies the constraint, that is, on the hydrostatic stress $T^H$. Here we assume the separate dependence

$$\tilde{\phi}(L^d, p) = \phi(L^d)\psi(p), \quad L^d \in \mathscr{L}^D,\ p \in \mathbb{R}, \tag{6-21}$$

where $\phi$ is a dissipation potential, $\psi$ is a smooth potential, and $p$ is the *hydrostatic pressure*

$$p = -\tfrac{1}{3} I \cdot T, \tag{6-22}$$

related to the hydrostatic stress by

$$T^H = -pI. \tag{6-23}$$

**6.2.3.** *The plastic stationarity condition.* At each point $x$, consider a deformation process $t \mapsto (f_t, F_t)$ from the reference placement, and a pressure process $t \mapsto p_t$. To minimize the total energy at the time $t$, take a family $\varepsilon \mapsto (\varepsilon v, \varepsilon \mathcal{L})$ of perturbations and consider the perturbed process $t \mapsto (f_\varepsilon, F_\varepsilon)$, with $f_\varepsilon$ and $F_\varepsilon$ as in (3-15). The perturbed plastic energy density is $\phi(F^d_\varepsilon)\psi(p_t)$, with $p_t$ the pressure at $t$ and

$$\phi(F^d_\varepsilon) = \phi(F^d_t) + \varepsilon \check{\nabla}\phi(F^d_t) \triangleright \mathcal{L}^d. \tag{6-24}$$

With the expansions (3-11) of $f_t$ and $F_t$, the perturbed total energy takes the form

$$\mathcal{E}_{\mathrm{tot}}(f_\varepsilon, F_\varepsilon, p_t) = \mathcal{E}_{\mathrm{tot}}(f_t, F_t, p_t) + \varepsilon \int_{\Omega_0} (\nabla\varphi(F_t)\cdot\mathcal{L} + \psi(p_t)\breve{\nabla}\phi(F_t^d) \triangleright \mathcal{L}^p)\, dV_0 \\ - \varepsilon\Big(\int_{\Omega_0} b_t\cdot v\, dV_0 + \int_{\partial\Omega_0} s_t \cdot v\, dA_0\Big) + o(\varepsilon). \quad (6\text{-}25)$$

From the stationarity condition (3-17), for $\mathcal{L}^d = 0$ we have $\mathcal{L} = \nabla v$, and the identification (3-19) of $\nabla\varphi(F_t)$ with the Piola stress $T_t$ follows. Moreover, by elimination of all terms in $v$ and subsequent localization, after recalling that $R_0^d = I$ implies $L^d = D^d$ we get the counterpart of the plastic stationarity condition (3-22),

$$T_t^D \cdot \mathcal{D}^d \le \psi(p_t)\breve{\nabla}\phi(tD^d) \triangleright \mathcal{D}^d, \quad (6\text{-}26)$$

to be satisfied by all $\mathcal{D}^d$ belonging to a given subset $\mathscr{S}_0^d$ of $\mathscr{S}^D$. With the same motivation used to establish (3-23) in the unconstrained case, for $\mathcal{D}^d = D^d$ inequality (6-26) reduces to the equality

$$T_t^D \cdot D^d = \psi(p_t)\breve{\nabla}\phi(tD^d) \triangleright D^d. \quad (6\text{-}27)$$

**6.2.4.** *The evolution law for the hydrostatic pressure.* From (3-19) we still have the incremental stress-strain relation (5-2), with $\mathbb{C}$ the elastic tensor $\nabla^2\varphi(I)$. For this tensor, the identities

$$\mathbb{C}[A] = \mathbb{C}[A^H + A^D] = (\mathbb{C}[A])^H + (\mathbb{C}[A])^D \quad (6\text{-}28)$$

hold for all $A$ in $\mathscr{L}$. For simplicity, we focus on the special case in which $\mathbb{C}$ maps the hydrostatic tensors into hydrostatic tensors[39]

$$\mathbb{C}[A^H] = (\mathbb{C}[A])^H. \quad (6\text{-}29)$$

Then (6-28) implies $\mathbb{C}[A^D] = (\mathbb{C}[A])^D$; that is, $\mathbb{C}$ also maps the deviators into deviators. Since the hydrostatic tensors are those parallel to the identity tensor $I$, from (6-29) and the linearity of $\mathbb{C}$ it follows that

$$\mathbb{C}[A^H] = 3kA^H, \quad (6\text{-}30)$$

with $k$ a positive material constant, called the *bulk modulus*.

For $N_0 \in \mathscr{S}^D$, in the incremental stress-strain relation (5-18) we have

$$\delta T = \mathbb{C}[\nabla u^H] + \mathbb{C}[\nabla u^D - \lambda^d N_0] = 3k\nabla u^H + \mathbb{C}[\nabla u^D - \lambda^d N_0], \quad (6\text{-}31)$$

and by effect of assumption (6-29) this equation splits into two parts

$$\delta T^H = 3k\nabla u^H, \qquad \delta T^D = \mathbb{C}[\nabla u^D - \lambda^d N_0]. \quad (6\text{-}32)$$

[39]This relation is satisfied by all orthotropic materials with cubic symmetry, and in particular by all isotropic materials; see, e.g., [Gurtin 1972, §26].

For the hydrostatic part, recalling the definition (6-22) of the hydrostatic pressure,

$$\delta p = -\tfrac{1}{3} I \cdot \delta T = -\tfrac{1}{3} I \cdot \delta T^H = -kI \cdot \nabla u^H = -kI \cdot \nabla u. \tag{6-33}$$

This is the evolution law for the hydrostatic pressure, according to which $p$ is determined by the hydrostatic part of the macroscopic deformation. We now proceed to the determination of the evolution law for the deviatoric stress.

**6.2.5.** *Yield conditions, elastic ranges, and admissible stresses.* At $t = 0$, from (6-26) we have

$$T_0^D \cdot \mathcal{D}^d \le \psi(p_0)\breve{\Phi}_0 \triangleright \mathcal{D}^d, \tag{6-34}$$

with $\breve{\Phi}_0$ as in (5-5). This inequality says that $T_0^D$, which is symmetric by the indifference condition (4-12)$_1$, belongs to a family of closed half-spaces $\mathcal{H}_{p_0}^N$ of $\mathscr{S}^D$, with normals $N$ in $\mathscr{S}_0^d$ and with distance from the origin proportional to $\psi(p_0)$. The intersection $\mathcal{C}_{p_0} = \psi(p_0)\mathcal{C}_0$ of all $\mathcal{H}_{p_0}^N$ is the *elastic range associated with the pressure* $p_0$, and the condition

$$T_0^D \in \psi(p_0)\, \mathcal{C}_0 \tag{6-35}$$

is the *yield condition for the pressure* $p_0$. Thus, with the initial deformation $(\iota, I)$ is associated a family of elastic ranges $p \mapsto \mathcal{C}_p = \psi(p)\, \mathcal{C}_0$, in which each $\mathcal{C}_p$ is a homothetic transformation of $\mathcal{C}_0$ with center at the origin and ratio $\psi(p)$.

In the space $\mathscr{S}$, each $\mathcal{C}_p$ belongs to the hyperplane lying at the (signed) distance $p$ from $\mathscr{S}^D$. Then the $\mathcal{C}_p$ are pairwise disjoint subsets of $\mathscr{S}$. Their union $\mathcal{K}_0$ is the *set of all admissible stresses* for the initial deformation.[40]

**6.2.6.** *Flow rule and consistency condition.* At $t = 0$, (6-27) reduces to

$$T_0^D \cdot D^d = \psi(p_0)\breve{\Phi}_0 \triangleright D^d. \tag{6-36}$$

Since both $D^d$ and $\mathcal{C}_{p_0}$ are in the space of the symmetric deviators, this equation tells us that $T_0^D$ is a boundary point of $\mathcal{C}_{p_0}$ relative to this space, and that $D^d$ belongs to the normal cone of $\mathcal{C}_{p_0}$ at $T_0^D$. Thus, the normality rule

$$D^p = \begin{cases} \lambda^d N_0,\ \lambda^d \ge 0 & \text{if } T_0^D \in \partial\mathcal{C}_{p_0}, \\ 0 & \text{if } T_0^D \in \mathcal{C}_{p_0} \setminus \partial\mathcal{C}_{p_0} \end{cases} \tag{6-37}$$

is established.

[40]The idea of considering as the set of admissible stresses a one-parameter family of elastic ranges is not new. Families of yield regions enclosed by a *bounding surface* [Dafalias and Popov 1975] were used to describe the work-hardening of metals [Phillips and Sierakowski 1965] and their response to cyclic loading [Mróz 1969]. Later, the same idea was applied to geomaterials by Vermeer and de Borst [1984] and by Roscoe and coworkers [Roscoe and Burland 1968; Roscoe and Poorooshasb 1963]. See also [Ziegler and Wehrli 1987, p. 223].

According to this rule, the plastic strain rate is zero at all interior points of $\mathcal{C}_{p_0}$. Since the interior points of $\mathcal{C}_{p_0}$ are also interior points of the set $\mathcal{K}_0$ of the admissible stresses, one is tempted to consider $\mathcal{K}_0$ as the real elastic range. This choice, which has been actually made in several plasticity models, is incompatible with the normality law, because the direction $N_0$ is normal to $\mathcal{C}_{p_0}$ but not, in general, to $\mathcal{K}_0$. This inconvenience has been circumvented by rejecting the normality law, that is, by adopting a nonassociated flow rule. Here we have shown that this choice contrasts with the result provided by energy minimization.[41]

For $\delta p \neq 0$, $D^p$ cannot be zero because (6-27) must be satisfied at $t > 0$. From this equation, with the expansions

$$\begin{aligned} T_t^D &= T_0^D + t\delta T^D + o(t), \\ \breve{\nabla}\phi(tD^p) &= \breve{\Phi}_0 + t\,\breve{\mathbb{K}}\{D^p\} + o(t), \\ \psi(p_t) &= \psi(p_0) + \psi'(p_0)\,\delta p + o(t), \end{aligned} \tag{6-38}$$

we get the separate conditions

$$\begin{aligned} T_0^D \cdot D^d &= \psi(p_0)\breve{\Phi}_0 \triangleright D^d, \\ \delta T^D \cdot D^d &= \psi(p_0)\breve{\mathbb{K}}\{D^d\} \triangleright D^d + \psi'(p_0)\,\delta p \breve{\Phi}_0 \triangleright D^p. \end{aligned} \tag{6-39}$$

The first equation is (6-34). Substituting it into the second equation, recalling the normality rule $D^d = \lambda^d N_0$ and setting

$$\kappa(p_0) = \psi'(p_0)/\psi(p_0), \tag{6-40}$$

we get the *consistency condition for isochoric plasticity*

$$\lambda^d\big(\delta T^D \cdot N_0 - \lambda^d \psi(p_0)\breve{\mathbb{K}}\{N_0\} \triangleright N_0 - \kappa(p_0)(T_0^D \cdot N_0)\,\delta p\big) = 0. \tag{6-41}$$

**6.2.7.** *The incremental response law and the hardening rule.* Comparing the consistency condition with $(6\text{-}32)_2$ multiplied by $N_0$ and recalling the expression (6-33) of $\delta p$, we find

$$\lambda^d(N_0, p_0) = \frac{\langle \mathbb{C}[N_0] \cdot \nabla u^D \rangle - \kappa(p_0)(T_0^D \cdot N_0)\,\delta p}{\mathbb{C}[N_0] \cdot N_0 + \psi(p_0)\breve{\mathbb{K}}\{N_0\} \triangleright N_0}. \tag{6-42}$$

[41] Another inconvenience of the nonassociated flow rules emerges from a comparison between isochoric plasticity and kinematic hardening. In both cases there is a family $t \mapsto \mathcal{C}_p$ of elastic ranges, and their union is the set $\mathcal{K}_0$ of the admissible stresses. However, in kinematic hardening the regions $\mathcal{C}_p$ are not pairwise disjoint. Then a boundary point of a region can be an interior point of another region, and therefore of $\mathcal{K}_0$. It is then inconceivable to restrict the plastic stretching to the boundary points of $\mathcal{K}_0$, since a nonnull plastic stretching is allowed at the boundary points of all $\mathcal{C}_p$. On the contrary, this is perfectly conceivable in isochoric plasticity, in which the boundary points of all $\mathcal{C}_p$ are also boundary points of $\mathcal{K}_0$. Thus, the choice of a nonassociated flow rule is possible in the second case but not in the first. Since there is no reason for choosing different flow rules in the two cases, the motivations for choosing nonassociated flow rules in the second case are not clear.

Then the incremental response law is (6-32)$_2$ with $\lambda^d$ as above.

At loading, $\mathbb{C}[N_0]\cdot\nabla u^D > 0$, the response is *hardening*, *softening*, or *perfectly plastic* if the product $\delta T^D\cdot N_0$ is positive, negative, or zero, respectively. From (6-41) and (6-42) we have

$$\delta T^D\cdot N_0 = h(N_0, p_0)\mathbb{C}[N_0]\cdot\nabla u^D + (1-h(N_0, p_0))\kappa(p_0)(T_0^D\cdot N_0)\delta p, \quad (6\text{-}43)$$

with the *hardening modulus*

$$h(N_0, p_0) = \frac{\psi(p_0)\,\breve{\mathbb{K}}\{N_0\}\triangleright N_0}{\mathbb{C}[N_0]\cdot N_0 + \psi(p_0)\,\breve{\mathbb{K}}\{N_0\}\triangleright N_0}. \quad (6\text{-}44)$$

Note that, assuming both the denominator and $\mathbb{C}[N_0]\cdot N_0$ are positive, we have $h(N_0, p_0) < 1$.

From (6-43) we see that $h(N_0, p_0)$ measures the evolution of the elastic range $\mathcal{C}_{p_0}$ at constant pressure, that is, under a purely deviatoric loading $\mathbb{C}[N_0]\cdot\nabla u^D > 0$. In this case the sign of $h(N_0, p_0)$, which assuming $\psi(p_0) > 0$ is the same of the sign of $\breve{\mathbb{K}}\{N_0\}\triangleright N_0$, is positive, negative, or zero when $\mathcal{C}_{p_0}$ expands, shrinks, or remains unchanged. In the three cases the response is hardening, softening, or perfectly plastic, respectively.

Under a purely hydrostatic loading, that is, for $\nabla u^D = 0$, from (6-43) we see that the pressure change $\delta p$ determines a change $\delta T^D$ of the deviatoric stress, such that

$$\delta T^D\cdot N_0 = (1-h(N_0, p_0))\,\kappa_0(p_0)(T_0^D\cdot N_0)\delta p. \quad (6\text{-}45)$$

This is the change of $T^D$ required to keep the total stress $T$ on the boundary of the elastic range during its evolution from $\mathcal{C}_{p_0}$ to $\mathcal{C}_{p_0+\delta p}$. Since $h(N_0, p_0) < 1$ and $T_0^D\cdot N_0 > 0$ by the dissipation inequality (5-15), $\delta T^D\cdot N_0$ has the same sign of $\delta p$ if $\kappa_0(p_0) > 0$ and the opposite sign if $\kappa_0(p_0) < 0$. In the first case, a positive $\delta p$ produces a dilatation of the elastic range, and in the second case it produces a contraction.

**6.3.** ***Isotropic isochoric plasticity.*** A special case of isochoric plasticity is the case in which the plastic energy density (6-21) depends only on the modulus of $L^d$:

$$\tilde{\phi}(tL^d, p_t) = \phi(t|L^d|)\psi(p_t). \quad (6\text{-}46)$$

The linearized forms of this energy corresponding to the Drucker–Prager and to the Mises yield conditions are helpful for understanding some basic differences between the plastic behaviors of metals and of geomaterials.

**6.3.1.** *The yield condition.* For an energy of the form (6-46) and with $\phi$ such that $\breve{\Phi}_0\triangleright\mathcal{D}^d = |\mathcal{D}^d|$, the plastic stationarity condition (6-34) reduces to

$$T_0^D\cdot\mathcal{D}^p \le \psi(p_0)|\mathcal{D}^p|, \quad (6\text{-}47)$$

and for any direction $N = \mathcal{D}^p/|\mathcal{D}^p|$ in $\mathscr{S}^{dD}$ we have

$$T_0^D \cdot N \leq \psi(p_0). \tag{6-48}$$

This inequality tells us that the generating half-spaces $\mathcal{H}_{p_0}^N$ of the elastic range $\mathcal{C}_{p_0}$ are all placed at the same distance $\psi(p_0)$ from the origin. If $\mathscr{S}^{dD} = \mathscr{S}^D$, then $\mathcal{C}_{p_0}$ is the ball of $\mathscr{S}^D$ centered at the origin and with radius $\psi(p_0)$, and at every boundary point $T_0^D$ of $\mathcal{C}_{p_0}$ the normal $N_0$ is parallel to $T_0^D$:

$$T_0^D = \psi(p_0) N_0. \tag{6-49}$$

Then, by the normality rule, $T_0^D$ has the same direction as $D^p = \lambda^d N_0$.

**6.3.2.** *The Drucker–Prager and the Mises conditions.* For the energies of the form (6-46), consider the special case in which

$$\phi(|L^d|) = |L^d|, \qquad \psi(p) = \alpha p + \beta, \quad \alpha, \beta > 0. \tag{6-50}$$

For such energies the radius $\psi(p)$ of $\mathcal{C}_p$ vanishes for $p = -\beta/\alpha$. Then the set $\mathcal{K}_0$ of the admissible stresses

$$\mathcal{K}_0 = \{T \in \mathscr{S} \mid T = T^D - pI,\ p \geq -\beta/\alpha,\ |T^D| \leq \alpha p + \beta\} \tag{6-51}$$

is a circular cone of $\mathscr{S}$ with vertex at $T = (\beta/\alpha)I$ and with axis on the hydrostatic axis. This is the set of the admissible stresses of the *Drucker–Prager yield condition*.

In the limit case $\alpha = 0$, the radius becomes equal to $\beta$, and $\mathcal{K}_0$ becomes the cylinder

$$\mathcal{K}_0 = \{T \in \mathscr{S} \mid T = T^D - pI,\ p \in \mathbb{R},\ |T^D| \leq \beta\}, \tag{6-52}$$

which is the set of the admissible stresses of the *Mises yield condition*.

**6.3.3.** *The plastic behavior of metals and geomaterials.* When comparing the behavior of metals and geomaterials, one sees that the latter exhibit a much lower strength in tension than in compression, while for metals the two strengths are of the same order of magnitude. Moreover, under increasing pressure the elastic range enlarges in geomaterials, and remains almost constant in metals.

These differences are captured by isotropic energies of the form (6-50). Indeed, in the Drucker–Prager cone the hydrostatic stress in tension cannot exceed the value $-\beta/\alpha$ attained at the vertex of the cone but is unbounded in compression, and the size of the elastic range increases with $p_R$. In Mises's cylinder, both tensile and compressive hydrostatic stresses are unbounded, and the size of the elastic range does not depend on the pressure. Therefore, the cone and the cylinder seem to be well suited to describe, at least in a first approximation, the behavior of geomaterials and of metals, respectively.

That the Mises criterion is appropriate to metals is almost universally recognized.[42] Large consensus was also initially met on adopting yield conditions of the Drucker–Prager type for geomaterials. In particular, favored by its analogy with Coulomb's theory of friction, the Mohr–Coulomb yield condition[43] was applied extensively in soil mechanics [Lubliner 1990, §6.1.3].[44] These models were coupled with normality in the deviatoric plane, and since the elastic range was identified with the set of the admissible stresses, this led to the adoption of nonassociated flow rules. In the present paper, in line with some earlier proposals [Ziegler and Wehrli 1987, p. 223; Srinivasa 2010], the associated character of these flow rules has been recovered regarding the hydrostatic pressure as an extra state variable. In this way, the assumption of normality in the deviatoric space is fully legitimated from the variational viewpoint.

Unfortunately, experimental evidence turned against the isochoricity assumption, since most geomaterials exhibit a form of inelastic volume change called *dilatancy*,[45] which fits neither normality with respect to the deviatoric plane nor normality with respect to the set of the admissible stresses.[46]

**6.4.** ***The Cam-clay model.*** An experiment-based model expressly conceived to describe the dilatancy of soils is the *Cam-clay model* [Roscoe et al. 1958; Roscoe and Poorooshasb 1963].[47] Here we show that this is a special case of the dilatational model described in Section 6.1.3.

**6.4.1.** *The plastic energy.* In the Cam-clay model we keep the irrotationality condition $R_0^d = I$ by which, due to condition (4-15), the plastic energy $\phi$ reduces to a function of the plastic stretching $D^d$. The assumed form of $\phi$ is

$$\phi(D^d) = \phi(\tilde{p}, \tilde{q}), \tag{6-53}$$

where

$$\tilde{p} = -\tfrac{1}{3} I \cdot D^d = -\tfrac{1}{3} I \cdot D^{dH}, \qquad \tilde{q} = |D^d + \tilde{p} I| = |D^{dD}|. \tag{6-54}$$

[42] "...Most modern discussions of plasticity (of metals) are based on generalizations and structural variations of the theory of Lévy, Mises, Prandtl, and Reuss..." [Gurtin et al. 2010, §76.1].

[43] For this condition, which can be regarded as a variant of the Drucker–Prager condition, see, e.g., [Lubliner 1990, §3.3.3].

[44] For other models of the Drucker–Prager type, see [Ziegler and Wehrli 1987, §VII.A, §VII.B].

[45] For plastic volume changes in geomaterials, see, e.g., [Lubliner 1990; Vermeer and de Borst 1984]. For metals, unexpected plastic volume changes were revealed by the experiments reported in [Wilson 2002].

[46] The volume increase due to normality with respect to $\mathcal{K}_0$ is measured by the *friction angle*, and the volume increase due to normality with respect to $\mathcal{C}_p$ is zero. The experiments show that the angle which measures the volume increase due to dilatancy has an intermediate value which is far both from the friction angle and from zero; see, e.g., [Lubliner 1990, §6.1.3], or [Vermeer and de Borst 1984, §2].

[47] The model was modified in [Roscoe and Burland 1968].

The dependence on the modulus of $D^{dD}$ describes an isotropic response to the deviatoric strain rates.

**6.4.2.** *The yield condition.* For an energy of the form (6-53), the elastic range $\mathcal{C}_0$ is a solid of revolution with axis on the hydrostatic axis. It is assumed that $\mathcal{C}_0$ is represented in the plane $(\tilde{p}, \tilde{q})$ by the line[48]

$$\tilde{q} = \chi(\tilde{p}), \qquad \tilde{p} \in (0, p^\dagger), \tag{6-55}$$

with

$$\chi(\tilde{p}) \in (0, q^\dagger), \quad \chi(0) = \chi(p^\dagger) = 0, \quad p^\dagger, q^\dagger < +\infty. \tag{6-56}$$

Let us determine the homogeneous dissipation potential $\phi^h$ associated with this elastic range.[49] Let $e^p$, $e^q$ be unit vectors in the directions of the axes $\tilde{p}$ and $\tilde{q}$. At a boundary point $(\tilde{p}, \tilde{q})$ of $\mathcal{C}_0$, the unit normal is

$$n(\tilde{p}) = \frac{e^q - \chi'(\tilde{p})\, e^p}{\sqrt{1 + \chi'^2(\tilde{p})}}, \tag{6-57}$$

and

$$d(\tilde{p}) = (\tilde{p} e^p + \chi(\tilde{p})\, e^q) \cdot n(\tilde{p}) = \frac{\chi(\tilde{p}) - \tilde{p}\chi'(\tilde{p})}{\sqrt{1 + \chi'^2(\tilde{p})}} \tag{6-58}$$

is the distance of $(\tilde{p}, \tilde{q})$ from the origin, measured in the direction of $n(\tilde{p})$. This is the value taken by the homogeneous potential $\phi^h$ at the boundary points of $\mathcal{C}_0$:[50]

$$\phi^h(\tilde{p}, \tilde{q}) = \phi^h(\tilde{p}, \chi(\tilde{p})) = d(\tilde{p}). \tag{6-59}$$

**6.4.3.** *Flow rule and hardening rule.* The flow rule assumed in the Cam-clay model is the normality rule. Accordingly, the plastic stretching at the boundary point $(\tilde{p}, \chi(\tilde{p}))$ of $\mathcal{C}_0$ has the direction $n(\tilde{p})$. Since $n(\tilde{p})$ has a nonnull component in the direction $e^p$, this implies that the plastic stretching is not isochoric. In particular, the volume change is positive (*dilatancy*) at points at which $\chi'(\tilde{p})$ is negative.

The homogeneous potential (6-59) corresponds to a perfectly plastic response. To get a hardening response, it is necessary to add to $\phi^h$ a nonhomogeneous part. For the quadratic potential

$$\phi(\lambda n(\tilde{p})) = (\lambda + \tfrac{1}{2}k\lambda^2) d(\tilde{p}), \quad \lambda \geq 0, \tag{6-60}$$

we have

$$\breve{\nabla}\phi(0) \triangleright n(\tilde{p}) = d(\tilde{p}), \qquad \breve{\nabla}^2\phi(0)\{n(\tilde{p})\} \triangleright n(\tilde{p}) = kd(\tilde{p}). \tag{6-61}$$

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[48]This line is a parabolic arc in the original Cam-clay model [Roscoe et al. 1958; Roscoe and Poorooshasb 1963] and a half-ellipse in the modified model [Roscoe and Burland 1968].

[49]Here we take advantage of the one-to-one correspondence between elastic ranges and homogeneous dissipation potentials; see Footnote 31.

[50]See Section 5.1.

Then denoting by $\tilde{t}_0 = (\tilde{p}, \tilde{q})$ and by $\delta\tilde{t}$ the representatives of $T_0$ and $\delta T$ on the plane $(\tilde{p}, \tilde{q})$, from the plastic stationarity condition (5-12) and from the consistency condition (5-17), we get

$$\tilde{t}_0 \cdot n(\tilde{p}) = d(\tilde{p}), \qquad \delta\tilde{t} \cdot n(\tilde{p}) = k\lambda^d d(\tilde{p}), \tag{6-62}$$

respectively. That is, for every direction $n$ the generating half-planes $\mathcal{H}_0^n$ of $\mathcal{C}_0$ undergo a translation proportional to the distance $d(\tilde{p})$ of the boundary point $\tilde{t}_0$ from the origin. This defines a new elastic range, obtained from $\mathcal{C}_0$ by a homothetic transformation with center at the origin and ratio $2k\lambda^d$. This is the evolution law for the elastic range currently assumed in the literature.[51] Clearly, a positive $k$ corresponds to hardening and a negative $k$ corresponds to softening.

## 7. Crystal plasticity

In materials with a crystalline structure, the plastic deformation consists of microslips occurring along some preferred directions on some preferred slip planes. A *plastic slip* is a plastic deformation of the form

$$L^d = \lambda^d s \otimes m, \quad \lambda^d \geq 0, \tag{7-1}$$

where $m$ is the unit normal to the slip plane, $\lambda^d$ is the slip intensity, and $s$, the slip direction, is a unit vector in the slip plane. The pair $(s, m)$ is a *slip system*, and the tensor $(s \otimes m)$ is the corresponding *Schmid tensor*. By the orthogonality of $s$ and $m$, $L^d$ is a deviator. Then the plastic deformation is isochoric; that is, crystal plasticity is a special case of isochoric plasticity.[52]

Below, I consider first the case of a single slip system, and then the case of a finite number of slip systems. Finally, I describe the *two-level shear* model based on the theory of structured deformations, in which the macroscopic features of plastic response can be reproduced without the use of nonsmooth potentials.

**7.1.** ***The single-slip model.*** The plastic slip (7-1) is a deformation of the form (4-3), $L^d = R_0^d D^d$, with[53]

$$R_0^d m = s, \qquad D^d = \lambda^d m \otimes m, \quad \lambda^d \geq 0. \tag{7-2}$$

What distinguishes this model from the preceding ones is precisely that the constitutive assumption $R_0^d = I$ is replaced by an assumption on the direction of $L^d$

[51] See, e.g., [Schofield and Wroth 1968].

[52] In crystal plasticity the energy is usually assumed to be independent of the hydrostatic pressure, that is, the function $\psi(p)$ in (6-21) is taken equal to one. For basic reference to the plasticity of crystals, see, e.g., [Gurtin et al. 2010, §102.1].

[53] In principle, $R_0^d$ may be any rotation which maps $m$ into $s$. For definiteness, one may add the prescription $R_0^d s = -m$, which characterizes $R_0^d$ as the rotation of amount $\pi/2$ about an axis perpendicular to both $s$ and $m$.

dictated by the geometry of the crystal.[54] Due to this peculiarity, the procedure for the determination of the incremental response law slightly departs from the path followed in the previous sections.

**7.1.1.** *The plastic stationarity condition.* From (7-1) and (7-2) and from the indifference conditions (4-14) and (4-15), neglecting the terms of order $o(t)$, for the plastic strain energy we have

$$\phi(F_t^d) = \phi(tL^d) = \phi(t\lambda^d s \otimes m) = \phi(t\lambda^d R_0^d m \otimes m) = \phi(t\lambda^d m \otimes m), \tag{7-3}$$

and since $m$ is fixed we may set

$$\phi(F_t^d) = \phi_m(t\lambda^d). \tag{7-4}$$

Thus, the only admissible plastic stretchings are those directed as $m \otimes m$, and the plastic energy is a function, depending on $m$, of their modulus $t\lambda^d$. Then from the definition of directional derivative we have

$$\begin{aligned} \nabla\phi(F_t^d) \triangleright s \otimes m &= \lim_{\varepsilon \to 0^+} \frac{\phi((t\lambda^d + \varepsilon)\, s \otimes m) - \phi(t\lambda^d s \otimes m)}{\varepsilon} \\ &= \lim_{\varepsilon \to 0^+} \frac{\phi_m(t\lambda^d + \varepsilon) - \phi_m(t\lambda^d)}{\varepsilon} = \phi_m'(t\lambda^d), \end{aligned} \tag{7-5}$$

and the plastic stationarity condition (3-22) takes the form

$$\tau_t \le \phi_m'(t\lambda^d), \tag{7-6}$$

where the *resolved shear stress*

$$\tau_t = T_t \cdot s \otimes m = T_t m \cdot s \tag{7-7}$$

is the tangential component in the direction $s$ of the stress vector $T_t m$ acting on the slip plane.

**7.1.2.** *Yield condition, flow rule, and consistency condition.* According to (7-1) and (7-2), the only direction allowed for the plastic stretching is $m \otimes m$. That is, the set of the admissible plastic stretchings is the half-line

$$\mathscr{S}_0^d = \{\lambda m \otimes m \mid \lambda \ge 0\}. \tag{7-8}$$

Then there is only one generating half-space $\mathcal{H}_0^{m\otimes m}$. It coincides with the elastic range $\mathcal{C}_0$, and the flow rule trivially says that the plastic stretching is directed as $m \otimes m$. In view of the expansions

$$\tau_t = \tau_0 + t\delta\tau + o(t), \qquad \phi_m'(t\lambda^d) = \phi_m'(0) + t\lambda^d \phi_m''(0) + o(t), \tag{7-9}$$

[54] Here only the case of $L^d$ directed as $s \otimes m$ is discussed. A more extended analysis in the context of the multiplicative decomposition was made by Reina and coauthors [Reina and Conti 2014; Reina et al. 2016].

from the plastic stationarity condition at $t = 0$ we have the inequality

$$\tau_0 \leq \phi'_m(0), \tag{7-10}$$

which shows that $\phi'_m(0)$ is the distance of the boundary of $\mathcal{C}_0$ from the origin. It also tells us that no plastic slip occurs as long as the resolved shear stress is below the *activation threshold* $\phi'_m(0)$. When this threshold is attained, a plastic slip may occur in the direction $s$. In this case (7-10) is satisfied as an equality, and from (7-6) and (7-9) we have

$$\delta\tau \leq \lambda^d \phi''_m(0). \tag{7-11}$$

This inequality becomes an equality when $\lambda^d > 0$. Then we have the consistency condition

$$\lambda^d(\delta\tau - \lambda^d \phi''_m(0)) = 0. \tag{7-12}$$

**7.1.3.** *Incremental response law and hardening rule.* From (7-12) and from the incremental stress-strain relation (5-2)

$$\delta\tau = \delta T \cdot (s \otimes m) = \mathbb{C}[\nabla u - \lambda^d s \otimes m] \cdot (s \otimes m), \tag{7-13}$$

by elimination of $\delta\tau$ we get

$$\lambda^d = \frac{\langle \mathbb{C}[\nabla u] \cdot (s \otimes m) \rangle}{\mathbb{C}[s \otimes m] \cdot (s \otimes m) + \phi''_m(0)}, \tag{7-14}$$

and substitution into (7-13) provides the incremental response law. In particular, for $\lambda^d > 0$ we have

$$\delta\tau = h_m \mathbb{C}[\nabla u] \cdot (s \otimes m), \tag{7-15}$$

with the hardening modulus

$$h_m = \frac{\phi''_m(0)}{\mathbb{C}[s \otimes m] \cdot (s \otimes m) + \phi''_m(0)}. \tag{7-16}$$

The numerator $\phi''_m(0)$ is positive if $\phi_m$ is strictly convex. On the contrary, nothing can be said a priori about the positiveness of the denominator. Indeed, because of the indifference condition (4-13), one can control only the restriction of $\mathbb{C}$ to the symmetric tensors. For all previous models this was enough, because only this restriction appears in the expressions (5-22) and (6-42) of $\lambda^d$. Here, from the indifference condition (4-13) we have

$$\begin{aligned}\mathbb{C}[s \otimes m] \cdot (s \otimes m) &= \mathbb{C}[(s \otimes m)^S] \cdot (s \otimes m)^S \\ &\quad + 2(s \otimes m)^W T_0 \cdot (s \otimes m)^S + (s \otimes m)^W T_0 \cdot (s \otimes m)^W.\end{aligned} \tag{7-17}$$

The two terms involving $T_0$ can be transformed into

$$-T_0 \cdot (s \otimes m)^W (2(s \otimes m)^S + (s \otimes m)^W) = \tfrac{1}{4} T_0 \cdot (3m \otimes m - s \otimes s). \tag{7-18}$$

Denoting by $c_0$ the smallest eigenvalue of $\mathbb{C}$ restricted to $\mathscr{S}$ and setting

$$\sigma_m = T_0 \cdot (m \otimes m), \qquad \sigma_s = T_0 \cdot (s \otimes s), \tag{7-19}$$

we get

$$\mathbb{C}[s \otimes m] \cdot (s \otimes m) \geq \tfrac{1}{2} c_0 + \tfrac{3}{4} \sigma_m - \tfrac{1}{4} \sigma_s. \tag{7-20}$$

Therefore, $\mathbb{C}[s \otimes m] \cdot (s \otimes m)$ is positive if $c_0 > 0$ and $\sigma_m$ and $\sigma_s$ are small with respect to $c_0$, and if $c_0 > 0$, the normal stress $\sigma_m$ acting on the slip plane is tensile, and the in-plane normal stress $\sigma_s$ is compressive.

**7.2.** ***The multislip model.*** A *multislip system* is defined as a finite set of slip systems $(s^\alpha, m^\alpha)$.[55] For each of them the plastic deformation has the form

$$L^{d\alpha} = \lambda^{d\alpha} s^\alpha \otimes m^\alpha, \quad \lambda^{d\alpha} \geq 0, \tag{7-21}$$

and the corresponding energy is

$$\phi^\alpha(L^{d\alpha}) = \phi^\alpha(\lambda^{d\alpha} m^\alpha \otimes m^\alpha) = \phi_{m^\alpha}(\lambda^{d\alpha}). \tag{7-22}$$

The total plastic deformation and the total energy are

$$L^p = \sum_\alpha \lambda^{d\alpha} s^\alpha \otimes m^\alpha, \qquad \phi(L^d) = \sum_\alpha \phi_{m^\alpha}(\lambda^{d\alpha}), \tag{7-23}$$

respectively. The gradient of $\phi$ is the homogeneous map which with every $\mathcal{L}^d = \sum_\beta \mu^\beta s^\beta \otimes m^\beta$ associates the number

$$\begin{aligned} \nabla\phi(F_t^d) \triangleright \mathcal{L}^d &= \nabla\phi\Big(\sum_\alpha \lambda^{d\alpha} s^\alpha \otimes m^\alpha\Big) \triangleright \sum_\beta \mu^\beta s^\beta \otimes m^\beta \\ &= \sum_\alpha \mu^\alpha \phi'_{m^\alpha}(\lambda^{d\alpha}), \end{aligned} \tag{7-24}$$

with the last identification preformed proceeding as in (7-5). The plastic stationarity condition (3-22) then takes the form

$$\sum_\alpha \mu^\alpha \tau_t^\alpha \leq \sum_\alpha \mu^\alpha \phi'_{m^\alpha}(t\lambda^{d\alpha}), \quad \mu^\alpha \geq 0, \tag{7-25}$$

where

$$\tau_t^\alpha = T_t \cdot (s^\alpha \otimes m^\alpha) \tag{7-26}$$

is the resolved shear stress for the slip system $(s^\alpha, m^\alpha)$. At $t = 0$, taking all $\mu^\alpha$ equal to zero except one, we get $n$ inequalities

$$\tau_0^\alpha \leq \phi'_{m^\alpha}(0), \tag{7-27}$$

[55] This model is based on Koiter's model of *singular yield surfaces* [1953]. See also [Martin and Reddy 1993].

one for each slip system. Each of them defines a half-space $\mathcal{H}_0^{N^\alpha}$ of $\mathscr{S}^D$ with normal $N^\alpha = m^\alpha \otimes m^\alpha$ and with boundary at the distance $\phi'_{m^\alpha}(0)$ from the origin. The intersection of the $\mathcal{H}_0^{N^\alpha}$ is the elastic range $\mathcal{C}_0$. It is a polyhedral convex subset of $\mathscr{S}^D$, whose faces have the normals $N^\alpha$ and whose vertices have normal cones consisting of positive combinations of the $N^\alpha$.

For each $\alpha$, the expansions (7-9) hold. When for some $\tau_0^\alpha$ the activation threshold $\phi'_{m^\alpha}(0)$ is reached, by the consistency condition in analogy with (7-12) we have

$$\lambda^{d\alpha}(\delta\tau^\alpha - \lambda^{d\alpha}\phi''_{m^\alpha}(0)) = 0. \tag{7-28}$$

If $T_0$ is on a face of $\mathcal{C}_0$, this occurs only for the $\alpha$ corresponding to that face. If $T_0$ is on a vertex of $\mathcal{C}_0$, this occurs for the $\alpha$ corresponding to the faces which concur at that vertex. Denoting by $\hat{\alpha}(T_0)$ the set of such $\alpha$, the normal cone at $T_0$ is

$$D^d = \sum_{\alpha\in\hat{\alpha}(T_0)} \lambda^{d\alpha} N^\alpha, \tag{7-29}$$

and the incremental relation (5-2) takes the form

$$\delta T = \mathbb{C}[\nabla u] - \sum_{\alpha\in\hat{\alpha}(T_0)} \lambda^{d\alpha}\mathbb{C}[N^\alpha]. \tag{7-30}$$

Then for all $\beta$ in $\hat{\alpha}(T_0)$,

$$\delta\tau^\beta = \delta T\cdot(s^\beta\otimes m^\beta) = \Bigl(\mathbb{C}[\nabla u] - \sum_{\alpha\in\hat{\alpha}(T_0)} \lambda^{d\alpha}\mathbb{C}[s^\alpha\otimes m^\alpha]\Bigr)\cdot(s^\beta\otimes m^\beta). \tag{7-31}$$

Together with the constraints $\lambda^{d\alpha} \geq 0$ and the complementarity conditions (7-28), these equations form a linear complementarity problem of the dimension of $\hat{\alpha}(T_0)$. In particular, if $T_0$ belongs to the relative interior of a face of $\mathcal{C}_{p_0}$, this dimension is one and we are back to the one-dimensional problem of the previous sections.

If the dimension is larger than one, the existence and uniqueness of the solution is guaranteed if the matrix $\{\mathbb{C}[s^\alpha\otimes m^\alpha]\cdot(s^\beta\otimes m^\beta)\}$ is positive definite. In this case the problem can be solved with the Gauss–Seidel iterative method or with any other nonlinear programming algorithm. But, like in the case of a single slip system, the positive definiteness is ensured only for suitable values of the initial stress $T_0$.

**7.3.** ***Periodic energies and two-level shears.*** In single crystals, a relative translation of an atomic unit along a slip plane maps the two halves of a crystal into a placement energetically indistinguishable from the initial one.[56] This suggests the

[56]See, e.g., [Gurtin et al. 2010, Figure 102.2].

use of periodic energies. In general, the energy $\phi$ is assumed to be of the form

$$\phi = \psi + \phi^{nh}, \tag{7-32}$$

with $\psi$ a smooth periodic potential and $\phi^{nh}$ a nonhomogeneous nonsmooth potential. Here we consider the simple case of $\phi^{nh} = 0$.[57]

A *two-level shear* is a deformation process $t \mapsto (f_t, F_t)$ in which the macroscopic deformation $f_t$ is a simple shear

$$f_t(x) = x + \bar{\gamma} t x\, s \otimes m, \quad \bar{\gamma} > 0, \tag{7-33}$$

and the deformation due to the disarrangements[58] is a single slip

$$F_t^d(x) = \gamma_t^d s \otimes m. \tag{7-34}$$

When $\phi$ reduces to a smooth potential $\psi$, the directional derivative $\check{\nabla}\phi$ reduces to the ordinary derivative $\nabla\psi$. Moreover, in the virtual strain rates $\mathcal{L}^d = \alpha s \otimes m$ the multiplier $\alpha$ is not anymore constrained to be positive. Then the stationarity condition (3-22) reduces to the equality

$$T_t \cdot (s \otimes m) = \nabla\psi(\gamma_t^d s \otimes m) \cdot (s \otimes m). \tag{7-35}$$

The left-hand side is the resolved shear stress (7-7). For the right-hand side, using the indifference condition (7-3) we define

$$\psi(F_t^d) = \psi(\gamma_t^d m \otimes m) = \psi_m(\gamma_t^d), \tag{7-36}$$

and proceeding as in (7-5) we find that the right-hand side of (7-35) is equal to $\psi_m'(\gamma_t^d)$. Then (7-35) reduces to

$$\tau_t = \psi_m'(\gamma_t^d). \tag{7-37}$$

On the other hand, for the energy without disarrangements $\varphi$, from (3-19), (7-33), and (7-34) we have

$$T_t = \nabla\varphi(F_t) = \nabla\varphi(\nabla f_t - F_t^d) = \nabla\varphi(I + (\bar{\gamma} t - \gamma_t^d) s \otimes m). \tag{7-38}$$

Multiplying by $(s \otimes m)$, on the left side we get again $\tau_t$. Then, after defining

$$\Phi_m^s(\bar{\gamma} t - \gamma_t^d) = \nabla\varphi(I + (\bar{\gamma} t - \gamma_t^d) s \otimes m) \cdot (s \otimes m), \tag{7-39}$$

we get

$$\tau_t = \Phi_m^s(\bar{\gamma} t - \gamma_t^d). \tag{7-40}$$

[57] For two-scale continua, the use of smooth potentials in the study of plasticity and fracture was initiated in [Choksi et al. 1999, §4, §5], and continued by Deseri and Owen [2002] for the plasticity of single crystals. The specular case $\phi = \phi^{nh}$ of a purely nonsmooth potential was advanced in [Del Piero 1998; 2018], without subsequent developments.

[58] In the presence of a smooth potential, I prefer to avoid calling $F_t^d$ a plastic deformation.

By elimination of $\gamma_t^d$ between this equation and (7-37), a relation between the shear stress $\tau_t$ and the macroscopic shear $\bar{\gamma}t$ is obtained. For example, in the case of $\Phi_m^s$ linear and $\psi_m$ trigonometric of period $p$,

$$\Phi_m^s(\xi) = k\xi, \qquad \psi_m(\xi) = \frac{k^d p}{2\pi}\left(1 - \cos\frac{2\pi\xi}{p}\right), \tag{7-41}$$

with $k$ and $k^d$ positive material constants, we get

$$\bar{\gamma}t = \frac{\tau_t}{k} + \frac{p}{2\pi}\sin^{-1}\left(\frac{\tau_t}{k^d}\right). \tag{7-42}$$

This determines a curve $\bar{\gamma}t = \mathcal{F}(\tau_t)$, in which $\mathcal{F}$ is a multivalued function with domain $(-k^d, k^d)$. The slope of the curve is

$$\frac{d\bar{\gamma}t}{d\tau_t} = \frac{1}{k} \pm \frac{p}{2\pi\sqrt{k^{d2} - \tau_t^2}}, \tag{7-43}$$

with the plus sign for the branch from the origin to its first intersection with the line $\tau_t = k^d$, the minus sign for the following branch up to the first intersection with the line $\tau_t = -k^d$, and so on.

The slope of the first branch is positive and increases from $1/k + p/2\pi k^d$ at $\tau_t = 0$ to $+\infty$ at $\tau_t = k^d$. The slope of the second branch increases from $-\infty$ at $\tau_t = k^d$, to zero at[59]

$$\tau_t = \sqrt{k^{d2} - \frac{k^2 p^2}{4\pi^2}}. \tag{7-44}$$

At this point the curve attains a local maximum. Then for a further increase of $\bar{\gamma}t$ there is no solution near the maximum point, and equilibrium for the increased $\bar{\gamma}t$ can be attained only jumping to another branch of the curve. This jump is a form of *catastrophic instability*, consisting of a sudden decrease of $\tau_t$ at constant $\bar{\gamma}t$.

Taking the macroscopic shear $\bar{\gamma}t$ as independent variable, the stress-strain response curve is $\tau_t = \mathcal{F}^{-1}(\bar{\gamma}t)$. The function $\mathcal{F}^{-1}$ is periodic with period $p$, since if $\bar{\gamma}t$ is a solution of (7-42) for some $\tau_t$, then $\bar{\gamma}t + np$ is a solution for the same $\tau_t$ for all $n$. In the equilibrium branch starting from $(\tau_t, \bar{\gamma}t) = (0, 0)$, $\tau_t$ increases with $\bar{\gamma}t$ up to the upper limit $k^d$, and then decreases up to the lower limit $-k^d$. When in the descending branch the local maximum of $\mathcal{F}$ is attained, $\gamma_t$ suddenly jumps downward at constant $\bar{\gamma}t$, to reach the next ascending branch of $\mathcal{F}^{-1}$. Due to the periodicity of $\mathcal{F}^{-1}$, the same jump occurs at each period. The resulting stress-strain

[59]This value is attained only if $2\pi k^d > kp$. But this condition is satisfied in practice, since $p$ is a very small length, of the order of the interatomic distance [Choksi et al. 1999, §4].

diagram then shows an initial increase of $\gamma_t$ from zero to $k^d$, followed by a periodic oscillation between $k^d$ and the value at which the maximum of $\mathcal{F}$ is attained.[60]

Macroscopically, this looks like a typical elastic-perfectly plastic response consisting of an initial growth followed by a horizontal plateau. A peculiarity of this model is that the plateau is made of microscopic oscillations.[61] By consequence, while in all previous models yielding is activated when a threshold determined by a nonsmooth potential is attained, in the present model yielding is due to catastrophic instability. When $\phi$ is a smooth potential there is no dissipation associated with the equilibrium branches, the whole dissipation being concentrated on the jumps.[62]

## Closure

I conclude with a summary of the main results.

(i) By imposing the nonnegativeness of the first variation of the energy, all constitutive elements of the theory become dependent on a single punctual inequality, the plastic stationarity condition.

(ii) From this condition the yield condition, the flow rule, the hardening rule, and the incremental response law can be deduced without any additional assumption.

(iii) A plastic energy made of a smooth potential plus a homogeneous dissipation potential determines a model of kinematic plasticity which includes Prager's kinematic hardening as a special case.

(iv) A plastic energy with the properties of a nonhomogeneous dissipation potential determines a model of dilatational hardening which includes isotropic hardening as a special case.

(v) The assumption of isochoricity of the plastic strain rate leads to the definition of a one-parameter family of pairwise disjoint elastic ranges, the parameter being the hydrostatic pressure. In each elastic range the normality rule holds. This renders unnecessary the nonassociated flow rules introduced by the many authors for which the elastic range is the union of all individual elastic ranges.

(vi) The Cam-clay model for soils is a particular case of isotropic dilatational plasticity.

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[60] See, e.g., [Choksi et al. 1999, Figure 6].

[61] This may reproduce the oscillations exhibited by testing machines operating with the "hard device", that is, by controlling the deformation. The oscillations are much less sensible in the "soft" testing machines, which control the applied load. This difference is well reproduced by the present model, see Figures 6 and 8 in [Choksi et al. 1999].

[62] See [Choksi et al. 1999, §4]. This alternative description of yielding led the present author to distinguish two types of yielding, which he called *reversible* and *irreversible* [Del Piero 1998; 2013].

(vii) Crystal plasticity is a special case of isochoric plasticity, with a particular form of the plastic stretching. The two-level shear model is a special case of crystal plasticity, with diffuse plastic dissipation replaced by concentrated catastrophic instability.

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