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HYPERBOLIC PREDATORS VS. PARABOLIC PREY*

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Abstract. We present a nonlinear predator–prey system consisting of a nonlocal conservation law for predators coupled with a parabolic equation for prey. The drift term in the predators' equation is a nonlocal function of the prey density, so that the movement of the predators can be directed towards regions with high prey density. Moreover, Lotka–Volterra type right hand sides describe the feeding. A theorem ensuring existence, uniqueness, continuous dependence of weak solutions, and various stability estimates is proved, in any space dimension. Numerical integrations show a few qualitative features of the solutions.

Key words. Nonlocal conservation laws, predatory–prey systems, mixed hyperbolic–parabolic problems.

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1. Introduction

Consider the following predator–prey model

$$\begin{cases} \partial_t u + \nabla\cdot(u\,v(w)) = (\alpha w - \beta)\,u, \\ \partial_t w - \mu\Delta w = (\gamma - \delta u)\,w, \end{cases} \tag{1.1}$$

where $u = u(t,x)$, respectively $w = w(t,x)$, is the predator, respectively prey, density at time $t\in\mathbb{R}^+$ and position $x\in\mathbb{R}^n$. Prey diffuse according to a parabolic equation, since $\mu > 0$. Here, γ is the prey birth rate and δ the prey mortality due to the predators. The predator density evolves according to a hyperbolic balance law, where the coefficient α in the source term accounts for the increase in the predator density due to feeding on prey, while β is the predator mortality rate. The flow $uv(w)$ accounts for the preferred predators' direction. The velocity v is in general a *nonlocal* and nonlinear function of the prey density. A typical choice can be

$$v(w) = \kappa\frac{\nabla(w*\eta)}{\sqrt{1+\|\nabla(w*\eta)\|^2}}, \tag{1.2}$$

meaning that predators move towards regions of higher concentrations of prey. Indeed, when η is a positive smooth mollifier with $\int_{\mathbb{R}^n}\eta\,\mathrm{d}x = 1$, the space convolution $(w(t)*\eta)(x)$ has the meaning of an average of the prey density at time t around position x. The denominator $\sqrt{1+\|\nabla(w*\eta)\|^2}$ is merely a smooth normalization factor, so that the positive parameter κ is the maximal predator speed.

Two key features of the model (1.1) are the following. First, while prey diffuse in all directions due to the Laplacian in the w equation, predators in (1.1) have a directed movement, for instance drifting towards regions where the prey density is higher. This allows, for instance, to describe predators chasing prey. Second, predators have a well

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defined horizon. Indeed, the radius of the support of η in (1.2) defines how far predators can *"feel"* the presence of prey and, hence, the direction in which they move.

The aim of this paper is to study the class of models (1.1) under suitable assumptions on v. We prove below existence, uniqueness, continuous dependence from the initial datum, and various stability estimates for the solutions to (1.1). Here, solutions are found in the space $\mathbf{L}^1 \cap \mathbf{L}^\infty \cap \mathbf{BV}$ for the predators and in $\mathbf{L}^1 \cap \mathbf{L}^\infty$ for the prey. Thus, solutions are here understood in the distributional sense, see definitions 2.1, 2.3, and 2.6. Moreover, all analytical results hold in any space dimension, the explicit dependence of the constants entering the estimates being duly reported in the proofs below. Besides, qualitative properties of solutions are shown by means of numerical integrations.

With reference to possible biological applications, the words *prey* and *predator* should be here understood in their widest sense. The diffusion in the second equation may well describe the evolution of a chemical substance or also of temperature. Indeed, setting for instance $\delta = 0$, the second equation decouples from the first, and the first one fits into [18, Formula (0.1)], see also [6, 7]. In this connection, we recall that the interest in nonlocal hyperbolic models is increasing in several fields.

Various multi D models devoted to crowd dynamics are considered in [7, Section 4] and in [6] in the case of a single population, in [8] for several populations. In these works, solutions are understood in the weak sense of Kružkov, see [19], and well posedness is proven in any space dimension.

Nonlocal models for aggregation and swarming are presented in [14, 15], where the existence of smooth or Lipschitz continuous solutions is proved in 1D and in 2D, the n dimensional case being considered in [16]. Due to the biological motivation, in these papers only one population is considered.

In structured population biology, the use of nonlocal models based on conservation laws is very common, also in a measure valued setting, see for example [1, 5, 12] and the references therein.

On the other hand, the use of purely parabolic equations in predator–prey models with spatial distributions is rather classical, see for instance [24, Section 1.2]. With respect to these models, the use of a first order differential operator in the predator density allows to describe the directed movement of predators and ensures that they have a finite propagation speed. Indeed, if the initial distribution of predators has compact support, then the region they occupy grows with finite speed and remains compact for all times, as proved below.

As analytical tools, in this paper we consider separately the equations

$$\partial_t u + \nabla \cdot (c(t,x)\, u) = b(t,x)\, u \qquad \text{and} \qquad \partial_t w - \mu \Delta w = a(t,x)\, w. \tag{1.3}$$

For the former, we exploit the classical results by Kružkov [19] and the more recent stability estimates proved in [9, 21]. The literature on the latter equation in (1.3) is vast; however, our considering it in $\mathbf{L}^1 \cap \mathbf{L}^\infty$ on all $\mathbb{R}^n$ seems to be somewhat unconventional, hence we provide detailed proofs of the necessary estimates. The two equations (1.3) are here studied following exactly the same template and analogous results are obtained. Once the necessary estimates for the solutions to (1.3) are proven, a fixed point argument allows us to prove the well posedness of (1.1) and several stability estimates.

The next section presents the analytical results: first the main theorem and then the propositions at its basis. Section 3 is devoted to sample numerical integrations of (1.1). All technical details are deferred to the final Section 4.

2. Analytical results

This section is devoted to the well posedness theorem that constitutes the main result of this paper. All proofs are deferred to Section 4.

Our first step is the rigorous definition of solution to (1.1).

DEFINITION 2.1. *Let $T>0$ be fixed. A solution to the system* (1.1) *on* $[0,T]$ *is a pair* $(u,w)\in\mathbf{C}^0([0,T];\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^2))$ *such that*

- *setting $a(t,x)=\gamma-\delta u(t,x)$, w is a weak solution to $\partial_t w-\mu\Delta w=aw$;*
- *setting $b(t,x)=\alpha w(t,x)-\beta$ and $c(t,x)=(v(w(t)))(x)$, u is a weak solution to $\partial_t u+\nabla\cdot(uc)=bu$.*

The extension to the case of the Cauchy problem is immediate. Below, in Definition 2.3, respectively in Definition 2.6, we state and use different definitions of solutions to the parabolic equation $\partial_t w-\mu\Delta w=aw$, respectively to the hyperbolic equation $\partial_t u+\nabla\cdot(uc)=bu$, and prove their equivalence in Lemma 2.4, respectively in Lemma 2.7.

Throughout, we work in the spaces

$$\begin{aligned}\mathcal{X} &= (\mathbf{L}^1\cap\mathbf{L}^\infty\cap\mathbf{BV})(\mathbb{R}^n;\mathbb{R}) \quad\times(\mathbf{L}^1\cap\mathbf{L}^\infty)(\mathbb{R}^n;\mathbb{R}) \quad\text{and}\\ \mathcal{X}^+ &= (\mathbf{L}^1\cap\mathbf{L}^\infty\cap\mathbf{BV})(\mathbb{R}^n;\mathbb{R}^+)\times(\mathbf{L}^1\cap\mathbf{L}^\infty)(\mathbb{R}^n;\mathbb{R}^+)\end{aligned}\tag{2.1}$$

with the norm

$$\|(u,w)\|_{\mathcal{X}}=\|u\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}+\|w\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}.\tag{2.2}$$

System (1.1) is defined by a few real parameters and by the map v, which is assumed to satisfy the following condition:

($\boldsymbol{v}$) $v\colon(\mathbf{L}^1\cap\mathbf{L}^\infty)(\mathbb{R}^n;\mathbb{R})\to(\mathbf{C}^2\cap\mathbf{W}^{1,\infty})(\mathbb{R}^n;\mathbb{R}^n)$ admits a constant K and an increasing map $C\in\mathbf{L}^\infty_{loc}(\mathbb{R}^+;\mathbb{R}^+)$ such that, for all $w,w_1,w_2\in(\mathbf{L}^1\cap\mathbf{L}^\infty)(\mathbb{R}^n;\mathbb{R})$,

$$\begin{aligned}\|v(w)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^n)}&\le K\,\|w\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})},\\ \|\nabla v(w)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n\times n})}&\le K\,\|w\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})},\\ \|v(w_1)-v(w_2)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^n)}&\le K\,\|w_1-w_2\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})},\\ \|\nabla(\nabla\cdot v(w))\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^n)}&\le C\left(\|w\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\right)\|w\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})},\\ \|\nabla\cdot(v(w_1)-v(w_2))\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}&\le C\left(\|w_2\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\right)\|w_1-w_2\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}.\end{aligned}$$

Above, the bound on the $\mathbf{L}^\infty$ norm of $v(w)$ by means of the $\mathbf{L}^1$ norm of w is typical of a nonlocal, e.g. convolution, operator. Indeed, Lemma 4.1 below ensures that under reasonable regularity conditions on the kernel η, the operator v in (1.2) satisfies **($\boldsymbol{v}$)**.

Relying solely on **($\boldsymbol{v}$)**, we state the main result of this paper.

THEOREM 2.2. *Fix $\alpha,\beta,\gamma,\delta\ge 0$ and $\mu>0$. Assume that v satisfies* **($\boldsymbol{v}$)**. *Then, there exists a map*

$$\mathcal{R}\colon\mathbb{R}^+\times\mathcal{X}^+\to\mathcal{X}^+$$

with the following properties:

1. ***$\boldsymbol{\mathcal{R}}$ is a semigroup:*** $\mathcal{R}_0=\mathrm{Id}$ *and* $\mathcal{R}_{t_2}\circ\mathcal{R}_{t_1}=\mathcal{R}_{t_1+t_2}$ *for all* $t_1,t_2\in\mathbb{R}^+$.

2. ***$\mathcal{R}$ solves* (1.1)*:*** *for all $(u_o,w_o)\in\mathcal{X}^+$, the map $t\to\mathcal{R}_t(u_o,w_o)$ solves the Cauchy Problem*

$$\begin{cases} \partial_t u+\nabla\cdot(u\,v(w))=(\alpha w-\beta)u \\ \partial_t w-\mu\Delta w=(\gamma-\delta u)w \\ u(0,x)=u_o(x) \\ w(0,x)=w_o(x) \end{cases}$$

in the sense of Definition 2.1. In particular, for all $(u_o,w_o)\in\mathcal{X}^+$ the map $t\to\mathcal{R}_t(u_o,w_o)$ is continuous in time.

3. ***Local Lipschitz continuity in the initial datum:*** *for all $r>0$ and for all $t\in\mathbb{R}^+$, there exists a positive $\mathcal{L}(t,r)$ such that for all $(u_1,w_1),(u_2,w_2)\in\mathcal{X}^+$ with*

$$\|u_i\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\le r, \qquad \mathrm{TV}(u_i)\le r, \qquad \|w_i\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\le r, \qquad \|w_i\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\le r$$

for $i=1,2$, the following estimate holds:

$$\|\mathcal{R}_t(u_1,w_1)-\mathcal{R}_t(u_2,w_2)\|_{\mathcal{X}}\le\mathcal{L}(t,r)\,\|(u_1,w_1)-(u_2,w_2)\|_{\mathcal{X}}.$$

4. ***Growth estimates:*** *for all $(u_o,w_o)\in\mathcal{X}^+$ and for all $t\in\mathbb{R}^+$, denote $(u,w)(t)=\mathcal{R}_t(u_o,w_o)$. Then,*

$$\begin{aligned} \|u(t)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} &\le \|u_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\exp\left(\alpha\frac{e^{\gamma t}-1}{\gamma}\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\right), \\ \|u(t)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} &\le \|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\exp\left((\alpha+K)\frac{e^{\gamma t}-1}{\gamma}\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\right), \\ \|w(t)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} &\le \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}e^{\gamma t}, \\ \|w(t)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} &\le \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}e^{\gamma t}. \end{aligned}$$

5. ***Propagation speed:*** *if $(u_o,w_o)\in\mathcal{X}^+$ is such that $\mathrm{spt}(u_o)\subseteq B(0,\rho_o)$, then, for all $t\in\mathbb{R}^+$,*

$$\mathrm{spt}(u(t))\subseteq B(0,\rho(t)) \qquad \textit{where} \qquad \rho(t)=\rho_o+K\,t\,e^{\gamma t}\|w_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}.$$

An explicit estimate of the Lipschitz constant $\mathcal{L}(t,r)$ is provided at (4.37).

Theorem 2.2 is proved through careful estimates on the parabolic problem

$$\begin{cases} \partial_t w-\mu\Delta w=a(t,x)\,w \\ w(t_o,x)=w_o(x) \end{cases} \tag{2.3}$$

and, separately, on the balance law

$$\begin{cases} \partial_t u+\nabla\cdot(c(t,x)\,u)=b(t,x)\,u \\ u(t_o,x)=u_o(x). \end{cases} \tag{2.4}$$

Our approaches to both the evolution equations (2.3) and (2.4) are identical. We recall below the key definitions, prove the basic well posedness results, and provide rigorous stability estimates, always referring to the spaces in (2.1) and with reference to the $\mathbf{L}^1$ norm.

To improve the readability of the statements below, we denote by $\mathcal{O}(t)$ an increasing smooth function of time t, depending on the space dimension n and on various norms of the coefficients a,μ in (2.3) and b,c in (2.4). All proofs are deferred to Section 4, where explicit estimates for all constants are provided.

Throughout, we fix $t_o, T \in \mathbb{R}^+$, with $T > t_o$, and denote

$$I = [t_o, T] \quad \text{and} \quad J = \left\{(t_1,t_2) \in I^2 \colon t_1 < t_2\right\}. \tag{2.5}$$

For completeness, we recall the following notions from the theory of parabolic equations. They are similar to various results in the wide literature on parabolic problems, see for instance [2, 23, 25], but here we are dealing with $\mathbf{L}^1$ solutions on the whole space.

Inspired by [25, Section 48.3], we give the following definition, where we use the notation (2.5).

Definition 2.3. *Let $a \in \mathbf{L}^\infty(I \times \mathbb{R}^n;\mathbb{R})$ and $w_o \in \mathbf{L}^1(\mathbb{R}^n;\mathbb{R})$. A* weak solution *to* (2.3) *is a function $w \in \mathbf{C}^0(I;\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}))$ such that, for all test functions $\varphi \in \mathbf{C}^1\left(I;\mathbf{C}^2_c(\mathbb{R}^n;\mathbb{R})\right)$*

$$\int_{t_o}^T \int_{\mathbb{R}^n} (w\,\partial_t\varphi + \mu w\,\Delta\varphi + a\,w\,\varphi)\,\mathrm{d}x\,\mathrm{d}t = 0 \tag{2.6}$$

and $w(t_o,x) = w_o(x)$.

The following lemma is similar to various results in the literature, see for instance [25, Section 48.3], and is here recalled for completeness. The heat kernel is denoted by $H_\mu(t,x) = (4\pi\mu t)^{-n/2}\exp\left(-\|x\|^2 \big/ (4\mu t)\right)$, where $t>0$, $x \in \mathbb{R}^n$, and $\mu > 0$ is fixed.

Lemma 2.4. *Let $a \in \mathbf{L}^\infty(I \times \mathbb{R}^n;\mathbb{R})$. Assume that $w_o \in \mathbf{L}^1(\mathbb{R}^n;\mathbb{R})$. Then,*

1. *any function w satisfying*

$$w(t,x) = (H_\mu(t-t_o) * w_o)(x) + \int_{t_o}^t (H_\mu(t-\tau) * (a(\tau)\,w(\tau)))(x)\,\mathrm{d}\tau \tag{2.7}$$

   solves (2.3) *in the sense of Definition 2.3;*

2. *any solution to* (2.3) *in the sense of Definition 2.3 satisfies* (2.7).

The well posedness of (2.3) is now proved.

Proposition 2.5. *Let $a \in \mathbf{L}^\infty(I \times \mathbb{R}^n;\mathbb{R})$. Then, the Cauchy problem* (2.3) *generates a map $\mathcal{P}\colon J \times \mathbf{L}^1(\mathbb{R}^n;\mathbb{R}) \to \mathbf{L}^1(\mathbb{R}^n;\mathbb{R})$ with the following properties:*

1. ***$\mathcal{P}$ is a Process:** $\mathcal{P}_{t,t} = \mathrm{Id}$ for all $t \in I$ and $\mathcal{P}_{t_2,t_3} \circ \mathcal{P}_{t_1,t_2} = \mathcal{P}_{t_1,t_3}$ for all $t_1,t_2,t_3 \in I$, with $t_1 \le t_2 \le t_3$.*

2. ***$\mathcal{P}$ solves** (2.3)**:** for all $w_o \in \mathbf{L}^1(\mathbb{R}^n;\mathbb{R})$, the function $t \to \mathcal{P}_{t_o,t} w_o$ solves the Cauchy problem* (2.3) *in the sense of Definition 2.3.*

3. ***Regularity in $\boldsymbol{w_o}$:** for all $(t_o,t) \in J$, the map $\mathcal{P}_{t_o,t}\colon \mathbf{L}^1(\mathbb{R}^n;\mathbb{R}) \to \mathbf{L}^1(\mathbb{R}^n;\mathbb{R})$ is linear and continuous, with*

$$\|\mathcal{P}_{t_o,t} w_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \le \mathcal{O}(t)\,\|w_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}.$$

4. *$\mathbf{L}^\infty$ **estimate:** for all $w_o \in (\mathbf{L}^1 \cap \mathbf{L}^\infty)(\mathbb{R}^n;\mathbb{R})$, for all $(t_o,t) \in J$,*

$$\|\mathcal{P}_{t_o,t} w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} \le \mathcal{O}(t)\,\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}.$$

5. ***Stability in a:*** *let* $a_1,a_2 \in \mathbf{L}^\infty(I\times\mathbb{R}^n;\mathbb{R})$ *with* $a_1-a_2\in\mathbf{L}^1(I\times\mathbb{R}^n;\mathbb{R})$ *and call* $\mathcal{P}^1,\mathcal{P}^2$ *the corresponding processes. Then, for all* $(t_o,t)\in J$ *and for all* $w_o\in(\mathbf{L}^1\cap\mathbf{L}^\infty)(\mathbb{R}^n;\mathbb{R})$,
$$\left\|\mathcal{P}^1_{t_o,t}w_o-\mathcal{P}^2_{t_o,t}w_o\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\leq\mathcal{O}(t)\,\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\,\|a_1-a_2\|_{\mathbf{L}^1([t_o,t]\times\mathbb{R}^n;\mathbb{R})}.$$
6. ***Positivity:*** *if* $w_o\in(\mathbf{L}^1\cap\mathbf{L}^\infty)(\mathbb{R}^n;\mathbb{R})$ *and* $w_o\geq 0$*, then* $\mathcal{P}_{t_o,t}w_o\geq 0$ *for all* $(t_o,t)\in J$.
7. ***Regularity in (t,x):*** *if* $w_o\in(\mathbf{L}^1\cap\mathbf{C}^1)(\mathbb{R}^n;\mathbb{R})$*, then* $(t,x)\to(\mathcal{P}_{t_o,t}w_o)(x)\in\mathbf{C}^1(I\times\mathbb{R}^n;\mathbb{R})$.
8. ***Regularity in time:*** *for all* $w_o\in\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})$*, the map* $t\to\mathcal{P}_{t_o,t}w_o$ *is in* $\mathbf{C}^0\big(I;\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})\big)$*, and, moreover, for every* $\vartheta\in]0,1[$ *and for all* $\tau,t_1,t_2\in I$ *with* $t_2\geq t_1\geq\tau>t_o$
$$\|\mathcal{P}_{t_o,t_2}w_o-\mathcal{P}_{t_o,t_1}w_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\leq\|w_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\left[\frac{n}{\tau-t_o}+\mathcal{O}(t_2)\right]|t_2-t_1|^\vartheta.$$
9. $\mathbf{W}^{1,1}$ ***estimate:*** *for all* $w_o\in\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})$*, for all* $(t_o,t)\in J$,
$$\|\nabla(\mathcal{P}_{t_o,t}w_o)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^n)}\leq\frac{\mathcal{O}(t)}{\sqrt{t-t_o}}\|w_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}.$$

We now follow the same template used in the preceding proposition and lemma, but referring to the hyperbolic problem (2.4). Similarly to [11, Section 4.3] and [27, Section 3.5], we give the following definition, where we used the notation (2.5).

DEFINITION 2.6. *Let* $b\in\mathbf{L}^\infty(I\times\mathbb{R}^n;\mathbb{R})$*,* $c\in\mathbf{L}^\infty(I\times\mathbb{R}^n;\mathbb{R}^n)$ *and* $u_o\in(\mathbf{L}^1\cap\mathbf{L}^\infty)(\mathbb{R}^n;\mathbb{R})$*. A* weak solution *to* (2.4) *is a function* $u\in\mathbf{C}^0(I;\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}))$ *such that for all test functions* $\varphi\in\mathbf{C}^1_c(\mathring{I}\times\mathbb{R}^n;\mathbb{R})$
$$\int_{t_o}^T\int_{\mathbb{R}^n}(u\partial_t\varphi+uc\cdot\nabla\varphi+bu\varphi)\,\mathrm{d}x\,\mathrm{d}t=0 \tag{2.8}$$
and $u(t_o,x)=u_o(x)$.

The following Lemma is analogous to Lemma 2.4, with the usual integral formula (2.7) replaced by integration along characteristics, see (2.9).

LEMMA 2.7. *Let* c *be such that* $c\in(\mathbf{C}^0\cap\mathbf{L}^\infty)(I\times\mathbb{R}^n;\mathbb{R}^n)$*,* $c(t)\in\mathbf{C}^1(\mathbb{R}^n;\mathbb{R}^n)$ $\forall t\in I$*,* $\nabla c\in\mathbf{L}^\infty(I\times\mathbb{R}^n;\mathbb{R}^{n\times n})$*. Assume that* $b\in\mathbf{L}^\infty(I\times\mathbb{R}^n;\mathbb{R})$ *and* $u_o\in(\mathbf{L}^1\cap\mathbf{L}^\infty)(\mathbb{R}^n;\mathbb{R})$*. Then*

1. *the function* u *defined by*
$$u(t,x)=u_o(X(t_o;t,x))\exp\left(\int_{t_o}^t\big(b(\tau,X(\tau;t,x))-\nabla\cdot c(\tau,X(\tau;t,x))\big)\,\mathrm{d}\tau\right), \tag{2.9}$$
where the map $t\mapsto X(t;t_o,x_o)$ *solves the Cauchy Problem*
$$\begin{cases}\dot X=c(t,X)\\ X(t_o)=x_o,\end{cases} \tag{2.10}$$
is a Kružkov solution to (2.4)*, i.e. for all* $k\in\mathbb{R}$ *and for all* $\varphi\in\mathbf{C}^1_c(\mathring{I}\times\mathbb{R}^n;\mathbb{R}^+)$,
$$\int_{t_o}^T\int_{\mathbb{R}^n}[(u-k)(\partial_t\varphi+c\cdot\nabla\varphi)+(bu-k\nabla\cdot c)\,\varphi]\,\mathrm{sgn}(u-k)\,\mathrm{d}x\,\mathrm{d}t\geq 0 \tag{2.11}$$
and $u(t_o,x)=u_o(x)$*, hence,* u *solves* (2.4) *in the sense of Definition 2.6;*

2. *any solution to* (2.4) *in the sense of Definition 2.6 coincides with* u *as defined in* (2.9).

Above, our choice of the Kružkov definition is motivated by our using the $\mathbf{L}^1$ bounds in [9, 19, 21] in the proofs of **3.** and **5.** below.

PROPOSITION 2.8. *We pose the assumptions:*

(b) $b\in(\mathbf{C}^1\cap\mathbf{L}^\infty)(I\times\mathbb{R}^n;\mathbb{R})$; $\nabla b\in\mathbf{L}^1(I\times\mathbb{R}^n;\mathbb{R}^n)$;

(c) $c\in(\mathbf{C}^2\cap\mathbf{L}^\infty)(I\times\mathbb{R}^n;\mathbb{R}^n)$; $\nabla c\in\mathbf{L}^\infty(I\times\mathbb{R}^n;\mathbb{R}^{n\times n})$; $\nabla(\nabla\cdot c)\in\mathbf{L}^1(I\times\mathbb{R}^n;\mathbb{R}^n)$.

Then, the Cauchy Problem (2.4) *generates a map* $\mathcal{H}\colon J\times(\mathbf{L}^1\cap\mathbf{L}^\infty\cap\mathbf{BV})(\mathbb{R}^n;\mathbb{R})\to(\mathbf{L}^1\cap\mathbf{L}^\infty\cap\mathbf{BV})(\mathbb{R}^n;\mathbb{R})$ *with the following properties:*

1. ***$\mathcal{H}$ is a process:*** $\mathcal{H}_{t,t}=\mathrm{Id}$ *for all* $t\in I$ *and* $\mathcal{H}_{t_2,t_3}\circ\mathcal{H}_{t_1,t_2}=\mathcal{H}_{t_1,t_3}$ *for all* $t_1,t_2,t_3\in I$, *with* $t_1\le t_2\le t_3$.
2. ***$\mathcal{H}$ solves*** (2.4)***:*** *for all* $u_o\in(\mathbf{L}^1\cap\mathbf{L}^\infty\cap\mathbf{BV})(\mathbb{R}^n;\mathbb{R})$, *the function* $t\to\mathcal{H}_{t_o,t}u_o$ *solves the Cauchy problem* (2.4) *in the sense of Definition 2.6.*
3. ***Regularity in u_o:*** *for all* $(t_o,t)\in J$ *the map* $\mathcal{H}_{t_o,t}\colon(\mathbf{L}^1\cap\mathbf{L}^\infty\cap\mathbf{BV})(\mathbb{R}^n;\mathbb{R})\to(\mathbf{L}^1\cap\mathbf{L}^\infty\cap\mathbf{BV})(\mathbb{R}^n;\mathbb{R})$ *is linear and continuous, with*
$$\|\mathcal{H}_{t_o,t}u_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\le\mathcal{O}(t)\,\|u_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}.$$
4. **$\mathbf{L}^\infty$ *estimate:*** *for all* $u_o\in(\mathbf{L}^1\cap\mathbf{L}^\infty\cap\mathbf{BV})(\mathbb{R}^n;\mathbb{R})$, *for all* $(t_o,t)\in J$,
$$\|\mathcal{H}_{t_o,t}u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\le\mathcal{O}(t)\,\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}.$$
5. ***Stability in b,c:*** *if* b_1,b_2 *satisfy* ***(b)*** *with* $b_1-b_2\in\mathbf{L}^1(I\times\mathbb{R}^n;\mathbb{R})$ *and* c_1,c_2 *satisfy* ***(c)*** *with* $\nabla\cdot(c_1-c_2)\in\mathbf{L}^1(I\times\mathbb{R}^n;\mathbb{R})$, *call* $\mathcal{H}^1,\mathcal{H}^2$ *the corresponding processes. Then, for all* $(t_o,t)\in J$ *and for all* $u_o\in(\mathbf{L}^1\cap\mathbf{L}^\infty\cap\mathbf{BV})(\mathbb{R}^n;\mathbb{R})$,
$$\begin{aligned}&\left\|\mathcal{H}^1_{t_o,t}u_o-\mathcal{H}^2_{t_o,t}u_o\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\\ &\le\mathcal{O}(t)\left(\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}+\mathrm{TV}(u_o)\right)\|c_1-c_2\|_{\mathbf{L}^1([t_o,t];\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^n))}\\ &\quad+\mathcal{O}(t)\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\left(\|b_1-b_2\|_{\mathbf{L}^1([t_o,t]\times\mathbb{R}^n;\mathbb{R})}+\|\nabla\cdot(c_1-c_2)\|_{\mathbf{L}^1([t_o,t]\times\mathbb{R}^n;\mathbb{R})}\right).\end{aligned}$$
6. ***Positivity:*** *if* $u_o\in(\mathbf{L}^1\cap\mathbf{L}^\infty\cap\mathbf{BV})(\mathbb{R}^n;\mathbb{R})$ *and* $u_o\ge0$, *then* $\mathcal{H}_{t_o,t}u_o\ge0$ *for all* $(t_o,t)\in J$.
7. ***Total variation bound:*** *if* $u_o\in(\mathbf{L}^1\cap\mathbf{L}^\infty\cap\mathbf{BV})(\mathbb{R}^n;\mathbb{R})$, *then, for all* $(t_o,t)\in J$,
$$\mathrm{TV}(\mathcal{H}_{t_o,t}u_o)\le\mathcal{O}(t)\left(\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}+\mathrm{TV}(u_o)\right).$$
8. ***Regularity in time:*** *for all* $u_o\in(\mathbf{L}^1\cap\mathbf{L}^\infty\cap\mathbf{BV})(\mathbb{R}^n;\mathbb{R})$, *the map* $t\to\mathcal{H}_{t_o,t}u_o$ *is in* $\mathbf{C}^{0,1}\left(I;\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})\right)$, *moreover for all* $t_1,t_2\in J$,
$$\begin{aligned}&\|\mathcal{H}_{t_o,t_2}u_o-\mathcal{H}_{t_o,t_1}u_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\\ &\le\mathcal{O}(t_2)\Big(\|u_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}+\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}+\mathrm{TV}(u_o)\Big)|t_2-t_1|.\end{aligned}$$
9. ***Finite propagation speed:*** *let* $(t_o,t)\in J$ *and* $u_o\in(\mathbf{L}^1\cap\mathbf{L}^\infty\cap\mathbf{BV})(\mathbb{R}^n;\mathbb{R})$ *have compact support* $\mathrm{spt}\,u_o$. *Then, also,* $\mathrm{spt}\,\mathcal{H}_{t_o,t}u_o$ *is compact.*

3. Numerical integrations

To illustrate some qualitative properties of the solutions to (1.1), we present the result of a few numerical integrations.

To integrate both equations we use the operator splitting algorithm to combine the differential operators and the source terms. The balance law is integrated by means of the Lax–Friedrichs scheme with dimensional splitting [22, Section 19.5], while its source term is solved using a second order Runge–Kutta method. For the parabolic equation, we use the forward finite differences algorithm and the usual Euler forward explicit method on the source term. We leave the proof of the convergence of this algorithm to the forthcoming work [26].

Notice that the numerical integration of (1.1) requires a convolution integral to be computed at each time step. This puts a constraint on the space mesh, which should be sufficiently small with respect to the radius of the support of the convolution kernel to allow a good approximation of this integral.

Here, we focus on the two-dimensional case, that is $n=2$, and use the vector field v in (1.2) with the compactly supported mollifier

$$\eta(x)=\hat{\eta}\left(\ell^2-\|x\|^2\right)^3\chi_{B(0,\ell)}(x) \quad \text{with } \hat{\eta}\in\mathbb{R}^+ \text{ such that } \int_{\mathbb{R}^2}\eta(x)\,\mathrm{d}x=1. \tag{3.1}$$

The analytical theory developed above is referred to as the Cauchy problem on the whole space $\mathbb{R}^2$. In both examples below, the numerical domain of integration is the rectangle $[-1,1]\times[-2,2]$. The necessary boundary conditions are different in the two cases and are specified below. The time step $(\Delta t)_P$ for the parabolic equation and the one $(\Delta t)_H$ for the hyperbolic part are chosen so that $(\Delta t)_P$ is of the order of $\left((\Delta t)_H\right)^2$. The time step for the hyperbolic equation complies with the usual CFL condition.

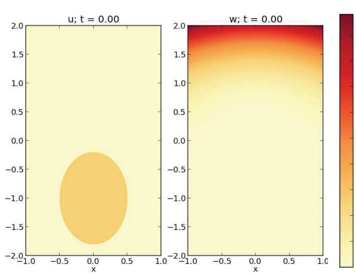
Below, we constrain both unknown functions u and w to remain equal to the initial datum all along the boundary, which is acceptable in the first equation since no wave in the solution to the balance law ever hits the numerical boundary. Concerning the second equation, the choice of these boundary conditions amounts to assume that the displayed solution is part of a solution defined on all $\mathbb{R}^2$ that gives a constant inflow into the computational domain.

3.1. Predators chasing prey. We present a situation in which the effect of the first order transport term in the predator equation is clearly visible, as well as the well known Lotka–Volterra type effect in which a species apparently almost disappears and then its density rises again.

We set v as in (1.2), η as in (3.1) and

$$\begin{array}{cccc} \alpha=2 & \beta=1 & \kappa=1 & \\ \gamma=1 & \delta=2 & \mu=0.5 & \ell=0.15 \end{array} \tag{3.2}$$

with initial datum (see figure below)



$$\begin{aligned} &u_0(x,y)=4\,\chi_A(x,y)\\ &w_0(x,y)=1.5\,y\max\{0,x^2+y^2-0.25\}\,\chi_B(x,y)\\ &\text{where}\\ &A=\{(x,y)\in\mathbb{R}^2\colon\ (2\,x)^2+(1.25\,(y+1))^2\le 1\}\\ &B=\{(x,y)\in\mathbb{R}^2\colon\ y\ge 0\}. \end{aligned} \tag{3.3}$$

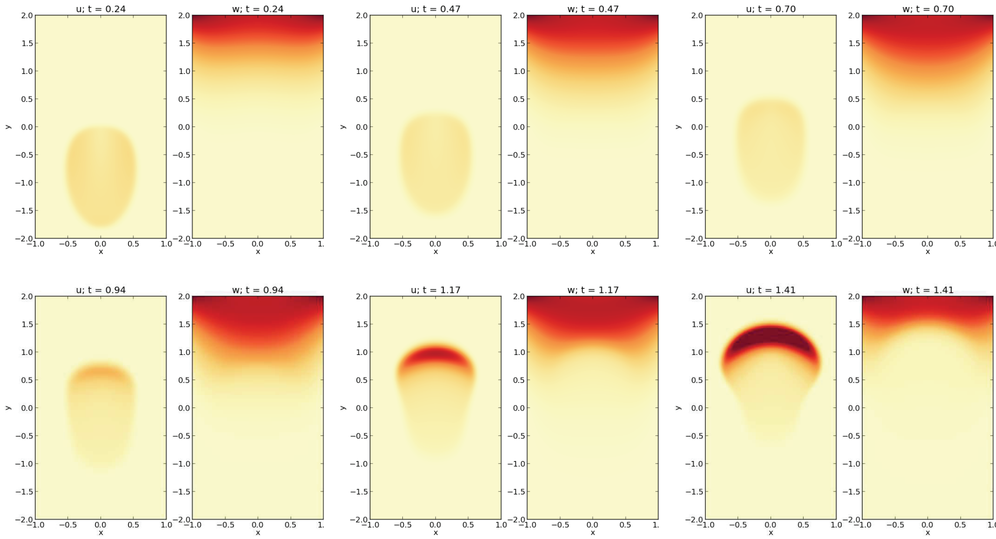


FIG. 3.1. *Numerical integration of* (1.1)–(3.1)–(3.2) *with initial datum* (3.3) *at times* $t = 0.24, 0.47, 0.70, 0.94, 1.17, 1.41$. *In each couple of figures, the predator density* u *is on the left and the prey density* w *is on the right. The colors range as in* (3.3) *in the interval* $[0, 15]$ *for* u *and* $[0, 14]$ *for* w. *First, predators decrease due to lack of nutrients. Thanks to diffusion, prey reach the zone where they are* "seen" *by predators. Then, clearly, predators are attracted towards prey and their density starts to increase. This solution was obtained with space mesh* $\Delta x = \Delta y = 0.005$.

The result of the numerical integration is in Figure 3.1. At first, the prey are outside the horizon of predators. Hence the latter decrease. Thanks to diffusion, some of the prey enter the region where predators feel their presence. This causes predators to move towards the highest prey density. Therefore, predators immediately increase and their effect on the prey population is clearly seen, as shown also by the graph of the integrals of u and w in Figure 3.2.

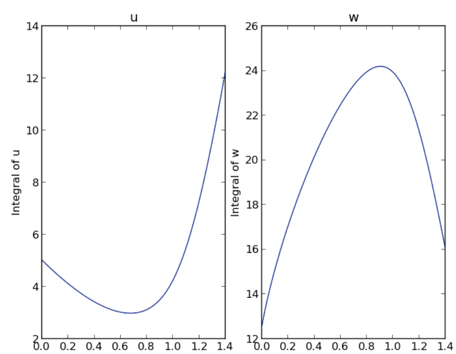


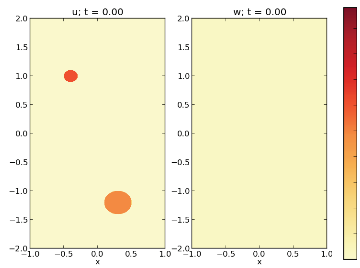
FIG. 3.2. *The integrals of* u, *left, and* w, *right, over the computational domain versus time;* u *and* w *are the numerical solutions to* (1.1)–(3.1)–(3.2).

We remark that, in the present setting, as time grows, undesired effects due to the presence of the boundary become relevant.

3.2. A dynamic equilibrium. In this case, the numerical solution displays an interesting asymptotic state in which the diffusion caused by the Laplacian in the prey equation counterbalances the first order nonlocal differential operator in the predator equation. The outcome is the onset of a discrete, quite regular, structure, see Figure 3.3. We set v as in (1.2), η as in (3.1) and

$$\begin{array}{llll} \alpha = 1 & \beta = 0.2 & \kappa = 1 & \multirow{2}{*}{$\ell = 0.25$} \\ \gamma = 0.4 & \delta = 24 & \mu = 0.5 & \end{array} \qquad (3.4)$$

with initial datum (see figure below)



$$\begin{aligned} & u_0(x,y) = 0.25\,\chi_C(x,y) + 0.2\,\chi_D(x,y) \\ & w_0(x,y) = 0.2 \\ & \text{where} \\ & C = \{(x,y) \in \mathbb{R}^2 : (x+0.4)^2 + (y-1)^2 < 0.01\} \\ & D = \{(x,y) \in \mathbb{R}^2 : (x-0.3)^2 + (y+1.2)^2 < 0.04\}. \end{aligned} \qquad (3.5)$$

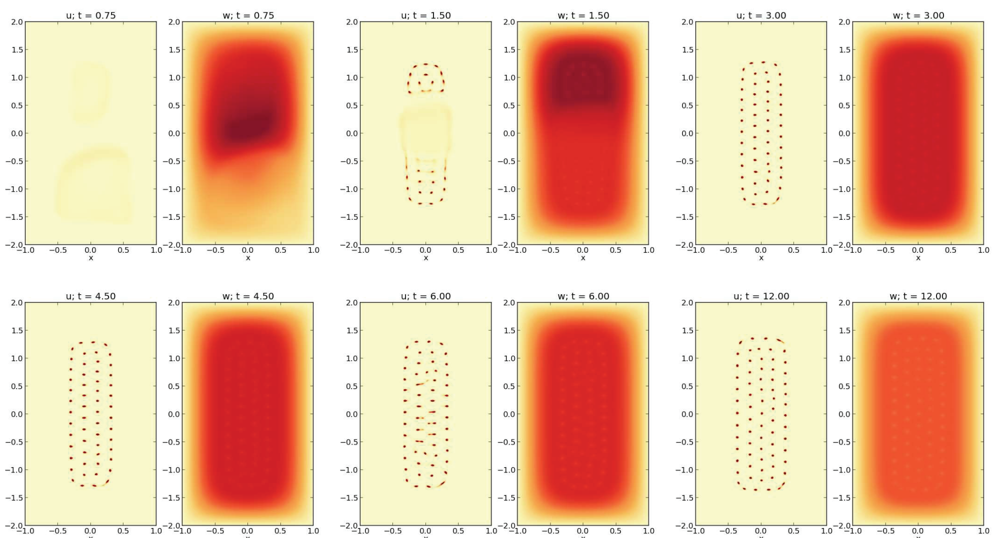


FIG. 3.3. *Numerical integration of* (1.1)–(3.1)–(3.4) *with initial datum* (3.5) *at times* $t = 0.75, 1.50, 3.00, 4.50, 6.00, 12.00$. *In each couple of figures, the predator density u is on the left and the prey density w is on the right. The colors range as in* (3.5) *in the interval* $[0, 0.40]$ *for u and* $[0.20, 0.24]$ *for w. First, predators decrease due to lack of nutrients and move towards the central region. Then, a discrete periodic pattern arises with predators focused in small regions regularly distributed along* 4 *columns and, at a later time, along* 5 *columns. This solution was obtained with space mesh* $\Delta x = \Delta y = 0.005$.

In this integration, predators first almost disappear, move towards the central part of the numerical domain and then start to increase. Slowly, a regular pattern arises.

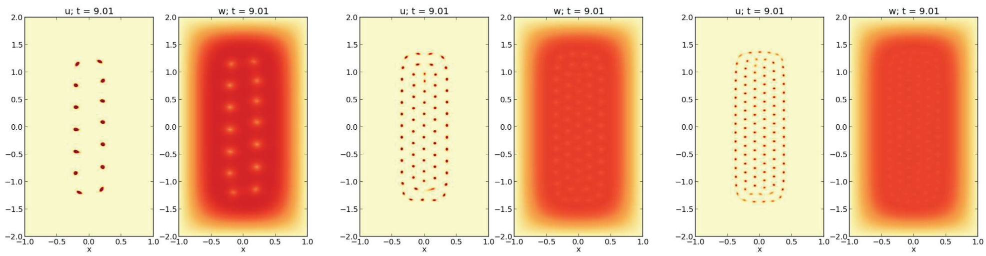


FIG. 3.4. *Solutions to* (1.1)–(3.1)–(3.4) *with initial datum* (3.5) *computed at time* $t = 9.01$ *with different values of* ℓ, *i.e. left* $\ell = 0.5$, *middle* $\ell = 0.25$ *and, right,* $\ell = 0.1875$. *As* ℓ *decreases, also the distance among peaks in the* u *density decreases and more peaks are possible. The color scale is as in Figure 3.3. These solutions were obtained with space mesh* $\Delta x = \Delta y = 0.0075$.

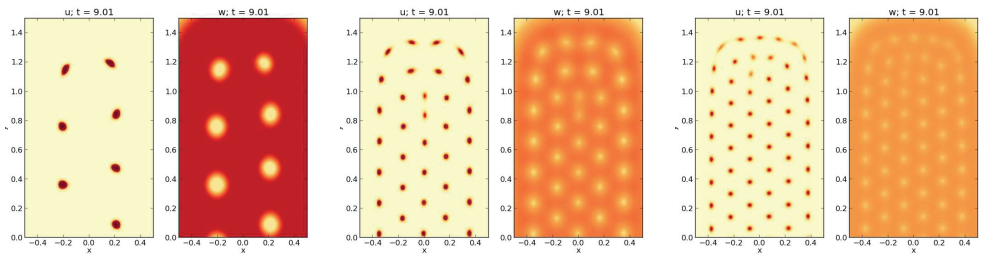


FIG. 3.5. *Zoom of Figure 3.4, restricted to the intervals* $x \in [-0.5, 0.5]$ *and* $y \in [0, 1.5]$. *Colors are as in* (3.5), *ranging in* $[0, 0.4]$ *for the* u *variable and in* $[0.225, 0.229]$ *for the* w *variable.*

Predators focus in small regions regularly distributed. These regions display a fairly stable behavior while passing from being arranged along 4 to along 5 columns, see Figure 3.3, second line.

From the analytical point of view, this pattern can be explained as a dynamic equilibrium between the first order non local operator present in the predator equation and the Laplacian in the prey equation. Where predators accumulate, their feeding on prey causes a *"hole"* in the prey density, see figures 3.4 and 3.5. As a consequence, the average gradient of the prey density, which directs the movement of predators, almost vanishes by symmetry considerations. Hence, predators almost do not move. At the same time, the diffusion of the prey keeps filling the *"holes"*, thus providing a persistent amount of nutrient to predators.

Coherently with this explanation, numerical integrations confirm that the above asymptotic state essentially depends on the size of the support of η. Indeed, the mean distance between pairwise nearest peaks in the density of u is slightly smaller than ℓ, see figures 3.4 and 3.5.

4. Technical details

4.1. Proofs related to $\partial_t w - \mu \Delta u = a(t,x) w$.

We use below the constant:

$$J_n = \frac{\Gamma\left((n+1)/2\right)}{\Gamma(n/2)} \tag{4.1}$$

where Γ is the Gamma function. Moreover, recall the classical estimates

$$\|H_\mu(t)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} = 1 \tag{4.2}$$

$$\|\nabla H_\mu(t)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} = \frac{1}{\sqrt{\mu t}}\,\frac{\Gamma((n+1)/2)}{\Gamma(n/2)} = \frac{J_n}{\sqrt{\mu t}} \tag{4.3}$$

$$\partial_t H_\mu(t,x) = \frac{1}{(4\pi\mu t)^{n/2}}\,\frac{\|x\|^2 - 2n\mu t}{4\mu t^2}\exp\left(-\frac{\|x\|^2}{4\mu t}\right)$$

$$\|\partial_t H_\mu(t)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \le \frac{n}{t}. \tag{4.4}$$

Proof of Lemma 2.4.
1. Since $H_\mu(t) \in \mathbf{C}^\infty(\mathbb{R}^n;\mathbb{R})$ and w satisfies (2.7), clearly $w(t) \in \mathbf{C}^\infty(\mathbb{R}^n;\mathbb{R})$ for all $t > t_o$. If $\varphi \in \mathbf{C}^1\left(I;\mathbf{C}^2_c(\mathbb{R}^n;\mathbb{R})\right)$ then $\int_{\mathbb{R}^n} \mu w \Delta\varphi \,\mathrm{d}x = \int_{\mathbb{R}^n} \mu \Delta w \varphi \,\mathrm{d}x$. Note first that $\mu\Delta H_\mu(t) = \partial_t H_\mu(t)$, and compute preliminarily

$$\begin{aligned}
\mu\Delta w(t,x) &= \mu\Delta\left((H_\mu(t-t_o) * w_o)(x)\right) + \int_{t_o}^t \mu\Delta\left((H_\mu(t-\tau) * (a(\tau)w(\tau)))(x)\right)\mathrm{d}\tau\\
&= (\mu\Delta H_\mu(t-t_o) * w_o)(x) + \int_{t_o}^t (\mu\Delta H_\mu(t-\tau) * (a(\tau)w(\tau)))(x)\,\mathrm{d}\tau\\
&= (\partial_t H_\mu(t-t_o) * w_o)(x) + \int_{t_o}^t (\partial_t H_\mu(t-\tau) * (a(\tau)w(\tau)))(x)\,\mathrm{d}\tau\\
&= \partial_t (H_\mu(t-t_o) * w_o)(x) + \int_{t_o}^t \partial_t (H_\mu(t-\tau) * (a(\tau)w(\tau)))(x)\,\mathrm{d}\tau\,.
\end{aligned}$$

Setting $\mathscr{H}(t,x) = \displaystyle\int_{t_o}^t (H_\mu(t-\tau) * (a(\tau)w(\tau)))(x)\,\mathrm{d}\tau$, the line above becomes

$$\mu\Delta w(t,x) = \partial_t (H_\mu(t-t_o) * w_o)(x) + \frac{\mathrm{d}}{\mathrm{d}t}\mathscr{H}(t,x) - a(t,x)w(t,x).$$

We are now able to prove (2.6):

$$\begin{aligned}
&\int_{t_o}^T \int_{\mathbb{R}^n} (w\partial_t\varphi + \mu\Delta w\varphi + aw\varphi)\,\mathrm{d}x\,\mathrm{d}t\\
&= \int_{t_o}^T \int_{\mathbb{R}^n} \Big[(H_\mu(t-t_o) * w_o)(x)\,\partial_t\varphi(t,x) + \mathscr{H}(t,x)\,\partial_t\varphi(t,x)\\
&\qquad + \partial_t\left((H_\mu(t-t_o) * w_o)(x)\right)\varphi(t,x) + \left(\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{H}(t,x)\right)\varphi(t,x) - a(t,x)\,w(t,x)\,\varphi(t,x)\\
&\qquad + a(t,x)\,w(t,x)\,\varphi(t,x)\Big]\,\mathrm{d}x\,\mathrm{d}t\\
&= \int_{t_o}^T \int_{\mathbb{R}^n} \frac{\mathrm{d}}{\mathrm{d}t}\left[\left((H_\mu(t-t_o) * w_o)(x) + \mathscr{H}(t,x)\right)\varphi(t,x)\right]\mathrm{d}x\,\mathrm{d}t\\
&= 0.
\end{aligned}$$

It is immediate to verify that if w satisfies (2.7), the initial condition holds.

2. Let w satisfy (2.7) and w_* be a weak solution to (2.3). Then, by the step above, the function $W = w - w_*$ is a weak solution to the linear Equation (2.3) with zero initial datum. By [13, Theorem 2.24], it follows that $W = 0$.

Proof of Proposition 2.5.

For all $t \in I$ denote $A(t) = \exp\left(\int_{t_o}^t \|a(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} \mathrm{d}\tau\right)$. The proofs of **1.–2.** are well known in the parabolic literature, see [2, 23]. By Lemma 2.4, recall that the solution $w(t,x) = (\mathcal{P}_{t_o,t} w_o)(x)$ to (2.3) satisfies (2.7).

3. Standard computations, using also (4.2), lead to:

$$\begin{aligned}
\|w(t)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \le{}& \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} H_\mu(t-t_o, x-\xi)\, |w_o(\xi)| \,\mathrm{d}\xi\, \mathrm{d}x \\
&+ \int_{\mathbb{R}^n} \int_{t_o}^t \int_{\mathbb{R}^n} H_\mu(t-\tau, x-\xi)\, |a(\tau,\xi)\, w(\tau,\xi)| \,\mathrm{d}\xi\, \mathrm{d}\tau\, \mathrm{d}x \\
\le{}& \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} + \int_{t_o}^t \|a(\tau)\, w(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \,\mathrm{d}\tau \\
\le{}& \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} + \int_{t_o}^t \|a(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} \|w(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \,\mathrm{d}\tau .
\end{aligned}$$

An application of Gronwall's lemma yields the thesis:

$$\|w(t)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \le A(t)\, \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}.$$

4. By (2.7) and (4.2),

$$\begin{aligned}
\|w(t)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} \le{}& \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} + \int_{t_o}^t \|a(\tau)\, w(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} \,\mathrm{d}\tau \\
\le{}& \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} + \int_{t_o}^t \|a(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} \|w(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} \,\mathrm{d}\tau .
\end{aligned}$$

An application of Gronwall Lemma gives the desired result:

$$\|w(t)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} \le A(t)\, \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}.$$

5. Denote $w_i(t) = \mathcal{P}^i_{t_o,t} w_o$ and $A_i(t) = \exp\left(\int_{t_o}^t \|a_i(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} \mathrm{d}\tau\right)$, for $i = 1,2$. Use (2.7), (4.2) and 4. above:

$$\begin{aligned}
&\|w_1(t) - w_2(t)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \\
\le{}& \int_{\mathbb{R}^n} \int_{t_o}^t \int_{\mathbb{R}^n} H_\mu(t-\tau, x-\xi)\, |a_1(\tau,\xi)\, w_1(\tau,\xi) - a_2(\tau,\xi)\, w_2(\tau,\xi)| \,\mathrm{d}\xi\, \mathrm{d}\tau\, \mathrm{d}x \\
\le{}& \int_{t_o}^t \|H_\mu(t-\tau)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \|a_1(\tau)\, w_1(\tau) - a_2(\tau)\, w_2(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \,\mathrm{d}\tau \\
={}& \int_{t_o}^t \|a_1(\tau)\, w_1(\tau) - a_2(\tau)\, w_2(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \,\mathrm{d}\tau \\
\le{}& \int_{t_o}^t \|a_1(\tau)\, w_1(\tau) - a_2(\tau)\, w_1(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \,\mathrm{d}\tau + \int_{t_o}^t \|a_2(\tau)\, w_1(\tau) - a_2(\tau)\, w_2(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \,\mathrm{d}\tau \\
\le{}& \int_{t_o}^t \|a_1(\tau) - a_2(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \|w_1(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} \,\mathrm{d}\tau \\
&+ \int_{t_o}^t \|a_2(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} \|w_1(\tau) - w_2(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \,\mathrm{d}\tau
\end{aligned}$$

$$\begin{aligned}
&\leq \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} \int_{t_o}^t A_1(\tau) \|a_1(\tau)-a_2(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \,\mathrm{d}\tau \\
&\quad + \int_{t_o}^t \|a_2(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} \|w_2(\tau)-w_1(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \,\mathrm{d}\tau\,.
\end{aligned}$$

An application of Gronwall's lemma yields the estimate:

$$\|w_1(t)-w_2(t)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \leq A_1(t)\, A_2(t)\, \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} \|a_1-a_2\|_{\mathbf{L}^1([t_o,t]\times\mathbb{R}^n;\mathbb{R})}.$$

6. Thanks to the $\mathbf{L}^\infty$ estimate at 4., we can apply [17, Chapter 2, Section 4, Theorem 9], regularizing the coefficient by means of point 5. above.

7. As is well known, note that (2.7) immediately ensures that w is of class $\mathbf{C}^1$.

8. By (2.7),

$$\begin{aligned}
\|w(t_2)-w(t_1)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \leq{}& \|(H_\mu(t_2-t_o)-H_\mu(t_1-t_o)) * w_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \\
&+ \int_{t_o}^{t_1} \|(H_\mu(t_2-s)-H_\mu(t_1-s)) * (a(s)w(s))\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \,\mathrm{d}s \\
&+ \int_{t_1}^{t_2} \|H_\mu(t_2-s) * (a(s)w(s))\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \,\mathrm{d}s
\end{aligned}$$

and we compute the three terms separately. The first one is the usual term of the heat equation, so that using (4.4),

$$\begin{aligned}
\|(H_\mu(t_2-t_o)-H_\mu(t_1-t_o)) * w_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} &\leq \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \int_{t_1-t_o}^{t_2-t_o} \|\partial_t H_\mu(s)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \,\mathrm{d}s \\
&\leq \frac{n}{\tau-t_o} \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} |t_2-t_1|.
\end{aligned}$$

Concerning the second term, use (4.4), point 3. above and follow the proof of [23, Proposition 4.2.4]: for every $\vartheta \in]0,1[$ we have

$$\begin{aligned}
&\int_{t_o}^{t_1} \|(H_\mu(t_2-s)-H_\mu(t_1-s)) * (a(s)w(s))\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \,\mathrm{d}s \\
&\leq \int_{t_o}^{t_1} \left\| \left(\int_{t_1-s}^{t_2-s} \partial_t H_\mu(\sigma) \,\mathrm{d}\sigma \right) * (a(s)w(s)) \right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \mathrm{d}s \\
&\leq \int_{t_o}^{t_1} \left\| \int_{t_1-s}^{t_2-s} \partial_t H_\mu(\sigma) \,\mathrm{d}\sigma \right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \|a(s)w(s)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \,\mathrm{d}s \\
&\leq \int_{t_o}^{t_1} \left(\int_{t_1-s}^{t_2-s} \|\partial_t H_\mu(\sigma)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \,\mathrm{d}\sigma \right) \|a(s)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} \|w(s)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \,\mathrm{d}s \\
&\leq \int_{t_o}^{t_1} \left(\int_{t_1-s}^{t_2-s} \frac{n}{\sigma} \,\mathrm{d}\sigma \right) \|a(s)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} A(s) \,\mathrm{d}s \\
&\leq \|a\|_{\mathbf{L}^\infty([t_o,t_1]\times\mathbb{R}^n;\mathbb{R})} \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} A(t_1) \int_{t_o}^{t_1} \frac{n}{(t_1-s)^\vartheta} \int_{t_1-s}^{t_2-s} \frac{1}{\sigma^{1-\vartheta}} \,\mathrm{d}\sigma \,\mathrm{d}s \\
&\leq \|a\|_{\mathbf{L}^\infty([t_o,t_1]\times\mathbb{R}^n;\mathbb{R})} \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} A(t_1) \int_{t_o}^{t_1} \frac{n}{\vartheta\,(t_1-s)^\vartheta} \left[(t_2-s)^\vartheta - (t_1-s)^\vartheta \right] \mathrm{d}s \\
&\leq \|a\|_{\mathbf{L}^\infty([t_o,t_1]\times\mathbb{R}^n;\mathbb{R})} \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} A(t_1) \int_{t_o}^{t_1} \frac{n}{\vartheta\,(t_1-s)^\vartheta} |t_2-t_1|^\vartheta \,\mathrm{d}s
\end{aligned}$$

$$\leq \|a\|_{\mathbf{L}^\infty([t_o,t_1]\times\mathbb{R}^n;\mathbb{R})} \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} A(t_1) \frac{n(t_1-t_o)^{1-\vartheta}}{\vartheta(1-\vartheta)} |t_2-t_1|^\vartheta,$$

where we used the inequality $(t_2-s)^\vartheta-(t_1-s)^\vartheta \leq (t_2-t_1)^\vartheta$. Concerning the third term, use (4.2) and point 3. above:

$$\begin{aligned}
&\int_{t_1}^{t_2} \|H_\mu(t_2-s)*(a(s)w(s))\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \mathrm{d}s\\
\leq& \int_{t_1}^{t_2} \|H_\mu(t_2-s)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \|a(s)w(s)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \mathrm{d}s\\
\leq& \|a\|_{\mathbf{L}^\infty([t_1,t_2]\times\mathbb{R}^n;\mathbb{R})} \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} A(t_2)|t_2-t_1|.
\end{aligned}$$

Summing up the above expressions, we obtain

$$\begin{aligned}
&\|w(t_2)-w(t_1)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\\
\leq& \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \left[\frac{n}{\tau-t_o} + \left(\frac{n(t_1-t_o)^{1-\vartheta}}{\vartheta(1-\vartheta)}+1\right) \|a\|_{\mathbf{L}^\infty([t_0,t_2]\times\mathbb{R}^n;\mathbb{R})} A(t_2)\right] |t_2-t_1|^\vartheta,
\end{aligned}$$

and the Hölder estimate at point 8. is proved. To prove the continuity in time, we are left to check the right continuity in t_o. To this aim, use (4.2) and point 3. above, introduce the variable $\zeta=\xi/(\mu t)^{n/2}$, and compute

$$\begin{aligned}
\|w(t)-w_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \leq& \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} H_\mu(t-t_o,\xi)|w_o(x-\xi)-w_o(x)|\mathrm{d}\xi\,\mathrm{d}x\\
&+\int_{t_o}^t \|H_\mu(t-s)*(a(s)w(s))\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \mathrm{d}s\\
\leq& \int_{\mathbb{R}^n} H_1(1,\zeta)\int_{\mathbb{R}^n} \left|w_o\left(x-(\mu t)^{\frac{n}{2}}\zeta\right)-w_o(x)\right|\mathrm{d}\zeta\,\mathrm{d}x\\
&+A(t)\|a\|_{\mathbf{L}^\infty([t_o,t]\times\mathbb{R}^n;\mathbb{R})}\|w_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}|t-t_o|.
\end{aligned}$$

Both terms above vanish as $t\to t_o$, (in the first use the Dominated Convergence Theorem), completing the proof of continuity in time.

9. Using (4.3), (2.7), the properties of the convolution and point 3. above,

$$\begin{aligned}
&\|\nabla w(t)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\\
\leq& \|\nabla H_\mu(t-t_o)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\\
&+\int_{t_o}^t \|\nabla H_\mu(t-\tau)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \|a(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} \|w(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \mathrm{d}\tau\\
\leq& \frac{J_n}{\sqrt{\mu(t-t_o)}} \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} + \|a\|_{\mathbf{L}^\infty(I\times\mathbb{R}^n;\mathbb{R})} \int_{t_o}^t \frac{J_n A(\tau)}{\sqrt{\mu(t-\tau)}} \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \mathrm{d}\tau\\
\leq& \frac{J_n}{\sqrt{\mu(t-t_o)}} \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \left(1+2(t-t_o)A(t)\|a\|_{\mathbf{L}^\infty(I\times\mathbb{R}^n\mathbb{R})}\right).
\end{aligned}$$

4.2. Proofs related to $\partial_t u+\nabla\cdot(c(t,x)u)=b(t,x)u$. We use below the constant

$$I_n = n\int_0^{\pi/2} (\cos\vartheta)^n \mathrm{d}\vartheta = n\frac{\Gamma((n+1)/2)\,\Gamma(1/2)}{2\Gamma((n+2)/2)}. \tag{4.5}$$

Proof of Lemma 2.7.

We recall [3, Section 3] and follow the proof of [7, Lemma 5.1].

1. Let $u_{o,n} \in \mathbf{C}^1(\mathbb{R}^n;\mathbb{R})$ approximate u_o in the sense that $\|u_{o,n} - u_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \to 0$ as $n \to +\infty$. Call u_n the corresponding quantity as given by (2.9). Then, $\|u_n - u\|_{\mathbf{L}^\infty(I;\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}))} \to 0$ as $n \to +\infty$, so that $u \in \mathbf{L}^\infty(I;\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}))$. Concerning the continuity in time, by (2.9) $u_n \in \mathbf{C}^0(I;\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}))$ and u is the uniform limit of the sequence u_n, hence $u \in \mathbf{C}^0(I;\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}))$.

Using the flow generated by (2.10), introduce the change of variable $y = X(t_o;t,x)$, so that $x = X(t;t_o,y)$. Denote its Jacobian by $J(t,y) = \det(\nabla_y X(t;0,y))$. Then, J solves

$$\frac{\mathrm{d}J(t,y)}{\mathrm{d}t} = \nabla\cdot c(t,X(t;t_o,y))\, J(t,y) \quad \text{with} \quad J(t_o,y) = 1.$$

Hence

$$J(t,y) = \exp\left(\int_{t_o}^{t} \nabla\cdot c(\tau, X(\tau;t_o,y))\,\mathrm{d}\tau\right),$$

so that (2.9) can be written as

$$u(t,x) = \frac{1}{J(t,y)}\, u_o(y)\,\mathcal{B}(t,y) \quad \text{where} \quad \begin{array}{rcl} \mathcal{B}(t,y) &=& \exp\left(\displaystyle\int_{t_o}^{t} b(\tau, X(\tau;t_o,y))\,\mathrm{d}\tau\right) \\ x &=& X(t;t_o,y) \end{array} \tag{4.6}$$

Let $k \in \mathbb{R}$ and $\varphi \in \mathbf{C}^1_c(\mathring{I} \times \mathbb{R}^n;\mathbb{R}^+)$. We prove (2.11) for u given as in (2.9):

$$\begin{aligned}
&\int_{t_o}^{T}\int_{\mathbb{R}^n} [(u-k)(\partial_t\varphi + c\cdot\nabla\varphi) + (bu - k\nabla\cdot c)\varphi]\,\mathrm{sgn}(u-k)\,\mathrm{d}x\,\mathrm{d}t \\
=& \int_{t_o}^{T}\int_{\mathbb{R}^n} \Bigg[\left(\frac{u_o(y)\mathcal{B}(t,y)}{J(t,y)} - k\right)\left(\partial_t\varphi(t,X(t;t_o,y)) + c(t,X(t;t_o,y))\cdot\nabla\varphi(t,X(t;t_o,y))\right) \\
&\quad + \left(b(t,X(t;t_o,y))\,\frac{u_o(y)\mathcal{B}(t,y)}{J(t,y)} - k\nabla\cdot c(t,X(t;t_o,y))\right)\varphi(t,X(t;t_o,y))\Bigg] \\
&\quad \times \mathrm{sgn}\left(\frac{u_o(y)\mathcal{B}(t,y)}{J(t,y)} - k\right) J(t,y)\,\mathrm{d}y\,\mathrm{d}t \\
=& \int_{t_o}^{T}\int_{\mathbb{R}^n} \Bigg[u_o(y)\mathcal{B}(t,y)\,\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t,X(t;t_o,y)) - k\,J(t,y)\,\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t,X(t;t_o,y)) \\
&\quad + \varphi(t,X(t;t_o,y))\,\frac{\mathrm{d}}{\mathrm{d}t}(u_o(y)\mathcal{B}(t,y)) - k\,\varphi(t,X(t;t_o,y))\,\frac{\mathrm{d}}{\mathrm{d}t}J(t,y)\Bigg] \\
&\quad \times \mathrm{sgn}\,(u_o(y)\mathcal{B}(t,y) - k\,J(t,y))\,\mathrm{d}y\,\mathrm{d}t \\
=& \int_{t_o}^{T}\int_{\mathbb{R}^n} \frac{\mathrm{d}}{\mathrm{d}t}\left[(u_o(y)\mathcal{B}(t,y) - k\,J(t,y))\,\varphi(t,X(t;t_o,y))\right] \\
&\quad \times \mathrm{sgn}\,(u_o(y)\mathcal{B}(t,y) - k\,J(t,y))\,\mathrm{d}y\,\mathrm{d}t \\
=& \int_{t_o}^{T}\int_{\mathbb{R}^n} \frac{\mathrm{d}}{\mathrm{d}t}\left(|u_o(y)\mathcal{B}(t,y) - k\,J(t,y)|\,\varphi(t,X(t;t_o,y))\right)\mathrm{d}y\,\mathrm{d}t \\
=& \,0.
\end{aligned}$$

It is immediate to verify that for u as in (2.9) the initial condition holds. By [19, Section 2], u is also a weak solution.

2. Let u be defined as in (2.9) and u_* be a weak solution to (2.4). Then, by the step above, the function $U = u - u_*$ is a weak solution to (2.4) with zero initial datum.

Fix $\tau \in]t_o, T]$, choose any $\varphi \in \mathbf{C}_c^1(\mathring{I} \times \mathbb{R}^n; \mathbb{R})$ and let $\beta_\varepsilon \in \mathbf{C}^1(I;\mathbb{R})$ such that $\beta_\varepsilon(t) = 1$ for $t \in [t_o + \varepsilon, \tau - \varepsilon]$, $\beta'_\varepsilon(t) \in [0, 2/\varepsilon]$ for $t \in [t_o, t_o + \varepsilon]$, $\beta'_\varepsilon(t) \in [-2/\varepsilon, 0]$ for $t \in [\tau - \varepsilon, \tau]$, and $\beta_\varepsilon(t) = 0$ for $t \in [\tau, T]$. Using the definition of weak solution,

$$
\begin{aligned}
0 &= \int_I \int_{\mathbb{R}^n} \left(U\,\partial_t(\varphi\beta_\varepsilon) + U c \cdot \nabla(\varphi\beta_\varepsilon) + bU\varphi\beta_\varepsilon\right) \mathrm{d}x\,\mathrm{d}t \\
&= \int_{t_o}^{\tau} \int_{\mathbb{R}^n} U\left(\partial_t\varphi + c\cdot\nabla\varphi + b\varphi\right)\beta_\varepsilon \,\mathrm{d}x\,\mathrm{d}t + \int_{t_o}^{t_o+\varepsilon} \beta'_\varepsilon \int_{\mathbb{R}^n} U\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_{\tau-\varepsilon}^{\tau} \beta'_\varepsilon \int_{\mathbb{R}^n} U\varphi\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{t_o}^{\tau} \beta_\varepsilon \int_{\mathbb{R}^n} U\left(\partial_t\varphi + c\cdot\nabla\varphi + b\varphi\right)\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^1 \varepsilon\beta'_\varepsilon(\varepsilon t + t_o) \int_{\mathbb{R}^n} U(\varepsilon t + t_o, x)\,\varphi(\varepsilon t + t_o, x)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_{\tau-\varepsilon}^{\tau} \beta'_\varepsilon(t) \int_{\mathbb{R}^n} U(t,x)\,\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

As $\varepsilon \to 0$, the first term converges to $\int_{t_o}^{\tau} \int_{\mathbb{R}^n} U\left(\partial_t\varphi + c\cdot\nabla\varphi + b\varphi\right)\mathrm{d}x\,\mathrm{d}t$. By the Dominated Convergence Theorem, the second term tends to 0, since $U(t_o) = 0$. Concerning the third term, note that

$$
\begin{aligned}
&\left| \int_{\tau-\varepsilon}^{\tau} \beta'_\varepsilon(t) \int_{\mathbb{R}^n} U(t,x)\,\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^n} U(\tau,x)\,\varphi(\tau,x)\,\mathrm{d}x \right| \\
=& \left| \int_{\tau-\varepsilon}^{\tau} \beta'_\varepsilon(t) \int_{\mathbb{R}^n} \left(U(t,x)\,\varphi(t,x) - U(\tau,x)\,\varphi(\tau,x)\right)\mathrm{d}x\,\mathrm{d}t \right| \to 0 \quad \text{as} \quad \varepsilon \to 0
\end{aligned}
$$

by the continuity of U in time and the smoothness of φ.

Choose $\eta \in \mathbf{C}_c^1(\mathbb{R}^n;\mathbb{R})$ and compute φ integrating backward along characteristics in

$$
\begin{cases}
\partial_t\varphi + c\cdot\nabla\varphi = -b\varphi \\
\varphi(\tau) = \eta
\end{cases}
$$

as in 3. in the proof of [7, Lemma 5.1]. Then,

$$
0 = \int_{t_o}^{\tau} \int_{\mathbb{R}^n} U\left(\partial_t\varphi + c\cdot\nabla\varphi + b\varphi\right)\mathrm{d}x\,\mathrm{d}t - \int_{\mathbb{R}^n} U(\tau,x)\,\varphi(\tau,x)\,\mathrm{d}x = -\int_{\mathbb{R}^n} U(\tau,x)\,\eta(x)\,\mathrm{d}x
$$

proving that $U(\tau)$ vanishes identically. By the arbitrariness of τ, $u = u_*$.

Proof of Proposition 2.8.

Define $g(t,x) = b(t,x) - \nabla\cdot c(t,x)$. The equation in (2.4) fits into the general form

$$
\partial_t u + \nabla\cdot f(t,x,u) = F(t,x,u) \quad \text{with} \quad
\begin{array}{rcl}
f(t,x,u) &=& c(t,x)\,u \\
F(t,x,u) &=& b(t,x)\,u.
\end{array}
$$

Points **1.** and **2.** follow, for instance, from [19, Theorem 1 and Theorem 2].

3. Start from [19, Formula (3.1)], since 0 solves the equation in (2.4), we get

$$
\left\|\mathcal{H}_{t_o,t} u_o\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \le \left\|u_o\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \exp\left(\|b\|_{\mathbf{L}^\infty([t_o,t]\times\mathbb{R}^n;\mathbb{R})}(t - t_o)\right).
$$

4. From (2.9) it easily follows that

$$\|\mathcal{H}_{t_o,t}u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} \leq \|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} \exp\Big(\|g\|_{\mathbf{L}^\infty([t_o,t]\times\mathbb{R}^n;\mathbb{R})}(t-t_o)\Big).$$

5. We refer to [9, Theorem 2.6], see also [10], as refined in [20, Proposition 2.10] and [21, Proposition 2.9]. Indeed, compute preliminarily

$$\begin{array}{rlrl}
\partial_u f(t,x,u) = c(t,x) & & \nabla\cdot f(t,x,u) = & (\nabla\cdot c(t,x))\, u \\
\partial_u \nabla f(t,x,u) = \nabla c(t,x) & & \nabla^2 f(t,x,u) = & \big(\nabla^2 c(t,x)\big)\, u \\
\partial_u F(t,x,u) = b(t,x) & & \nabla F(t,x,u) = & (\nabla b(t,x))\, u \\
F(t,x,u)-\nabla\cdot f(t,x,u) = (b(t,x)-\nabla\cdot c(t,x))u & & = & g(t,x)u \\
\partial_u (F(t,x,u)-\nabla\cdot f(t,x,u)) = b(t,x)-\nabla\cdot c(t,x) & & = & g(t,x) \\
\nabla(F(t,x,u)-\nabla\cdot f(t,x,u)) = (\nabla b(t,x)-\nabla(\nabla\cdot c(t,x)))u & & = & \nabla g(t,x)u
\end{array} \tag{4.7}$$

and introduce the quantities

$$\begin{aligned}
\kappa_o^* &= (2n+1)\|\nabla c_1\|_{\mathbf{L}^\infty([t_o,t]\times\mathbb{R}^n;\mathbb{R}^{n\times n})} + \|b_1\|_{\mathbf{L}^\infty([t_o,t]\times\mathbb{R}^n;\mathbb{R})} \\
\kappa^* &= \|b_1\|_{\mathbf{L}^\infty([t_o,t]\times\mathbb{R}^n;\mathbb{R})} + \|\nabla\cdot(c_1-c_2)\|_{\mathbf{L}^\infty([t_o,t]\times\mathbb{R}^n;\mathbb{R})} \\
\kappa_1^* &= \|b_1\|_{\mathbf{L}^\infty([t_o,t]\times\mathbb{R}^n;\mathbb{R})} + \|\nabla\cdot(c_1-c_2)\|_{\mathbf{L}^\infty([t_o,t]\times\mathbb{R}^n;\mathbb{R})} + (2n+1)\|\nabla c_1\|_{\mathbf{L}^\infty([t_o,t]\times\mathbb{R}^n;\mathbb{R}^{n\times n})}.
\end{aligned}$$

For $i=1,2$, let b_i satisfy **(b)** with $b_1-b_2\in\mathbf{L}^1(I\times\mathbb{R}^n;\mathbb{R})$ and c_i satisfy **(c)** with $\nabla\cdot(c_1-c_2)\in\mathbf{L}^1(I\times\mathbb{R}^n;\mathbb{R})$. Then, it is immediate to check that the requirements in [20, Section 2] and in [21, Section 2] hold. Indeed, with obvious notation,

$$\left.\begin{array}{r}
c_i\in\mathbf{L}^\infty(I\times\mathbb{R}^n;\mathbb{R}^n);\, c_i\in\mathbf{C}^2(I\times\mathbb{R}^n;\mathbb{R}^n);\, b_i\in\mathbf{C}^1(I\times\mathbb{R}^n;\mathbb{R}) \\
(b_i-\nabla\cdot c_i)\in\mathbf{L}^\infty(I\times\mathbb{R}^n;\mathbb{R})
\end{array}\right\}\Rightarrow \textbf{(H1*)}$$

$$\left.\begin{array}{r}
\nabla c_1\in\mathbf{L}^\infty(I\times\mathbb{R}^n;\mathbb{R}^{n\times n});\, b_1\in\mathbf{L}^\infty(I\times\mathbb{R}^n;\mathbb{R}); \\
\nabla b_1-\nabla(\nabla\cdot c_1)\in\mathbf{L}^1(I\times\mathbb{R}^n;\mathbb{R}^n)
\end{array}\right\}\Rightarrow \textbf{(H2*)}$$

$$\left.\begin{array}{r}
c_1-c_2\in\mathbf{L}^\infty(I\times\mathbb{R}^n;\mathbb{R}^n) \\
(b_1-b_2)\in\mathbf{L}^\infty(I\times\mathbb{R};\mathbb{R});\, ((b_1-b_2)-\nabla\cdot(c_1-c_2))\in\mathbf{L}^1(I\times\mathbb{R}^n;\mathbb{R})
\end{array}\right\}\Rightarrow \textbf{(H3*)}$$

Note also that

$$\frac{\kappa_o^* e^{\kappa_o^* t}-\kappa^* e^{\kappa * t}}{\kappa_o^*-\kappa^*} = e^{\kappa_o^* t} + \frac{\kappa^*\left(e^{\kappa_o^* t}-e^{\kappa^* t}\right)}{\kappa_o^*-\kappa^*} \leq (1+\kappa^* t)e^{\kappa_1^* t} \leq e^{\kappa^* t}e^{\kappa_1^* t}.$$

Applying [20, Proposition 2.10] and [21, Proposition 2.9] we now obtain

$$\begin{aligned}
&\left\|\mathcal{H}^1_{t_o,t}u_o-\mathcal{H}^2_{t_o,t}u_o\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \\
&\leq e^{(\kappa^*+\kappa_1^*)(t-t_o)}\|c_1-c_2\|_{\mathbf{L}^1([t_o,t];\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^n))} \\
&\quad\times\left[\mathrm{TV}(u_0)+I_n\int_{t_o}^t e^{-\kappa_o^*(\tau-t_o)}\left\|\mathcal{H}^1_{t_o,\tau}u_o\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\|\nabla g_1(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^n)}\,\mathrm{d}\tau\right] \\
&\quad+e^{\kappa^*(t-t_o)}\int_{t_o}^t\|g_1(\tau)-g_2(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\max_{i=1,2}\left\|\mathcal{H}^i_{t_o,\tau}u_o\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\mathrm{d}\tau
\end{aligned}$$

where $\left\|\mathcal{H}^i_{t_o,t}(u_o)\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}$ is estimated in 4., $g_i=b_i-\nabla\cdot c_i$, I_n is as in (4.5).

6. Directly follows from (2.9).

7. By assumptions, b satisfies **(b)** and c satisfies **(c)**, hence, by (4.7), both **(H1*)** and **(H2*)** hold. From [9, Theorem 2.5] or [21, Theorem 2.2], it directly follows that

$$\mathrm{TV}(\mathcal{H}_{t_o,t}u_o) \le \mathrm{TV}(u_o) e^{\kappa_o^*(t-t_o)} + I_n \int_{t_o}^t e^{\kappa_o^*(t-\tau)} \|\nabla g(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^n)} \|\mathcal{H}_{t_o,\tau}u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} \mathrm{d}\tau \tag{4.8}$$

where $\|\mathcal{H}_{t_o,t}(u_o)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}$ is estimated in 4., I_n is as in (4.5) and

$$\kappa_o^* = (2n+1)\|\nabla c\|_{\mathbf{L}^\infty([t_o,t]\times\mathbb{R}^n;\mathbb{R}^{n\times n})} + \|b\|_{\mathbf{L}^\infty([t_o,t]\times\mathbb{R}^n;\mathbb{R})}.$$

8. Assume that $t_1 < t_2$. From Definition 2.6, we have that

$$\int_{t_i}^T \int_{\mathbb{R}^n} (u\partial_t\varphi + u c\cdot\nabla\varphi + b u\varphi)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^n} \varphi(t_i,x)\,u(t_i,x)\,\mathrm{d}x = 0. \tag{4.9}$$

Following the proof of [11, Theorem 4.3.1], let $\varphi(t,x) = \chi(t)\psi(x)$ with $\chi \in \mathbf{C}_c^1(\mathring{I};\mathbb{R})$, $\chi(t) = 1$ for $t \in [t_1,t_2]$, $\psi \in \mathbf{C}_c^1(\mathbb{R}^n;\mathbb{R})$ with $|\psi(x)| \le 1$ for $x \in \mathbb{R}^n$. Subtract (4.9) for $i=1$ from (4.9) for $i=2$, use [4, Proposition 3.2] and the estimates at 3., 4., and 7. to obtain:

$$\begin{aligned}
&\|u(t_2)-u(t_1)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \\
={}& \sup_{\psi\in\mathbf{C}_c^1, |\psi(x)|\le 1} \int_{\mathbb{R}^n} \psi(x)\,(u(t_2,x)-u(t_1,x))\,\mathrm{d}x \\
={}& \sup_{\psi\in\mathbf{C}_c^1, |\psi(x)|\le 1} \int_{t_1}^{t_2}\int_{\mathbb{R}^n} (u(t,x)\,c(t,x)\cdot\nabla\psi(x) + b(t,x)\,u(t,x)\,\psi(x))\,\mathrm{d}x\,\mathrm{d}t \\
={}& \sup_{\psi\in\mathbf{C}_c^1, |\psi(x)|\le 1} \int_{t_1}^{t_2}\int_{\mathbb{R}^n} [-\nabla(u(t,x)c(t,x))\,\psi(x) + b(t,x)u(t,x)\psi(x)]\,\mathrm{d}x\,\mathrm{d}t \\
\le{}& \int_{t_1}^{t_2}\int_{\mathbb{R}^n} [\|\nabla(u(t,x)c(t,x))\| + |b(t,x)u(t,x)|]\,\mathrm{d}t\,\mathrm{d}x \\
\le{}& \int_{t_1}^{t_2}\left[\int_{\mathbb{R}^n}\|u(t,x)\nabla c(t,x)\|\,\mathrm{d}x + \int_{\mathbb{R}^n}\|c(t,x)\|\,\mathrm{d}(\nabla u(t))\,(x) + \int_{\mathbb{R}^n}|b(t,x)u(t,x)|\,\mathrm{d}x\right]\mathrm{d}t \\
\le{}& \int_{t_1}^{t_2} \|u(t)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\left[\|b(t)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} + \|\nabla c(t)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n\times n})}\right]\mathrm{d}t \\
&\quad + \int_{t_1}^{t_2} \|c(t)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^n)}\,\mathrm{TV}(u(t))\,\mathrm{d}t \\
\le{}& \|u_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} e^{\|b\|_{\mathbf{L}^\infty([t_o,t_2]\times\mathbb{R}^n;\mathbb{R})}(t_2-t_o)} \\
&\quad \times\left[\|b\|_{\mathbf{L}^\infty([t_1,t_2]\times\mathbb{R}^n;\mathbb{R})} + \|\nabla c\|_{\mathbf{L}^\infty([t_1,t_2]\times\mathbb{R}^n;\mathbb{R}^{n\times n})}\right]|t_2-t_1| \\
&\quad + \|c\|_{\mathbf{L}^\infty([t_1,t_2]\times\mathbb{R}^n;\mathbb{R}^n)} \int_{t_1}^{t_2} \mathrm{TV}(u_o) e^{\kappa_o^*(t-t_o)}\,\mathrm{d}t \\
&\quad + I_n \|c\|_{\mathbf{L}^\infty([t_1,t_2]\times\mathbb{R}^n;\mathbb{R}^n)} \int_{t_1}^{t_2}\int_{t_o}^{t} e^{\kappa_o^*(t-\tau)} \|\nabla g(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^n)} \|u(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\,\mathrm{d}\tau\,\mathrm{d}t \\
\le{}& |t_2-t_1| \left[\|c\|_{\mathbf{L}^\infty([t_1,t_2]\times\mathbb{R}^n;\mathbb{R}^n)}\,\mathrm{TV}(u_o) e^{\kappa_o^*(t_2-t_o)} \right. \\
&\quad + \|u_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} e^{\|b\|_{\mathbf{L}^\infty([t_o,t_2]\times\mathbb{R}^n;\mathbb{R})}(t_2-t_o)} \left[\|b\|_{\mathbf{L}^\infty([t_1,t_2]\times\mathbb{R}^n;\mathbb{R})} + \|\nabla c\|_{\mathbf{L}^\infty([t_1,t_2]\times\mathbb{R}^n;\mathbb{R}^{n\times n})}\right]
\end{aligned}$$

$$+I_n \|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} \|c\|_{\mathbf{L}^\infty([t_1,t_2]\times\mathbb{R}^n;\mathbb{R}^n)}$$
$$\times \int_{t_o}^{t_2} \|\nabla g(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^n)} e^{\kappa_o^*(t_2-\tau)+\|g\|_{\mathbf{L}^\infty([t_o,\tau]\times\mathbb{R}^n;\mathbb{R})}(\tau-t_o)} \mathrm{d}\tau \Bigg]$$

proving point 8.

9. Directly follows from (2.9), since, by **(c)**, the speed of characteristics is bounded.

4.3. Proof of Theorem 2.2.

We are going to construct a solution to (1.1) as limit of a Cauchy sequence of approximate solutions. For $r>0$, we introduce the domain

$$\mathcal{X}_r = \left\{ (u,w) \in \mathcal{X}^+ \colon \begin{array}{ll} \|u\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} \le r, & \mathrm{TV}(u) \le r \\ \|w\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} \le r, & \|w\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \le r \end{array} \right\}.$$

Choose an initial datum

$$(u_o,w_o) \in \mathcal{X}_r \quad \text{with moreover} \quad w_o \in (\mathbf{C}^1 \cap \mathbf{W}^{1,1})(\mathbb{R}^n;\mathbb{R}^+), \tag{4.10}$$

and set, for $t\in[0,T]$, $(u_0(t),w_0(t)) = (u_o,w_o)$. For $i\in\mathbb{N}$, define for $(t,x)\in[0,T]\times\mathbb{R}^n$,

$$a_{i+1}(t,x) = \gamma - \delta\, u_i(t,x), \quad b_{i+1}(t,x) = \alpha\, w_i(t,x) - \beta, \quad \text{and} \quad c_{i+1}(t,x) = v\left(w_i(t)\right)(x),$$

and let (u_{i+1},w_{i+1}) be such that

$$\begin{cases} \partial_t u_{i+1} + \nabla\cdot(c_{i+1}u_{i+1}) = b_{i+1}\, u_{i+1} \\ \partial_t w_{i+1} - \mu\Delta w_{i+1} = a_{i+1}\, w_{i+1} \\ u_{i+1}(0) = u_o \\ w_{i+1}(0) = w_o. \end{cases}$$

Claim 0: For all $i\in\mathbb{N}$,
C0.1 (u_i,w_i) is well defined and in $\mathbf{L}^1([0,T];\mathcal{X}^+)$;
C0.2 $w_i \in \mathbf{C}^1([0,T]\times\mathbb{R}^n;\mathbb{R})$ and $\nabla w_i \in \mathbf{L}^1([0,T]\times\mathbb{R}^n;\mathbb{R}^n)$;
C0.3 $a_{i+1}\in\mathbf{L}^\infty([0,T]\times\mathbb{R}^n;\mathbb{R})$;
C0.4 b_{i+1} satisfies **(b)** with $I=[0,T]$;
C0.5 c_{i+1} satisfies **(c)** with $I=[0,T]$.

Proof of Claim 0. We prove it by induction.
Case $i=0$**:** is immediate by (4.10) and by the above definition of a_1,b_1,c_1.
From $i-1$ **to** i**:** Assume now that C0.1, … , C0.5 are all satisfied up to the i-th iteration. Then, Proposition 2.5 and Proposition 2.8 can now be applied, proving C0.1. Moreover, by 7. and 9. in Proposition 2.5, also C0.2 holds. Furthermore, the estimate at 3. and 4. in Proposition 2.8 ensure that C0.3 holds. Moreover, C0.1 and C0.2 directly imply C0.4 and, together with $\boldsymbol{(v)}$, also C0.5, completing the proof of the present claim.

Hence, thanks to $\boldsymbol{(v)}$, it clearly follows that:

$$a_{i+1}-a_i,\, b_{i+1}-b_i,\, \nabla\cdot(c_{i+1}-c_i) \in \mathbf{L}^1(I\times\mathbb{R}^n;\mathbb{R}). \tag{4.11}$$

In the next two claims we particularize the $\mathbf{L}^1$ and $\mathbf{L}^\infty$ estimates in 3. and 4. of Propositions 2.5 and 2.8, thanks to the explicit expressions of a, b and c.

Claim 1: For all $i\in\mathbb{N}$, if w_i is defined up to time $\hat{T}$, then for all $t\in[0,\hat{T}]$,

$$\|w_i(t)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\leq\|w_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\,e^{\gamma t}\quad\text{and}\quad\|w_i(t)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\leq\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\,e^{\gamma t}.$$

Proof of Claim 1. Assume $i>0$, the case $i=0$ being obvious. By (2.7),

$$\begin{aligned}
w_i(t,x)&=\int_{\mathbb{R}^n}H_\mu(t,x-\xi)\,w_o(\xi)\,\mathrm{d}\xi+\int_0^t\int_{\mathbb{R}^n}H_\mu(t-\tau,x-\xi)\,(\gamma-\delta\,u_{i-1}(\tau,\xi))\,w_i(\tau,\xi)\,\mathrm{d}\xi\,\mathrm{d}\tau\\
&\leq\int_{\mathbb{R}^n}H_\mu(t,x-\xi)\,w_o(\xi)\,\mathrm{d}\xi+\int_0^t\int_{\mathbb{R}^n}\gamma\,H_\mu(t-\tau,x-\xi)\,w_i(\tau,\xi)\,\mathrm{d}\xi\,\mathrm{d}\tau\,.
\end{aligned}$$

By Gronwall's lemma and (4.2):

$$\begin{aligned}
w_i(t,x)&\leq\left(\int_{\mathbb{R}^n}H_\mu(t,x-\xi)\,w_o(\xi)\,\mathrm{d}\xi\right)\exp\left(\int_0^t\int_{\mathbb{R}^n}\gamma\,H_\mu(t-\tau,x-\xi)\,\mathrm{d}\xi\,\mathrm{d}\tau\right)\\
&=e^{\gamma t}\int_{\mathbb{R}^n}H_\mu(t,x-\xi)\,w_o(\xi)\,\mathrm{d}\xi\,.
\end{aligned}$$

The proof of the claim follows.

Claim 2: For all $i\in\mathbb{N}$, if u_i is defined up to time $\hat{T}$, then for all $t\in[0,\hat{T}]$,

$$\begin{aligned}
\|u_i(t)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}&\leq\|u_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\exp\left(\alpha\,\frac{e^{\gamma t}-1}{\gamma}\,\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\right)\\
&\leq\|u_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\exp\left(\alpha\,t\,e^{\gamma t}\,\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\right)\\
\|u_i(t)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}&\leq\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\exp\left((\alpha+K)\,\frac{e^{\gamma t}-1}{\gamma}\,\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\right)\\
&\leq\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\exp\left((\alpha+K)\,t\,e^{\gamma t}\,\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\right).
\end{aligned}$$

Proof of Claim 2. Assume $i>0$, the case $i=0$ being obvious. By $\boldsymbol{(v)}$ and Claim 0, we can apply Lemma 2.7, and by (2.9) we obtain

$$u_i(t,x)\leq u_o\,(X(0;t,x))\exp\int_0^t(\alpha\,w_{i-1}\,(\tau,X(\tau;t,x))-\nabla\cdot c_i\,(\tau,X(\tau;t,x))\,\mathrm{d}\tau)\,. \tag{4.12}$$

To obtain the $\mathbf{L}^1$ estimate, we adopt the notation in (4.6) with $t_o=0$, $b=\alpha w_{i-1}$ and $c=c_i$, so that, using Claim 1 above, we have

$$\begin{aligned}
\|u_i(t)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}&\leq\int_0^t\frac{1}{J(\tau,y)}\,u_o(y)\,\mathcal{B}(\tau,y)\,J(\tau,y)\,\mathrm{d}\tau\\
&\leq\|u_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\exp\left(\int_0^t\alpha\,\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\,e^{\gamma\tau}\,\mathrm{d}\tau\right)\\
&=\|u_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\exp\left(\alpha\,\frac{e^{\gamma t}-1}{\gamma}\,\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\right).
\end{aligned}$$

The $\mathbf{L}^\infty$ estimate is obtained from (4.12) using $\boldsymbol{(v)}$ and Claim 1:

$$\|u_i(t)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\leq\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\exp\int_0^t\Big(\alpha\,\|w_{i-1}(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}+K\,\|w_{i-1}(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\Big)\,\mathrm{d}\tau$$

$$
\begin{aligned}
&\le \|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} \exp \int_0^t (\alpha+K)\,\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\, e^{\gamma\tau}\,\mathrm{d}\tau \\
&= \|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} \exp\left((\alpha+K)\,\frac{e^{\gamma t}-1}{\gamma}\,\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\right),
\end{aligned}
$$

completing the proof of Claim 2.

Claim 3: For all $i\in\mathbb{N}$, if u_i is defined up to time $\hat{T}$, then for all $t\in[0,\hat{T}]$ we have $\mathrm{TV}(u_i(t))\le\mathcal{F}(t)$, where $\mathcal{F}\in\mathbf{C}^0([0,\hat{T}];\mathbb{R}^+)$ depends only on the hypotheses and on the initial data, being in particular independent of i.

Proof of Claim 3. Apply 7. in 2.8 in the explicit form (4.8) to obtain

$$
\begin{aligned}
\mathrm{TV}(u_i(t)) &\le \left[\mathrm{TV}(u_o) + I_n \int_0^t \|\nabla(b_i-\nabla\cdot c_i)(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^n)} \|u_i(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\,\mathrm{d}\tau\right] e^{\kappa_o^* t} \\
&\le \mathrm{TV}(u_o)e^{\kappa_o^* t} \\
&\quad + I_n\left[\|\nabla b_i\|_{\mathbf{L}^1([0,t]\times\mathbb{R}^n;\mathbb{R}^n)} + \|\nabla\nabla\cdot c_i\|_{\mathbf{L}^1([0,t]\times\mathbb{R}^n;\mathbb{R}^n)}\right]\|u_i\|_{\mathbf{L}^\infty([0,t]\times\mathbb{R}^n;\mathbb{R})} e^{\kappa_o^* t}.
\end{aligned}
$$

All terms in the right hand side above are estimated by means of quantities independent of i. More precisely, using ***(v)*** and Claim 1:

$$
\begin{aligned}
\kappa_o^* &= (2n+1)\|\nabla c_i\|_{\mathbf{L}^\infty([0,t]\times\mathbb{R}^n;\mathbb{R}^{n\times n})} + \|b_i\|_{\mathbf{L}^\infty([0,t]\times\mathbb{R}^n;\mathbb{R})} \\
&\le ((2n+1)K+\alpha)\,\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} e^{\gamma t} + \beta.
\end{aligned}
$$

Observe that $\|\nabla b_i\|_{\mathbf{L}^1([0,t]\times\mathbb{R}^n;\mathbb{R}^n)} = \alpha\,\|\nabla w_{i-1}\|_{\mathbf{L}^1([0,t]\times\mathbb{R}^n;\mathbb{R}^n)}$. Recall the proof of 9. in Proposition 2.5. Note preliminarily that, since $a_i(\tau,x)=\gamma-\delta\,u_{i-1}(\tau,x)$, by Claim 2

$$
\begin{aligned}
\|a_i\|_{\mathbf{L}^\infty([0,t]\times\mathbb{R}^n;\mathbb{R})} &\le \gamma+\delta\,\|u_{i-1}\|_{\mathbf{L}^\infty([0,t]\times\mathbb{R}^n;\mathbb{R})} \\
&\le \gamma+\delta\,\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} \exp\left((\alpha+K)\,\frac{e^{\gamma t}-1}{\gamma}\,\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\right). \\
&\le \gamma+\delta\,\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} \exp\Big((\alpha+K)\,\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\, t\, e^{\gamma t}\Big).
\end{aligned}
\tag{4.13}
$$

Use Claim 1, (4.13), and the expression (4.1) of J_n in the proof of 9. in Proposition 2.5:

$$
\begin{aligned}
&\|\nabla w_{i-1}(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^n)} \\
\le{}& \|\nabla H_\mu(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \\
&+ \int_0^\tau \|\nabla H_\mu(\tau-s)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\, \|a_{i-1}(s)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\, \|w_{i-1}(s)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\,\mathrm{d}s \\
\le{}& \|\nabla H_\mu(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \\
&+ \|a_{i-1}\|_{\mathbf{L}^\infty([0,\tau]\times\mathbb{R}^n;\mathbb{R})} \int_0^\tau \|\nabla H_\mu(\tau-s)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\, e^{\gamma s}\, \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\,\mathrm{d}s \\
\le{}& \|\nabla H_\mu(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} + \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\, e^{\gamma\tau} \int_0^\tau \|\nabla H_\mu(\tau-s)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\,\mathrm{d}s \\
&\times\left(\gamma+\delta\,\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} e^{(\alpha+K)\,\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\tau\, e^{\gamma\tau}}\right) \\
\le{}& \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\left[\frac{J_n}{\sqrt{\mu\tau}} + e^{\gamma\tau}\left(\gamma+\delta\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} e^{(\alpha+K)\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\tau\, e^{\gamma\tau}}\right)\int_0^\tau \frac{J_n}{\sqrt{\mu(\tau-s)}}\,\mathrm{d}s\right]
\end{aligned}
$$

$$\leq \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \frac{J_n}{\sqrt{\mu\tau}} \left[1 + 2\tau e^{\gamma\tau} \left(\gamma + \delta \|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} e^{(\alpha+K)\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} \tau e^{\gamma\tau}}\right)\right].$$

Therefore

$$\begin{aligned} &\|\nabla b_i\|_{\mathbf{L}^1([0,t]\times\mathbb{R}^n;\mathbb{R}^n)} \\ \leq\; & \frac{\alpha J_n \sqrt{t}}{\sqrt{\mu}} \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \left[1 + 2t e^{\gamma t} \left(\gamma + \delta \|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} e^{(\alpha+K)\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} t e^{\gamma t}}\right)\right]. \end{aligned} \tag{4.14}$$

Recall that $\nabla(\nabla\cdot c_i) = \nabla(\nabla \cdot v(w_{i-1}))$. By **(*v*)** and Claim 1,

$$\begin{aligned} \|\nabla(\nabla\cdot c_i)\|_{\mathbf{L}^1([0,t]\times\mathbb{R}^n;\mathbb{R}^n)} &\leq \left\| C\left(\|w_{i-1}\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\right) \|w_{i-1}\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \right\|_{\mathbf{L}^1([0,t];\mathbb{R})} \\ &\leq \left\| C\left(\|w_{i-1}\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\right)\right\|_{\mathbf{L}^\infty([0,t];\mathbb{R})} \|w_{i-1}\|_{\mathbf{L}^1([0,t]\times\mathbb{R}^n;\mathbb{R})} \\ &\leq C\left(\|w_{i-1}\|_{\mathbf{L}^\infty([0,t];\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}))}\right) \|w_{i-1}\|_{\mathbf{L}^1([0,t]\times\mathbb{R}^n;\mathbb{R})} \\ &\leq C\left(\|w_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} e^{\gamma t}\right) \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} t e^{\gamma t}. \end{aligned} \tag{4.15}$$

Concerning $\|u_i\|_{\mathbf{L}^\infty([0,t]\times\mathbb{R}^n;\mathbb{R})}$, it is bounded by Claim 2, completing the proof of Claim 3.

The sequence of approximate solutions we construct belongs to the set

$$X = \left\{(u,w) \in \mathbf{L}^1([0,T];\mathcal{X}^+) \colon \operatorname{TV}(u(t)) \leq \mathcal{F}(t) \text{ for all } t \in [0,T]\right\}, \tag{4.16}$$

which is a complete metric space with the distance

$$\begin{aligned} d((u_1,w_1),(u_2,w_2)) &= \|u_2 - u_1\|_{\mathbf{L}^1([0,T];\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}))} + \|w_2 - w_1\|_{\mathbf{L}^1([0,T];\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}))} \\ &= \int_0^T \int_{\mathbb{R}^n} \left(|u_2(t,x) - u_1(t,x)| + |w_2(t,x) - w_1(t,x)|\right) \mathrm{d}x\, \mathrm{d}t. \end{aligned}$$

We now prove that there exist positive T and $\mathcal{K}(T,r)$ such that for all $i \in \mathbb{N}$,

$$d((u_{i+1},w_{i+1}),(u_i,w_i)) \leq T\mathcal{K}(T,r)\, d((u_i,w_i),(u_{i-1},w_{i-1})). \tag{4.17}$$

By (4.11), recall the proof of 5. in Proposition 2.5 and apply the $\mathbf{L}^\infty$ estimate in Claim 1:

$$\begin{aligned} \|w_{i+1}(t) - w_i(t)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} &\leq \int_0^t \|a_{i+1}(\tau) - a_i(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \|w_{i+1}(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} \mathrm{d}\tau \\ &\quad + \int_0^t \|a_i(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} \|w_{i+1}(\tau) - w_i(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \mathrm{d}\tau \\ &\leq \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} \int_0^t e^{\gamma\tau} \|a_{i+1}(\tau) - a_i(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \mathrm{d}\tau \\ &\quad + \int_0^t \|a_i(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} \|w_{i+1}(\tau) - w_i(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \mathrm{d}\tau. \end{aligned}$$

Apply Gronwall Lemma:

$$\begin{aligned} &\|w_{i+1}(t) - w_i(t)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \\ \leq\; & \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} e^{\gamma t} \|a_{i+1} - a_i\|_{\mathbf{L}^1([0,t]\times\mathbb{R}^n;\mathbb{R})} \exp\left(\|a_i\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}))}\right) \end{aligned}$$

$$\leq \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} e^{\gamma t} \|a_{i+1}-a_i\|_{\mathbf{L}^1([0,t]\times\mathbb{R}^n;\mathbb{R})} \exp\Big(t\|a_i\|_{\mathbf{L}^\infty([0,t]\times\mathbb{R}^n;\mathbb{R})}\Big).$$

Hence, recalling (4.13), we obtain

$$\begin{aligned}
&\|w_{i+1}(t)-w_i(t)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\\
&\leq \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\,\delta\,\|u_i-u_{i-1}\|_{\mathbf{L}^1([0,t];\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}))}\\
&\qquad\times\exp\left(2\gamma t+\delta t\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\exp\left((\alpha+K)\,\frac{e^{\gamma t}-1}{\gamma}\,\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\right)\right);\\
&\|w_{i+1}-w_i\|_{\mathbf{L}^1([0,T];\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}))}\\
&\leq \delta T\,\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\,\|u_i-u_{i-1}\|_{\mathbf{L}^1([0,T];\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}))}\\
&\qquad\times\exp\left(2\gamma T+\delta T\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\exp\left((\alpha+K)\,\frac{e^{\gamma T}-1}{\gamma}\,\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\right)\right)\\
&\leq T\,\mathcal{K}_w(T,r)\,\|u_i-u_{i-1}\|_{\mathbf{L}^1([0,T];\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}))},
\end{aligned}\tag{4.18}$$

since $(u_o,w_o)\in\mathcal{X}_r$, where

$$\mathcal{K}_w(T,r)=\delta\,r\exp\left[2\gamma T+\delta T\,r\exp\left((\alpha+K)\,\frac{e^{\gamma T}-1}{\gamma}\,r\right)\right].\tag{4.19}$$

We now pass to estimate $\|u_{i+1}-u_i\|_{\mathbf{L}^1([0,T];\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}))}$. To this aim, by (4.11), we start from 5. in Proposition 2.8 and use Claim 2 above:

$$\begin{aligned}
&\|u_{i+1}(t)-u_i(t)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\\
&\leq e^{\kappa^* t}e^{\kappa_1^* t}\,\|c_{i+1}-c_i\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^n))}\\
&\quad\times\left[\mathrm{TV}(u_0)+I_n\int_0^t e^{-\kappa_o^*\tau}\|u_{i+1}(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\,\|\nabla b_{i+1}(\tau)-\nabla(\nabla\cdot c_{i+1}(\tau))\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^n)}\,\mathrm{d}\tau\right]\\
&\quad+e^{\kappa^* t}\int_0^t \|b_{i+1}(\tau)-b_i(\tau)-\nabla\cdot(c_{i+1}(\tau)-c_i(\tau))\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\\
&\quad\times\max\left\{\|u_{i+1}(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})},\|u_i(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\right\}\mathrm{d}\tau\\
&\leq e^{\kappa^* t}e^{\kappa_1^* t}\,\|c_{i+1}-c_i\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^n))}\left[\mathrm{TV}(u_o)+I_n\,\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\right.\\
&\quad\left.\times\int_0^t e^{(\alpha+K)\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\tau e^{\gamma\tau}}\,\|\nabla b_{i+1}(\tau)-\nabla(\nabla\cdot c_{i+1}(\tau))\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^n)}\,\mathrm{d}\tau\right]\\
&\quad+e^{\kappa^* t}\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\\
&\quad\times\int_0^t e^{(\alpha+K)\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\tau e^{\gamma\tau}}\,\|b_{i+1}(\tau)-b_i(\tau)-\nabla\cdot(c_{i+1}(\tau)-c_i(\tau))\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\,\mathrm{d}\tau.
\end{aligned}$$

We proceed estimating all terms appearing in the inequality above. Begin by 5. in Proposition 2.8, **(v)** and Claim 1:

$$\begin{aligned}
\kappa^* &= \|b_{i+1}\|_{\mathbf{L}^\infty([0,t]\times\mathbb{R}^n;\mathbb{R})}+\|\nabla\cdot(c_{i+1}-c_i)\|_{\mathbf{L}^\infty([0,t]\times\mathbb{R}^n;\mathbb{R})}\\
&\leq \alpha\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}e^{\gamma t}+\beta+\|\nabla\cdot c_{i+1}\|_{\mathbf{L}^\infty([0,t]\times\mathbb{R}^n;\mathbb{R})}+\|\nabla\cdot c_i\|_{\mathbf{L}^\infty([0,t]\times\mathbb{R}^n;\mathbb{R})}\\
&\leq \alpha\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}e^{\gamma t}+\beta+2K\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}e^{\gamma t}\\
&=\lambda\qquad\text{where}\qquad \lambda=(\alpha+2K)\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}e^{\gamma t}+\beta;
\end{aligned}\tag{4.20}$$

$$\begin{aligned}\kappa_1^* &= \|b_{i+1}\|_{\mathbf{L}^\infty([0,t]\times\mathbb{R}^n;\mathbb{R})} + \|\nabla\cdot(c_{i+1}-c_i)\|_{\mathbf{L}^\infty([0,t]\times\mathbb{R}^n;\mathbb{R})} + (2n+1)\|\nabla c_{i+1}\|_{\mathbf{L}^\infty([0,t]\times\mathbb{R}^n;\mathbb{R})}\\ &\le (\alpha+2K)\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} e^{\gamma t} + \beta + (2n+1)K\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} e^{\gamma t}\\ &= \lambda_1 \qquad \text{where} \qquad \lambda_1 = (\alpha+(2n+3)K)\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} e^{\gamma t} + \beta. \end{aligned} \tag{4.21}$$

By $\boldsymbol{(v)}$ and Claim 1,

$$\begin{aligned}\|c_{i+1}-c_i\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^n))} &= \|v(w_i)-v(w_{i-1})\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^n))}\\ &\le K\|w_i-w_{i-1}\|_{\mathbf{L}^1([0,t];\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}))}.\end{aligned} \tag{4.22}$$

By (4.14) and (4.15), we obtain

$$\begin{aligned}&\int_0^t e^{(\alpha+K)\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\tau e^{\gamma\tau}} \|\nabla b_{i+1}(\tau) - \nabla(\nabla\cdot c_{i+1}(\tau))\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^n)}\,\mathrm{d}\tau\\ &\le e^{(\alpha+K)\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} t e^{\gamma t}} \|\nabla b_{i+1} - \nabla(\nabla\cdot c_{i+1})\|_{\mathbf{L}^1([0,t]\times\mathbb{R}^n;\mathbb{R}^n)}\\ &\le e^{(\alpha+K)\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} t e^{\gamma t}} \left[\|\nabla b_{i+1}\|_{\mathbf{L}^1([0,t]\times\mathbb{R}^n;\mathbb{R}^n)} + \|\nabla(\nabla\cdot c_{i+1})\|_{\mathbf{L}^1([0,t]\times\mathbb{R}^n;\mathbb{R}^n)}\right].\\ &\le e^{(\alpha+K)\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} t e^{\gamma t}} \left[C\left(\|w_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} e^{\gamma t}\right)\|w_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} t e^{\gamma t}\right.\\ &\qquad \left. +\alpha\|w_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} J_n \frac{\sqrt{t}}{\sqrt{\mu}}\left(1+2te^{\gamma t}\left(\gamma+\delta\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} e^{(\alpha+K)\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} t e^{\gamma t}}\right)\right)\right]\\ &= \boldsymbol{M}_t(u_o,w_o),\end{aligned} \tag{4.23}$$

where, for brevity, we set

$$\begin{aligned}&\boldsymbol{M}_t(u_o,w_o)\\ &= \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} e^{(\alpha+K)\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} t e^{\gamma t}} \sqrt{t}\left[C\left(\|w_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} e^{\gamma t}\right)\sqrt{t}e^{\gamma t}\right.\\ &\qquad \left. + J_n\frac{\alpha}{\sqrt{\mu}}\left(1+2te^{\gamma t}\left(\gamma+\delta\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} e^{(\alpha+K)\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} t e^{\gamma t}}\right)\right)\right] \qquad (4.24)\\ &\le r e^{(\alpha+K) r t e^{\gamma t}} \sqrt{t}\left[C\left(r e^{\gamma t}\right)\sqrt{t}e^{\gamma t} + J_n\frac{\alpha}{\sqrt{\mu}}\left(1+2te^{\gamma t}\left(\gamma+\delta r e^{(\alpha+K) r t e^{\gamma t}}\right)\right)\right], \qquad (4.25)\end{aligned}$$

since $(u_o,w_o)\in\mathcal{X}_r$. Pass to

$$\begin{aligned}&\int_0^t e^{(\alpha+K)\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\tau e^{\gamma\tau}} \|b_{i+1}(\tau)-b_i(\tau)-\nabla\cdot(c_{i+1}(\tau)-c_i(\tau))\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\,\mathrm{d}\tau\\ &\le e^{(\alpha+K)\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} t e^{\gamma t}} \int_0^t \|b_{i+1}(\tau)-b_i(\tau)-\nabla\cdot(c_{i+1}(\tau)-c_i(\tau))\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\,\mathrm{d}\tau.\end{aligned} \tag{4.26}$$

In particular, by the definition of b_i, $\boldsymbol{(v)}$ and Claim 1, we have

$$\begin{aligned}&\int_0^t \|b_{i+1}(\tau)-b_i(\tau)-\nabla\cdot(c_{i+1}(\tau)-c_i(\tau))\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\,\mathrm{d}\tau\\ &\le \int_0^t \|b_{i+1}(\tau)-b_i(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\,\mathrm{d}\tau + \int_0^t \|\nabla\cdot(c_{i+1}(\tau)-c_i(\tau))\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\,\mathrm{d}\tau\\ &\le \alpha\int_0^t \|w_i(\tau)-w_{i-1}(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\,\mathrm{d}\tau\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t C\left(\|w_{i-1}(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\right)\|w_i(\tau)-w_{i-1}(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\,\mathrm{d}\tau\\
\le{}&\alpha\|w_i-w_{i-1}\|_{\mathbf{L}^1([0,t];\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}))}\\
&+\left\|C\left(\|w_{i-1}(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\right)\right\|_{\mathbf{L}^\infty([0,t];\mathbb{R})}\|w_i-w_{i-1}\|_{\mathbf{L}^1([0,t];\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}))}\\
\le{}&\left(\alpha+C\left(\|w_{i-1}\|_{\mathbf{L}^\infty([0,t]\times\mathbb{R}^n;\mathbb{R})}\right)\right)\|w_i-w_{i-1}\|_{\mathbf{L}^1([0,t];\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}))}\\
\le{}&\left(\alpha+C\left(\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}e^{\gamma t}\right)\right)\|w_i-w_{i-1}\|_{\mathbf{L}^1([0,t];\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}))}.
\end{aligned}\tag{4.27}
$$

Use (4.20), (4.21), (4.22), (4.23), (4.26), and (4.27) to bound $\|u_{i+1}(t)-u_i(t)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}$:

$$
\begin{aligned}
&\|u_{i+1}(t)-u_i(t)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\\
\le{}&e^{\lambda t}\|w_i-w_{i-1}\|_{\mathbf{L}^1([0,t];\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}))}\Big[K e^{\lambda_1 t}\left[\mathrm{TV}(u_o)+I_n\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\boldsymbol{M}_t(u_o,w_o)\right]\\
&+\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\left(\alpha+C\left(\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}e^{\gamma t}\right)\right)e^{(\alpha+K)\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}te^{\gamma t}}\Big]
\end{aligned}
$$

and then, recalling (4.20), (4.21), and (4.24),

$$
\begin{aligned}
&\|u_{i+1}-u_i\|_{\mathbf{L}^1([0,T];\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}))}\\
\le{}&T\|w_i-w_{i-1}\|_{\mathbf{L}^1([0,T];\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}))}\exp\left((\alpha+2K)\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}Te^{\gamma T}+\beta T\right)\\
&\times\Bigg\{K\exp\left((\alpha+(2n+3)K)\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}Te^{\gamma T}+\beta T\right)\Big\{\mathrm{TV}(u_o)\\
&+I_n\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\|w_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\sqrt{T}e^{(\alpha+K)\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}Te^{\gamma T}}\\
&\times\Big[J_n\frac{\alpha}{\sqrt{\mu}}\left(1+2Te^{\gamma T}\left(\gamma+\delta\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}e^{(\alpha+K)\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}Te^{\gamma T}}\right)\right)\\
&+C\left(\|w_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}e^{\gamma T}\right)\sqrt{T}e^{\gamma T}\Big]\Big\}\\
&+\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\left(\alpha+C\left(\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}e^{\gamma T}\right)\right)e^{(\alpha+K)\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}Te^{\gamma T}}\Bigg\}\\
\le{}&T\mathcal{K}_u(T,r)\|w_i-w_{i-1}\|_{\mathbf{L}^1([0,T];\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}))},
\end{aligned}\tag{4.28}
$$

where, by (4.25),

$$
\begin{aligned}
\mathcal{K}_u(T,r)={}&re^{\left((2\alpha+3K)re^{\gamma T}+\beta\right)T}\Bigg\{Ke^{\left(2(n+1)Kre^{\gamma T}+\beta\right)T}\\
&\times\Big\{1+I_n r\sqrt{T}e^{(\alpha+K)rTe^{\gamma T}}\Big[\frac{J_n\alpha}{\sqrt{\mu}}\left(1+2Te^{\gamma T}\left(\gamma+\delta re^{(\alpha+K)rTe^{\gamma T}}\right)\right)\\
&+C\left(re^{\gamma T}\right)\sqrt{T}e^{\gamma T}\Big]\Big\}+\alpha+C\left(re^{\gamma T}\right)\Bigg\}.
\end{aligned}\tag{4.29}
$$

Therefore, the above definition of $\mathcal{K}_u(T,r)$, together with (4.18), (4.19) and (4.28), yields

$$
d((u_{i+1},w_{i+1}),(u_i,w_i))
$$

$$
\begin{aligned}
&= \|u_{i+1}-u_i\|_{\mathbf{L}^1([0,T];\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}))} + \|w_{i+1}-w_i\|_{\mathbf{L}^1([0,T];\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}))}\\
&\le T\max\{\mathcal{K}_u(T,r),\mathcal{K}_w(T,r)\}\left[\|u_i-u_{i-1}\|_{\mathbf{L}^1([0,T];\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}))} + \|w_i-w_{i-1}\|_{\mathbf{L}^1([0,T];\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}))}\right]\\
&= T\mathcal{K}(T,r)\,d((u_i,w_i),(u_{i-1},w_{i-1}))
\end{aligned}
$$

where

$$
\mathcal{K}(T,r) = \max\{\mathcal{K}_u(T,r),\mathcal{K}_w(T,r)\}, \tag{4.30}
$$

proving (4.17).

For any positive r, we can now choose T_r so that $T_r\mathcal{K}(T_r,r)<1$. The sequence (u_i,w_i) converges in the space X defined in (4.16) to a limit, say, (u_*,w_*). By construction, see Claim 0, both u_* and w_* attain non negative values. We now check that (u_*,w_*) solves (1.1) in the sense of Definition 2.1.

Clearly, $(u_*,w_*)(0)=(u_o,w_o)$. Moreover, by the above construction, we have that for any $\varphi\in\mathbf{C}_c^\infty(]0,T_r[\times\mathbb{R}^n;\mathbb{R})$

$$
\begin{aligned}
\int_0^{T_r}\int_{\mathbb{R}^n}\left(w_i\,\partial_t\varphi+\mu\,w_i\,\Delta\varphi+(\gamma-\delta\,u_{i-1})\,w_i\,\varphi\right)\mathrm{d}x\,\mathrm{d}t&=0\\
\int_0^{T_r}\int_{\mathbb{R}^n}\left(u_i\,\partial_t\varphi+u_i\,v(w_{i-1})\cdot\nabla\varphi+(\alpha\,w_{i-1}-\beta)\,u_i\,\varphi\right)\mathrm{d}x\,\mathrm{d}t&=0.
\end{aligned}
$$

Thanks to the $\mathbf{L}^\infty$ bounds proved in claims 1 and 2, we can apply the Dominated Convergence Theorem, ensuring that (u_*,w_*) is a weak solution to (1.1) for $t\in[0,T_r]$.

For all $t\in[0,T_r]$, we define $\mathcal{R}_{0,t}(u_o,w_o)=(u_*,w_*)(t)$.

Consider now a couple of initial data $(u_{1,o},w_{1,o}),(u_{2,o},w_{2,o})$ satisfying (4.10). For $t\in[0,T_r]$, we know that $(u_i,w_i)(t)=\mathcal{R}_{0,t}(u_{i,o},w_{i,o})$ solves (1.1) with initial datum $(u_{i,o},w_{i,o})$ in distributional sense. For $(t,x)\in[0,T_r]\times\mathbb{R}^n$ we define for $i=1,2$

$$
a_i(t,x)=\gamma-\delta\,u_i(t,x),\quad b_i(t,x)=\alpha\,w_i(t,x)-\beta,\quad\text{and}\quad c_i(t,x)=v\,(w_i(t))\,(x).
$$

Using the operators $\mathcal{P}$ of Proposition 2.5 and $\mathcal{H}$ of Proposition 2.8, observe that $w_i(t)=\mathcal{P}^i_{0,t}w_{i,o}$ and $u_i(t)=\mathcal{H}^i_{0,t}u_{i,o}$, for $i=1,2$. Moreover, note that $\mathcal{P}^1_{0,t}w_{2,o}$ is the solution to (2.3) with a_1 in the source term and initial datum $w_{2,o}$, while $\mathcal{H}^1_{0,t}u_{2,o}$ is the solution to (2.4) with coefficients b_1,c_1 and initial datum $u_{2,o}$. We compute $\|\mathcal{R}_{0,t}(u_{1,o},w_{1,o})-\mathcal{R}_{0,t}(u_{2,o},w_{2,o})\|_{\mathcal{X}}$ as defined in (2.2):

$$
\begin{aligned}
&\|\mathcal{R}_{0,t}(u_{1,o},w_{1,o})-\mathcal{R}_{0,t}(u_{2,o},w_{2,o})\|_{\mathcal{X}}\\
&= \left\|\mathcal{H}^1_{0,t}u_{1,o}-\mathcal{H}^2_{0,t}u_{2,o}\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} + \left\|\mathcal{P}^1_{0,t}w_{1,o}-\mathcal{P}^2_{0,t}w_{2,o}\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\\
&\le \left\|\mathcal{H}^1_{0,t}u_{1,o}-\mathcal{H}^1_{0,t}u_{2,o}\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} + \left\|\mathcal{H}^1_{0,t}u_{2,o}-\mathcal{H}^2_{0,t}u_{2,o}\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} && (4.31)\\
&\quad + \left\|\mathcal{P}^1_{0,t}w_{1,o}-\mathcal{P}^1_{0,t}w_{2,o}\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} + \left\|\mathcal{P}^1_{0,t}w_{2,o}-\mathcal{P}^2_{0,t}w_{2,o}\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}. && (4.32)
\end{aligned}
$$

Compute each term of (4.31) separately. Since $\mathcal{H}^1_{0,t}$ is linear, by 3. in Proposition 2.8 and its particularization in Claim 2, the first term in (4.31) is estimated by

$$
\begin{aligned}
\left\|\mathcal{H}^1_{0,t}u_{1,o}-\mathcal{H}^1_{0,t}u_{2,o}\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} &\le \|u_{1,o}-u_{2,o}\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\exp\left(\alpha\,\|w_{1,o}\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\,t\,e^{\gamma t}\right)\\
&\le \|u_{1,o}-u_{2,o}\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\exp\left(\alpha\,r\,t\,e^{\gamma t}\right). && (4.33)
\end{aligned}
$$

Concerning the second term in (4.31), recall 5. in Proposition 2.8 and adapt the estimates above for $\|u_{i+1}(t)-u_i(t)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}$, using $\boldsymbol{M}_t$ as defined in (4.24)–(4.25) and

$$\begin{aligned}
\Theta\ &= \Big((\alpha+K)\,\|w_{1,o}\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}+K\|w_{2,o}\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\Big)\,e^{\gamma t}+\beta,\\
\Theta_1&= \Big((\alpha+2\,(n+1)\,K)\,\|w_{1,o}\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}+K\|w_{2,o}\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\Big)\,e^{\gamma t}+\beta.
\end{aligned}$$

obtaining

$$\begin{aligned}
&\left\|\mathcal{H}^1_{0,t}u_{2,o}-\mathcal{H}^2_{0,t}u_{2,o}\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\\
&\leq e^{\Theta t}\int_0^t \left\|\mathcal{P}^1_{0,\tau}w_{1,o}-\mathcal{P}^2_{0,\tau}w_{2,o}\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\,\mathrm{d}\tau\\
&\quad\times\Bigg[K\,e^{\Theta_1 t}\left[\mathrm{TV}\,(u_{2,o})+I_n\,\|u_{2,o}\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\,\boldsymbol{M}_t(u_{1,o},w_{1,o})\right]\\
&\quad+\|u_{2,o}\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\,\Big(\alpha+C\Big(\|w_{2,o}\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\,e^{\gamma t}\Big)\Big)\,e^{(\alpha+K)\,t\,e^{\gamma t}\max_{i=1,2}\|w_{i,o}\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}}\Bigg]\\
&\leq \mathcal{K}_u(t,r)\int_0^t \left\|\mathcal{P}^1_{0,\tau}w_{1,o}-\mathcal{P}^2_{0,\tau}w_{2,o}\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\,\mathrm{d}\tau\,,
\end{aligned}\tag{4.34}$$

where $\mathcal{K}_u(t,r)$ is defined in (4.29).

Pass to (4.32). Since the map $\mathcal{P}^1_{0,t}$ is linear, by 3. in Proposition 2.5 and its particularization in Claim 1, we have the following estimate for the first term in (4.32):

$$\left\|\mathcal{P}^1_{0,t}w_{,o}-\mathcal{P}^1_{0,t}w_{2,o}\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\leq \|w_{1,o}-w_{2,o}\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\,e^{\gamma t}.\tag{4.35}$$

Concerning the second term in (4.32), recall 5. in Proposition 2.5 and adapt the estimates above for $\|w_{i+1}(t)-w_i(t)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}$ to obtain

$$\begin{aligned}
&\left\|\mathcal{P}^1_{0,t}w_{2,o}-\mathcal{P}^2_{0,t}w_{2,o}\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\\
&=\delta\,\|w_{2,o}\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\,\exp\left(2\gamma t+\delta t\,\|u_{2,o}\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\,e^{(\alpha+K)\,\|w_{2.o}\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\,\frac{e^{\gamma t}-1}{\gamma}}\right)\\
&\quad\times\int_0^t\left\|\mathcal{H}^1_{0,\tau}u_{1,o}-\mathcal{H}^2_{0,\tau}u_{2,o}\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\,\mathrm{d}\tau\\
&\leq\mathcal{K}_w(t,r)\int_0^t\left\|\mathcal{H}^1_{0,\tau}u_{1,o}-\mathcal{H}^2_{0,\tau}u_{2,o}\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\,\mathrm{d}\tau
\end{aligned}\tag{4.36}$$

with $\mathcal{K}_w(t,r)$ as in (4.19). We rewrite (4.31)–(4.32) using (4.33)–(4.34)–(4.35)–(4.36):

$$\begin{aligned}
&\|\mathcal{R}_{0,t}(u_{1,o},w_{1,o})-\mathcal{R}_{0,t}(u_{2,o},w_{2,o})\|_{\mathcal{X}}\\
&=\exp\left(\alpha r\,t\,e^{\gamma t}\right)\,\|u_{1,o}-u_{2,o}\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}+\mathcal{K}_u(t,r)\int_0^t\left\|\mathcal{P}^1_{0,\tau}w_{1,o}-\mathcal{P}^2_{0,t}w_{2,o}\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\,\mathrm{d}\tau\\
&\quad+e^{\gamma t}\,\|w_{1,o}-w_{2,o}\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}+\mathcal{K}_w(t,r)\int_0^t\left\|\mathcal{H}^1_{0,\tau}u_{1,o}-\mathcal{H}^2_{0,t}u_{2,o}\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\,\mathrm{d}\tau\\
&\leq e^{(\gamma+\alpha r\,e^{\gamma t})\,t}\,\Big(\|u_{1,o}-u_{2,o}\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}+\|w_{1,o}-w_{2,o}\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\Big)\\
&\quad+\mathcal{K}(t,r)\int_0^t\|\mathcal{R}_{0,\tau}(u_{1,o},w_{1,o})-\mathcal{R}_{0,\tau}(u_{2,o},w_{2,o})\|_{\mathcal{X}}\,\mathrm{d}\tau
\end{aligned}$$

and $\mathcal{K}(t,r)$ is as in (4.30). An application of Gronwall Lemma yields: for all $t\in[0,T_r]$

$$\begin{aligned}&\|\mathcal{R}_{0,t}(u_{1,o},w_{1,o})-\mathcal{R}_{0,t}(u_{2,o},w_{2,o})\|_{\mathcal{X}}\le\mathcal{L}(t,r)\,\|(u_{1,o},w_{1,o})-(u_{2,o},w_{2,o})\|_{\mathcal{X}},\\ &\text{where}\qquad \mathcal{L}(t,r)=\exp\left[\left(\gamma+\alpha r e^{\gamma t}+\mathcal{K}(t,r)\right)t\right]\qquad\text{with}\qquad \mathcal{K}(t,r)\text{ as in (4.30)},\end{aligned}\tag{4.37}$$

proving point 3. under condition (4.10).

Denote $(u(t),w(t))=\mathcal{R}_{0,t}(u_o,w_o)$ and define

$$a(t,x)=\gamma-\delta u(t,x),\quad b(t,x)=\alpha w(t,x)-\beta,\quad\text{and}\quad c(t,x)=v\left(w(t)\right)(x).\tag{4.38}$$

Then, with the notation in Proposition 2.5 and Proposition 2.8, by construction

$$u(t)=\mathcal{H}_{0,t}u_o\text{ and }w(t)=\mathcal{P}_{0,t}w_o,\text{ so that }\mathcal{R}_{0,t}(u_o,w_o)=(\mathcal{P}_{0,t}w_o,\mathcal{H}_{0,t}u_o).\tag{4.39}$$

We now extend the map $t\to\mathcal{R}_{0,t}(u_o,w_o)$ to the whole time axis. Indeed, define

$$T_*=\sup\left\{T\in\mathbb{R}^+:\mathcal{R}_{0,t}(u_o,w_o)\text{ is defined for all }t\in[0,T]\right\}.$$

The previous construction ensures that $\mathcal{R}_{0,t}(u_o,w_o)$ is defined at least for all $t\in[0,T_r]$, hence the supremum above is well defined. Assume now that $T_*<+\infty$. By 8. in Proposition 2.8, the first component of the map $t\to\mathcal{R}_{0,t}(u_o,w_o)$ is Lipschitz continuous with a constant uniformly bounded by a function of r on any bounded time interval. By 8. in Proposition 2.5 the second component of the map $t\to\mathcal{R}_{0,t}(u_o,w_o)$ is uniformly continuous on, say $[0,T_r/2]$ and Hölder continuous with exponent $\vartheta=1/2$ for $t\in[T_r/2,T_*[$, the Hölder constant being bounded by a function of r. Hence, the map $t\to\mathcal{R}_{0,t}(u_o,w_o)$ is uniformly continuous for $t\in[0,T_*[$ and the pair $(U,W)=\lim_{t\to T_*^-}(u(t),w(t))$ is well defined and in $\mathcal{X}_{r_*}$ for a suitable r_*. Moreover, since $W=w(T_*)=\mathcal{P}_{0,T_*}w_o$, we have that (U,W) also satisfies (4.10) by 7. and 9. in Proposition 2.5. Repeating the construction above, we show that the Cauchy problem consisting of (1.1) with initial datum $(u,w)(T_*)=(U,W)$ admits a solution defined on $[T_*,T_*+T_{r_*}]$, which contradicts the choice of T_*, unless $T_*=+\infty$. Define $a,b,$ and c as in (4.38). Then, by 8. in Proposition 2.5 and 8. in Proposition 2.8, (4.39) directly ensures the continuity in time of $\mathcal{R}$, for all $t\in\mathbb{R}^+$. This completes the proof of point 2. under condition (4.10).

Choose now a general initial datum $(u_o,w_o)\in\mathcal{X}^+$, so that $(u_o,w_o)\in\mathcal{X}_r$ for a suitable $r>0$. Let ρ_n be a sequence of mollifiers with $\rho_n\in\mathbf{C}_c^\infty(\mathbb{R}^n;\mathbb{R}^+)$ and $\int_{\mathbb{R}^n}\rho_n(x)\,\mathrm{d}x=1$. Then, the sequence of initial data $(u_o,w_o*\rho_n)$ is in $\mathcal{X}_r$, satisfies (4.10), and converges to (u_o,w_o) in $\mathcal{X}$. By (4.37), we can uniquely extend $\mathcal{R}$ through the limit $\mathcal{R}_{0,t}(u_o,w_o)=\lim_{n\to+\infty}\mathcal{R}_{0,t}(u_o,w_o*\rho_n)$ to all $\mathcal{X}_r$ and for all $t\in\mathbb{R}^+$, completing the proof of point 2. and of point 3. for all $t\in\mathbb{R}^+$. Note that the positivity of the solution follows from Claim 0. The $\mathbf{L}^1$ and $\mathbf{L}^\infty$ estimates at point 4. now follow from Claims 1 and 2. Again, the continuity in time of the map $\mathcal{R}$ so extended directly follows from 8. in Proposition 2.5 and 8. in Proposition 2.8.

We now prove that $\mathcal{R}$ is a process. For any $(u_o,w_o)\in\mathcal{X}^+$, use the notation (4.38) and observe that 1. in Proposition 2.5, 1. in Proposition 2.8 and (4.39) ensure that the map $\mathcal{R}$ is a process. Note however that (1.1) is autonomous, hence definitions 2.3, 2.6, and 2.1 ensure that $\mathcal{R}$ is a semigroup, proving point 1.

Point 5. follows from 9. in Proposition 2.8, $(\boldsymbol{v})$, and the $\mathbf{L}^1$ estimate in 4. above.

LEMMA 4.1. *Let η be such that $\nabla\eta\in(\mathbf{C}^2\cap\mathbf{W}^{2,1}\cap\mathbf{W}^{1,\infty})(\mathbb{R}^n;\mathbb{R}^n)$. Then, the map v defined in* (1.2) *satisfies* $(\boldsymbol{v})$ *with*

$$K=\max\left\{2\kappa\|\nabla\eta\|_{\mathbf{W}^{2,1}},2\kappa\|\nabla\eta\|_{\mathbf{W}^{1,\infty}},3\|\nabla\eta\|_{\mathbf{W}^{1,\infty}},\frac{48}{25\sqrt{5}}\|\nabla\eta\|_{\mathbf{W}^{2,1}}\right\},\tag{4.40}$$

$$C(\xi) = K\,(1+K\xi) \quad \textit{for all } \xi \in \mathbb{R}^+.$$

In the proof below we use the Euclidean norm $\|v\|_{\mathbb{R}^n} = \sqrt{\sum_{i=1}^n (v_i)^2}$ on vectors in $\mathbb{R}^n$ and the operator norm $\|A\|_{\mathbb{R}^{n\times n}} = \sup_{v\colon \|v\|_{\mathbb{R}^n}=1} \|Av\|_{\mathbb{R}^n}$ on $n\times n$ matrices.

Proof. The bound on $\|v(w)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}$ is ensured by

$$\|v(w)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^n)} \le \kappa \|\nabla\eta\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^n)} \|w\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}, \tag{4.41}$$

see also [6, Lemma 3.1]. To estimate $\nabla v(w)$, use the identity $\nabla(f\,v) = f\,\nabla v + v\otimes\nabla f$:

$$\begin{aligned}
\nabla v(w) &= \kappa \frac{1}{\left(1+\|w*\nabla\eta\|^2\right)^{1/2}} \nabla(w*\nabla\eta) + \kappa\,(w*\nabla\eta)\otimes\nabla\frac{1}{\left(1+\|w*\nabla\eta\|^2\right)^{1/2}}\\
&= \kappa\frac{w*\nabla^2\eta}{\left(1+\|w*\nabla\eta\|^2\right)^{1/2}} - \kappa\,(w*\nabla\eta)\otimes\frac{\left(w*\nabla^2\eta\right)(w*\nabla\eta)}{\left(1+\|w*\nabla\eta\|^2\right)^{3/2}}.
\end{aligned}$$

Recall that $\|v_1\otimes v_2\|_{\mathbb{R}^{n\times n}} \le \|v_1\|_{\mathbb{R}^n}\|v_2\|_{\mathbb{R}^n}$ and $\|Av\|_{\mathbb{R}^n} \le \|A\|_{\mathbb{R}^{n\times n}}\|v\|_{\mathbb{R}^n}$, hence

$$\begin{aligned}
&\|\nabla v(w)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n\times n})}\\
&\le \kappa\frac{\left\|w*\nabla^2\eta\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n\times n})}}{\sqrt{1+\|w*\nabla\eta\|^2}}\\
&\quad+\kappa\left\|\frac{w*\nabla\eta}{\sqrt{1+\|w*\nabla\eta\|^2}}\otimes\frac{w*\nabla^2\eta}{\sqrt{1+\|w*\nabla\eta\|^2}}\,\frac{w*\nabla\eta}{\sqrt{1+\|w*\nabla\eta\|^2}}\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n\times n})}\\
&\le \kappa\left\|w*\nabla^2\eta\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n\times n})}\\
&\quad+\kappa\left\|\frac{w*\nabla\eta}{\sqrt{1+\|w*\nabla\eta\|^2}}\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^n)}\left\|w*\nabla^2\eta\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n\times n})}\left\|\frac{w*\nabla\eta}{\sqrt{1+\|w*\nabla\eta\|^2}}\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^n)}\\
&\le 2\kappa\left\|w*\nabla^2\eta\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n\times n})}\\
&\le 2\kappa\left\|\nabla^2\eta\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^{n\times n})}\|w\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})},
\end{aligned}\tag{4.42}$$

proving also that $v(w)\in\mathbf{W}^{1,\infty}(\mathbb{R}^n;\mathbb{R}^n)$. Pass now to

$$\begin{aligned}
&v(w_1)-v(w_2)\\
&=\kappa\frac{(w_1-w_2)*\nabla\eta}{\sqrt{1+\|w_1*\nabla\eta\|^2}}+\kappa\,(w_2*\nabla\eta)\left(\frac{1}{\sqrt{1+\|w_1*\nabla\eta\|^2}}-\frac{1}{\sqrt{1+\|w_2*\nabla\eta\|^2}}\right).
\end{aligned}$$

Using the elementary inequality $\left|(1+x^2)^{1/2}-(1+y^2)^{1/2}\right| \le |x-y|$ we obtain:

$$\begin{aligned}
&v(w_1)-v(w_2)\\
&=\kappa\frac{(w_1-w_2)*\nabla\eta}{\sqrt{1+\|w_1*\nabla\eta\|^2}}+\kappa\frac{w_2*\nabla\eta}{\sqrt{1+\|w_2*\nabla\eta\|^2}}\,\frac{\sqrt{1+\|w_2*\nabla\eta\|^2}-\sqrt{1+\|w_1*\nabla\eta\|^2}}{\sqrt{1+\|w_1*\nabla\eta\|^2}};
\end{aligned}$$

$$
\begin{aligned}
&\left\|v(w_1)-v(w_2)\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^n)}\\
&\le \kappa\left\|(w_1-w_2)*\nabla\eta\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^n)}+\kappa\left|\left\|w_1*\nabla\eta\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^n)}-\left\|w_2*\nabla\eta\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^n)}\right|\\
&\le 2\kappa\left\|(w_1-w_2)*\nabla\eta\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^n)}\\
&\le 2\kappa\left\|w_1-w_2\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\left\|\nabla\eta\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^n)}.
\end{aligned}
\tag{4.43}
$$

Compute the divergence of $v(w)$ as follows:

$$
\nabla\cdot v(w)=\kappa\frac{w*(\Delta\eta)}{\sqrt{1+\left\|w*\nabla\eta\right\|^2}}\left(1-\frac{(w*\nabla\eta)\,(w*\nabla\eta)}{1+\left\|w*\nabla\eta\right\|^2}\right)=\kappa\frac{w*(\Delta\eta)}{\left(1+\left\|w*\nabla\eta\right\|^2\right)^{3/2}}.
\tag{4.44}
$$

We now compute the gradient of (4.44):

$$
\nabla\left(\nabla\cdot v(w)\right)=\kappa\frac{w*\nabla\Delta\eta}{\left(1+\left\|w*\nabla\eta\right\|^2\right)^{3/2}}-3\kappa\,(w*\Delta\eta)\frac{w*\nabla^2\eta}{\left(1+\left\|w*\nabla\eta\right\|^2\right)^{2}}\frac{w*\nabla\eta}{\sqrt{1+\left\|w*\nabla\eta\right\|^2}}
$$

so that

$$
\begin{aligned}
&\left\|\nabla\left(\nabla\cdot v(w)\right)\right\|_{\mathbf{L}^1(\mathbb{R}^n;\ \mathbb{R}^n)}\\
&\le\kappa\left\|w*\nabla\Delta\eta\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^n)}+3\kappa\left\|w*\Delta\eta\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\left\|w*\nabla^2\eta\right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^{n\times n})}\\
&\le\kappa\left\|w\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\left[\left\|\nabla\Delta\eta\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^n)}+3\left\|\Delta\eta\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\left\|\nabla^2\eta\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n\times n})}\left\|w\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\right].
\end{aligned}
\tag{4.45}
$$

Consider now

$$
\begin{aligned}
&\nabla\cdot\left(v(w_1)-v(w_2)\right)\\
=&\kappa\frac{(w_1-w_2)*\Delta\eta}{\left(1+\left\|w_1*\nabla\eta\right\|^2\right)^{3/2}}+\kappa(w_2*\Delta\eta)\left(\frac{1}{\left(1+\left\|w_1*\nabla\eta\right\|^2\right)^{3/2}}-\frac{1}{\left(1+\left\|w_2*\nabla\eta\right\|^2\right)^{3/2}}\right).
\end{aligned}
$$

Using the inequality $\left|(1+x^2)^{-3/2}-(1+y^2)^{-3/2}\right|\le\frac{48}{25\sqrt{5}}|x-y|$,

$$
\begin{aligned}
&\left\|\nabla\cdot\left(v(w_1)-v(w_2)\right)\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\\
&\le\kappa\left\|(w_1-w_2)*\Delta\eta\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}+\frac{48}{25\sqrt{5}}\kappa\left\|w_2*\Delta\eta\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\left\|(w_1-w_2)*\nabla\eta\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^n)}\\
&\le\kappa\left\|w_1-w_2\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\left\|\Delta\eta\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\\
&\qquad+\frac{48}{25\sqrt{5}}\kappa\left\|w_2\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\left\|\Delta\eta\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\left\|w_1-w_2\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\left\|\nabla\eta\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^n)}\\
&\le\kappa\left\|w_1-w_2\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\left\|\Delta\eta\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\left(1+\frac{48}{25\sqrt{5}}\left\|\nabla\eta\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^n)}\left\|w_2\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\right).
\end{aligned}
\tag{4.46}
$$

Setting K as in (4.40), the inequalities above become:

$$
\begin{array}{rll}
\left\|v(w)\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^n)} & \le K\left\|w\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} & \text{from (4.41)}\\
\left\|\nabla v(w)\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} & \le K\left\|w\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} & \text{from (4.42)}\\
\left\|v(w_1)-v(w_2)\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^n)} & \le K\left\|w_1-w_2\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} & \text{from (4.43)}\\
\left\|\nabla\left(\nabla\cdot v(w)\right)\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^n)} & \le K\left(1+K\left\|w\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\right)\left\|w\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} & \text{from (4.45)}\\
\left\|\nabla\cdot\left(v(w_1)-v(w_2)\right)\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} & \le K\left(1+K\left\|w_2\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\right)\left\|w_1-w_2\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} & \text{from (4.46)}
\end{array}
$$

completing the proof. □

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