Annali di Matematica (2013) 192:407–421
DOI 10.1007/s10231-011-0230-1

A compactness result for differentiable functions with an application to boundary value problems

Filomena Cianciaruso · Vittorio Colao · Giuseppe Marino · Hong-Kun Xu

Received: 27 June 2011 / Accepted: 28 October 2011 / Published online: 13 November 2011

Abstract We characterize the relative compactness of subsets of the space $\mathcal{BC}^m([0, +\infty[; E)$ of bounded and m-differentiable functions defined on $[0, +\infty[$ with values in a Banach space E. Moreover, we apply this characterization to prove the existence of solutions of a boundary value problem in Banach spaces.

Keywords Compactness · Ascoli–Arzela · Differentiable function · Boundary value problem

Mathematics Subject Classification (2000) 46E15 · 46B50 · 34B40

1 Introduction

One of the most celebrated results on compactness characterization for continuous functions is the Ascoli–Arzela Theorem for continuous functions defined on a compact topological space Q. It has extensively been applied to many fields of mathematical analysis, such as

F. Cianciaruso · V. Colao · G. Marino (✉)
Dipartimento di Matematica, Universitá della Calabria, 87036 Arcavacata di Rende, CS, Italy
e-mail: gmarino@unical.it

F. Cianciaruso
e-mail: cianciaruso@unical.it

V. Colao
e-mail: colao@mat.unical.it

H.-K. Xu
Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung 80424, Taiwan
e-mail: xuhk@math.nsysu.edu.tw

H.-K. Xu
Department of Mathematics, College of Science, King Saud University,
P.O. Box 2455, Riyadh 11451, Saudi Arabia

functional analysis, operator theory, nonlinear analysis, differential, and integral equations. Moreover, this compactness theorem has been extended to more general settings.

Bartle [3] characterized, in several ways, the relatively compact subsets of the space $\mathcal{BC}(Q)$ of real bounded and continuous functions defined on a topological space Q. For example, we have the following result.

Theorem 1.1 [3]. *The following statements are equivalent for a bounded subset* $F \subseteq \mathcal{BC}(Q)$:

(1) *F is conditionally (strongly) compact;*
(2) *F is equicontinuous on Q;*
(3) *If F_0 is a denumerable subset of F and if $\{q_n\}$ is a sequence in Q for which $\{f(q_n))\}$ converges for each $f \in F_0$, then the convergence is uniform on F_0;*
(4) *For any positive ϵ there is a partition $Q = \cup_{i=1}^{n} A_i$, such that if q', q'' belong to the same A_i then $|f(q') - f(q'')| < \epsilon,\ f \in F$.*

Avramescu [1] considered the space $\mathcal{C}_l([0, +\infty[; \mathbb{R}^n)$ of the continuous functions defined on $[0, +\infty[$ with values in $\mathbb{R}^n$, which admit finite limit at infinity.

More recently, De Pascale, Lewicki, and Marino [5] gave sufficient conditions for relative compactness of subsets of the spaces $\mathcal{BC}(Q)$ and $\mathcal{BC}(Q; \mathbb{R}^n)$ with some applications to boundary value problems.

On the same line of [1], Xiao, Kim, and Huang [7] gave a necessary and sufficient condition to characterize the relative compactness of the subsets of the space

$$\mathcal{C}_*^m([0, +\infty[; E) = \{x \in \mathcal{C}^m([0, +\infty[; E) : \lim_{t \to +\infty} x^{(k)}(t) = 0 \quad \text{for all } k = 0, \dots, m\},$$

where E is a real Banach space. Moreover, they gave an application to a class of second-order boundary value problems in Banach spaces.

It is the purpose of this paper to characterize the relatively compact subsets of the space $\mathcal{BC}^m([0, +\infty[; E)$ of bounded and m-differentiable functions defined on $[0, +\infty[$ with values in a Banach space E. Moreover, we shall use this characterization to prove an existence theorem of solutions of a second-order boundary value problem in Banach spaces.

2 Preliminaries

In this paper, we shall denote by E a Banach space and by $\mathcal{BC}^m([0, +\infty[; E)$ $(m \geq 0)$ the space of all functions that are continuous and bounded with their derivatives up to (and including) the m-th order, from $[0, +\infty[$ to E, i.e.,

$$\mathcal{BC}^m([0, +\infty[; E) := \{f \in \mathcal{C}([0, +\infty[, E) : \forall k = 0, \dots, m, \\ f^{(k)} \text{ is bounded and continuous}\}$$

We equip $\mathcal{BC}^m([0, +\infty[; E)$ with the norm:

$$\|y\|_{\mathcal{BC}^m} = \begin{cases} \|y\|_\infty := \sup_{t \in [0, +\infty[} \|y(t)\|_E, & \text{if } m = 0, \\ \sum_{k=0}^{m} \|y^{(k)}\|_\infty, & \text{if } m > 0. \end{cases}$$

It is worth noting that $(\mathcal{BC}^m([0,+\infty[;E), \|\cdot\|_m)$ is a Banach space. For a subset H of $\mathcal{BC}^m([0,+\infty[;E)$, we define

$$H^{(0)} = H, \;\; H^{(k)} = \left\{ y^{(k)} : y \in H \right\} \subset \mathcal{BC}^{m-k}([0,+\infty[;E), \;\; (1 \le k \le m);$$
$$H^{(k)}(t) = \left\{ y^{(k)}(t) : y \in H \right\} \subset E, \;\; (0 \le k \le m, \; t \in [0,+\infty[).$$

In the sequel, we shall denote by α, β, and γ the Kuratowski, Istratescu, and Hausdorff measures of noncompactness, respectively. Namely, if M is a bounded subset of a Banach space X, then

$$\alpha(M) := \inf\{\epsilon > 0 : \text{ there exists } M_1, \ldots, M_n \subset M, \; \mathrm{diam} M_i < \epsilon, \; i = 0, \ldots, n, \; M \subset \bigcup_{k=0}^{n} M_k\};$$
$$\beta(M) := \sup\{\epsilon > 0 : \text{ there exists } (x_n)_{n=1}^{\infty} \subset M \text{ such that } \|x_n - x_m\|_E \ge \epsilon, \; n \ne m\};$$
$$\gamma(M) := \inf\left\{\epsilon > 0 : \text{ there exist balls } B_1, \ldots, B_n \text{ of radius } \epsilon \text{ such that } M \subset \bigcup_{k=0}^{n} B_k \right\}.$$

It is well known (see for example [2]) that

- $\gamma \le \alpha \le \beta \le 2\gamma$;
- M is relatively compact if, and only if, $\phi(M) = 0$ for $\phi \in \{\alpha, \beta, \gamma\}$.

For a fixed element $l \in E$, let $\mathcal{C}_l([0,+\infty[;E)$ be the space of all continuous functions from $[0,+\infty[$ to E, converging to l as $t \to \infty$. For this space, compactness can be characterized as follows:

Proposition 2.1 *Let E be a Banach space. A subset $G \subset \mathcal{C}_l([0,+\infty[;E)$ is relatively compact if and only if the following conditions are satisfied:*

(1) *G is bounded;*
(2) *G is equicontinuous;*
(3) *for any $t_0 \in [0,+\infty[$, $G(t_0)$ is a relatively compact set in E;*
(4) *as $t \to +\infty$, $x(t)$ converges to l uniformly for $x \in G$.*

Proof The proof is essentially given in [1] for functions with values in $\mathbb{R}^n$. Condition (2.1) ensures its validity on $\mathcal{C}_l([0,+\infty[;E)$, where E is a general Banach space. □

We need the following lemma in the last section. Here and throughout the paper, the integral which appears is the Bochner integral.

Lemma 2.2 [6] *Let E be a Banach space and $H \subset \mathcal{C}([0,+\infty[;E)$. If H is a countable set and there exists a positive function $\rho \in \mathcal{L}^1(0,+\infty)$ such that $\|u(t)\|_E \le \rho(t)$ for all $t \in [0,+\infty[$, $u \in H$, then the function*

$$t \mapsto \alpha\left(\{u(t) : u \in H\}\right)$$

is integrable on $[0,+\infty[$ and

$$\alpha\left(\left\{\int_0^{+\infty} u(t)\mathrm{d}t : u \in H\right\}\right) \le 2\int_0^{+\infty} \alpha\left(\{u(t) : u \in H\}\right)\mathrm{d}t. \tag{2.1}$$

3 A compactness result

In this section, we introduce our main result. In particular, we shall prove a compactness result for bounded functions with bounded derivatives on unbounded sets.

Theorem 3.1 *A subset* $H \subset \mathcal{BC}^m([0,+\infty[;E)$ *is relatively compact if, and only if, the following conditions are satisfied:*

(C1) $H^{(m)}$ *is relatively compact in* $\mathcal{BC}([0,+\infty[;E)$*;*
(C2) *for all* $k = 0,\dots,m-1$ *and for all* $\epsilon > 0$, *there exists* $M = M_\epsilon > 0$ *such that*

$$\phi_{\mathcal{BC}([M,+\infty[;E)}\left(H^{(k)}_{|[M,+\infty[}\right) < \epsilon \quad \textit{for } \phi \in \{\alpha,\beta,\gamma\}.$$

Proof If H is relatively compact in $\mathcal{BC}^m([0,+\infty[;E)$, then $H^{(k)}$ is relatively compact in $\mathcal{BC}([0,+\infty[;E)$ for any $k = 0,\dots,m$; therefore, (C1) and (C2) follow.

To prove the converse, we assume that (C1) and (C2) hold. In order to show that H is relatively compact in $\mathcal{BC}^m([0,+\infty[;E)$, we will use induction over j to prove that $H^{(m-j)}$ is relatively compact in $\mathcal{BC}^j([0,+\infty[;E)$.

If $j = 0$, the statement follows by assumption (C1). Assume that $H^{(m-j+1)}$ is relatively compact in $\mathcal{BC}^{j-1}([0,+\infty[;E)$. We now show that $H^{(m-j)}$ must be also relatively compact in $\mathcal{BC}^j([0,+\infty[;E)$. Suppose on the contrary that $H^{(m-j)}$ is not relatively compact in $\mathcal{BC}^j([0,+\infty[;E)$; then, since $H^{(m-j)}$ is either unbounded or $\beta_{\mathcal{BC}^j([0,+\infty[;E)}\left(H^{(m-j)}\right) > 0$, there exists a sequence $(f_n)_{n\in\mathbb{N}}$ such that, for all $p \neq q$,

$$\|f_p - f_q\|_{\mathcal{BC}^j} = \sum_{k=0}^{j} \left\| f_p^{(m-k)} - f_q^{(m-k)} \right\|_\infty \geq 4\eta \tag{3.1}$$

for some fixed $\eta > 0$.

From the inductive hypothesis, it follows that there exists a subsequence $(f_{n_r})_{r\in\mathbb{N}} \subset H$ such that $(f_{n_r}^{(m-j+1)})_{r\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{BC}^{j-1}([0,+\infty[;E)$. Therefore, there exists $r_0 \in \mathbb{N}$ such that

$$\|f_{n_p} - f_{n_q}\|_{\mathcal{BC}^{j-1}} = \sum_{k=0}^{j-1} \left\| f_{n_p}^{(m-k)} - f_{n_q}^{(m-k)} \right\|_\infty < \eta \quad \text{for all } p,q \geq r_0. \tag{3.2}$$

It follows from (3.1) and (3.2) that

$$\left\| f_{n_p}^{(m-j)} - f_{n_q}^{(m-j)} \right\|_\infty = \sup_{t\in[0,+\infty[} \left\| f_{n_p}^{(m-j)}(t) - f_{n_q}^{(m-j)}(t) \right\|_E \geq 3\eta$$
$$\text{for all } p,q \geq r_0,\ p \neq q.$$

Then, for every $p \neq q$, there exists $t_{pq} \in [0,+\infty[$ such that

$$\left\| f_{n_p}^{(m-j)}(t_{pq}) - f_{n_q}^{(m-j)}(t_{pq}) \right\|_E \geq 2\eta.$$

We note that the sequence $\{t_{pq}\} \subset [0,\infty[$ is bounded. Indeed, let $M_0 \in [0,+\infty[$ be such that

$$\beta_{\mathcal{BC}([M_0,+\infty[;E)}\left(H^{(m-j)}_{|[M_0,+\infty[}\right) < \eta \tag{3.3}$$

and suppose the existence of a subsequence $(t_{p_j q_j})_{j\in\mathbb{N}} \subset [0, +\infty[$ such that $t_{p_i q_i} \geq i M_0$. Then,

$$\left\| f_{n_{p_i}}^{(m-j)} - f_{n_{q_i}}^{(m-j)} \right\|_\infty \geq \left\| f_{n_{p_i}}^{(m-j)}(t_{p_i q_i}) - f_{n_{q_i}}^{(m-j)}(t_{p_i q_i}) \right\|_E \geq 2\eta$$

which contradicts (3.3). Thus, there exists a constant $N > 0$ such that $t_{pq} < N$ for any p and q.

On the other hand, by choosing a further subsequence if necessary, we have

$$\left\| f_{n_p}^{(m-j)}(M_0) - f_{n_q}^{(m-j)}(M_0) \right\|_E \leq \left\| f_{n_p}^{(m-j)} - f_{n_q}^{(m-j)} \right\|_\infty < \eta \text{ for any } n_p \neq n_q.$$

Therefore,

$$\begin{aligned}
\eta &= 2\eta - \eta \\
&\leq \| f_{n_p}^{(m-j)}(t_{pq}) - f_{n_q}^{(m-j)}(t_{pq}) \|_E - \| f_{n_p}^{(m-j)}(M_0) - f_{n_q}^{(m-j)}(M_0) \|_E \\
&\leq \left\| \left(f_{n_p}^{(m-j)}(t_{pq}) - f_{n_p}^{(m-j)}(M_0) \right) - \left(f_{n_q}^{(m-j)}(t_{pq}) - f_{n_q}^{(m-j)}(M_0) \right) \right\|_E \\
&= \left\| \int_{t_{pq}}^{M_0} \left(f_{n_p}^{(m-j+1)}(t) - f_{n_q}^{(m-j+1)}(t) \right) \mathrm{d}t \right\|_E \\
&\leq |M_0 - t_{pq}| \| f_{n_p}^{(m-j+1)} - f_{n_q}^{(m-j+1)} \|_\infty \\
&\leq (M_0 + N) \| f_{n_p}^{(m-j+1)} - f_{n_q}^{(m-j+1)} \|_\infty.
\end{aligned}$$

It turns out that $\{ f_{n_k}^{(m-j+1)} \}$ contains no convergent subsequences in the space $\mathcal{BC}^{m-(j-1)}([0, +\infty[; E)$. □

Corollary 3.2 *A subset $H \subset \mathcal{BC}^m([0, +\infty[; E)$ is relatively compact if, and only if, there exists $L > 0$ such that the following conditions are satisfied:*

(C3) $H^{(m)}_{|[0,L]}$ *is relatively compact in $\mathcal{C}([0, L]; E)$;*

(C4) *for all $k = 0, \ldots, m$, $H^{(k)}_{|[L,+\infty[}$ is relatively compact in $\mathcal{BC}([L, +\infty; E)$.*

Proof For the "if" part, note that conditions (C3) and (C4) imply condition (C1) of Theorem 3.1. Furthermore, (C4) is stronger than (C2). The converse "only if" part is indeed trivial. □

4 An application to boundary value problems

In this section, we apply Theorem 3.1 to a second-order boundary value problem in a Banach space E. More precisely, we shall prove the existence of a solution $y \in \mathcal{C}^2([0, +\infty[; E)$ for the second-order boundary value problem

$$\text{(P)} \qquad \begin{cases} y'' + m^2 y' = f(t, y) \text{ a.e. on } [0, +\infty[\\ y(0) = \xi, \quad \lim\limits_{t\to+\infty} y(t) \in L \end{cases}$$

where $m \in \mathbb{R}$, $f : [0, +\infty[\times E \to E$, $\xi \in E$, $L \subseteq E$.

Our approach consists in converting (P) to an equivalent fixed point problem that involves upper semi-continuous multivalued operators. To this end, let us recall some results on these operators we shall use in the sequel.

Definition 4.1 Let E, F be Banach spaces and $\emptyset \neq D \subset E$. A multivalued operator $T : D \to 2^F \setminus \emptyset$ is said to be *upper semi-continuous* (*usc* briefly) if $T^{-1}(A)$ is closed in D whenever $A \subset F$ is closed. Equivalently, T is *usc* if $(y_n)_{n\in\mathbb{N}} \subset D$, $A \subset F$ is closed, $y_n \to y \in D$ and $Ty_n \cap A \neq \emptyset$ for all $n \in \mathbb{N}$, then also $Ty \cap A \neq \emptyset$.

We need the following characterization of the upper-semicontinuity for multivalued operators:

Theorem 4.2 (*see, for example,* [4, Proposition 1.2]) *Let $D \neq \emptyset$ be a subset of a Banach space E and $T : D \to 2^E \setminus \emptyset$ have closed values. Then, if $graph(T)$ is closed and $T(D)$ is relatively compact, T is usc.*

Now, we can state the following fixed point theorem for multivalued operators:

Theorem 4.3 (Bohnenluest–Karlin (1950) [8]) *Let D be a nonempty closed convex subset of E and let $T : D \to 2^D$ be a multivalued usc operator. Suppose that*

(1) *$T(x)$ is nonempty, compact and convex for all $x \in D$;*
(2) *$T(D)$ is relatively compact.*

Then, there exists at least one $y \in D$ such that $y \in Ty$.

We begin to transform the boundary value problem (P) into an equivalent fixed point problem for a multivalued operator.

Lemma 4.4 *Let $f : [0, +\infty[\times E \to E$ be a continuous function and assume that*

(1) *there exist two positive functions $p, q \in \mathcal{L}^1(0, +\infty)$ such that:*

$$\|f(t, y)\|_E \leq p(t) + q(t)\|y\|_E \quad \textit{for every } t \geq 0, y \in E; \tag{4.1}$$

(2)

$$\lim_{t\to+\infty} e^{-m^2 t} \int_0^t e^{m^2 s} p(s) ds = 0, \tag{4.2}$$

$$\lim_{t\to+\infty} e^{-m^2 t} \int_0^t e^{m^2 s} q(s) ds = 0.$$

Define the multivalued operator T on $\mathcal{C}^1([0, +\infty, E)$ by

$$(Ty)(t) := \Bigg\{ l + (\xi - l)e^{-m^2 t} - \frac{1}{m^2} \int_t^{+\infty} f(s, y(s))ds + \frac{e^{-m^2 t}}{m^2} \int_0^{+\infty} f(s, y(s))ds - \frac{e^{-m^2 t}}{m^2} \int_0^t e^{m^2 s} f(s, y(s))ds : l \in L \Bigg\}.$$

Then, problem (P) *is equivalent to the fixed point problem:*

Find $y_0 \in \mathcal{C}^1([0, +\infty, E)$ such that $y_0 \in Ty_0$.

Proof Setting $u = y'$ in the second-order differential equation

$$y'' + m^2 y' = f(t, y) \text{ a.e. on } [0, +\infty[,$$

we obtain

$$u' + m^2 u = f\left(t, \int_0^t u(\sigma)\mathrm{d}\sigma + \xi\right)$$

which admits the general solution

$$u(t) = \mathrm{e}^{-m^2 t}\left(k + \int_0^t \mathrm{e}^{m^2 s} f\left(s, \int_0^s u(\sigma)\mathrm{d}\sigma + \xi\right)\right)\mathrm{d}s, \quad k \in \mathbb{R}.$$

By integrating again, we derive

$$\begin{aligned}
y(t) &= -\frac{k}{m^2}\mathrm{e}^{-m^2 t} + \int_0^t \mathrm{e}^{-m^2 \tau} \int_0^\tau \mathrm{e}^{m^2 s} f(s, y(s))\mathrm{d}s\mathrm{d}\tau + c \\
&= c - \frac{k}{m^2}\mathrm{e}^{-m^2 t} + \int_0^t \mathrm{e}^{m^2 s} f(s, y(s)) \int_s^t \mathrm{e}^{-m^2 \tau}\mathrm{d}\tau\mathrm{d}s \\
&= c - \frac{k}{m^2}\mathrm{e}^{-m^2 t} + \frac{1}{m^2}\int_0^t (1 - \mathrm{e}^{m^2(s-t)}) f(s, y(s))\mathrm{d}s \\
&= c - \frac{k}{m^2}\mathrm{e}^{-m^2 t} + \frac{1}{m^2}\int_0^t f(s, y(s))\mathrm{d}s - \frac{\mathrm{e}^{-m^2 t}}{m^2}\int_0^t \mathrm{e}^{m^2 s} f(s, y(s))\mathrm{d}s.
\end{aligned}$$

Imposing the boundary conditions and by (4.1) and (4.2), we obtain

$$\begin{cases} c - \frac{k}{m^2} = \xi, \\ c + \dfrac{1}{m^2}\displaystyle\int_0^{+\infty} f(s, y(s))\mathrm{d}s, \in L \end{cases}$$

which finally yields

$$y(t) \in \left\{ l + (\xi - l)\mathrm{e}^{-m^2 t} - \frac{1}{m^2}\int_t^{+\infty} f(s, y(s))\mathrm{d}s + \frac{\mathrm{e}^{-m^2 t}}{m^2}\int_0^{+\infty} f(s, y(s))\mathrm{d}s \right.$$
$$\left. -\frac{\mathrm{e}^{-m^2 t}}{m^2}\int_0^t \mathrm{e}^{m^2 s} f(s, y(s))\mathrm{d}s,\ l \in L \right\} = (Ty)(t).$$

Now it is not hard, proceeding backward in the argument above, to see that the inclusion $y \in Ty$ implies that y solves Problem (P). □

Theorem 4.5 *Assume that $f : [0, +\infty[\times E \to E$ is continuous and $f(t, B_r)$ is relatively compact for any $r > 0$ and for any $t \geq 0$, where $B_r \subset E$ is the closed ball centered at 0 and with radius r. Moreover, we suppose that*

(1) *L is a nonempty convex compact subset of E;*
(2) *there exist two positive functions* $p, q \in \mathcal{L}^1(0, +\infty)$ *such that:*

(a) $\|f(t, y)\|_E \leq p(t) + q(t)\|y\|_E$ *for every* $t \geq 0$, $y \in E$;
(b) $Q := \|q\|_{\mathcal{L}^1(0,+\infty)} < \dfrac{m^2}{2m^2+3}$;

(3) *there hold the relations:*

$$\lim_{t\to+\infty} \mathrm{e}^{-m^2 t} \int_0^t \mathrm{e}^{m^2 s} p(s)\mathrm{d}s = 0, \tag{4.3}$$

$$\lim_{t\to+\infty} \mathrm{e}^{-m^2 t} \int_0^t \mathrm{e}^{m^2 s} q(s)\mathrm{d}s = 0.$$

Then, there exists at least a solution of Problem (P).

Proof Set

$$P := \|p\|_{\mathcal{L}^1(0,+\infty)},$$

$$r_0 := \frac{\max_{l\in L}\left(\|l\|_E + \|\xi - l\|_E\right) + m^2 \max_{l\in L}\|\xi - l\|_E + \left(2 + \frac{3}{m^2}\right)P}{1 - \left(2 + \frac{3}{m^2}\right)Q}$$

and

$$D := \{y \in \mathcal{BC}^1([0, +\infty[; E) : \|y\|_{\mathcal{BC}^1} \leq r_0\}.$$

The idea of the proof is to use Theorem 4.3 to ensure the existence of a fixed point y_0 for the multivalued operator T introduced in Lemma 4.4, so that y_0 solves the problem (P).

We shall divide the proof into four steps.

Step 1 T maps D to 2^D, so that $\{Ty\}$ is uniformly bounded for all $y \in D$.

Proof of Step 1 Let $y \in D$ and $z \in Ty$ be fixed; then, there exists $l \in L$ such that

$$z(t) = l + (\xi - l)\mathrm{e}^{-m^2 t} - \frac{1}{m^2}\int_t^{+\infty} f(s, y(s))\mathrm{d}s + \frac{\mathrm{e}^{-m^2 t}}{m^2}\int_0^{+\infty} f(s, y(s))\mathrm{d}s - \frac{\mathrm{e}^{-m^2 t}}{m^2}\int_0^t \mathrm{e}^{m^2 s} f(s, y(s))\mathrm{d}s. \tag{4.4}$$

Then, for fixed $t \in [0, +\infty[$,

$$\|z(t)\|_E = \left\| l + (\xi - l)\mathrm{e}^{-m^2 t} - \frac{1}{m^2}\int_t^{+\infty} f(s, y(s))\mathrm{d}s + \frac{\mathrm{e}^{-m^2 t}}{m^2}\int_0^{+\infty} f(s, y(s))\mathrm{d}s - \frac{\mathrm{e}^{-m^2 t}}{m^2}\int_0^t \mathrm{e}^{m^2 s} f(s, y(s))\mathrm{d}s \right\|_E$$

$$\leq \|l\|_E + \|\xi - l\|_E + \frac{3}{m^2} \int_0^{+\infty} \|f(s, y(s))\|_E \mathrm{d}s$$

$$\leq \|l\|_E + \|\xi - l\|_E + \frac{3}{m^2} \int_0^{+\infty} (p(s) + r_0 q(s))\, \mathrm{d}s$$

$$\leq \|l\|_E + \|\xi - l\|_E + \frac{3}{m^2} (P + Qr_0)\,.$$

From this last relation, it follows that

$$\|z\|_\infty \leq \|l\|_E + \|\xi - l\|_E + \frac{3}{m^2} (P + Qr_0)\,.$$

Moreover,

$$\|z'(t)\|_E = \left\| m^2(l-\xi)\mathrm{e}^{-m^2 t} + \mathrm{e}^{-m^2 t} \int_0^t \mathrm{e}^{m^2 s} f(s, y(s))\mathrm{d}s \right.$$

$$\left. -\mathrm{e}^{-m^2 t} \int_0^{+\infty} f(s, y(s))\mathrm{d}s \right\|_E$$

$$\leq m^2 \|\xi - l\|_E + 2 \int_0^{+\infty} \|f(s, y(s))\|_E \mathrm{d}s$$

$$\leq m^2 \|\xi - l\|_E + 2 \int_0^{+\infty} (p(s) + r_0 q(s))\mathrm{d}s$$

$$\leq m^2 \|\xi - l\|_E + 2(P + Qr_0),$$

which implies that

$$\|z'\|_\infty \leq m^2 \|\xi - l\|_E + 2(P + Qr_0). \tag{4.5}$$

Finally, we get

$$\|z\|_{\mathcal{BC}^1} \leq \max_{l \in L} (\|l\|_E + \|\xi - l\|_E) + m^2 \max_{l \in L} \|\xi - l\|_E + \left(2 + \frac{3}{m^2}\right)(P + Qr_0)$$

$$= r_0 - \left(2 + \frac{3}{m^2}\right) Qr_0 + \left(2 + \frac{3}{m^2}\right) Qr_0 = r_0$$

for any $z \in D$. □

Step 2 For any fixed $y \in D$, Ty is compact and convex.

Proof of Step 2 Let $(z_n)_{n\in\mathbb{N}}$ be a sequence in Ty and let $(l_n)_{n\in\mathbb{N}} \subset L$ be such that

$$z_n(t) = l_n + (\xi - l_n)\mathrm{e}^{-m^2 t} - \frac{1}{m^2} \int_t^{+\infty} f(s, y(s))\mathrm{d}s + \frac{\mathrm{e}^{-m^2 t}}{m^2} \int_0^{+\infty} f(s, y(s))\mathrm{d}s$$

for all $t \in [0, +\infty[$.

Since only the first two terms depend on $\{l_n\}$, we can rewrite this last expression as

$$z_n(t) = l_n + (\xi - l_n)e^{-m^2 t} + (Fy)(t),$$

where

$$(Fy)(t) := -\frac{1}{m^2}\int_t^{+\infty} f(s, y(s))ds + \frac{e^{-m^2 t}}{m^2}\int_0^{+\infty} f(s, y(s))ds. \tag{4.6}$$

By compactness assumption on L,, we may extract a subsequence $(l_{n_k})_{k\in\mathbb{N}}$ such that $\lim_{k\to+\infty} l_{n_k} = l \in L$, thus obtaining

$$\lim_{k\to+\infty} z_{n_k}(t) = l + (\xi - l)e^{-m^2 t} + (Fy)(t) =: z(t) \in (Ty)(t),$$

for any fixed $t \in [0, +\infty[$. It turns out that

$$\|z_{n_k} - z\|_{\mathcal{BC}^1} \leq (2 + m^2)\|l_{n_k} - l\|_E \to 0 \quad \text{as } k \to \infty.$$

This verifies that Ty is compact.

The convexity of Ty easily follows from (4.6) and the convexity assumption on L. □

Step 3 $T(D)$ is relatively compact.

Proof of Step 3 To prove this step, we shall apply Theorem 3.1, i.e., we have to show that

(a) The set $\mathcal{A} := \{z' : z \in T(D)\}$ is relatively compact in $\mathcal{BC}([0, +\infty[; E)$;
(b) For every $\epsilon > 0$ there exists $M > 0$ such that

$$\gamma_{\mathcal{BC}([M,+\infty[;E)}\left((T(D))\big|_{[M,+\infty[}\right) < \epsilon.$$

(a) Let $(z_n)_{n\in\mathbb{N}} \subseteq T(D)$, $(y_n)_{n\in\mathbb{N}} \subseteq D$ and $(l_n)_{n\in\mathbb{N}} \subseteq L$ be such that

$$z_n(t) = l_n + (\xi - l_n)e^{-m^2 t} + (Fy_n)(t),$$

hence

$$z_n'(t) = m^2(l_n - \xi)e^{-m^2 t} + e^{-m^2 t}\int_0^t e^{m^2 s} f(s, y_n(s))ds - e^{-m^2 t}\int_0^{+\infty} f(s, y_n(s))ds.$$

Using Proposition 2.1, we will prove that $(z_n')_{n\in\mathbb{N}}$ is relatively compact in $\mathcal{BC}([0, +\infty[; E)$. Note that the boundedness of $\mathcal{A}$ follows from **Step 1**. Moreover, $\{z_n', n \in \mathbb{N}\}$ is equicontinuous. Indeed, for fixed $\epsilon > 0$ and $n \in \mathbb{N}$, we have

$$\begin{aligned}\|z_n'(t_1)-z_n'(t_2)\|_E &\le m^2\|l_n-\xi\|_E\left|e^{-m^2t_1}-e^{-m^2t_2}\right| \\ &\quad+\left|e^{-m^2t_1}-e^{-m^2t_2}\right|\int_0^{t_1} e^{-m^2s}\|f(s,y_n(s))\|_E ds \\ &\quad+e^{-m^2t_2}\left|\int_{t_1}^{t_2}\|f(s,y_n(s))\|_E ds\right| \\ &\quad+\left|e^{-m^2t_1}-e^{-m^2t_2}\right|\int_0^{+\infty}\|f(s,y_n(s)\|_E ds \\ &\le m^2\max_{l\in L}\|l-\xi\|_E\left|e^{-m^2t_1}-e^{-m^2t_2}\right| \\ &\quad+2\left|e^{-m^2t_1}-e^{-m^2t_2}\right|\int_0^{+\infty}\|f(s,y_n(s))\|_E ds \\ &\quad+\left|\int_{t_1}^{t_2}\|f(s,y_n(s))\|_E ds\right| \\ &\le m^2\max_{l\in L}\|l-\xi\|_E\left|e^{-m^2t_1}-e^{-m^2t_2}\right| \\ &\quad+2\left|e^{-m^2t_1}-e^{-m^2t_2}\right|(P+r_0Q)+\left|\int_{t_1}^{t_2}(p(s)+r_0q(s))ds\right|.\end{aligned}$$

Set

$$C:=\max\{m^2\max_{l\in L}\|l-\xi\|_E,2(P+r_0Q)\}$$

and choose $\delta>0$ such that, for any $t_1,t_2\in[0,+\infty[$ with $|t_1-t_2|<\delta$, we have

$$\left|e^{-m^2t_1}-e^{-m^2t_2}\right|<\frac{\epsilon}{3C}\quad\text{and}\quad\left|\int_{t_1}^{t_2}(p(s)+r_0q(s))ds\right|<\frac{\epsilon}{3}.$$

Consequently, we obtain

$$\|z_n'(t_1)-z_n'(t_2)\|_E<\epsilon\quad\text{for all } n\in\mathbb{N}.$$

Now, we verify the condition (2.1) of Proposition 2.1, that is, for any fixed $t_0\in[0,1]$, the set $\{z_n'(t_0),n\in\mathbb{N}\}$ is relatively compact in E. Let α be the Kuratowski measure of noncompactness. Then, we have

$$\begin{aligned}&\alpha\left(\left\{z_n'(t_0),n\in\mathbb{N}\right\}\right)\\ &=\alpha\left(\left\{m^2(l_n-\xi)e^{-m^2t_0}+e^{-m^2t_0}\int_0^{t_0}e^{m^2s}f(s,y_n(s))ds\right.\right.\end{aligned}$$

$$-\mathrm{e}^{-m^2 t_0}\int_0^{+\infty} f(s,y_n(s))\mathrm{d}s, n\in\mathbb{N}\Big\}\Big)$$

$$\le \alpha\left(\left\{m^2(l_n-\xi)\mathrm{e}^{-m^2 t_0} : n\in\mathbb{N}\right\}\right)+\alpha\left(\left\{\mathrm{e}^{-m^2 t_0}\int_0^{t_0}\mathrm{e}^{m^2 s} f(s,y_n(s))\mathrm{d}s : n\in\mathbb{N}\right\}\right)$$

$$+\alpha\left(\left\{\mathrm{e}^{-m^2 t_0}\int_0^{+\infty} f(s,y_n(s))\mathrm{d}s, n\in\mathbb{N}\right\}\right).$$

By compactness of L and Lemma 2.2, we obtain

$$\begin{aligned}
&\alpha\left(\left\{z_n'(t_0), n\in\mathbb{N}\right\}\right)\\
&\le 2\mathrm{e}^{-m^2 t_0}\int_0^{t_0}\alpha\left(\left\{\mathrm{e}^{m^2 s} f(s,y_n(s)),\ n\in\mathbb{N}\right\}\right)\mathrm{d}s\\
&\quad +2\mathrm{e}^{-m^2 t_0}\int_0^{+\infty}\alpha\left(\left\{f(s,y_n(s)), n\in\mathbb{N}\right\}\right)\mathrm{d}s\\
&\le 2\mathrm{e}^{-m^2 t_0}\int_0^{t_0}\mathrm{e}^{m^2 s}\alpha\left(\left\{f(s,w),\ w\in B_{r_0}\right\}\right)\mathrm{d}s\\
&\quad +2\mathrm{e}^{-m^2 t_0}\int_0^{+\infty}\alpha\left(\left\{f(s,w), w\in B_{r_0}\right\}\right)\mathrm{d}s.
\end{aligned}$$

Since by hypothesis $\alpha\left(\left\{f(t,B_{r_0})\right\}\right)=0$ for any $t\in[0,+\infty[$, we can conclude that

$$\alpha\left(\left\{z_n'(t_0), n\in\mathbb{N}\right\}\right)=0,$$

i.e., $(z_n'(t_0))_{n\in\mathbb{N}}$ is relatively compact in E for any fixed t_0.

Finally, from (4.3), it follows that $\lim_{n\to+\infty} z_n'(t)=0$ uniformly on n.

(b) Let $\epsilon>0$ be fixed and $l_1,\ldots,l_n$ be such that $L\subseteq\bigcup_{i=1}^{n} B\left(l_i,\frac{\epsilon}{2}\right)$. By (4.2), there exists $M_\epsilon>0$ such that for any $t\ge M_\epsilon$

$$\max_{l\in L}\|\xi-l\|_E\mathrm{e}^{-m^2 t}+\frac{1}{m^2}\int_t^{+\infty}(p(s)+r_0 q(s))\,\mathrm{d}s$$

$$+\frac{\mathrm{e}^{-m^2 t}}{m^2}\int_0^{+\infty}(p(s)+r_0 q(s))\,\mathrm{d}s+\frac{\mathrm{e}^{-m^2 t}}{m^2}\int_0^{t}\mathrm{e}^{m^2 s}(p(s)+r_0 q(s))\,\mathrm{d}s<\frac{\epsilon}{2}. \quad (4.7)$$

Let $z\in T(D)\big|_{[M_\epsilon,+\infty[}$ be fixed and let $l\in L$ and $y\in D$ be such that

$$z(t)=l+(\xi-l)\mathrm{e}^{-m^2 t}+(Fy)(t).$$

Let i be such that $\|l - l_i\|_E \leq \frac{\epsilon}{2}$; then for every $t \geq M_\epsilon$ we have

$$\begin{aligned}\|z(t) - l_i\|_E &\leq \|z(t) - l\|_E + \|l - l_i\|_E \\ &< \|\xi - l\|_E \, \mathrm{e}^{-m^2 t} + \frac{1}{m^2} \int_t^{+\infty} \|f(s, y(s))\|_E \mathrm{d}s \\ &\quad + \frac{\mathrm{e}^{-m^2 t}}{m^2} \int_0^{+\infty} \|f(s, y(s))\|_E \mathrm{d}s + \frac{\mathrm{e}^{-m^2 t}}{m^2} \int_0^t \mathrm{e}^{m^2 s} \|f(s, y(s))\|_E \mathrm{d}s + \frac{\epsilon}{2} \\ &\leq \max_{l \in L} \|\xi - l\|_E \, \mathrm{e}^{-m^2 t} + \frac{1}{m^2} \int_t^{+\infty} (p(s) + r_0 q(s)) \mathrm{d}s \\ &\quad + \frac{\mathrm{e}^{-m^2 t}}{m^2} \int_0^{+\infty} (p(s) + r_0 q(s)) \mathrm{d}s + \frac{\mathrm{e}^{-m^2 t}}{m^2} \int_0^t \mathrm{e}^{m^2 s} (p(s) + r_0 q(s)) \mathrm{d}s + \frac{\epsilon}{2}.\end{aligned}$$

Then, from (4.7), it follows that

$$\sup_{t \in [M_\epsilon, +\infty[} \|z(t) - l_i\|_E < \epsilon,$$

i.e.,

$$\gamma_{\mathcal{BC}([M_\epsilon, +\infty[; E)} \left((T(D))\big|_{[M_\epsilon, +\infty[} \right) < \epsilon.$$

□

Step 4 T is upper semicontinuous.

Proof of Step 4 Using Theorem 4.2, we need to show that $Graph(T(D))$ is closed, that is, if $(y_n)_{n \in \mathbb{N}} \subseteq D$ is a sequence converging to an element $y \in D$, $(z_n)_{n \in \mathbb{N}}$ is a sequence converging to z and such that $z_n \in T y_n$ for any $n \in \mathbb{N}$, then $z \in T y$.
For this purpose, let $(l_n)_{n \in \mathbb{N}}$ such that

$$z_n(t) = l_n + (\xi - l_n) \mathrm{e}^{-m^2 t} + (F y_n)(t).$$

Let $(l_{n_k})_{k \in \mathbb{N}}$ be a subsequence of $(l_n)_{n \in \mathbb{N}}$ converging to $l \in L$. Let $\tilde{z} \in T y$ be defined by

$$\tilde{z}(t) = l + (\xi - l) \mathrm{e}^{-m^2 t} + (F y)(t).$$

We claim that $\tilde{z} = z$.

To see this, fix $\epsilon > 0$. Since $z_{n_k} \to z$ by hypothesis, there exists $k_0 \in \mathbb{N}$ such that if $k \geq k_0$, then

$$\|z_{n_k} - z\|_\infty < \epsilon/3. \tag{4.8}$$

Moreover, by following the proof of **Step 2**, we can fix $k_1 \in \mathbb{N}$ such that for any $k \geq k_1$

$$\sup_{t \in [0, +\infty[} \|l_{n_k} + (\xi - l_{n_k}) \mathrm{e}^{-m^2 t} + (F y)(t) - (l + (\xi - l) \mathrm{e}^{-m^2 t} + (F y)(t))\|_E < \epsilon/3. \tag{4.9}$$

On the other hand, for any fixed $t \in [0, +\infty[$ and by (4.6), we get

$$
\begin{aligned}
&\|(Fy_n)(t) - (Fy)(t)\|_E \\
&= \Bigg\| -\frac{1}{m^2} \int_t^{+\infty} f(s, y_n(s))\mathrm{d}s + \frac{\mathrm{e}^{-m^2 t}}{m^2} \int_0^{+\infty} f(s, y_n(s))\mathrm{d}s \\
&\quad - \left(-\frac{1}{m^2} \int_t^{+\infty} f(s, y(s))\mathrm{d}s + \frac{\mathrm{e}^{-m^2 t}}{m^2} \int_0^{+\infty} f(s, y(s))\mathrm{d}s \right) \Bigg\|_E \\
&\leq \frac{1}{m^2} \int_t^{+\infty} \|f(s, y_n(s)) - f(s, y(s))\|_E \mathrm{d}s \\
&\quad + \frac{1}{m^2} \int_0^{+\infty} \|f(s, y_n(s)) - f(s, y(s))\|_E \mathrm{d}s.
\end{aligned}
$$

It turns out that

$$
\|Fy_n - Fy\|_\infty \leq \frac{2}{m^2} \|f(\cdot, y_n(\cdot)) - f(\cdot, y(\cdot))\|_{\mathcal{L}^1(0,+\infty)}. \tag{4.10}
$$

Note that

$$
\lim_{k \to +\infty} f(t, y_{n_k}(t)) = f(t, y(t))
$$

and that, by **Step 1** and by hypothesis (2)(a), $f(\cdot, y_n(\cdot))$ is dominated by $p(\cdot) + r_0 q(\cdot) \in \mathcal{L}^1(0, +\infty)$. Hence, we can apply the Lebesgue dominated convergence Theorem and (4.10) to deduce that there exists $k_2 \in \mathbb{N}$ such that

$$
\|Fy_{n_k} - Fy\|_\infty < \epsilon/3, \tag{4.11}
$$

for any $k \geq k_2$.

To put an end to this step, we observe that if $\bar{k} \geq \max\{k_0, k_1, k_2\}$, then for any $k \geq \bar{k}$ and for any $t \in [0, +\infty[$ fixed, we get, by (4.8), (4.9) and (4.11),

$$
\begin{aligned}
\|z(t) - \tilde{z}(t)\|_E &\leq \|z_{n_k} - z\|_\infty + \|z_{n_k}(t) - \tilde{z}(t)\| \\
&\leq \epsilon/3 + \sup_{t \in [0,+\infty[} \|l_{n_k} + (\xi - l_{n_k})\mathrm{e}^{-m^2 t} \\
&\qquad + (Fy)(t) - (l + (\xi - l)\mathrm{e}^{-m^2 t} + (Fy)(t))\|_E + \|Fy_{n_k} - Fy\|_\infty \\
&< \epsilon.
\end{aligned}
$$

Since $\epsilon > 0$ is arbitrary, we get then $z = \tilde{z} \in Ty$. □

To bring us to a conclusion, we observe that Theorem 4.3 ensures the existence of a fixed point y_0 of the multivalued mapping T, and y_0 is a solution to the boundary value problem (P). □

Remark 4.6 Note that the hypothesis that $f(t, B_r)$ be compact for any choice of $t \in [0, +\infty[$ and $r > 0$ can be weakened by assuming only that $f(t, B_{r_0})$ is compact for any $t \in [0, +\infty[$, where r_0 is given in the proof of Theorem 4.5.

Acknowledgments F. Cianciaruso, V. Colao, and G. Marino were supported by Ministero dell'Universitá e della Ricerca of Italy. H.K. Xu was supported in part by NSC 97-2628-M-110-003-MY3 (Taiwan). He extended his appreciation to the Deanship of Scientific Research at King Saud University for funding the work through the research group project No RGP-VPP-087.

References

1. Avramescu, C.: Sur l'existence des solutions convergentes des systémes d'équations différentielles non linéaires. Ann. Mat. Pura Appl. **4**, 147–168 (1969)
2. Banas, J., Goebel, K.: Measures of noncompactness in banach spaces. In: Lecture Notes in Pure and Applied Mathematics, vol. 60. Dekker, New York (1980)
3. Bartle, R.G.: On compactness in functional analysis. Trans. Am. Math. Soc. **79**, 35–57 (1955)
4. Deimling, K.: Multivalued Differential Equations. Walter de Gruyter, Berlin (1992)
5. De Pascale, E., Lewicki, G., Marino, G.: Some conditions for compactness in $\mathcal{BC}(Q)$ and their application to boundary value problems. Analysis **22**, 21–32 (2002)
6. Liu, Z., Liu, L., Wu, Y., Zhao, J.: Unbounded solutions of a boundary value problem for abstract nth-order differential equations on a infinite interval. J. Appl. Math. Stoch. Anal. 11. Art. ID 589480 (2008)
7. Xiao, Y.B., Kim, J.K., Huang, N.J.: A generalization of Ascoli–Arzela theorem with an application. Nonlinear Funct. Anal. Appl. **11**(2), 305–317 (2006)
8. Zeidler, E.: Nonlinear Functional Analysis and Its Applications I: Fixed-Point Theorems. Springer, Berlin (1985)