

## OPTIMALITY CONDITIONS AND DUALITY IN MULTIOBJECTIVE PROGRAMMING WITH $(\Phi, \rho)$ -INVEXITY\*

Massimiliano FERRARA

*Dept. of Historical, Law, Economics and Social Sciences  
University Mediterranea of Reggio Calabria, ITALY  
massimiliano.ferrara@unirc.it*

Maria Viorica STEFANESCU

*Academy of Economic Studies-Bucharest, Department of Mathematics  
Romania*

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**Abstract:**  $(\Phi, \rho)$ -invexity has recently been introduced with the intent of generalizing invex functions in mathematical programming. Using such conditions we obtain new sufficiency results for optimality in multiobjective programming and extend some classical duality properties.

**Keywords:** Multiobjective programming, invexity, duality.

### 1. INTRODUCTION

The theory of mathematical programming grew remarkably after generalized convexity had been used in the settings of optimality conditions and duality theory. Hanson [5] showed that both weak duality and Kuhn-Tucker sufficiency for optimum hold when convexity was replaced by a weaker condition. This condition, called invexity by Craven [2], was further studied for more general problems and was a source of a vast literature. After the works of Hanson and Craven, other types of differentiable functions have been introduced with the intent of generalizing invex functions from different points

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of view. Hanson and Mond [6] introduced the concept of  $F$ -convexity and Jeyakumar [3] generalized Vial's  $\rho$ -convexity ([12]) introducing the concept of  $\rho$ -invexity. The concept of generalized  $(F, \rho)$  convexity, introduced by Preda [11] is in turn an extension of the above properties and was used by several authors to obtain relevant results.

A large literature was developed around generalized invexity and its applications in mathematical programming and variational problem. Following the line of above cited papers, several authors have extended the basic theoretical results in multiobjective programming. From the more recent literature we refer to Xu [14] and Ojha and Mukherjee [10] for duality under generalized  $(F, \rho)$ -convexity, Mishra [9] and Yang et al. [15] for duality under second order  $F$ -convexity, Liang et al. [7] and Hachimi and Aghezzaf [4] for optimality criteria and duality involving  $(F, \alpha, \rho, d)$ -convexity or generalized  $(F, \alpha, \rho, d)$ -type functions. A common feature of all these extensions of  $F$  convexity is the sublinearity of the scale function and, when this is the case, the linearity with respect to the parameters. The  $(F, \rho)$  convexity was recently generalized to  $(\Phi, \rho)$ -invexity by Caristi, Ferrara and Stefanescu [1], and we will use this concept to extend some theoretical results of multiobjective programming. We begin by introducing some notation for vector inequalities.

For  $x = (x_1, x_2, \dots, x_r)$ ,  $y = (y_1, y_2, \dots, y_r) \in \mathfrak{R}^r$ , where  $\mathfrak{R}^r$  stands for an Euclidean space, the following order notation will be used:

$$x \geq y \text{ if and only if } x_i \geq y_i \text{ for all } i = 1, 2, \dots, r$$

$$x > y \text{ if and only if } x_i > y_i \text{ for all } i = 1, 2, \dots, r$$

and  $x \geq_{not} y$ ,  $x >_{not} y$  are the negations of  $x \geq y$ , respectively,  $x > y$ . The problem to be considered here is the multiobjective programming problem:

$$(VP): \min f(x), \quad g(x) \leq 0, \quad x \in X_0$$

where  $X_0$  is a nonvoid open subset of  $\mathfrak{R}^n$ ,  $f = (f_1, f_2, \dots, f_p): X_0 \mapsto \mathfrak{R}^p$ ,  $g = (g_1, g_2, \dots, g_m): X_0 \mapsto \mathfrak{R}^m$ ,  $f_i, i \in \{1, 2, \dots, p\}$  and  $g_j, j \in \{1, 2, \dots, m\}$  are assumed to be differentiable on  $X_0$ . The symbol "min" is used with the generic meaning of finding solutions of one of the types defined below.

Let  $X$  be the set of all feasible solutions of  $(VP)$ ,

$$X = \{x \in X_0 \mid g(x) \leq 0\}$$

**Definition 1.**  $a \in X$  is said to be a weakly efficient solution of  $(VP)$  if there is no  $x \in X$  such that  $f(x) < f(a)$ .

**Definition 2.**  $a \in X$  is said to be an efficient solution of  $(VP)$  if there is no  $x \in X$  such that  $f(x) \leq f(a)$ ,  $f(x) \neq f(a)$ .

**Definition 3.**  $a \in X$  is said to be a properly efficient solution of  $(VP)$  if it is efficient and there exists a positive constant  $K$  such that for each  $x \in X$  and for each

$i \in \{1, 2, \dots, p\}$  satisfying  $f_i(x) < f_i(a)$ , there exists at least one  $j \in \{1, 2, \dots, p\}$  such that  $f_j(a) < f_j(x)$  and  $f_i(a) - f_i(x) \leq K(f_j(x) - f_j(a))$ .

Denoting by  $WE(VP)$ ,  $E(VP)$  and  $PE(VP)$  the sets of all weakly efficient, efficient, respectively, properly efficient solutions of  $(VP)$ , we have:

$$WE(VP) \supseteq E(VP) \supseteq PE(VP)$$

For readers' convenience, let us write the definitions of  $(\Phi, \rho)$ -invexity from [1]. Let:  $\varphi: X_0 \mapsto \mathfrak{R}$  be a differentiable function ( $X_0 \subseteq \mathfrak{R}^n$ ),  $X \subseteq X_0$  and  $a \in X_0$ .

In the next definitions, an element of the  $(n+1)$ -dimensional Euclidean space  $\mathfrak{R}^{n+1}$  is represented as the ordered pair  $(y, r)$ , with  $y \in \mathfrak{R}^n$  and  $r \in \mathfrak{R}$ ,  $\rho$  is a real number and  $\Phi$  is a real-valued function defined on  $X_0 \times X_0 \times \mathfrak{R}^{n+1}$ , such that  $\phi(x, a, \cdot)$  is convex on  $\mathfrak{R}^{n+1}$  and  $\phi(x, a, (0, r)) \geq 0$  for every  $(x, a) \in X_0 \times X_0$  and  $r \in \mathfrak{R}_+$ .

**Definition 4.**  $\varphi$  is said to be  $(\Phi, \rho)$ -invex. at  $a$ , with respect to  $X$ , if

$$\varphi(x) - \varphi(a) \geq \Phi(x, a, (\nabla \varphi(a), \rho)), \forall x \in X \quad (1)$$

**Definition 5.**  $\varphi$  is said to be pseudo  $(\Phi, \rho)$ -invex at  $a$ , with respect to  $X$ , if  $\varphi(x) - \varphi(a) \geq 0$  whenever  $\Phi(x, a, (\nabla \varphi(a), \rho)) \geq 0$  for some  $x \in X$ .

**Definition 6.**  $\varphi$  is said to be strictly pseudo  $(\Phi, \rho)$ -invex at  $a$ , with respect to  $X$ , if  $\varphi(x) - \varphi(a) > 0$  whenever  $\Phi(x, a, (\nabla \varphi(a), \rho)) \geq 0$  for some  $x \in X$ ,  $x \neq a$ .

**Definition 7.**  $\varphi$  is said to be quasi  $(\Phi, \rho)$ -invex at  $a$ , with respect to  $X$ , if  $\Phi(x, a, (\nabla \varphi(a), \rho)) \leq 0$ , whenever  $\varphi(x) - \varphi(a) \leq 0$  for some  $x \in X$ .

**Definition 8**  $\varphi$  is said to be semistrict quasi  $(\Phi, \rho)$ -invex at  $a$ , with respect to  $X$ , if  $\Phi(x, a, (\nabla \varphi(a), \rho)) < 0$  whenever  $\varphi(x) - \varphi(a) < 0$  for some  $x \in X$ .

$\varphi$  is  $(\Phi, \rho)$ -invex (pseudo  $(\Phi, \rho)$ -invex, strictly pseudo  $(\Phi, \rho)$ -invex, quasi  $(\Phi, \rho)$ -invex, semistrict quasi  $(\Phi, \rho)$ -invex) on  $X_0$  if it is  $(\Phi, \rho)$ -invex (respectively pseudo  $(\Phi, \rho)$ -invex, strictly pseudo  $(\Phi, \rho)$ -invex, quasi  $(\Phi, \rho)$ -invex, semistrict quasi  $(\Phi, \rho)$ -invex) at  $a$ , for every  $a \in X_0$ .

Obviously,  $(\Phi, \rho)$ -invexity implies pseudo, quasi and semistrict quasi  $(\Phi, \rho)$ -invexity and strict pseudo  $(\Phi, \rho)$ -invexity implies pseudo  $(\Phi, \rho)$ -invexity.

**Remark 1.** For  $\Phi(x, a, (y, \rho)) = F(x, a, y) + \rho d^2(x, a)$ , where  $F$  is sublinear in the third argument, the above definitions turn to the corresponding versions of  $(F, \rho)$ -convexity.

Therefore the  $(\Phi, \rho)$ -invexity generalizes  $(F, \rho)$ -convexity; the set of scale functions which define the respective property for a given function is strictly larger in the former case than in the latter. Hence all results of the next two sections extend similar results obtained under  $(F, \rho)$ -convexity:

Everywhere in the following,  $(\Phi, \rho)$ -invexity with respect to  $X$  (or some of its generalizations defined above) will be required for functions involved in a multiobjective programming problem, where  $X$  is the set of feasible solutions. To simplify the terminology we will omit the formal reference to  $X$  in all statements. So that, we shall phrase simply “ $f_i$  are  $(\Phi, \rho)$ -invex at  $a$ ” but we will understand “ $f_i$  are  $(\Phi, \rho)$ -invex at  $a$ , with respect to  $X$ ”.

## 2. OPTIMALITY CONDITIONS

Likewise in the smooth scalar optimization, the Kuhn-Tucker conditions are necessary and/or sufficient conditions for optimality in multiobjective programming, if some additional conditions are fulfilled. Particularly, the sufficiency of Kuhn-Tucker conditions represents not only one of the most important theoretical achievements, but also a fundamental practical result.

Unlike the one-objective case, in multiobjective programming efficient algorithms are missing. Therefore, the basic technique for solution consists in finding Kuhn-Tucker points and checking their optimality. Hence, any extension of the sufficiency results under weaker conditions actually represents both theoretical and technical progress in multiobjective programming.

As we have pointed out in the previous section,  $(\Phi, \rho)$ -invexity follows several consecutive extensions of Hanson's invexity. Since each type of invexity is characterized by a specific class of scale functions, the larger this class is, the more general is the respective property. Then, it is obvious that  $(\Phi, \rho)$ -invexity strictly generalizes  $(F, \rho)$ -convexity, which in turn is strictly weaker than both  $F$ -convexity and  $\rho$ -invexity and each one of these two concepts generalizes Hanson's invexity.

However, a question rises when the invexity is required for proving sufficiency results. Is such a result really extended when a more general invexity condition replaces another invexity condition? According to Martin ([8]), in the scalar mathematical programming, any generalized invexity turns to Hanson's invexity when the sufficiency of Kuhn-Tucker conditions holds. In other words, whenever the objective function and all active restriction functions satisfy simultaneously the same generalized invexity at a Kuhn-Tucker point which is an optimum solution, then all these functions should satisfy the usual invexity, too (quasi-invexity for the restrictions). This is not the case in multiobjective programming, and our examples below show that the sufficiency of Kuhn-Tucker conditions can be proved under  $(\Phi, \rho)$ -invexity, even if Hanson's invexity is not satisfied. Therefore, the results of this section are real extensions of the similar results known in the literature.

The main results of this section are the next four theorems concerning sufficient conditions for optimality under various  $(\Phi, \rho)$ -invexities defined in Section 1.

For the sake of completeness, we will also prove a necessity result under  $(\Phi, \rho)$ -invexity.

We recall first the basic definition of the Kuhn-Tucker conditions.

**Definition 9.** The triple  $(a, \mu, \lambda)$ , where  $a \in X$ ,  $\mu \in \mathfrak{R}_+^p$ ,  $\mu \neq 0$ , and  $\lambda \in \mathfrak{R}_+^m$  is said to be a Kuhn- Tucker point of (VP) if

$$\sum_{i=1}^p \mu_i \nabla f_i(a) + \sum_{j=1}^m \lambda_j \nabla g_j(a) = 0 \quad (2)$$

and

$$\sum_{j=1}^m \lambda_j g_j(a) = 0 \quad (3)$$

The next four theorems establish the sufficiency of the Kuhn-Tucker conditions for the optimality of a feasible solution. For a feasible solution  $a$  denote by  $J(a)$  the set of active restrictions:

$$J(a) = \{j \in \{1, 2, \dots, m\} \mid g_j(a) = 0\}.$$

**Theorem 1.** Let  $(a, \mu, \lambda)$  be a Kuhn-Tucker point of (VP). Assume that for each  $i \in \{1, 2, \dots, p\}$   $f_i$  is pseudo  $(\Phi, \rho_{0i})$  – invex at  $a$ , for each  $j \in J(a)$ ,  $g_j$  is quasi  $(\Phi, \rho_j)$  – invex at  $a$  and  $\sum_{i=1}^p \mu_i \rho_{0i} + \sum_{j \in J(a)} \lambda_j \rho_j \geq 0$ . Then,  $a \in WE(VP)$ .

**Proof:** By way of contradiction suppose that  $a \notin WE(VP)$ . Then, there exists a feasible solution  $x \in X$ , such that  $f(x) < f(a)$ .

$$\text{Set } w_0 = \frac{1}{\sum_{i=1}^p \mu_i + \sum_{j \in J(a)} \lambda_j}, \quad w_i = w_0 \mu_i, \quad i = 1, 2, \dots, p, \quad v_j = w_0 \lambda_j, \quad j = 1, 2, \dots, m.$$

$$\text{Obviously, } \sum_{i=1}^p w_i + \sum_{j \in J(a)} v_j = 1,$$

$$\sum_{i=1}^p w_i \rho_{0i} + \sum_{j \in J(a)} v_j \rho_j \geq 0 \quad \text{and} \quad \sum_{i=1}^p w_i \nabla f_i(a) + \sum_{j \in J(a)} v_j \nabla g_j(a) = 0.$$

Hence,

$$\begin{aligned} 0 &\leq \Phi\left(x, a, \left(\sum_{i=1}^p w_i \nabla f_i(a) + \sum_{j \in J(a)} v_j \nabla g_j(a)\right), \left(\sum_{i=1}^p w_i \rho_{0i} + \sum_{j \in J(a)} v_j \rho_j\right)\right) \leq \\ &\sum_{i=1}^p w_i \Phi(x, a, (\nabla f_i(a), \rho_{0i})) + \sum_{j \in J(a)} v_j \Phi(x, a, (\nabla g_j(a), \rho_j)) \end{aligned}$$

For each  $j \in J(a)$ ,  $g_j(x) - g_j(a) = g_j(x) \leq 0$ , and since  $g_j$  is quasi  $(\Phi, \rho_j)$  – invex at  $a$ , it follows that  $\sum_{j \in J(a)} v_j \Phi(x, a, (\nabla g_j(a), \rho_j)) \leq 0$ .

Therefore,

$$\sum_{i=1}^p w_i \Phi(x, a, (\nabla f_i(a), \rho_{0i})) \geq 0 \quad (4)$$

so that  $\Phi(x, a, (\nabla f_i(a), \rho_{0i})) \geq 0$  for at least one  $i \in \{1, 2, \dots, p\}$ . Then,  $f_i(x) - f_i(a) < 0$ , from the pseudo  $(\Phi, \rho_{0i})$ - invexity of  $f_i$ . This contradicts the initial assumption. ■

**Remark 2.** The inequality (4) holds for every feasible solution  $x$  whenever each  $g_j, j \in J(a)$  is quasi  $(\Phi, \rho_j)$ - invex at  $a$  and the inequality  $\sum_{i=1}^p \mu_i \rho_{0i} + \sum_{j \in J(a)} \lambda_j \rho_j \geq 0$  is satisfied.

**Theorem 2.** Let  $(a, \mu, \lambda)$  be a Kuhn-Tucker point of (VP). Assume that for each  $i \in \{1, 2, \dots, p\}$   $f_i$  is semistrict quasi  $(\Phi, \rho_{0i})$ - invex at  $a$ , for each  $j \in J(a)$ ,  $g_j$  is quasi  $(\Phi, \rho_j)$ - invex at  $a$  and  $\sum_{i=1}^p \mu_i \rho_{0i} + \sum_{j \in J(a)} \lambda_j \rho_j \geq 0$ . Then,  $a \in WE(VP)$ .

**Proof:** Suppose that  $a \notin WE(VP)$ , so that  $f(x) < f(a)$  for some  $x \in X$ .

As well as in the previous proof, the inequality  $\sum_{i=1}^p w_i \Phi(x, a, (\nabla f_i(a), \rho_{0i})) \geq 0$  results from the definition of  $\Phi$  and the quasi  $(\Phi, \rho_{0i})$ - invexity of  $g_j$  (Remark 1.) But, since  $\mu \geq 0$ ,  $\mu \neq 0$ , and  $f_i$  are semistrict quasi  $(\Phi, \rho_{0i})$ - invex at  $a$  the strict inequality  $\sum_{i=1}^p w_i \Phi(x, a, (\nabla f_i(a), \rho_{0i})) < 0$  should be satisfied. The contradiction proves the theorem. ■

**Theorem 3.** Let  $(a, \mu, \lambda)$  be a Kuhn-Tucker point of (VP). Assume that for each  $i \in \{1, 2, \dots, p\}$   $f_i$  is strictly pseudo  $(\Phi, \rho_{0i})$ - invex at  $a$ , for each  $j \in J(a)$ ,  $g_j$  is quasi  $(\Phi, \rho_j)$ - invex at  $a$  and  $\sum_{i=1}^p \mu_i \rho_{0i} + \sum_{j \in J(a)} \lambda_j \rho_j \geq 0$ . Then,  $a \in E(VP)$ .

**Proof:** Let  $x$  be any feasible solution. Following the same line as in the proof of Theorem 1, we obtain that  $\Phi(x, a, (\nabla f_i(a), \rho_{0i})) \geq 0$  for at least one  $i \in \{1, 2, \dots, p\}$ . Then, the assumptions on  $f_i$  imply  $f_i(x) > f_i(a)$ . Therefore,  $a \in E(VP)$ . ■

**Theorem 4.** Let  $(a, \mu, \lambda)$  be a Kuhn-Tucker point of (VP) where  $\mu > 0$ . Assume that for each  $i \in \{1, 2, \dots, p\}$   $f_i$  is  $(\Phi, \rho_{0i})$ - invex at  $a$ , for each  $j \in J(a)$ ,  $g_j$  is quasi  $(\Phi, \rho_j)$ - invex at  $a$  and  $\sum_{i=1}^p \mu_i \rho_{0i} + \sum_{j \in J(a)} \lambda_j \rho_j \geq 0$ . Then,  $a \in PE(VP)$ .

**Proof:** First, let us show that  $a \in E(VP)$ .

Suppose that this is not true. Then,  $\sum_{i=1}^p \mu_i (f_i(x) - f_i(a)) < 0$ .

Since (4) is satisfied by  $x$ , the  $(\Phi, \rho_{0i})$ - invexity of  $f_i$  implies the inequality  $\sum_{i=1}^p \mu_i (f_i(x) - f_i(a)) \geq 0$ , in contradiction with the above one.

Now, we show that  $a \in PE(VP)$ .

In the contrary, for each positive  $K$ , there exists  $x \in X$  and  $i \in \{1, 2, \dots, p\}$ , such that  $f_i(a) - f_i(x) > 0$  and

$$f_i(a) - f_i(x) > K(f_j(x) - f_j(a)) \quad (5)$$

for every  $i \in \{1, 2, \dots, p\}$  satisfying  $f_j(x) > f_j(a)$ .

Particularly, take  $K = (p-1) \max_{1 \leq k, \ell \leq p, k \neq \ell} \frac{\mu_k}{\mu_\ell}$ .

Since (5) holds for every  $j \neq i$  we have

$$f_i(a) > f_i(x) > (p-1) \frac{\mu_j}{\mu_i} (f_j(x) - f_j(a)), \forall j \neq i$$

Then, the invexity assumptions imply:

$$-\Phi(x, a, (\nabla f_i(a), \rho_{0i})) > (p-1) \frac{\mu_j}{\mu_i} \Phi(x, a, (\nabla f_i(a), \rho_{0j})), \forall j \neq i$$

or, equivalently,

$$\frac{1}{p-1} \mu_i \Phi(x, a, (\nabla f_i(a), \rho_{0i})) + \mu_j \Phi(x, a, (\nabla f_i(a), \rho_{0j})) < 0, \forall j \neq i$$

Summing these inequalities it results

$$\mu_i \Phi(x, a, (\nabla f_i(a), \rho_{0i})) + \sum_{j \neq i} \mu_j \Phi(x, a, (\nabla f_i(a), \rho_{0j})) < 0,$$

or,

$$\sum_{j \neq i} \mu_j \Phi(x, a, (\nabla f_j(a), \rho_{0j})) < 0,$$

contradicting (4). ■

The simple examples below show that the above results are real extensions of previously known results concerning sufficiency of Kuhn-Tucker conditions.

**Example 1.** Let be the four-objective programming problem

$$\min(f_1(x), f_2(x), f_3(x), f_4(x)), \quad g(x) \leq 0, \quad x \in X_0$$

where

$$\begin{aligned} X_0 &= (-1, 1) \times (-1, 1) \subseteq \mathfrak{R}^2, \quad g(x) = -x_1 x_2, \quad f_1(x) = (x_1 - 1)(x_2 - 2), \\ f_2(x) &= (x_1 - 1)(x_2 + 3), \quad f_3(x) = (x_1 + 2)(x_2 - 1) \\ f_4(x) &= -(x_1 + 1)^2 - (x_2 + 1)^2. \end{aligned}$$

It is easy to see that the triple  $(a, \mu, \lambda)$ , where  $a = (0, 0) \in X$ ,  $\mu = (1, 1, 1, 0)$ ,  $\lambda = 1$ , is a Kuhn-Tucker point.

Setting

$$\Phi(x, a, (y, r)) = (2^{r+y_1+y_2} - 1) |(x_1 - a_1)(x_2 - a_2)| + \langle y, x - a \rangle$$

and  $\rho_{01} = 4$ ,  $\rho_{02} = -1$ ,  $\rho_{03} = 0$ ,  $\rho_{04} = 4$  and  $\rho = -1$ , one can verify that  $f_i$  are strictly pseudo  $(\Phi, \rho_{0i})$ -invex at  $a$  (with respect to  $X$ ) and  $g$  is quasi  $(\Phi, \rho)$ -invex at  $a$ .

Since  $\sum_{i=1}^4 \mu_i \rho_{0i} + \lambda \rho = 2 > 0$ , all assumptions of Theorem 3 hold, hence  $a$  is an efficient solution.

On the other hand, Hanson's invexity is neither verified at  $a$  for  $f_1, f_2, f_3$ , and  $f_4$  (nor for  $f_i$ ,  $i = 1, 2, 3, 4$  and  $g$ ). Assuming the contrary, the following four inequalities:

$$-2\eta_1(x, a) - \eta_2(x, a) \leq f_1(x) - f_1(a)$$

$$3\eta_1(x, a) - \eta_2(x, a) \leq f_2(x) - f_2(a)$$

$$-\eta_1(x, a) + 2\eta_2(x, a) \leq f_3(x) - f_3(a)$$

$$-2\eta_1(x, a) - 2\eta_2(x, a) \leq f_4(x) - f_4(a)$$

should be satisfied for all  $x \in X$  and some function  $(\eta_1, \eta_2)$ . Obviously, this is an impossibility (Take, for instance  $x = (1/2, 1/2)$ ).

**Example 2:** Let be the two-objective programming problem

$$\min(f_1(x), f_2(x)), \quad g(x) \leq 0, \quad x \in X_0$$

where  $X_0, g, X$  and  $f_1$  are as in the above and  $f_2(x) = (x_1 + 1)(x_2 + 2)$ . The triple  $(a, \mu, \lambda)$  where  $a = (0, 0) \in X$ ,  $\mu = (1, 1)$ ,  $\lambda = 1/2$ , is a Kuhn-Tucker point.

The functions  $f_1, f_2, g$  are  $(\Phi, \rho)$ -invex at  $a$  (with respect to  $X$ ). Indeed, defining  $\Phi : X_0 \times X_0 \times \mathfrak{R}^3 \mapsto \mathfrak{R}$  by

$$\Phi(x, a, (y, r)) = 2(2^r - 1) |(x_1 - a_1)(x_2 - a_2)| + \langle y, x - a \rangle$$

we can easily verify that for every  $x \in X$  the following relations hold:

$$\Phi(x, a, (\nabla f_1(a), 1/2)) \leq x_1 x_2 - 2x_1 - x_2 = f_1(x) - f_1(a)$$

$$\Phi(x, a, (\nabla f_2(a), 1/2)) \leq x_1 x_2 + 2x_1 + x_2 = f_2(x) - f_2(a)$$

$$\Phi(x, a, (\nabla g(a), -1)) = -x_1 x_2 = g(x) - g(a)$$

Therefore,  $f_1, f_2$ , and  $g$  are  $(\Phi, \rho)$ -invex at  $a$  for the parameter values  $\rho_{01} = \rho_{02} = 1/2$ , respectively,  $\rho = -1$ .

Since  $\sum_{i=1}^2 \mu_i \rho_{0i} + \lambda \rho = 1/2 \geq 0$ , all assumptions of Theorem 4 hold, hence  $a$  is a proper efficient solution.



On the other hand, Hanson's invexity is not verified at  $a$  for  $f_1, f_2$ , and  $g$ . Assuming the contrary, the following three inequalities:

$$-2\eta_1(x, a) - \eta_2(x, a) \leq f_1(x) - f_1(a)$$

$$-2\eta_1(x, a) - \eta_2(x, a) \leq f_2(x) - f_2(a)$$

$$0 \leq g(x) - g(a)$$

should be satisfied for all  $x \in X$  and some function  $(\eta_1, \eta_2) \triangleright$  and that is impossible.

Now we will show that  $(\Phi, \rho)$ - invexity can be used instead of other generalized convexity properties for proving the necessity of Kuhn-Tucker conditions.

**Theorem 5.** *Let  $a$  be an efficient solution of (VP). Assume that Slater's constraint qualification is satisfied for all restrictions indexed in  $J(a)$ . If, for each  $j \in J(a)$ ,  $g_j$  is semistrict quasi  $(\Phi, \rho_j)$ - invex at  $a$ , for some  $\rho_j \geq 0$ , then there exist  $\mu \in \mathfrak{R}_+^p, \mu \neq 0$  and  $\lambda \in \mathfrak{R}_+^m$  such that (2) and (3) are verified by  $(a, \mu, \lambda)$ .*

**Proof:** As it is known  $a \in E(VP)$  if and only if  $a$  is optimal for all problems:

$$(P_i) : \min f_i(x), \quad x \in X_0, \quad g(x) \leq 0, \quad f_k(x) \leq f_k(a) \forall k \neq i$$

$$i = 1, 2, \dots, n.$$

Let  $x^* \in X$ , satisfying Slater's conditions:  $g_j(x^*) < 0, \forall_j \in J(a)$ .

Fix  $i$ . The differentiability assumptions ensure the existence of Fritz-John multipliers of the scalar problem  $(P_i)$ . Thus, there exist  $w^i \in \mathfrak{R}_+^p$  and  $v^i \in \mathfrak{R}_+^m$  such that

$$\sum_{\ell=1}^p w_\ell^i \nabla f_\ell(a) + \sum_{j=1}^m v_j^i \nabla g_j(a) = 0 \tag{6}$$

$$\sum_{j=1}^m v_j^i \nabla g_j(a) = 0 \tag{7}$$

$$\sum_{\ell=1}^p w_\ell^i + \sum_{j=1}^m v_j^i > 0 \tag{8}$$

We are going to prove that  $w_\ell^i > 0$  for at least one  $\ell$ .

Otherwise, if  $w^i = 0$ , it follows from (8) that  $v_0^i = \sum_{j=1}^m v_j^i > 0$ . Setting  $\lambda_j^i = v_j^i / v_0^i, j = 1, 2, \dots, m$ , the equality (6) becomes  $\sum_{j \in J(a)} \lambda_j^i \nabla g_j(a) = 0$ .

Then, it results from the properties of  $\Phi$  that:

$$0 \leq \Phi \left( x^*, a, \left( \sum_{j \in J(a)} \lambda_j^i \nabla g_j(a), \sum_{j \in I(a)} \lambda_j^i \rho_j \right) \right) \leq \sum_{j \in J(a)} \lambda_j^i \Phi(x^*, a, (\nabla g_j(a), \rho_j))$$

On the other hand,  $g_j(x^*) < 0, j \in J(a)$ , according to Slater's conditions. Then, because each  $g_j$  is semistrict quasi  $(\Phi, \rho_j)$ - invex,  $\sum_{j \in J(a)} \lambda_j^i \Phi(x^*, a, (\nabla g_j(a), \rho_j)) < 0$  so that we have arrived to a contradiction.

Finally, observe that the conclusions of the theorem hold for  $\mu_k = \sum_{i=1}^p w_k^i, k = 1, 2, \dots, p$  and  $\lambda_j = \sum_{i=1}^p v_j^i, j = 1, 2, \dots, m$ . ■

### 3. DUALITY

Two types of duality are considered here, Wolfe duality and Mond-Weir duality. In both cases we prove that the weak duality property and the direct duality theorem hold under conditions introduced in the first section.

Let us consider first the Wolfe-type dual problem:

$$(WVP) : \max \left( f(y) + \sum_{j=1}^m \lambda_j g_j(y) e \right),$$

$$\sum_{i=1}^p \mu_i \nabla f_i(y) + \sum_{j=1}^m \lambda_j \nabla g_j(y) = 0, (y, \mu, \lambda) \in X_0 \times \mathfrak{R}_+^p \times \mathfrak{R}_+^m, \sum_{i=1}^p \mu_i = 1.$$

where  $e$  stands for the vector of  $\mathfrak{R}^p$  whose components are all ones, and "max" requires solutions defined in a manner analogous to those of problem (VP) by reversing the inequalities.

The following weak duality result holds.

**Theorem 6.** Let  $x$  be a feasible solution of (VP) and  $(y, \mu, \lambda)$  a feasible solution of (WVP). Assume that for each  $i \in \{1, 2, \dots, p\}$ ,  $f_i$  is  $(\Phi, \rho_{0i})$ - invex at  $y$ , for each  $j \in \{1, 2, \dots, m\}$ ,  $g_j$  is  $(\Phi, \rho_j)$ - invex at  $y$  and  $\sum_{i=1}^p \mu_i \rho_{0i} + \sum_{j=1}^m \lambda_j \rho_j \geq 0$ .

Then,  $f(x) <_{not} f(y) + \sum_{j=1}^m \lambda_j g_j(y) e$ .

**Proof:** Setting  $w$  and  $v$  as in the proof of Theorem 1, we obtain first

$$0 \leq \sum_{i=1}^p w_i \Phi(x, y, (\nabla f_i(y), \rho_{0i})) + \sum_{j=1}^m v_j \Phi(x, y, (\nabla g_j(y), \rho_j))$$

and hence,

$$0 \leq \sum_{i=1}^p \mu_i \Phi(x, y, (\nabla f_i(y), \rho_{0i})) + \sum_{j=1}^m \lambda_j \Phi(x, y, (\nabla g_j(y), \rho_j)) \quad (9)$$

Since each  $f_i$  is  $(\Phi, \rho_{0i})$ - invex at  $y$  and each  $g_j$  is  $(\Phi, \rho_j)$ - invex at  $y$  it follows that

$$0 \leq \sum_{i=1}^p \mu_i (f_i(x) - f_i(y)) + \sum_{j=1}^m \lambda_j (g_j(x) - g_j(y)) \quad (10)$$

and, since  $\sum_{j=1}^m \lambda_j g_j(x) \leq 0$ , it results

$$\sum_{i=1}^p \mu_i (f_i(x) - f_i(y)) - \sum_{j=1}^m \lambda_j g_j(y) \geq 0$$

This inequality proves the theorem. ■

Now we prove a direct duality result.

**Theorem 7.** Let  $a$  be an efficient solution of (VP). Assume that Slater's constraint qualification is satisfied for all restrictions indexed in  $J(a)$ . If, for each  $i \in \{1, 2, \dots, p\}$ ,  $f_i$  is  $(\Phi, \rho_{0i})$ - invex on  $X_0$ , and for each  $j \in \{1, 2, \dots, m\}$ ,  $g_j$  is  $(\Phi, \rho_{0j})$ - invex on  $X_0$ , where  $\rho_{0i}, \rho_{0j} \geq 0$ , then there exist  $\mu$  and  $\lambda$  such that  $(y, \mu, \lambda)$  is a weakly efficient solution of (WVP).

**Proof:** By Theorem 5, there exists  $\mu \in \mathfrak{R}_+^p$  and  $\lambda \in \mathfrak{R}_+^m$  such that  $(a, \mu, \lambda)$  verifies (2) and (3). Obviously, since  $\mu \neq 0$ , we can assume that  $\sum_{i=1}^p \mu_i = 1$  ( Otherwise,  $\mu$  and  $\lambda$  should be redefined dividing their components by  $\sum_{i=1}^p \mu_i > 0$  .)

Thus,  $(a, \mu, \lambda)$  is a feasible solution of (WVP).

To prove that it is weakly efficient, suppose, by way of contradiction, that there exists a feasible solution  $(y, \mu', \lambda')$  of (WVP) such that

$$f(a) + \sum_{j=1}^m \lambda_j g_j(a) e < f(y) + \sum_{j=1}^m \lambda'_j g_j(y) e.$$

But, since  $\sum_{j=1}^m \lambda_j g_j(a) = 0$  by (3), and the invexity conditions are satisfied for  $y$ , one contradicts the previous theorem. ■

Consider now the Mond-Weir dual of (VP).

$$(MWVP) : \max f(y),$$

$$\sum_{i=1}^p \mu_i \nabla f_i(y) + \sum_{j=1}^m \lambda_j \nabla g_j(y) = 0, \quad \sum_{j=1}^m \lambda_j g_j(y) = 0, \quad (y, \mu, \lambda) \in X_0 \times \mathfrak{R}_+^p \times \mathfrak{R}_+^m,$$

$$\sum_{i=1}^p \mu_i = 1.$$

The next results represent the counterpart of the duality results established in the above.

**Theorem 8.** Let  $x$  be a feasible solution of (VP) and  $(y, \mu, \lambda)$  a feasible solution of (MWVP). Assume that for each  $i \in \{1, 2, \dots, p\}$ ,  $f_i$  is semistrict quasi  $(\Phi, \rho_{0i})$ - invex at  $y$ , for each  $j \in \{1, 2, \dots, m\}$ ,  $g_j$  is  $(\Phi, \rho_j)$ - invex at  $y$  and  $\sum_{i=1}^p \mu_i \rho_{0i} + \sum_{j=1}^m \lambda_j \rho_j \geq 0$ . Then,  $f(x) <_{\text{not}} f(y)$

**Proof:** Suppose that  $f(x) < f(y)$  for some feasible solutions  $x$  and  $(y, \mu, \lambda)$ . As in the above, the properties of  $\Phi$  and the equality  $\sum_{i=1}^p \mu_i \nabla f_i(y) + \sum_{j=1}^m \lambda_j \nabla g_j(y) = 0$ , produce the inequality (9). Now, since  $g_j(x) \leq 0$  and  $\sum_{j=1}^m \lambda_j g_j(y) = 0$ , the  $(\Phi, \rho_j)$ - invexity of  $g_j$  implies that

$$0 \leq \sum_{i=1}^p \mu_i \Phi(x, y, (\nabla f_i(y), \rho_{0i})) + \sum_{j=1}^m \lambda_j (g_j(x) - g_j(y)) \leq \sum_{i=1}^p \mu_i \Phi(x, y, (\nabla f_i(y), \rho_{0i}))$$

Finally, we arrive to a contradiction, because the initial assumption and the semistrict quasi  $(\Phi, \rho_{0i})$ - invexity of  $f_i$  implies the strict inequality  $\sum_{i=1}^p \mu_i \Phi(x, y, (\nabla f_i(y), \rho_{0i})) < 0$ . ■

The proof of the following theorem is similar to those of Theorem 7 and is based on the previous result.

**Theorem 9.** Let  $a$  be an efficient solution of (VP). Assume that Slater's constraint qualification is satisfied for all restrictions indexed in  $J(a)$ . If, for each  $i \in \{1, 2, \dots, p\}$ ,  $f_i$  is semistrict quasi  $(\Phi, \rho_{0i})$ -invex on  $X_0$ , and for each  $j \in \{1, 2, \dots, m\}$ ,  $g_j$  is  $(\Phi, \rho_j)$ - invex on  $X_0$ , where  $\rho_{0i}, \rho_j \geq 0$ , then there exist  $\mu$  and  $\lambda$  such that  $(a, \mu, \lambda)$  is a weakly efficient solution of (MWVP).

**Remark 3.** An example in [1] shows that duality properties in scalar mathematical programming hold under  $(\Phi, \rho)$ - invexity even if usual invexity is not fulfilled. Obviously this conclusion is still valid for multiobjective programming, and then our results really extend similar results involving other types of invexity.

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