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The set-indexed bandit problem

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Abstract

Motivated by spatial problems of allocations, we give a proof of the existence of an optimal solution to a set-indexed formulation of the bandit problem. The proof is based on a compactization of collections of fuzzy stopping sets and fuzzy optional increasing paths, and a construction of set-indexed integrals. © 2002 Elsevier Science B.V. All rights reserved.

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1. Formulation of the problem

The aim of this paper is to present a formulation of the Bandit problem in the framework of set-indexed processes, and to prove the existence of an optimal solution to this problem. The multi-armed bandit problem was already studied by several authors. We refer to Mandelbaum (1987, 1988) in which time is replaced by a partially ordered set, or, more recently, Kaspi and Mandelbaum (1998) for motivation and description of the state of art.

In this section, we give the notation and state the main existence result. In order to prove it, we need two kinds of tools: the first one are fuzzy stopping sets and fuzzy optional increasing paths that will be the subject of the next section. The other tool is a kind of set-indexed integration; it will be developed in Section 3. In Section 4, we study an extension of the Baxter–Chacon topology (see Baxter and Chacon, 1977) and we prove that fuzzy sets and fuzzy optional increasing paths are compact. The last section is devoted to the proof of the main theorem.

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Our framework can be applied to many different examples. A simple motivation in which a natural model is provided by processes indexed by a collection of subsets of a general space was already suggested by Gittins (1989) in the following close example: A prospector looking for gold in some region, has to decide where to look and if he prospects for any size of piece of land must repeatedly take further decisions in the light of his successes to date. His decision problem is how to allocate the prospection of the region in a sequential manner, so as to maximize his likely reward. This problem is spatio-temporal since at each "instant", the prospector has to decide where to excavate. Another kind of example is concerned with the theory of optimal search for the location of a specific object. Another example is related with aerial photography (see Ivanoff and Merzbach, 2002), in which at each time, an airplane pilot must decide what is the optimal direction permitting better pictures. The instantaneous decisions maker has to take into consideration different random constraints like clouds, draughts, etc.

Our notation follows that of Ivanoff and Merzbach (2000).

Let (T,d) be a compact complete metric space. Throughout the whole paper, if $\mathscr{A}^*$ is any class of subsets of T, $\mathscr{A}^*(u)$ denotes the class of finite union of elements in $\mathscr{A}^*$; "$\subsetneq$" indicates strict inclusion and "$\overline{(\cdot)}$" and "$(\cdot)^\circ$" denotes, respectively, the closure and the interior of a set. $\overline{\mathscr{A}^*(u)}$ denotes the class of countable intersections of elements in $\mathscr{A}^*(u)$. Under an assumption that is always satisfied when T is a metric space, it is proved in Aletti (2001) that $\overline{\mathscr{A}^*(u)}$ is equal to the class of the closures of countable unions of elements in $\mathscr{A}^*(u)$, when $\mathscr{A}^*$ is an indexing collection as defined below.

Definition 1. A nonempty class $\mathscr{A}$ of compact, connected subsets of T is called an *indexing collection* if it satisfies the following:

(1) $\emptyset, T \in \mathscr{A}$, and $A^\circ \neq A$ if $A \neq \emptyset$ or T and $A \in \mathscr{A}$.

(2) $\mathscr{A}$ is closed under arbitrary intersections and if $A, B \neq \emptyset$ and $A, B \in \mathscr{A}$, then $A \cap B \neq \emptyset$. If $(A_i)_{i\in\mathbb{N}}$ is increasing and $A_i \in \mathscr{A}$, then $\overline{\bigcup_i A_i} \in \mathscr{A}$.

(3) $\sigma(\mathscr{A}) = \mathfrak{B}_T$, where $\mathfrak{B}_T$ is the collection of all Borel sets of T.

(4) *Separability from above*

There exists an increasing sequence of finite subclasses $\mathscr{A}_n \subseteq \mathscr{A}$, $\mathscr{A}_n = \{A_1^n, \ldots, A_{k_n}^n\}$ closed under intersections with $\emptyset, T \in \mathscr{A}_n$, and a sequence of functions $g_n : \mathscr{A} \to \mathscr{A}_n(u)$ such that:

(a) g_n preserves arbitrary intersections and finite unions (i.e. $g_n(\bigcap_{A\in\mathscr{A}'} A) = \bigcap_{A\in\mathscr{A}'} g_n(A)$ for any $\mathscr{A}' \subseteq \mathscr{A}$, and if $\bigcup_{i=1}^k A_i = \bigcup_{j=1}^m A'_j$, then $\bigcup_{i=1}^k g_n(A_i) = \bigcup_{j=1}^m g_n(A'_j)$),

(b) for each $A \in \mathscr{A}$, $A \subseteq (g_n(A))^\circ$,

(c) $g_n(A) \subseteq g_m(A)$ if $n \geqslant m$,

(d) for each $A \in \mathscr{A}$, $A = \bigcap_n g_n(A)$,

(e) if $A \in \mathscr{A}$ and $A' \in \mathscr{A}_n$, then $g_n(A) \cap A' \in \mathscr{A}_n$,

(f) $g_n(\emptyset) = \emptyset \ \forall n$.

Remark 1. Denote $\emptyset' = \bigcap_n \bigcap_{A\in\mathscr{A}_n, A\neq\emptyset} A = \bigcap_{A\in\mathscr{A}, A\neq\emptyset} A$. (The second equality is a result of separability.) We have that $\emptyset' \in \mathscr{A}$ and $\emptyset' \neq \emptyset$, by the finite intersection property for compact sets.

Remark 2. Since g_n preserves finite unions, $A \subseteq B \Rightarrow g_n(A) \subseteq g_n(B)$. As well, g_n has a unique extension to $\mathscr{A}(u)$ which is defined as follows: for $B \in \mathscr{A}(u)$, $g_n(B) = \bigcup_{A \in \mathscr{A}, A \subseteq B} g_n(B)$. If $B \in \overline{\mathscr{A}(u)}$, we define $g_n(B) = \bigcap_{B' \in \mathscr{A}(u), B \subseteq B'} g_n(B')$. In Ivanoff and Merzbach (2000) it is proved that $g_n : \overline{\mathscr{A}(u)} \to \mathscr{A}_n(u)$ is well defined. It preserves countable intersections and finite unions and $B \subseteq (g_n(B))^{\circ}$.

We define the Hausdorff metric on the nonempty compact subsets of T : $d_H(A,B) = \inf\{\varepsilon: B \subseteq A^{\varepsilon}$ and $A \subseteq B^{\varepsilon}\}$, where $A^{\varepsilon} = \{t \in T: d(t,A) \leqslant \varepsilon\}$, for any $\varepsilon > 0$ and $d(t,A) = \inf\{d(t,s): s \in \mathscr{A}\}$.

We require that $\mathscr{A}$ be compact in the Hausdorff metric.

Remark 3. For the sake of simplicity, we required that T is compact and $T \in \mathscr{A}$. Using Alexandroff compactification, all our results will still hold supposing only that T is locally compact.

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space. A *set-indexed filtration* is a family $\{\mathscr{F}_A,\ A \in \mathscr{A}\}$ of sub-σ-algebras of $\mathscr{F}$ which is:
- complete ($\mathscr{F}_{\emptyset}$ contains all the $\mathbb{P}$-null sets);
- increasing (if $A \subseteq B$, $A, B \in \mathscr{A}$, then $\mathscr{F}_A \subseteq \mathscr{F}_B$);
- outer-continuous ($\mathscr{F}_{\cap_{n=1}^{\infty} A_n} = \bigcap_{n=1}^{\infty} \mathscr{F}_{A_n}$, where $\{A_n\}$ is a decreasing sequence in $\mathscr{A}$).

Remark 4. Let $\mathscr{F}^r_B := \bigcap_n \vee_{A \in \mathscr{A}, A \subseteq g_n(B)} \mathscr{F}_A$. In Ivanoff and Merzbach (2000, Lemma 1.4.1) it is stated that $\{\mathscr{F}^r_B,\ B \in \overline{\mathscr{A}(u)}\}$ is (complete) monotone outer continuous and $\mathscr{F}^r_A = \mathscr{F}_A$, for any $A \in \mathscr{A}$. This allows us to naturally speak about a filtration indexed by $\overline{\mathscr{A}(u)}$.

Definition 2. A random set $\xi : \Omega \to \mathscr{A}(u)$ is called a *stopping set* if $\{A \subseteq \xi\} \in \mathscr{F}_A$, for any $A \in \mathscr{A}$. A random set $\xi : \Omega \to \overline{\mathscr{A}(u)}$ is called a *generalized stopping set* if $\{A \subseteq \xi\} \in \mathscr{F}_A$, for any $A \in \mathscr{A}$. An *optional increasing path* (o.i.p.) is a family $\boldsymbol{\xi} = \{\xi_t,\ t \in [0,1]\}$ of generalized stopping sets s.t.

(1) $\xi_0 = \emptyset'$,
(2) $\xi_1 = T$,
(3) $\xi_s \subseteq \xi_t$ if $s \leqslant t$ a.s.,
(4) $\forall \omega \in \Omega,\ \ \xi_t(\omega) \subseteq g_n(g_n(\xi_s(\omega))) =: g_n^2(\xi_s)(\omega)$, if $s \leqslant t < s + 2^{-\beta(n)}$, where $\lim_{n\to\infty} \beta(n) = \infty$ a.s.

The set of all o.i.p.'s will be denoted by $\boldsymbol{\Xi}$.

Remark 5. Following Ivanoff and Merzbach (2000), our definition is "opposite" to that of Kurtz (1980) and Hürzeler (1985),[2] and hence this paper is not a direct extension of Dalang (1990). The formulation of the bandit problem will be done here for set-indexed processes.

Condition 4 relates to $|Z_u| = u$ in Dalang (1990). It states that the speed of o.i.p.'s growing is bounded. It will be necessary in Lemma 18 as $|Z_u| = u$ is in Dalang (1990).

[2] We require that $\{A \subseteq \xi\} \in \mathscr{F}_A$ instead of requiring $\{\xi \subseteq A\} \in \mathscr{F}_A$.

Following the formulation of the bandit problem given in Dalang (1990), let $X = \{X_A: A\in\overline{\mathscr{A}(u)}\}$ be a real-valued set-indexed process defined on $(\Omega,\mathscr{F},\mathbb{P})$ with upper-semicontinuous sample paths such that $E[\sup_{A\in\overline{\mathscr{A}(u)}}|X_A|] < \infty$ and let $V=\{V_s,\ s\in[0,1]\}$ be a bounded nonnegative right-continuous process with nondecreasing sample paths ($\mathscr{A}$ is an indexing collection).

Main Theorem 1. *For each* $\xi\in\Xi$, *set* $R(\xi) = E(\int_{[0,1]} X_{\xi_t}\mathrm{d}V_t)$. *Then there exists an optional increasing path* $\xi^*\in\Xi$ *s.t.* $R(\xi^*) = \sup_{\xi\in\Xi}R(\xi)$, *i.e.*, ξ^* *is an optimal o.i.p.*

This theorem will be proved in the last section.

2. Fuzzy stopping sets

Definition 3. Let $\mathscr{B}$ be the Borel sets of $[0,1]$. A map $\tau:\Omega\times[0,1]\to\overline{\mathscr{A}(u)}$ will be called a *fuzzy set* if it is $\mathscr{F}\otimes\mathscr{B}$-measurable and

(i) *nondecreasing* in the second variable: $s\leqslant t$ implies $\tau(\cdot,s)\subseteq\tau(\cdot,t)$;

(ii) *inner continuous* in the second variable: if $t_n\to t$, $t_n\leqslant t$, then

$$\overline{\bigcup_n \tau(\cdot,t_n)} = \tau(\cdot,t);$$

(iii) *outer continuous at* $0:\tau(\cdot,0) = \bigcap_{t>0}\tau(\cdot,t)$;

and will be called a *fuzzy stopping set* if it is a fuzzy set and

$$\{(\omega,r)\colon A\subseteq\tau(\omega,r)\}\in\mathscr{F}_A\otimes\mathscr{B}$$

for any $A\in\mathscr{A}$.

Also, any function on Ω will be considered defined on $\Omega\times[0,1]$ in the obvious way.

This implies that a stopping set η may be seen as a fuzzy stopping set (where $\xi(\omega,v) = \eta(\omega)$).

Let $\xi:\Omega\times[0,1]\to\mathscr{A}$ be nondecreasing, inner continuous and outer continuous at 0. We denote for every $v\in[0,1]$

$$\xi_v:\Omega\to\mathscr{A}(u),\quad \xi_v(\omega) = \xi(\omega,v).$$

Proposition 1. *Let* $\xi:\Omega\times[0,1]\to\overline{\mathscr{A}(u)}$ *be nondecreasing, inner continuous and outer continuous at* 0. ξ *is a fuzzy stopping set iff* ξ_v *is a generalized stopping set for each* $v\in[0,1]$. *In particular, taking* $\mathscr{F}_A=\mathscr{F}$, ξ *is a fuzzy set iff* ξ_v *is* $\mathscr{F}$*-measurable for each* $v\in[0,1]$.

Proof. If $A\nsubseteq\xi(\omega,r)$, then there exists $\varepsilon>0$ s.t. $A\nsubseteq\xi(\omega,r+\varepsilon)$ by inner continuity. Hence we have that

$$\begin{aligned}\{(\omega,r)\colon A\nsubseteq\xi(\omega,r)\} &= \bigcup_{v\in\mathbb{Q}\cap[0,1]}(\{\omega\colon A\nsubseteq\xi(\omega,v)\}\times[0,v])\\ &= \bigcup_{v\in\mathbb{Q}\cap[0,1]}(\{\omega\colon A\nsubseteq\xi_v(\omega)\}\times[0,v]). \qquad\square\end{aligned}$$

Given a fuzzy set ξ, define the "probability" $P_\xi(\omega,\cdot)$ on $\overline{\mathscr{A}(u)}$ by

$$P_\xi(\omega,A) = \begin{cases} \inf\{v\in[0,1]\colon A\subseteq(\xi(\omega,v))^\circ\} & \text{if } A\subseteq(\xi(\omega,1))^\circ, \\ 1 & \text{if } A\nsubseteq(\xi(\omega,1))^\circ. \end{cases} \tag{1}$$

Remark 6. By inner continuity, when $t>0$ we have that

$$B\subseteq(\xi(\omega,t))^\circ \quad\Leftrightarrow\quad t>P_\xi(\omega,B), \quad B\in\overline{\mathscr{A}(u)} \tag{2}$$

and hence

$$\xi(\omega,t) = \begin{cases} \overline{\bigcup\{A\in\mathscr{A}\colon P_\xi(\omega,A)<t\}} & \text{if } t>0, \\ \overline{\bigcup\{A\in\mathscr{A}\colon P_\xi(\omega,A)=0\}} & \text{if } t=0 \end{cases} \tag{3}$$

by Definitions 1(1) and 3(iii).

Proposition 2. *P_ξ satisfies the following properties*:
- takes values in $[0,1]$: $P_\xi(\omega,\emptyset)=0$, $P_\xi(\omega,T)=1$,
- is nondecreasing: *If $A\subseteq B$, then $P_\xi(\omega,A)\leqslant P_\xi(\omega,B)$*,
- is outer continuous: *If $A=\bigcap_n A_n$, then $P_\xi(\omega,A_n)\to P_\xi(\omega,A)$. (In fact, if $A\subseteq B^\circ$, then there exists k_0 s.t. $A_k\subseteq B^\circ$ for any $k\geqslant k_0$.)*

Denote by $\mathfrak{B}$ (resp. $\mathfrak{B}'$) the set of the all nondecreasing, outer continuous processes as $\{P(\cdot,A),\ A\in\mathscr{A}\}$ taking values in $[0,1]$ which are measurable (resp. adapted).

Proposition 2 states that $P_\xi\in\mathfrak{B}$ (resp. $P_\xi\in\mathfrak{B}'$) when ξ is a fuzzy (stopping) set. The following proposition states the converse:

Proposition 3. *Let $P\in\mathfrak{B}'$ (resp. $P\in\mathfrak{B}'$). Then the random set*

$$\xi(\omega,t) = \begin{cases} \mathrm{Lev}_{\omega,t} & \text{if } t>0, \\ \bigcap_s \mathrm{Lev}_{\omega,s} & \text{if } t=0, \end{cases} \tag{4}$$

where

$$\mathrm{Lev}_{\omega,t} = \overline{\bigcup\{A\in\mathscr{A}\colon P(\omega,A)<t\}}$$

is a fuzzy (stopping) set and $\mathbb{P}[P_\xi(\cdot,A)=P(\cdot,A),\text{ any } A]=1$.

Proof. It is a consequence of (2). □

Corollary 4. *$P\in\mathfrak{B}'$ corresponds to a stopping set iff $P(\cdot,A)\in\{0,1\}$.*

The conditions that make a family of stopping sets be an optional increasing path may be transported on $\prod_{[0,1]}\mathfrak{B}'$ to fuzzy optional increasing paths in the following way:

Definition 4. A *fuzzy optional increasing path* is a family $\boldsymbol{P}=\{P^t,\ t\in[0,1]\}\in\prod_{[0,1]}\mathfrak{B}'$ s.t.

(1) $P^t \in \mathfrak{B}'$ $\forall t \in [0,1]$,
(2) $P^0(\omega, A) = 1$, for any $A \in \mathscr{A}$,
(3) $P^1(\omega, A) = 0$, for any $A \in \mathscr{A}$, $A \neq T$,
(4) $P^s(\omega, \cdot) \geqslant P^t(\omega, \cdot)$ if $s \leqslant t$,
(5) $P^t(\omega, g_n^2(A)) \geqslant P^s(\omega, A)$ if $s \leqslant t < s + 2^{-\beta(n)}$.
We denote by $\mathfrak{P}'$ the set of all the fuzzy optional increasing paths.

Lemma 5. *$\{P^t,\ t \in [0,1]\} \in \mathfrak{P}'$ corresponds to an optional increasing path iff $P^t(\cdot, A) \in \{0,1\}$ for any $t \in [0,1]$.*

Proof. It is a consequence of Corollary 4. □

Remark 7. We only underline that, if $s \leqslant t < s + 2^{-\beta(n)}$, we have

$$P^t(\omega, A) \leqslant P^s(\omega, A) \leqslant P^t(\omega, g_n^2(A)). \tag{5}$$

3. Integration on $\overline{\mathscr{A}(u)}$

The set-indexed bandit problem is expressed here as the maximization of a function ϕ_X defined on the set of the optional increasing paths by

$$(\xi_t)_{t\in[0,1]} \mapsto E\left[\int_{\mathbb{R}_+} X_{\xi_t}\,\mathrm{d}V_t\right],$$

where $\{X_A,\ A \in \overline{\mathscr{A}(u)}\}$ is an (sufficient regular) integrable set-indexed stochastic process and $\{V_t,\ t \in [0,1]\}$ is a bounded process with nondecreasing sample paths. We note that

$$E\left[\int_{\mathbb{R}_+} X_{\xi_t}\,\mathrm{d}V_t\right] = E\left[\int_{\mathbb{R}_+}\left(\int_{[0,1]} X_{\xi_t(\cdot,v)}\,\mathrm{d}v\right)\mathrm{d}V_t\right],$$

since the fuzzy set obtained by a stopping set is constant.

We give a definition of integration on $\overline{\mathscr{A}(u)}$ s.t.

$$\int f(A)P_\xi(\cdot, \mathrm{d}A) = \int_{[0,1]} f(\xi(\cdot, v))\,\mathrm{d}v,$$

so that $\phi_X : \Xi \to \mathbb{R}$ will be extended to an affine function $\Phi_X : \mathfrak{P}' \to \mathbb{R}$ by setting:

$$(P^t)_{t\in[0,1]} \mapsto E\left[\int_{\mathbb{R}_+}\left(\int_{\mathscr{A}} X_A P^t(\cdot, \mathrm{d}A)\right)\mathrm{d}V_t\right].$$

Let f be any bounded Borel function on $\mathscr{A}$ and ξ be a fuzzy set. We denote

$$f\hat{}(\xi)(\omega) = \int_{[0,1]} \max(f[\xi(\omega, v)], 0)\,\mathrm{d}v,$$

$$f\check{}(\xi)(\omega) = \int_{[0,1]} \max(-f[\xi(\omega, v)], 0)\,\mathrm{d}v.$$

Definition 5. We say that a function $f : \mathscr{A} \to \mathbb{R}$ is *integrable* (and we denote it by $f \in \mathfrak{L}^1_{\mathfrak{B}}[\mathscr{A}]$) if, for any fuzzy set ξ, $f\hat{}(\xi)(\omega) < +\infty$ and $f\check{}(\xi)(\omega) < +\infty$, and we define

$$f(\xi)(\omega) := f\hat{}(\xi)(\omega) - f\check{}(\xi)(\omega). \tag{6}$$

Let $A, B \in \overline{\mathscr{A}(u)}$. We define

$$\chi_B(A) = \begin{cases} 0 & \text{if } B \subseteq (A)^\circ, \\ 1 & \text{otherwise,} \end{cases} \qquad \mathbf{1}_A(B) = \begin{cases} 1 & \text{if } B \subseteq A, \\ 0 & \text{otherwise} \end{cases}$$

so that, for any $B \in \mathscr{A}$

$$\chi_B(\xi(\omega, v)) = \begin{cases} 0 & \text{if } B \subseteq (\xi(\omega, v))^\circ, \\ 1 & \text{otherwise,} \end{cases} \qquad \mathbf{1}_{(\xi(\omega,v))^\circ}(B) = \begin{cases} 1 & \text{if } B \subseteq (\xi(\omega, v))^\circ, \\ 0 & \text{otherwise.} \end{cases}$$

Following this definition and (2), we have that

$$P(\omega, B) = \int_{[0,1]} \chi_B(\xi(\omega, v))\,\mathrm{d}v = \int_{[0,1]} (1 - \mathbf{1}_{(\xi(\omega,v))^\circ}(B))\,\mathrm{d}v.$$

Remark 8. Let I be an interval of $\mathbb{R}_+$ of the form $[0, s]$. It is natural to denote

$$\chi_I(t) = \begin{cases} 0 & \text{if } I \subseteq [0, t), \\ 1 & \text{otherwise,} \end{cases} \qquad \mathbf{1}_I(t) = \begin{cases} 1 & \text{if } [0, t] \subseteq I, \\ 0 & \text{otherwise} \end{cases}$$

and this fact explains our choice.

Let $f : \mathscr{A} \to \mathbb{R}_+ \cup \{+\infty\}$ be a nonnegative extended real function on $\mathscr{A}$. Let $\sqsupset_f : \mathbb{R} \to \overline{\mathscr{A}(u)}$,

$$\sqsupset_f(t) := \overline{\bigcup \{A \in \mathscr{A}\colon\ f(A) < t\}}.$$

We have $\sqsupset_f(s) \subseteq \sqsupset_f(t)$ when $s \leqslant t$.

Definition 6. We say that a bounded nonnegative real function $f : \overline{\mathscr{A}(u)} \to \mathbb{R}$ is *regular* iff $f(A) = \lim_n f_n(A)$, for any $A \in \overline{\mathscr{A}(u)}$, where

$$f_n(A) = \sum_{i=0}^{n2^n - 1} i2^{-n}[\chi_{\sqsupset_f([i+1]2^{-n})}(A) - \chi_{\sqsupset_f(i2^{-n})}(A)].$$

We denote by $\mathscr{L}[\mathscr{A}]$ the monotone class generated by the linear combinations of regular functions which are integrable.

Proposition 6. $C(\overline{\mathscr{A}(u)}) \cap \mathfrak{L}^1_{\mathfrak{B}}[\mathscr{A}] \subseteq \mathscr{L}[\mathscr{A}]$ *($C(\overline{\mathscr{A}(u)})$ is the set of continuous function on $\overline{\mathscr{A}(u)}$).*

Proof. It is sufficient to prove that, given a set $B \in \overline{\mathscr{A}(u)}$, the function $\mathbf{1}_B(A)$ belongs to $\mathscr{L}[\mathscr{A}]$ (in fact, the monotone class generated by linear combinations of $\mathbf{1}_B$, $B \in \overline{\mathscr{A}(u)}$ contains both $C(\overline{\mathscr{A}(u)})$ and $\mathfrak{L}^1_{\mathfrak{B}}[\mathscr{A}]$).

For any $n \in \mathbb{N}$, we define

$$\begin{aligned} f_n^B(A) &= \min_{\substack{A^* \in \mathscr{A}_n(u) \\ A^* \not\subseteq (g_n(B))}} \chi_{A^*}(g_n(A)) \\ &= 1 - \max_{\substack{A^* \in \mathscr{A}_n(u) \\ A^* \not\subseteq g_n(B)}} [1 - \chi_{A^*}(g_n(A))] \\ &= 1 - f_n'^B(A). \end{aligned}$$

We note that, if $A \subseteq B$, then $A^* \not\subseteq g_n(A)$, for any $A^* \not\subseteq g_n(B)$, and hence $\chi_{A^*}(g_n(A)) = 1$ and $f_n^B(A) = 1$, for any $n \in \mathbb{N}$.

If $A \not\subseteq B$, there will be an $n \in \mathbb{N}$ s.t. $g_n(A) \not\subseteq g_n(B)$. Then $f_n^B(A) = 0$ and so $\mathbf{1}_B = \inf f_n^B = 1 - \sup f_n'^B$.

Since $\mathscr{A}_n \subseteq \mathscr{A}_{n+1}$, then $\{f_n^B, \ n \in \mathbb{N}\}$ is a monotone sequence of functions on $\mathscr{A}$ and then we have only to prove that $f_n'^B \in \mathscr{L}[\mathscr{A}]$, for each $n \in \mathbb{N}$. Since χ_{A^*} is a regular function, then $f_n'^B \in \mathscr{L}[\mathscr{A}]$. □

Proposition 7. *The space $\mathscr{L}[\mathscr{A}]$ is the space of the integrable functions on $\overline{\mathscr{A}(u)}$.*

Proof. The proof follows by Proposition 6 and the fact that

$$\chi_B(A) = \sup_{\substack{f \in C(\overline{\mathscr{A}(u)}) \\ 0 \leqslant f \leqslant 1}} f(A)$$

$$f(A) = 0 \text{ when } B \subseteq A^\circ$$

since $\mathfrak{L}^1_{\mathfrak{B}}[\mathscr{A}]$ is a monotone class. □

- Let f be a bounded positive regular function on $\mathscr{A}$ and ξ be a fuzzy set. We define

$$\int f(A) P_\xi(\omega, \mathrm{d}A)$$

$$:= \lim_n \sum_{i=0}^{n2^n - 1} i2^{-n} [P_\xi(\omega, \sqsupset_f([i+1]2^{-n})) - P_\xi(\omega, \sqsupset_f(i2^{-n}))].$$

We have

$$f(\xi)(\omega)=\int_{[0,1]}\left(\lim_n f_n(\xi(\omega,v))\right)\mathrm{d}v$$

$$=\lim_n\sum_{i=0}^{n2^n-1} i2^{-n}\int_{[0,1]}[\chi_{\sqsupset_f([i+1]2^{-n})}(\xi(\omega,v))-\chi_{\sqsupset_f(i2^{-n})}(\xi(\omega,v))]\,\mathrm{d}v$$

$$=\lim_n\sum_{i=0}^{n2^n-1} i2^{-n}[P_\xi(\omega,\sqsupset_f([i+1]2^{-n}))-P_\xi(\omega,\sqsupset_f(i2^{-n}))]$$

$$=\int f(A)P_\xi(\omega,\mathrm{d}A). \tag{7}$$

- It is not difficult to show that the extension (via linear combinations and monotone class, as usual) of the integral operator to $\mathscr{L}[\mathscr{A}]$ gives coherent definitions and all those definitions are well posed. We only underline that, for any $f\in\mathscr{L}[\mathscr{A}]$, $f(\xi)(\omega)=\int f(A)P_\xi(\omega,\mathrm{d}A)$ and $\int f(A)P_\xi(\omega,\mathrm{d}A)$ is a linear function on $\mathfrak{B}'$.
- If ξ does not depend on the second variable for some ω_0 ($\xi(\omega_0,v)=A$, for any $v\in[0,1]$), we have

$$f(\xi)(\omega_0)=\int_{[0,1]} f(\xi(\omega_0,v))\,\mathrm{d}v=f(A).$$

Then $\Phi_X:\mathfrak{P}'\to\mathbb{R}$, where

$$\Phi_X((P^t)_{t\in[0,1]})=E\left[\int_{\mathbb{R}_+}\left(\int_{\mathscr{A}} X(A)\,P^t(\cdot,\mathrm{d}A)\right)\mathrm{d}V_t\right]$$

is an affine function on $\mathfrak{P}'$ that extends $\phi_X:\Xi\to\mathbb{R}$.

4. Compactness of fuzzy stopping sets and fuzzy o.i.p.

We wish to study the convergence of generalized stopping sets and of the associated stopping variables, as in Baxter and Chacon (1977).

Let $\mathscr{G}$ be a fixed σ-algebra contained in $\mathscr{F}$.

Definition 7. Let $\{\xi_n,\ n\in\mathbb{N}\}$ and ξ be fuzzy sets. ξ_n will be said to *converge weakly to* ξ *on* $\mathscr{G}$ ($\xi_n\overset{\mathscr{G}}{\Rightarrow}\xi$) if the distribution of $\xi_n(\omega,v)$ on $G\times[0,1]$ converges weakly to that of $\xi(\omega,v)$ on $G\times[0,1]$ for any $G\in\mathscr{G}$. That is, if we define $f(\xi_n)$ and $f(\xi)$ as in (6),

$$\xi_n\overset{\mathscr{G}}{\Rightarrow}\xi\quad\Leftrightarrow\quad E(f(\xi_n)|\mathscr{G})\overset{\text{a.s.}}{\to}\mathbb{E}(f(\xi)|\mathscr{G})\quad\text{when } n\to\infty$$

for each $f\in C(\overline{\mathscr{A}(u)})\cap\mathscr{L}[\mathscr{A}]$, where $C(\overline{\mathscr{A}(u)})$ is the set of continuous functions on $\overline{\mathscr{A}(u)}$.

Let $\Gamma = \Gamma(\{\mathscr{F}_A\})$ be the set of all fuzzy stopping sets. For any $Y \in \mathfrak{L}_1(\Omega, \mathscr{G}, \mathbb{P})$ and any $f \in C(\overline{\mathscr{A}(u)})$, let $\iota(Y, f) : \Gamma \to \mathbb{R}$ be defined by $\iota(Y, f)(\xi) = E[Yf(\xi)]$. Let I the collection of all such ι and $\mathscr{T} = \mathscr{T}(\mathscr{G})$ be the topology on Γ generated by all ι in I.

Proposition 8. *$\mathscr{T}$ is generated by $\iota(\chi_G, f)$, obtained as G runs over an $\mathfrak{L}_1$-dense subset of $\mathscr{G}$ and f runs over a sup-norm-dense subset of $C(\overline{\mathscr{A}(u)}) \cap \mathscr{L}[\mathscr{A}]$.*

Corollary 9. *$\xi_n \overset{\mathscr{G}}{\Rightarrow} \xi$ if and only if $\xi_n \overset{\mathscr{T}}{\to} \xi$.*

For any $Y \in \mathfrak{L}_1(\Omega, \mathscr{F}, \mathbb{P})$ and $f \in \mathscr{L}[\mathscr{A}]$, let

$$\alpha_\xi(Y, f) := \iota(Y, f)(\xi) = E[Yf(\xi)]$$

α_ξ will be called the distribution map for ξ. By (7), we have

$$\begin{aligned}\alpha_\xi(Y, f) &= E[Yf(\xi)] \\ &= E\left[Y\left(\int_{[0,1]} f(\xi_v)\,\mathrm{d}v\right)\right] \\ &= E\left[Y\left(\int f(A)P_\xi(\cdot, \mathrm{d}A)\right)\right].\end{aligned}$$

For $H \in \mathscr{F}$ write $\alpha_\xi(\chi_H, f)$ as $\alpha_\xi(H, f)$ and similarly for $K \in \overline{\mathscr{A}(u)}$ let $\alpha_\xi(Y, \chi_K) = \alpha_\xi(Y, K)$.

Proposition 10. *Let $f \in \mathscr{L}[\mathscr{A}]$ such that $f(A) \neq 0$ only if $A \subseteq B$ (we will also write that $f \equiv 0$ on $(B, T]$). If ξ is a fuzzy stopping set, then $\int f(A)P_\xi(\cdot, \mathrm{d}A)$ is $\mathscr{F}_B$-measurable. Conversely, if $P \in \mathfrak{B}$ and $\int f(A)P(\cdot, \mathrm{d}A)$ is $\mathscr{F}_B$-measurable for every function $f \in \mathscr{L}[\mathscr{A}] \cap C(\overline{\mathscr{A}(u)})$ with support on $[0, B]$, then $P \in \mathfrak{B}'$.*

Proof. It is a consequence of Proposition 1 and the fact that

$$P_\xi(\cdot, B^*) = \int_{[0,1]} \chi_{B^*}(\xi(\cdot, v))\,\mathrm{d}v. \qquad \square$$

Remark 9. For any fuzzy set ξ, let us temporally restrict the distribution map α_ξ to $\mathfrak{L}_1(\Omega, \mathscr{F}, \mathbb{P}) \times \{\mathscr{L}[\mathscr{A}] \cap C(\overline{\mathscr{A}(u)})\}$. Then α_ξ has the following properties:

(i) α_ξ is a bilinear function on $\mathfrak{L}_1(\Omega, \mathscr{F}, \mathbb{P}) \times \{\mathscr{L}[\mathscr{A}] \cap C(\overline{\mathscr{A}(u)})\}$.

(ii)
- $Y \geqslant 0$, $f \geqslant 0$ imply $\alpha_\xi(Y, f) \geqslant 0$.
- $\alpha_\xi(1, 1) = 1$.
- $|\alpha_\xi(Y, f)| \leqslant \|Y\|_1 \|f\|_\infty$ for all Y, f.

(iii) $\alpha_\xi(Y, f) = \alpha_\xi(E[Y|\mathscr{F}_B, f)$ for any $Y \in \mathfrak{L}_1(\Omega, \mathscr{F}, \mathbb{P})$ and any function $f \in \mathscr{L}[\mathscr{A}] \cap C(\overline{\mathscr{A}(u)})$ with support on $[0, B]$.

Lemma 11. *Let α be a map satisfying* (i)–(iii) *in Remark* 9. *Then there exists a unique $(\mathbb{P} \times \lambda_{[0,1]})$ fuzzy stopping set ξ s.t. $\alpha_\xi = \alpha$.* (*Note*: *$\lambda_{[0,1]}$ denotes the Lebesgue measure on* $[0, 1]$.)

Proof. For any $Y \in \mathfrak{L}_1(\Omega, \mathscr{F}, \mathbb{P})$ with $0 \leqslant Y \leqslant 1$,

$$\|Y\|_1 + \|1 - Y\|_1 = 1 = \alpha(1, 1) = \alpha(Y, 1) + \alpha(1 - Y, 1)$$

and $\|Y\|_1 \geqslant \alpha(Y, 1)$, $\|1 - Y\|_1 \geqslant \alpha(1 - Y, 1)$. Hence $\alpha(Y, 1) = \|Y\|_1$. By continuity, $\alpha(Y, 1) = E[Y]$ for any $Y \in \mathfrak{L}_1(\Omega, \mathscr{F}, \mathbb{P})$.

For any $Y \in \mathfrak{L}_1(\Omega, \mathscr{F}, \mathbb{P})$, the map $\alpha(Y, \cdot)$ is linear on $\mathscr{L}[\mathscr{A}] \cap C(\overline{\mathscr{A}(u)})$ and hence $\mu(B) = \alpha(Y, \mathbf{1}_B)$ defines a bounded Borel measure on $\mathscr{B}_T$. $\mu(T) = E[Y]$, and if $Y \geqslant 0$ then $\mu(\cdot) \geqslant 0$. One may extend α to all $Y \in \mathfrak{L}_1(\Omega, \mathscr{F}, \mathbb{P})$, and all functions $f \in \mathscr{L}[\mathscr{A}]$ by defining $\alpha(Y, f) = \int_T f(A)\, \alpha(Y, \mathrm{d}A)$. Clearly α is bilinear, $Y \geqslant 0$, $f \geqslant 0$ imply $\alpha(Y, f) \geqslant 0$, $\alpha(Y, 1) = E[Y]$ and $|\alpha(Y, f)| \leqslant \|Y\|_1 \, \|f\|_\infty$.

Using the outer continuity of $\{\mathscr{F}_A\}$ and the fact that $\alpha(Y, \cdot)$ is continuous under bounded pointwise limits, it is easy to see that (iii) in Remark 9 holds for all $Y \in \mathfrak{L}_1(\Omega, \mathscr{F}, \mathbb{P})$ and any bounded Borel function f s.t. $f = 0$ on $(B, T]$.

For B in a countable base of $\overline{\mathscr{A}(u)}$, let $c(B)$ be a bounded $\mathscr{F}_B$-measurable function such that $\alpha(Y, B) = E[Yc(B)]$ for any $Y \in \mathfrak{L}_1(\Omega, \mathscr{F}, \mathbb{P})$. We may choose $c(B)$ s.t. $c(T) = 1$, $c(B) \geqslant 0$. Clearly, $c(B) \leqslant c(B^*)$ whenever $B \subseteq B^*$ ($\mathbb{P}$-a.s.), so we may assume that $c(B) \leqslant c(B^*)$ everywhere, by replacing $c(B^*)$ by $\sup\{c(B)\colon B \subseteq B^*\}$.

Let $P(\omega, \cdot)$ be the probability on $\overline{\mathscr{A}(u)}$ such that

$$P(\omega, B) = \inf\{c(B^*, \omega)\colon B \subset B^*\}$$

for each $B \in \overline{\mathscr{A}(u)}$ ($P(\omega, T) = 1$, $P(\omega, \emptyset) = 0$). By outer continuity, $P(\cdot, B)$ is $\mathscr{F}_B$-measurable for any $B \in \mathscr{A}$ and hence $P \in \mathfrak{B}'$. By Proposition 3, there exists ξ s.t. $P_\xi(\omega, B) = P(\omega, B)$. For any $Y \in \mathfrak{L}_1(\Omega, \mathscr{F}, \mathbb{P})$ and $B \in \overline{\mathscr{A}(u)}$

$$\begin{aligned}
\alpha(Y, B) &= \inf_{B \subset B^*, B^* \in \mathscr{A}_n(u)} \alpha(Y, B^*) \\
&= \inf_{B \subset B^*,\, B^* \in \mathscr{A}_n(u)} \{E[Yc(B^*)]\} \\
&= E\left[Y \inf_{B \subset B^*,\, B^* \in \mathscr{A}_n(u)} \{c(B^*)\} \right] \\
&= E[YP_\xi(\cdot, B)] \\
&= \alpha_\xi(Y, B).
\end{aligned}$$

Hence $\alpha = \alpha_\xi$. ξ is unique by Lemma 12, so the lemma is proved. □

Let $\Gamma = \Gamma(\{\mathscr{F}_A\})$ be the set of fuzzy stopping sets, and let $\Lambda = \Lambda(\{\mathscr{F}_A\})$ be the set of maps α satisfying (i)–(iii) in Remark 9. There exists a natural map $\Theta : \Gamma \to \Lambda$ which takes each member of Γ to its distribution map

$$\Theta(\xi) = \alpha. \tag{8}$$

Lemma 11 says that if $\{\mathscr{F}_A,\ A \in \overline{\mathscr{A}(u)}\}$ is outer continuous then Θ is an onto map, and Θ is one-to-one ($\mathbb{P} \times \lambda_{[0,1]}$-a.s.).

It is easy to check that $\xi(\omega, v)$ is the intersection of the largest sets B such that $\int f(A)P_\xi(\omega, \mathrm{d}A) < v+\varepsilon$ for some nonincreasing f in $C(\overline{\mathscr{A}(u)}) \cap \mathscr{L}[\mathscr{A}]$ with $0 \leqslant f \leqslant 1$ and $f = 0$ on $[B, T]$. Let ζ be another fuzzy set. Clearly the following statements are equivalent:

- $\xi(\omega, \cdot) \subseteq \zeta(\omega, \cdot)$.
- $P_\xi(\omega, \cdot) \geqslant P_\zeta(\omega, \cdot)$.
- $\int f(A)P_\xi(\omega, \mathrm{d}A) \geqslant \int f(A)P_\zeta(\omega, \mathrm{d}A)$, for all nonincreasing functions $f \in \{C(\overline{\mathscr{A}(u)}) \cap \mathscr{L}[\mathscr{A}]\}$.

Lemma 12. *Let ξ and ζ be fuzzy sets. Suppose $\alpha_\xi(Y, f) \geqslant \alpha_\zeta(Y, f)$ for every $Y \geqslant 0$ in $\mathfrak{L}_1(\Omega, \mathscr{F}, \mathbb{P})$ and every nonincreasing f in $\{C(\overline{\mathscr{A}(u)}) \cap \mathscr{L}[\mathscr{A}]\}$. Then $\xi \subseteq \zeta$ ($\mathbb{P} \times \lambda_{[0,1]}$-a.s.).*

Proof. Choose a countable dense set D of nonincreasing functions in $C(\overline{\mathscr{A}(u)}) \cap \mathscr{L}[\mathscr{A}]$. For any $f \in D$,

$$\int Y \left[\int f(A)P_\xi(\omega, \mathrm{d}A) \right] \mathrm{d}\mathbb{P} \geqslant \int Y \left[\int f(A)P_\zeta(\omega, \mathrm{d}A) \right] \mathrm{d}\mathbb{P}$$

for all $Y \geqslant 0$ in $\mathfrak{L}_1(\Omega, \mathscr{F}, \mathbb{P})$. Hence

$$\int f(A)P_\xi(\cdot, \mathrm{d}A) \geqslant \int f(A)P_\zeta(\cdot, \mathrm{d}A), \quad \mathbb{P}\text{-a.s.}$$

Let $\Omega_1 \in \mathscr{F}$ with $\mathbb{P}[\Omega \setminus \Omega_1] = 0$ s.t.

$$\int f(A)P_\xi(\omega, \mathrm{d}A) \geqslant \int f(A)P_\zeta(\omega, \mathrm{d}A)$$

for all $f \in D$, $\omega \in \Omega_1$. Then the same inequality holds on Ω_1 for all nonincreasing $f \in \{C(\overline{\mathscr{A}(u)}) \cap \mathscr{L}[\mathscr{A}]\}$. Then, as stated above, $\xi \subseteq \zeta$ on Ω_1, so the lemma is proved. □

Lemma 13. *Let ξ and ζ be fuzzy sets. Let $\mathscr{G}$ be a sub-σ-algebra of $\mathscr{F}$. The following conditions are equivalent*:

(i) *$\alpha_\xi(Y, f) = \alpha_\zeta(Y, f)$, for all $Y \in \mathfrak{L}_1(\Omega, \mathscr{G}, \mathbb{P})$, $f \in \{C(\overline{\mathscr{A}(u)}) \cap \mathscr{L}[\mathscr{A}]\}$.*

(ii) *$E(f(\xi)|\mathscr{G}) = E(f(\zeta)|\mathscr{G})$, for any $f \in C(\overline{\mathscr{A}(u)}) \cap \mathscr{L}[\mathscr{A}]$.*

The proof is immediate.

Lemma 14. *Let ξ be a fuzzy set. Let $\mathscr{G}$ be a sub-σ-algebra of $\mathscr{F}$. Then there exists a unique* ($\mathbb{P} \times \lambda_{[0,1]}$-*a.s.*) *$\mathscr{G}$-fuzzy set ζ for which* (ii) *in Lemma* 13 *holds.*

Proof. Restrict α_ξ to $\mathfrak{L}_1(\Omega, \mathscr{G}, \mathbb{P}) \times C(T)$. Apply Lemma 11 with $\mathscr{F} = \mathscr{F}_A = \mathscr{G}$ to obtain ζ. Apply Lemma 13 to conclude that (ii) holds. Uniqueness follows also from Lemma 11 with $\mathscr{F} = \mathscr{F}_A = \mathscr{G}$.

It is easy to check that if ξ happens to be a fuzzy stopping set then ζ can be chosen to be a $\{\mathscr{F}_A \cap \mathscr{G}\}$-fuzzy stopping set provided that $\{\mathscr{F}_A,\ A \in \overline{\mathscr{A}(u)}\}$ is outer continuous and

$$E[E[\,\cdot\,|\mathscr{G}]\,|\,\mathscr{F}_A] = E[\,\cdot\,|\mathscr{G}_A \cap \mathscr{G}] \quad \text{for all } A \in \overline{\mathscr{A}(u)}. \qquad \square$$

Theorem 15. *If* $\{\mathscr{F}_A,\ A \in \overline{\mathscr{A}(u)}\}$ *is outer continuous then* $(\Gamma, \mathscr{T})$ *is compact.*

Proof. Let Θ be the map defined in (8). For each $Y \in \mathfrak{L}_1(\Omega, \mathscr{F}, \mathbb{P})$ and $f \in C(T)$, let $\psi : \Lambda \to \mathbb{R}$ be defined by $\psi(Y, f)(\alpha) = \alpha(Y, f)$. Let Ψ be the set of all such ψ. Let $\mathscr{C}$ be the topology on Λ generated by all ψ in Ψ. It is easy to see that $\mathscr{T} = \Theta^{-1}(\mathscr{C})$. $\mathscr{C}$ is compact, by the same argument used to prove that unit ball in the dual of a normed linear space is compact. Since Θ is onto by Lemma 11, $\mathscr{T}$ is also compact. $\square$

Theorem 16. *The set* $\mathfrak{B}$ (*resp.* $\mathfrak{B}'$) *has the following properties*:

(i) $\mathfrak{B}$ (*resp.* $\mathfrak{B}'$) *is convex.*

(ii) *The set of the extremal elements in* $\mathfrak{B}'$ *is the set of those which correspond to the set of the generalized stopping set.*

(iii) $\mathfrak{B}$ (*resp.* $\mathfrak{B}'$) *is compact in the topology induced by* $(\Gamma, \mathscr{T})$.

Proof.

(i) The convex combination of two (adapted) increasing outer continuous processes with values in $[0,1]$ is an (adapted) increasing outer continuous process with values in $[0,1]$.

(ii) Suppose that there exists A s.t. $\mathbb{P}[\{\omega\colon 0 < P(\omega, A) < 1\}] > 0$. Then we may choose a convenient λ such that if

$$P'(\omega, B) = \frac{P(\omega, B) \wedge \lambda}{\lambda}$$

and

$$P''(\omega, B) = \frac{(P(\omega, B) - \lambda) \vee 0}{1 - \lambda}$$

for any $B \in \mathscr{A}$, then P' and P'' are distinct. We have $P = \lambda P' + (1 - \lambda) P''$. Moreover, P' and P'' are nondecreasing, compatible and outer continuous, since P is. By Corollary 4, if P does not correspond to a stopping set, it cannot be one of the extremal elements of $\mathfrak{B}'$.

(iii) It is a consequence of Theorem 15. $\square$

Proposition 17. *The set* $\mathfrak{P}'$ *has the following properties*:

(i) $\mathfrak{P}'$ *is convex.*

(ii) *The set of the extremal elements in* $\mathfrak{P}'$ *is the set of those which correspond to the set of the optional increasing path.*

(iii) $\mathfrak{P}'$ *is compact for the product topology.*

Proof.

(i) It is a consequence of Theorem 16 and the fact that a convex combination is a linear function.

(ii) If $\mathbf{P}=\{P^t,\ t\in[0,1]\}\in\mathfrak{P}'$, we can define two elements ${}^1\mathbf{P}$ and ${}^2\mathbf{P}$ by the formulas

$$^1P^t(\omega,A)=\min(2P^t(\omega,A),1),$$

$$^2P^t(\omega,A)=\max(2P^t(\omega,A)-1,0).$$

It is easy to check that ${}^1\mathbf{P},{}^2\mathbf{P}\in\mathfrak{P}'$ and $2\mathbf{P}={}^1\mathbf{P}+{}^1\mathbf{P}$ and so $\mathbf{P}$ can be an extremal element in $\mathfrak{P}'$ only if $\mathbf{P}={}^1\mathbf{P}={}^2\mathbf{P}$. Then $P^t(\omega,A)\in\{0,1\}$ and the proof is finished via Lemma 5.

(iii) It is sufficient to note that $\mathfrak{P}'$ is a closed subset of $\prod_{[0,1]}\mathfrak{B}'$ and thus is compact (as in Dalang, 1990, this set is closed since it is defined by (1)–(5) of Definition 4). □

5. Proof of the Main Theorem

Let $\mathscr{M}$ denotes the deterministic elements of $\mathfrak{P}'$, i.e., $\boldsymbol{\pi}=\{P^t,\ t\in[0,1]\}\in\mathscr{M}$ provided each $\pi^t:\overline{\mathscr{A}(u)}\to[0,1]$ is a nondecreasing outer continuous function such that $\pi^t(T)=1$ and

- $\pi^s(\cdot)\geqslant\pi^t(\cdot)$ if $s\leqslant t$,
- $\pi^t(g_n^2(A))\geqslant\pi^s(A)$ if $s\leqslant t<s+2^{-\beta(n)}$.

As in (5), we may affirm that if $s\leqslant t<s+2^{-\beta(n)}$,

$$\pi^t(A)\leqslant\pi^s(A)\leqslant\pi^t(g_n^2(A)). \tag{9}$$

Let x be an integrable function on $\overline{\mathscr{A}(u)}$, and let $\boldsymbol{\pi}(x):[0,1]\to\mathbb{R}$ be defined by

$$\boldsymbol{\pi}(x)(t)=\begin{cases}\displaystyle\int_T x(A)\,\pi^t(\mathrm{d}A) & \text{when } t\in[0,1),\\ x(T) & \text{when } t=1.\end{cases}$$

Lemma 18. *If $x:\overline{\mathscr{A}(u)}\to\mathbb{R}$ is a continuous function on $\overline{\mathscr{A}(u)}$, then $\{\boldsymbol{\pi}(x),\ \boldsymbol{\pi}\in\mathscr{M}\}$ is an equicontinuous set of functions on $[0,1]$, i.e., for all $\varepsilon>0$, there exists $\delta>0$ such that if $|t-\bar t|<\delta$ then*

$$|\boldsymbol{\pi}(x)(t)-\boldsymbol{\pi}(x)(\bar t)|<\varepsilon \tag{10}$$

for any $\boldsymbol{\pi}\in\mathscr{M}$.

Proof. Since a continuous function on a compact set is uniformly continuous, then (10) has to be checked for all fixed $t\in[0,1]$. The case $t=1$ follows immediately from the continuity of x at 1, so we assume $t\in[0,1)$. Now

$$\begin{aligned}|\boldsymbol{\pi}(x)(t)-\boldsymbol{\pi}(x)(\bar t)|&=\left|\int_{\mathscr{A}}x(A)[\pi^t(\mathrm{d}A)-\pi^{\bar t}(\mathrm{d}A)]\right|\\&=\left|\int_{[0,1]}[x(\xi^t(v))-x(\xi^{\bar t}(v))]\,\mathrm{d}v\right|.\end{aligned} \tag{11}$$

We must prove that for any $n \in \mathbb{N}$, there exists $k \in \mathbb{N}$ s.t.

$$|t-\bar{t}| < 2^{-k}, \quad \boldsymbol{\pi} \in \mathscr{M} \;\Rightarrow\; \left| \int_T x(A)[\pi^t(\mathrm{d}A) - \pi^{\bar{t}}(\mathrm{d}A)] \right| < 2^{-n}.$$

Since x is uniformly continuous on a compact set by Remark 3, then $d(A,B) < \delta_\varepsilon$ implies $|x(A)-x(B)| < \varepsilon$ and hence, by (11), we can prove that $|t-\bar{t}| < 2^{-k}$ implies $d(\xi^t(v), \xi^{\bar{t}}(v)) < \delta$ for any $v \in [0,1]$.

First, let us choose $N \in \mathbb{N}$ s.t. $\sup_A d(A, g_N^2(A)) < \delta$. By (9) and (4), if $k = \beta(N)$, $|t-\bar{t}| < 2^{-k}$ implies $d(\xi^t(v), \xi^{\bar{t}}(v)) < \delta$ for any $v \in [0,1]$. □

The proof of the following result will follow that of Dalang (1990). We will write it for completeness.

Lemma 19. *Suppose* $E[\sup_{A\in\overline{\mathscr{A}(u)}} |X_A|] < \infty$. *Then* Φ_X *is continuous on* $\mathfrak{P}'$.

Proof. Let V^k denote the increasing process defined by

$$V_t^k = \sum_{j=1}^{2^k} \Delta V_j^k I_{\{j2^{-k} \leqslant t\}} \quad \text{where } \Delta V_j^k = V_{j2^{-k}} - V_{(j-1)2^{-k}}.$$

Since $E[\sup_{A\in\overline{\mathscr{A}(u)}} |X_A|] < \infty$, it is not difficult to see using Lemma 18 that

$$\sup_{\pi\in\Xi_A} \left| \int_{[0,1]} \pi(X)(t)\,\mathrm{d}V_t - \int_{[0,1]} \pi(X)(t)\,\mathrm{d}V_t^k \right| \underset{k\to\infty}{\longrightarrow} 0 \quad \mathbb{P}\text{-a.s.}$$

Fix $\varepsilon > 0$. Again since $E[\sup_{A\in\overline{\mathscr{A}(u)}} |X_A|] < \infty$, it follows that we may choose $k \in \mathbb{N}$ s.t.

$$E\left[\sup_{\pi\in\Xi_A} \left| \int_{[0,1]} \pi(X)(t)\,\mathrm{d}V_t - \int_{[0,1]} \pi(X)(t)\,\mathrm{d}V_t^k \right| \right] < \frac{\varepsilon}{3}. \tag{12}$$

Fix $\mathbf{P} \in \mathfrak{P}'$, and define an open set $\mathscr{O}$ of $\mathfrak{P}'$ by

$$\mathscr{O} = \left\{ \tilde{\mathbf{P}} \in \mathfrak{P}' \colon |E[(\tilde{\mathbf{P}}(X)(j2^{-k}) - \mathbf{P}(X)(j2^{-k}))\Delta V_j^k]| < \frac{\varepsilon}{3} 2^{-k},\ j = 1,\ldots,2^k \right\}.$$

Using (12), we see that for $\tilde{\mathbf{P}} \in \mathscr{O}$,

$$\begin{aligned} &\left| E\left[\int_{[0,1]} \tilde{\mathbf{P}}(X)(t)\,\mathrm{d}V_t \right] - E\left[\int_{[0,1]} \mathbf{P}(X)(t)\,\mathrm{d}V_t \right] \right| \\ &\quad < 2\frac{\varepsilon}{3} + \left| E\left[\int_{[0,1]} \tilde{\mathbf{P}}(X)(t)\,\mathrm{d}V_t^k \right] - E\left[\int_{[0,1]} \mathbf{P}(X)(t)\,\mathrm{d}V_t^k \right] \right| < \varepsilon \end{aligned}$$

by the definitions of $\mathscr{O}$ and V^k. This completes the proof. □

Proof of the Main Theorem. This proof is now similar to that of Dalang (1990); Mazziotto and Millet (1987).

Note that $(\overline{\mathscr{A}(u)}, d)$ is a metric space s.t. its topology has a countable space (and hence, by Bourbaki (1966, IX, Section 2.8 Proposition 12), it is homeomorphic to a subspace of $I^{\mathbb{N}}$, where I is the interval $[0,1]$ in $\mathbb{R}$).

Then, by the proof of Dalang (1989, Proposition 7.1), there is a nonincreasing sequence of processes $(X_A^n)_{n\in\mathbb{N}}$ (where $E[\sup_{A\in\overline{\mathscr{A}(u)}}|X_A^n|]<\infty$) s.t.

$$X_A(\omega)=\lim_{n\to\infty}\downarrow X_A^n(\omega),\quad \forall A\in\overline{\mathscr{A}(u)},\ \forall\omega\in\Omega.$$

Since $\mathrm{d}V_t$ is a nonnegative measure, we obtain

$$\lim_{n\to\infty}\downarrow \Phi_{X^n}(\mathbf{P})=\Phi_X(\mathbf{P}),\quad \forall\mathbf{P}\in\mathfrak{P}'.$$

By Lemma 19, this shows that Φ is upper semicontinuous on $\mathfrak{P}'$. Hence Φ attains its maximum on $\mathfrak{P}'$ and since Φ is affine, this maximum is attained at an extremal element of $\mathfrak{P}'$ (see Bourbaki, 1981, II.58, Proposition 1). Proposition 17 completes the proof. □

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