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Discrete Fourier Calculus and Graph Reconstruction

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Building on results of E. Barletta et al., [4], we give several applications of discrete Fourier calculus and the convolution product of functions defined on binary codes and generalize a result by R.P. Stanley, [17], on the vertex-switching reconstruction problem.

KEYWORDS: Hamming graph, combinatorial Laplacian, discrete Fourier transform, fundamental solution, unlabelling operator, vertex-switching reconstruction problem

1. Introduction

A linear *code* is a linear space over $\mathbb{Z}_2$ (the binary field). Given a code V a vector $v \in V$ is a *codeword*. For any finite dimensional code $\mathbb{Z}_2^N$ the *weight* $M(v)$ of a codeword $v \in \mathbb{Z}_2^N$ is the number of ones in v. The use of codes in graph theory is prompted by the fact that as long as a graph G of order n is recognized by looking at its edge set, G may be identified with a codeword in $\mathbb{Z}_2^{n(n-1)/2}$ and $|E(G)|$ (the number of edges of G) is the weight of that codeword. Precisely, let $\mathcal{G}_n$ be the set of all graphs on the vertices $\{x_1, \cdots, x_n\}$. Let $K^n \in \mathcal{G}_n$ be the complete graph on $\{x_1, \cdots, x_n\}$ and let us denote the edges of K^n by $E(K^n) = \{e_1, \cdots, e_N\}$ where $N = |E(K^n)| = \binom{n}{2} = n(n-1)/2$. Then the map

$$k : \mathcal{G}_n \to \mathbb{Z}_2^N, \quad k(G) = \big(\chi_{E(G)}(e_1), \cdots, \chi_{E(G)}(e_N)\big), \quad G \in \mathcal{G}_n, \tag{1}$$

is a bijection, thus providing the identification mentioned above. Here $\chi_{E(G)} : \{e_1, \cdots, e_N\} \to \mathbb{Z}_2$ is the characteristic function of the edge set of $G \in \mathcal{G}_n$ (thought of as $\mathbb{Z}_2$-valued, rather than real valued).

The code $\mathbb{Z}_2^N$ may also be thought of as a finite graph G_N whose vertices are the codewords $X \in \mathbb{Z}_2^N$ and two codewords X, Y are joined by an edge if the Hamming distance $H(X, Y)$ is 1. We adopt the notations and conventions in [8] and [7] (see also our Section 2 for a supply of definitions and basic results used throughout the paper). There is a natural $\mathbb{Z}_2$-analog to the Fourier transform (cf. [7], p. 325) i.e. an invertible linear map $\mathcal{F} : \mathcal{D}(G_N) \to \mathcal{D}(G_N)$ (where $\mathcal{D}(G_N)$ is the space of all functions $f : \mathbb{Z}_2^N \to \mathbb{R}$) which possesses many of the properties of the classical Fourier transform e.g. transforms the convolution product of functions on $\mathbb{Z}_2^N$ into the ordinary product of the transformed functions (cf. e.g. Lemma 2 in [4], p. 15). P. Diaconis & R. L. Graham, [7], used $\mathcal{F}$ to characterize the invertibility of the Radon transform $R_S : \mathcal{D}(G_N) \to \mathcal{D}(G_N)$ based on translates of a set $S \subseteq V(G_N)$: R_S is invertible if and only if $\mathcal{F}(\chi_S) \neq 0$ everywhere. This is the *Diaconis-Graham lemma*, cf. Lemma 6 in Appendix A to this paper. It also turns out that $\mathcal{F}$ transforms finite differences $D_\alpha f(X) = f(X + e_\alpha) - f(X)$ (cf. Section 2.2 below) into products with $a_\alpha(X) = (-1)^{X_\alpha} - 1$ of the transformed function, thus making $\mathcal{F}$ suitable for applications to *combinatorial PDEs* on G_N (in a way similar to the use of the classical Fourier transform). Indeed the discrete Fourier transform on G_N was used (cf. [4]) to solve the initial value problems for the combinatorial analogs to the heat, wave and Cattaneo equations on a Hamming graph. As a continuation of the ideas in [4] we address two types of applications of the Fourier transform on $\mathbb{Z}_2^N$. One is to discuss discrete analogs to the notion of fundamental solution for the *combinatorial Laplacian* Δ defined by (5) below. In particular we start a calculus (Fourier transforms, convolution products, etc.) with distributions on a Hamming graph i.e. linear maps $T : \mathcal{D}(G_N) \to \mathbb{R}$ (cf. Section 3). Another application concerns reconstruction problems in graph theory (cf. [5]).

Let $G \in \mathcal{G}_n$. If $x \in V(G)$ is a vertex let $G(x)$ be the graph obtained from G by *switching* at x i.e. by deleting all edges of G incident to x and inserting all possible edges incident to x originally not in G

$$V(G(x)) = V(G) = \{x_1, \cdots, x_n\},$$
$$E(G(x)) = [E(G) \setminus N_G(x)] \cup [N_{K^n}(x) \setminus N_G(x)],$$

where $N_G(x) = \{e \in E(G) : x \in e\}$ is the set of all edges of G incident to x. The *vertex-switching reconstruction problem* (cf. [17]) is: can G be reconstructed from $G(x_1), \cdots, G(x_n)$? That is, if H is another graph on $\{x_1, \cdots, x_n\}$ such that $G(x_i) \approx H(x_i)$ (graph isomorphisms) for any $1 \leq i \leq n$ then are G and H isomorphic? This is a variation of the Kelly-Ulam conjecture (cf. [5]) where vertex-deletion is replaced by vertex-switching. A technique introduced

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by R. P. Stanley, [16], led him to a positive solution when $n \not\equiv 0 \pmod 4$ cf. Corollary 1 below. Remarkably a key ingredient in Stanley's proof of Corollary 1 is the discrete Fourier calculus on $\mathbb{Z}_2^N$ and the Diaconis-Graham lemma mentioned before. Let $U : L^2(\mathcal{G}_n) \to L^2(\mathcal{G}_n)$ be the *unlabelling operator* (see Section 4.1) where $L^2(\mathcal{G}_n)$ is the space of all functions $F : \mathcal{G}_n \to \mathbb{R}$ (which turns out to be isomorphic to $\mathcal{D}(\mathbb{Z}_2^{n(n-1)/2})$). The usefulness of the unlabelling operator (due to [17], p. 134) comes from the fact that $U(\chi_{\{G\}}) = U(\chi_{\{H\}})$ if and only if the graphs G and H are isomorphic (cf. (4) in [17]). We show that when $G(x_i) \approx H(x_i)$ for any $1 \le i \le n$ then

$$U(\chi_{\{H\}}) = U(\chi_{\{G\}}) + F_{GH} \tag{2}$$

where $F = F_{GH} : \mathcal{G}_n \to \mathbb{R}$ is given by

$$F(g) = 2^{-n(n-1)/2} \sum_{\mathcal{F}(\chi_S)(X)=0} (-1)^{X\cdot k(g)} \sum_{\sigma\in\sigma_n} \left[(-1)^{X\cdot k(\sigma\cdot H)} - (-1)^{X\cdot k(\sigma\cdot G)}\right]$$

for any $g \in \mathcal{G}_n$. Cf. our Theorem 2. Here $S \subset \mathbb{Z}_2^{n(n-1)/2}$ corresponds under the isomorphism (1) to the set of all claws $K_{1,n-1}$ on the vertices $\{x_1, \cdots, x_n\}$. It follows that

Corollary 1. (R. P. Stanley, [17]) *Let $G, H \in \mathcal{G}_n$ be two graphs on the vertices $\{x_1, \cdots, x_n\}$ so that $G(x_i) \approx H(x_i)$ for any $1 \le i \le n$. If $n \not\equiv 0 \pmod 4$ then $G \approx H$.*

Proof. Stanley's arguments show that $\{X \in \mathbb{Z}_2^{n(n-1)/2} : \mathcal{F}(\chi_S)(X) = 0\}$ is empty when $n \not\equiv 0 \pmod 4$. Then (by (2)) G and H have the same unlabellings. Q.e.d.

Any $X \in \mathbb{Z}_2^{n(n-1)/2}$ may be thought of as an upper triangular matrix $X = [X_{ij}]_{1\le i,j\le n} \in \mathcal{M}_n(\mathbb{Z}_2)$ (with $X_{ij} = 0$ for $1 \le j \le i \le n$). Let

$$A_n = \left\{ X \in \mathcal{M}_n(\mathbb{Z}_2) : X_{ij} = X_{ji}, \quad X_{ii} = 0, \quad \sum_{i=1}^n \prod_{j=1}^n (-1)^{X_{ij}} = 0 \right\}. \tag{3}$$

One novelty brought by us (within the vertex switching reconstruction problem) is the explicit form of F in (2). It implies that

Corollary 2. *Let $G, H \in \mathcal{G}_n$ be two graphs on the vertices $\{x_1, \cdots, x_n\}$ such that $G(x_i) \approx H(x_i)$ for any $1 \le i \le n$. If*

$$|I_G| \sum_{g\in \mathrm{Orb}(G)} (-1)^{X\cdot k(g)} = |I_H| \sum_{g\in \mathrm{Orb}(H)} (-1)^{X\cdot k(g)}$$

for any $X \in A_n$ then $G \approx H$.

Here $\mathrm{Orb}(G)$ is the orbit of $G \in \mathcal{G}_n$ with respect to the natural action on $\mathcal{G}_n$ of the group of permutations of order $n!$ and I_G is the isotropy group of G.

The paper is organized as follows. In Section 2 we recall the needed notions of graph theory, as they apply to Hamming graphs. Section 3 continues the work in [4] and solves the problem

$$\Delta u = \delta \quad \text{in} \quad B_N(0), \quad u = 0, \quad \text{on} \quad \partial B_N(0).$$

Section 4 is devoted to a generalization (cf. Theorem 2) of a result by R. P. Stanley, [17]. The linear algebra methods introduced in [16] and [7] were the source of inspiration for both [4] and the present paper. The observation (cf. (2)) that when $n \equiv 0 \pmod 4$ (the case not covered by [17], cf. Corollary 1) it is true at least that $\chi_{\{H\}} - \chi_{\{G\}}$ lies in the fibre $U^{-1}(F_{GH})$ of the unlabelling operator U suggests that an application of the theory of Hilbert spaces with a reproducing kernel (as developed by N. Aronszajn, [2], and S. Saitoh, [14]) may shed light on the vertex switching reconstruction problem. Section 4.3 takes a step in this direction by showing that the general theory in [14] can indeed be applied to organize (cf. Proposition 7) the range of U as a (finite dimensional) Hilbert space $\mathcal{H}_K$ with a reproducing kernel. Then (by a result in [14]) there is a unique element $f_{GH} \in U^{-1}(F_{GH})$ of smallest norm (coinciding with $\|F_{GH}\|_{\mathcal{H}_K}$). However the role of f_{GH} (in the reconstruction of G from the $G(x_i)$'s) is unclear as yet.

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2. Definitions and Basic Results

2.1 Graphs.

Let $G = (V(G), E(G))$ be a connected, finite graph, without multiple edges, where $V(G)$ is the vertex set and $E(G) \subseteq [V(G)]^2$ the edge set. Let $\mathcal{D}(G)$ be the space of all functions $f : V(G) \to \mathbb{R}$ endowed with the inner product

$$(f, g) = \sum_{x\in V(G)} f(x)g(x), \quad f, g \in \mathcal{D}(G). \tag{4}$$

Let $A \subseteq V(G)$. The *average* of $f \in \mathcal{D}(G)$ over A is given by $M_A(f) = (1/|A|)\sum_{X\in A} f(X)$ where $|A|$ denotes the cardinality of the set A. The *combinatorial Laplacian* is

$$(\Delta f)(x) = f(x) - \frac{1}{m(x)} \sum_{y\in N(x)} f(y), \quad f \in \mathcal{D}(G), \quad x \in V(G). \tag{5}$$

Here $m(x) = |N(x)|$ and $N(x) = \{y \in V(G) : d(x, y) = 1\}$ is the set of all neighbors of x. A function $f \in \mathcal{D}(G)$ is *harmonic* if $\Delta f = 0$. Of course any such f satisfies the mean value property $f(x) = M_{N(x)}(f)$ for any $x \in V(G)$ and many results in potential theory carry over to this discrete setting (cf. e.g. M. Kanai, [9], p. 231–234, P. M. Soardi, [15]). We set $\mathbf{H}^0(G) = \{f \in \mathcal{D}(G) : \Delta f = 0\}$. The dimension $b_0(G) = \dim_{\mathbb{R}} \mathbf{H}^0(G)$ is the *first Betti number* of G.

A *distribution* on G is a linear map $T : \mathcal{D}(G) \to \mathbb{R}$. Let $\mathcal{D}(G)^*$ be the space of all distributions on G. If $A \subseteq V(G)$ then any function $f : A \to \mathbb{R}$ defines a distribution (of *function type*) $\tilde{f} \in \mathcal{D}(G)^*$ given by

$$\tilde{f}(\varphi) = \sum_{x\in A} f(x)\varphi(x), \quad \varphi \in \mathcal{D}(G).$$

If $L : \mathcal{D}(G) \to \mathcal{D}(G)$ is a linear map then LT is the distribution on G defined by $(LT)(\varphi) = T(L^*\varphi)$ for any $\varphi \in \mathcal{D}(G)$, where L^* is the adjoint of L i.e. $(L^*\varphi, \psi) = (\varphi, L\psi)$. We often write $\langle T, \varphi\rangle = T(\varphi)$. The product of a function $f \in \mathcal{D}(G)$ and a distribution T is given by $\langle fT, \varphi\rangle = T(f\varphi)$.

Let $\sigma(G)$ be the *spectrum* of Δ i.e. $\lambda \in \sigma(G)$ if $\Delta u = \lambda u$ for some $u \neq 0$. The *magnification* of G is

$$h(G) = \inf\left\{\frac{|\partial X|}{\mathrm{Vol}(X)} : X \subseteq V(G),\ |X| < \infty\right\}$$

where $\partial X = \{\{x, y\} \in E(G) : x \in X \text{ and } y \notin X\}$ and $\mathrm{Vol}(X) = \sum_{x\in X} m(x)$. The *Cheeger estimate* is that

$$\sigma(G) \subset [1 - \sqrt{1 - h(G)^2},\ 1 + \sqrt{1 - h(G)^2}] \subset [0, 2],$$

cf. [1]. New Cheeger type estimates were obtained by H. Urakawa, [19]. See also [3]. Eigenvalue problems for the combinatorial Laplacian occur in a natural way. For instance let

$$i\hbar\partial_n\Psi = -\frac{\hbar^2}{2m}\,\Delta\Psi + V(x)\Psi \tag{6}$$

be the *combinatorial Schrödinger equation* with the unknown function $\Psi : V(G) \times \mathbb{Z}_+ \to \mathbb{C}$ (the *discrete wave function*) where $\partial_n\Psi(x, n) = \Psi(x, n + 1) - \Psi(x, n)$ for any $x \in V(G)$ and $n \in \mathbb{Z}_+$. Also $V : V(G) \to \mathbb{R}$ is a given function (the *discrete potential*), $m > 0$ and $\hbar$ are constants ($\hbar = h/(2\pi)$ and h is Planck's constant), and $\mathbb{Z}_+ = \{0, 1, 2, \cdots\}$. The following result is then immediate:

Proposition 1. *Let $E > 0$ be a constant and $y : V(G) \to \mathbb{R}$ a solution to*

$$-\frac{\hbar^2}{2m}\,\Delta y + (V(x) + E)y = 0. \tag{7}$$

Then the discrete wave function

$$\Psi(x, n) = C\left(1 + \frac{Ei}{\hbar}\right)^n y(x) \quad (C \in \mathbb{R})$$

is a solution to (6).

Let us look for a solution of the form $\Psi(x, n) = y(x)u(n)$ where $y : V(G) \to \mathbb{R}$ and $u : V(G) \to \mathbb{C}$. Substitution into (6) and separation of variables lead to

$$i\hbar\,\frac{\partial_n u}{u} = -\frac{\hbar^2}{2m}\,\frac{\Delta y}{y} + V(x)$$

hence there is a constant $E > 0$ such that

$$i\hbar\,\partial_n u + E\,u(n) = 0 \tag{8}$$

and $y(x)$ satisfies (7) (the $\mathbb{Z}_+$-*independent Schrödinger equation*). The $\mathbb{Z}_+$-part (8) is a recurrence relation yielding $u(n) = (1 + Ei/\hbar)^n u(0)$. Finally when $V(x) = 0$ the equation (7) becomes $\Delta y = \lambda y$ (where $\lambda = 2mE/\hbar^2$), thus leading to an example of eigenvalue problem for the combinatorial Laplacian. A detailed study of the solutions to (6) is relegated to a further paper.

2.2 Hamming graphs.

Let us specialize the discussion to the case of Hamming graphs, our main example through the present paper. Let G_N be the *Hamming graph* of order 2^N i.e. $V(G_N) = \mathbb{Z}_2^N$ and two vertices $X, Y \in V(G_N)$ are adjacent if $H(X, Y) = 1$. Here $\mathbb{Z}_2$ is the binary field and $H(X, Y)$ is the *Hamming distance* between X and Y i.e. the number of coordinates where

X and Y disagree. The *Hamming ball* of center $X_0 \in V(G_N)$ and radius $r > 0$ is $B_r(X_0) = \{X \in V(G_N) : H(X_0, X) < r\}$ and its *boundary* is $\partial B_r(X_0) = \{X \in V(G_N) : H(X_0, X) = r\}$. Also we set $\overline{B}_r(X_0) = B_r(X_0) \cup \partial B_r(X_0)$. Then $V(G_N) = \overline{B}_N(0)$. If $\{e_\alpha : 1 \le \alpha \le N\}$ is the canonical basis in $\mathbb{Z}_2^N$ then $\partial B_1(X_0) = \{X_0 + e_\alpha : 1 \le \alpha \le N\}$. Note that $\partial B_r(X) = \emptyset$ unless $r \in \{1, \cdots, N\}$. Clearly in a Hamming graph $m(X) = N$, i.e. G_N is N-regular. The combinatorial Laplacian on G_N may be written as

$$\Delta\varphi(X) = \varphi(X) - \frac{1}{N}\sum_{\alpha=1}^{N} \varphi(X + e_\alpha), \quad X \in V(G_N),$$

for any $\varphi \in \mathcal{D}(G_N)$. Then $\mathbf{H}^0(G_N) = \mathbb{R}$ (see Corollary 4). Given $\varphi \in \mathcal{D}(G_N)$ we set

$$D_\alpha\varphi(X) = \varphi(X + e_\alpha) - \varphi(X), \quad X \in V(G_N).$$

The (*discrete*) *gradient* of φ is the function $D\varphi : V(G_N) \to \mathbb{R}^N$ given by $D\varphi = (D_1\varphi, \cdots, D_N\varphi)$. By a result of E. Barletta et al., [4]

$$\Delta\varphi = -\frac{1}{N}\sum_{\alpha=1}^{N} D_\alpha\varphi.$$

Note that

$$\begin{aligned}(D_\alpha^*\varphi, \psi) = (\varphi, D_\alpha\psi) &= \sum_X \varphi(X)(D_\alpha\psi)(X) \\ &= \sum_X \varphi(X)[\psi(X + e_\alpha) - \psi(X)] \\ &= \sum_Y \varphi(Y - e_\alpha)\psi(Y) - \sum_X \varphi(X)\psi(X) = (D_\alpha\varphi, \psi)\end{aligned}$$

because of $-e_\alpha = e_\alpha$ in $\mathbb{Z}_2$. In particular the combinatorial Laplacian is self-adjoint i.e. $\Delta^* = \Delta$. Also given $T \in \mathcal{D}(G_N)^*$ the *derivative* $D_\alpha T$ is the distribution given by $(D_\alpha T)(\varphi) = T(D_\alpha\varphi)$ for any $\varphi \in \mathcal{D}(G_N)$.

We end this section with a comment on discrete wave functions $\Psi(X, n)$ on a Hamming graph. It is a natural question whether

$$\sum_{X \in V(G_N)} |\Psi(X, n)|^2 = \text{const.} \tag{9}$$

as a function of n when Ψ satisfies

$$\partial_n\Psi = \frac{i\hbar}{2m}\,\Delta\Psi \tag{10}$$

(the *free* combinatorial Schrödinger equation). We establish the following negative result:

Proposition 2. *Any discrete wave function $\Psi(X, n)$ satisfying* (9)–(10) *is constant as a function of n.*

We shall use the inner product

$$(f, g) = \sum_{X \in V(G_N)} f(X)\overline{g(X)} \tag{11}$$

for any $f, g : V(G) \to \mathbb{C}$ (coinciding with (4) on real valued functions). Then Proposition 2 follows from

Lemma 1. *For any solution $\Psi(X, n)$ to* (10)

$$\partial_n \sum_{X \in V(G_N)} |\Psi(X, n)|^2 = \|\partial_n\Psi(\cdot, n)\|^2, \quad n \in \mathbb{Z}_+. \tag{12}$$

Here $\|f\| = (f, f)^{1/2}$. Indeed if (9) holds then the left-hand side of (12) vanishes hence $\partial_n\Psi(X, n) = 0$ i.e. $\Psi(X, n) = \Psi(X, 0)$ for any $n \in \mathbb{Z}_+$. Q.e.d.

To prove Lemma 1 we compute

$$\partial_n \sum_X |\Psi(X, n)|^2 =$$

(by adding and subtracting $\Psi(X, n)\overline{\Psi}(X, n + 1)$)

$$\begin{aligned}&= \sum_X \{(\partial_n\Psi)(X, n)\overline{\Psi}(X, n + 1) + \Psi(X, n)(\partial_n\overline{\Psi})(X, n)\} \\ &= \frac{i\hbar}{2m}\sum_X \{(\Delta\Psi)(X, n)\overline{\Psi}(X, n + 1) - \Psi(X, n)(\Delta\overline{\Psi})(X, n)\}\end{aligned}$$

$$= \frac{i\hbar}{2m}\{(\Delta\Psi(\cdot,n),\Psi(\cdot,n+1)) - (\Psi(\cdot,n),\Delta\Psi(\cdot,n))\}.$$

As Δ is a real operator its ($\mathbb{C}$-linear) extension to $\mathbb{C}$-valued functions is self-adjoint with respect to (11) as well. Thus

$$\partial_n \sum_X |\Psi(X,n)|^2 = \frac{i\hbar}{2m}(\Delta\Psi(\cdot,n),\partial_n\Psi(\cdot,n))$$

and (10) implies (12).

3. Fundamental Solutions

3.1 Discrete electrostatic fields.

Given $q > 0$ the *potential* of the (discrete) electrostatic field created by the point charge q at $X_0 \in V(G_3)$ is the function $U : V(G_3) \setminus \{X_0\} \to \mathbb{R}$ given by

$$U(X) = \frac{q}{R(X)}, \quad R(X) = H(X_0, X), \quad X \in V(G_3) \setminus \{X_0\},$$

and the (*discrete*) *electrostatic field* created by q is

$$E = -DU. \tag{13}$$

A priori the definition (13) lacks of rigor. Indeed given $f : A \subseteq V(G_3) \to \mathbb{R}$ the derivative $D_\alpha f$ is only defined when $X + e_\alpha \in A$ for any $X \in A$. As in the classical theory of electrostatics the rigorous formulation is in terms of distributions. The potential U defines a distribution $\tilde{U}$ on G_3 given by

$$\tilde{U}(\varphi) = \sum_{X \neq X_0} U(X)\varphi(X), \quad \varphi \in \mathcal{D}(G_3).$$

Then (13) may be correctly written as $E = -D\tilde{U}$ and E is an $\mathbb{R}^N$-valued distribution on G_3.

Let $\rho \in \mathcal{D}(G_3)^*$ be a fixed distribution on G_3 referred to as the *density* of electric charge and let us postulate that

$$\Delta U = -4\pi\rho. \tag{14}$$

Then (13)–(14) are combinatorial analogs to the fundamental equations of electrostatics (cf. e.g. [10], p. 197). In particular (14) is the (*combinatorial*) *Poisson equation*. Can one solve (14) (say in $\mathcal{D}(G_3)^*$)? The classical solution (in continuous mathematics) is written as a convolution among the fundamental solution for the Laplace operator and the given density ρ. As we shall shortly see, on a Hamming graph there are several discrete analogs to the notion of a fundamental solution, none of which is entirely satisfactory. We close this section by showing that

Proposition 3. *Let $U : V(G_N) \setminus \{X_0\} \to \mathbb{R}$ be the function given by $U(X) = 1/H(X_0, X)$ for any $X \neq X_0$ and let $\tilde{U}$ be the distribution* (*of function type*) *determined by U. Then $\Delta\tilde{U} = \tilde{f}$ where*

$$f(X) = \begin{cases} -1/N, & X = X_0 \\ (2N-1)/(2N), & X \in S_1(X_0) \\ \dfrac{(N-2)R(X)^2 - N}{(R(X)^2 - 1)R(X)}, & X \in B_N(X_0) \setminus \overline{B}_1(X_0) \\ (N-2)/[N(N-1)], & X \in S_N(X_0) \end{cases}$$

for any $X \in V(G_N)$ where $S_r(X_0) = \partial B_r(X_0)$.

Proof. For any $\varphi \in \mathcal{D}(G_N)$

$$\Delta\tilde{U}(\varphi) = \tilde{U}(\Delta\varphi) = -\frac{1}{N}\sum_{X \neq X_0} U(X)\sum_{\alpha=1}^{N}(D_\alpha\varphi)(X)$$
$$= \sum_{X \neq X_0} U(X)\varphi(X) - \frac{1}{N}\sum_{X \neq X_0} U(X)\sum_{\alpha=1}^{N}\varphi(X + e_\alpha).$$

Note that

Lemma 2. *For each $\alpha \in \{1, \cdots, N\}$*

$$R(X + e_\alpha) = R(X) \pm 1, \quad X \in A_\alpha^\pm,$$

where $A_\alpha^+ = \{X \in V(G_N) : X_\alpha = X_{0,\alpha}\}$ and $A_\alpha^- = V(G_N) \setminus A_\alpha^+$. Also $X_0 + e_\alpha \in A_\alpha^-$ and $X_0 + e_\beta \in A_\alpha^+$ for any $\beta \neq \alpha$. In particular i) $\cup_{\alpha=1}^N (A_\alpha^- \setminus \{X_0 + e_\alpha\}) = V(G_N) \setminus \overline{B}_1(X_0)$ and ii) $\cup_{\alpha=1}^N A_\alpha^+ = B_N(X_0)$.

Proof.

$$\cup_{\alpha=1}^{N}\big(A_\alpha^- \setminus \{X_0+e_\alpha\}\big) = \big(\cup_\alpha A_\alpha^-\big) \setminus \partial B_1(X_0) = \big(V(G_N)\setminus\{X_0\}\big)\setminus \partial B_1(X_0)$$

and (i) is proved. Moreover codewords with at least one coordinate coinciding with a coordinate of X_0 are at Hamming distance $\leq N-1$ from X_0 thus proving (ii).

Let us go back to the proof of Proposition 3. We have

$$\sum_{X\neq X_0} U(X)\sum_{\alpha=1}^{N}\varphi(X+e_\alpha) = \sum_\alpha \sum_{Y\neq X_0+e_\alpha} U(Y-e_\alpha)\varphi(Y)$$
$$= \sum_\alpha \sum_{Y\in A_\alpha^-\setminus\{X_0+e_\alpha\}} \frac{\varphi(Y)}{R(Y)-1} + \sum_\alpha \sum_{Y\in A_\alpha^+} \frac{\varphi(Y)}{R(Y)+1}$$
$$= \sum_{X\in \overline{B}_N(X_0)\setminus \overline{B}_1(X_0)} \frac{\varphi(X)}{R(X)-1} + \sum_{X\in B_N(X_0)} \frac{\varphi(X)}{R(X)+1}$$

hence

$$\Delta\tilde{U}(\varphi) = -\frac{1}{N}\varphi(X_0) + \frac{2N-1}{2N}\sum_{X\in S_1(X_0)}\varphi(X) + \sum_{X\in B_N(X_0)\setminus B_1(X_0)} \frac{(N-2)R(X)^2-N}{(R(X)^2-1)R(X)}\,\varphi(X) + \frac{N-2}{N(N-1)}\sum_{X\in S_N(X_0)}\varphi(X).$$

3.2 Fundamental solutions.

Let $G=(X\cup\partial X, E\cup\partial E)$ be a finite *graph-with-boundary* i.e. 1) $E\subseteq E(X,X)$, 2) $\partial E\subseteq E(X,\partial X)$, and 3) $\{x\in V(G): m(x)=1\}\subseteq \partial X$, cf. [11]. Here for any $A\subseteq V(G)$ and $B\subseteq V(G)$ we denote by $E(A,B)$ the set of all edges $e=xy\in E(G)$ such that $x\in A$ and $y\in B$. Let $O\in V(G)$ be a fixed vertex. A discrete analog to the Dirac distribution (*concentrated at O*) is $\delta_O: V(G)\to\mathbb{R}$ given by

$$\delta_O(x) = \begin{cases} 1, & x=O \\ 0, & x\neq O \end{cases} \quad x\in V(G).$$

Then a *fundamental solution* for Δ is a solution u to the problem

$$\begin{cases} \Delta u = \delta_O & \text{in } X, \\ u=0 & \text{on } \partial X. \end{cases} \tag{15}$$

Following the ideas in [4] we shall solve (15) on G_N which is a graph-with-boundary $S_N(0)=\{\mathbf{1}_N\}$ where $\mathbf{1}_N=(1,\cdots,1)\in\mathbb{Z}_2^N$. We may state the following:

Theorem 1. *There is a unique solution $\mathcal{E}$ to*

$$\begin{cases} \Delta u = \delta & \text{in } B_N(0), \\ u=0 & \text{on } \partial B_N(0), \end{cases} \tag{16}$$

given by

$$\mathcal{E}(X) = \frac{N}{2^N}\sum_{Y\neq 0}\frac{\big(1+(-1)^{X\cdot Y}\big)\big(1-(-1)^{\mathbf{1}_N\cdot Y}\big)}{N-\sum_{\alpha=1}^{N}(-1)^{Y_\alpha}},$$

for any $X\in V(G_N)$. Here $\delta=\delta_0$.

Here $X\cdot Y=\sum_{\alpha=1}^{N}X_\alpha Y_\alpha\in\mathbb{Z}_2$. The *convolution product* on $\mathcal{D}(G_N)$ is given by

$$(f*g)(X) = \sum_{Y\in V(G_N)} f(X-Y)g(Y), \quad X\in V(G_N),$$

for any $f,g\in\mathcal{D}(G_N)$. To give an example let $v\in\mathbb{Z}_2^N$ and let us consider $E_v(X)=(-1)^{v\cdot X}$. Then

$$(E_vf)*(E_vg) = E_v(f*g), \quad f,g\in\mathcal{D}(G_N).$$

As another example let $a\in\mathbb{Z}_2$ and $f_a(X)=(-1)^{-aX^2}$ (the binary analog of a Gaussian distribution). Here $X^2=X\cdot X$. Let us set $\Lambda_0=\{X\in V(G_N): X\cdot X=0\}$ (the *null cone* in $\mathbb{Z}_2^N$). Clearly $\Lambda_0=\{X\in\mathbb{Z}_2^N: M(X)\in 2\mathbb{Z}\}$. Here for any

$Y \in \mathbb{Z}_2^N$ we set $M(Y) = |\{\alpha \in \{1, \cdots, N\} : Y_\alpha \neq 0\}|$ (the weight of Y as a codeword). Also we consider $\Lambda_1 = \{X \in V(G_N) : X \cdot X = 1\}$ (so that $\Lambda_0 = V(G_N) \setminus \Lambda_1$). We have (as $2a = 0$ in $\mathbb{Z}_2$)

$$\begin{aligned}(f_a * f_b)(X) &= \sum_Y f_a(X - Y) f_b(Y) \\ &= f_a(X) \sum_Y (-1)^{(a+b)Y^2} = f_a(X) \left(|\Lambda_0| + \sum_{Y \in \Lambda_1} (-1)^{a+b} \right)\end{aligned}$$

or

$$f_a * f_b = (|\Lambda_0| + (-1)^{a+b} |\Lambda_1|) f_a.$$

As an elementary byproduct we may state

Proposition 4. $|\Lambda_0| = |\Lambda_1| = 2^{N-1}$.

This is an immediate consequence of the fact that Λ_0 is a subgroup of $\mathbb{Z}_2^N$ of index 2. However, it may also be derived by using the previously introduced techniques. Indeed, as the convolution product is commutative $(|\Lambda_0| - |\Lambda_1|)(f_1 - f_0) = 0$ hence the conclusion follows (otherwise $f_1(X) = 1$, a contradiction for $X \in \Lambda_1$).

A simple application of Theorem 1 is

Corollary 3. *Let $g \in \mathcal{D}(G_N)$ and $u = g * \mathcal{E}$. Then*

$$\Delta u(X) = g(X) - g(X - \mathbf{1}_N)$$

for any $X \in V(G_N)$.

To prove Theorem 1 we need to recall the *discrete Fourier transform* $\mathcal{F} : \mathcal{D}(G_N) \to \mathcal{D}(G_N)$ given by

$$(\mathcal{F} f)(X) = \hat{f}(X) = \sum_{Y \in V(G_N)} (-1)^{X \cdot Y} f(Y), \quad X \in V(G_N).$$

See [7] and [17]. Among its properties the discrete Fourier transform is invertible and $\mathcal{F}^{-1} = 2^{-N} \mathcal{F}$. Also $\mathcal{F}(f * g) = \hat{f}\, \hat{g}$ for any $f, g \in \mathcal{D}(G_N)$, cf. e.g. Lemma 2 in [4], p. 15.

Proof of Theorem 1. Let u be a solution to the problem (16). Then $\Delta u(\mathbf{1}_N) = c$ where we set $c = -(1/N) \sum_{\alpha=1}^N u(\mathbf{1}_N + e_\alpha) \in \mathbb{R}$. Consequently $\Delta u = f$ where

$$f(X) = \begin{cases} \delta(X), & X \in B_N(0) \\ c, & X \in \partial B_N(0) \end{cases} \quad X \in V(G_N). \tag{17}$$

Then

$$\hat{f}(X) = (-1)^{X \cdot \mathbf{1}_N} c + \sum_{Y \in B_N(0)} (-1)^{X \cdot Y} \delta(Y) = 1 + (-1)^{X \cdot \mathbf{1}_N} c$$

for any $X \in V(G_N)$. By a result in [4] (cf. Lemma 3, p. 17)

$$\mathcal{F}(\Delta f) = \frac{1}{N}\, a \hat{f} \tag{18}$$

where $a(X) = N - \sum_{\alpha=1}^N (-1)^{X_\alpha}$. By applying the Fourier transform to $\Delta u = f$

$$\frac{1}{N}\, a(X)\, \hat{u}(X) = 1 + (-1)^{X \cdot \mathbf{1}_N} c$$

hence for $X = 0$ it follows that $c = -1$. Thus $\hat{f}(X) = 1 - (-1)^{X \cdot \mathbf{1}_N}$ so that

$$\hat{u}(X) = \begin{cases} N \dfrac{1 - (-1)^{X \cdot \mathbf{1}_N}}{a(X)} & X \neq 0, \\ \lambda & X = 0, \end{cases}$$

where $\lambda = \hat{u}(0)$. Applying the inverse Fourier transform

$$\begin{aligned} u(X) &= 2^{-N} \sum_Y (-1)^{X \cdot Y} \hat{u}(Y) \\ &= 2^{-N} \left\{ \lambda + N \sum_{Y \neq 0} (-1)^{X \cdot Y} \frac{1 - (-1)^{Y \cdot \mathbf{1}_N}}{a(Y)} \right\} \end{aligned}$$

hence for $X = \mathbf{1}_N$

$$\lambda = N \sum_{Y \neq 0} \frac{1 - (-1)^{\mathbf{1}_N \cdot Y}}{a(Y)}.$$

Q.e.d.

Proof of Corollary 3. We compute $h = \Delta(g * \mathcal{E})$. Applying the Fourier transform (by (18))

$$\hat{h} = \frac{1}{N}\, a\, \mathcal{F}(g * \mathcal{E}) = \frac{1}{N}\, a\, \hat{g}\, \hat{\mathcal{E}} = \hat{g}\, \mathcal{F}(\Delta \mathcal{E}) = \hat{g}\, \hat{f}$$

where f is given by (17) with $c = -1$. Applying $\mathcal{F}^{-1}$

$$h(X) = (g * f)(X) = \sum_{Y} g(X - Y) f(Y) = g(X) - g(X - \mathbf{1}_N).$$

Q.e.d.

With the same methods we reobtain the following result (well known to hold for any connected regular graph)

Corollary 4. $\beta_0(G_N) = 1$.

Proof. Let $u \in \mathbf{H}^0(G_N)$. Applying the Fourier transform to $\Delta u = 0$ gives $(1/N)\, a\, \hat{u} = 0$ i.e. $\hat{u}(X) = 0$ for any $X \neq 0$. Then applying the inverse Fourier transform gives $u(X) = 2^{-N} \hat{u}(0)$ for any $X \in V(G_N)$. Q.e.d.

There is yet another combinatorial analog to the notion of fundamental solution, suggested by the form of the fundamental solution to the Laplacian on $\mathbb{R}^3$ namely $1/r$ where $r(X) = X \cdot X$ for any $X \in V(G_N) \setminus \Lambda_0$, i.e. $r = \chi_{\Lambda_1}$. We may state the following:

Proposition 5. *Let $H : V(G_N) \to \mathbb{R}$ be given by*

$$H(X) = \begin{cases} 1, & X \in \Lambda_0 \\ -1, & X \in \Lambda_1 \end{cases} \quad X \in V(G_N).$$

Then $\Delta(1/r) = -H$ in distributional sense. Moreover given $g \in \mathcal{D}(G_N)$ if $u = (1/r) * g$ then $\Delta u = -H * g$.

Proof. For any $\varphi \in \mathcal{D}(G_N)$

$$\big(\Delta\, 1/r\big)(\varphi) = \sum_{X \in \Lambda_1} \frac{\Delta \varphi(X)}{r(X)} = \sum_{X \in \Lambda_1} \left\{ \varphi(X) - \frac{1}{N} \sum_{\alpha=1}^{N} \varphi(X + e_\alpha) \right\}$$

and since $X + e_\alpha \in \Lambda_1$ if and only if $X \in \Lambda_0$ (providing another proof of Corollary 4)

$$\sum_{X \in \Lambda_1} \sum_{\alpha=1}^{N} \varphi(X + e_\alpha) = \sum_{\alpha} \sum_{X + e_\alpha \in \Lambda_1} \varphi(X) = N \sum_{X \in \Lambda_0} \varphi(X)$$

hence $\Delta(1/r) = -\tilde{H}$.

The convolution product on the second statement of Proposition 5 is in distributional sense. Precisely, we introduce the following notions. The *tensor product* of the distributions $T \in \mathcal{D}(G_N)^*$ and $S \in \mathcal{D}(G_M)$ is given by

$$(T \otimes S)(\varphi) = \langle T\,,\, X \mapsto S(\varphi(X, \cdot)) \rangle, \quad \varphi \in \mathcal{D}(G_{N+M}).$$

The *convolution product* of the distributions T, S on G_N is given by

$$(T * S)(\varphi) = \langle T \otimes S, \varphi^{\Delta} \rangle, \quad \varphi \in \mathcal{D}(G_N),$$

where $\varphi^{\Delta}(X, Y) = \varphi(X + Y)$. For distributions of function type the two notions of convolution product coincide i.e.

$$\tilde{f} * \tilde{g} = \widetilde{f * g}$$

for any $f, g \in \mathcal{D}(G_N)$. Indeed

$$\begin{aligned} \langle \tilde{f} * \tilde{g}, \varphi \rangle = \langle \tilde{f} \otimes \tilde{g}, \varphi^{\Delta} \rangle &= \sum_{X \in V(G_N)} f(X) \tilde{g}(\varphi^{\Delta}(X, \cdot)) \\ &= \sum_{X,Y} f(X) g(Y) \varphi(X + Y) = \sum_{X,Z} f(X) g(Z - X) \varphi(Z) \\ &= \sum_{Z} (f * g)(Z) \varphi(Z) = \langle \widetilde{f * g}, \varphi \rangle. \end{aligned}$$

Q.e.d.

We shall also need the Fourier transform on distributions $\mathcal{F} : \mathcal{D}(G_N)^* \to \mathcal{D}(G_N)^*$ given by

$$\langle \mathcal{F}(T)\,,\, \varphi \rangle = \hat{T}(\varphi) = T(\hat{\varphi}), \quad \varphi \in \mathcal{D}(G_N).$$

Again for distributions of function type

$$\langle \mathcal{F}(\tilde{f}), \varphi \rangle = \langle \tilde{f}, \hat{\varphi} \rangle = \sum_{X \in V(G_N)} f(X)\hat{\varphi}(X)$$
$$= \sum_X f(X) \sum_Y (-1)^{X \cdot Y} \varphi(Y) = \sum_Y \hat{f}(Y)\varphi(Y) = \langle \widetilde{\mathcal{F}(f)}, \varphi \rangle$$

i.e. $\mathcal{F}(\tilde{f}) = \widetilde{\mathcal{F}(f)}$ for any $f \in \mathcal{D}(G_N)$.

Lemma 3. *For any* $T \in \mathcal{D}(G_N)^*$

(19) $$\mathcal{F}(D_\alpha T) = a_\alpha \hat{T}, \quad 1 \le \alpha \le N,$$

where $a_\alpha(X) = (-1)^{X_\alpha} - 1$. *In particular* $\mathcal{F}(\Delta T) = (1/N)\, a\, \hat{T}$. *Moreover*

(20) $$\mathcal{F}(T * \tilde{f}) = \hat{f}\, \hat{T}$$

for any $f \in \mathcal{D}(G_N)$.

Proof. To prove (19)

$$\langle \mathcal{F}(D_\alpha T), \varphi \rangle = \langle D_\alpha T, \hat{\varphi} \rangle = T(D_\alpha \hat{\varphi}).$$

On the other hand (by Lemma 3 in [4], p. 17)

$$\mathcal{F}(D_\alpha \hat{\varphi}) = a_\alpha \mathcal{F}(\hat{\varphi}) = 2^N a_\alpha \varphi$$

hence

(21) $$D_\alpha \hat{\varphi} = \mathcal{F}(a_\alpha \varphi).$$

Finally (by (21))

$$\langle \mathcal{F}(D_\alpha T), \varphi \rangle = \langle T, \mathcal{F}(a_\alpha \varphi) \rangle = \langle \hat{T}, a_\alpha \varphi \rangle = \langle a_\alpha \hat{T}, \varphi \rangle.$$

In particular

$$\mathcal{F}(\Delta T) = -\frac{1}{N} \sum_{\alpha=1}^{N} \mathcal{F}(D_\alpha T) = -\frac{1}{N} \sum_\alpha a_\alpha \hat{T} = \frac{1}{N}\, a\, \hat{T}.$$

To prove (20)

$$\langle \mathcal{F}(T * \tilde{f}), \varphi \rangle = \langle T * \tilde{f}, \hat{\varphi} \rangle = \langle T \otimes \tilde{f}, \hat{\varphi}^\Delta \rangle = \langle T, \psi \rangle$$

where

$$\psi(X) = \tilde{f}(\hat{\varphi}^\Delta(X, \cdot)) = \sum_{Y \in V(G_N)} f(Y)\hat{\varphi}^\Delta(X, Y)$$
$$= \sum_Y f(Y)\hat{\varphi}(X + Y) = \sum_{Y,Z} (-1)^{(X+Y)\cdot Z} f(Y)\varphi(Z)$$
$$= \sum_Z (-1)^{X \cdot Z} \varphi(Z) \sum_Y (-1)^{Y \cdot Z} f(Y) = \sum_Z (-1)^{X \cdot Z} \varphi(Z)\hat{f}(Z)$$

i.e. $\psi = \mathcal{F}(\varphi \hat{f})$. Hence

$$\langle \mathcal{F}(T * \tilde{f}), \varphi \rangle = \langle T, \mathcal{F}(\varphi\, \hat{f}) \rangle = \langle \hat{T}, \varphi\, \hat{f} \rangle = \langle \hat{f}\, \hat{T}, \varphi \rangle.$$

Q.e.d.

At this point we may prove the second statement in Proposition 5. Let us set $T = \Delta((1/r) * g)$. Then (by (20))

$$\hat{T} = \frac{1}{N}\, a\, \mathcal{F}\left(\frac{1}{r} * g\right) = \frac{1}{N}\, a\, \hat{g}\, \mathcal{F}\left(\frac{1}{r}\right)$$
$$= \hat{g}\, \mathcal{F}(\Delta(1/r)) = -\hat{g}\, \mathcal{F}(\tilde{H}) = -\mathcal{F}(\tilde{H} * \tilde{g})$$

i.e. T is the distribution determined by the function $-H * g$. Q.e.d.

4. The Vertex-Switching Reconstruction Problem

4.1 On a result by R. P. Stanley.

Let $\mathcal{G}_n$ be the set of all graphs on the vertices $\{x_1, \cdots, x_n\}$. There is a natural action of the permutation group σ_n (of order $n!$) on $\mathcal{G}_n$. Indeed let $\sigma \in \sigma_n$ and $G \in \mathcal{G}_n$. We define a graph $\sigma \cdot G$ on $\{x_1, \cdots, x_n\}$ by indicating its edge set i.e.

$$E(\sigma \cdot G) = \{\sigma \cdot e : e \in E(G)\}$$

where $\sigma \cdot e = \{x_{\sigma(i)}, x_{\sigma(j)}\}$ for any edge $e = \{x_i, x_j\}$ of G. Let $L^2(\mathcal{G}_n)$ be the linear space of all functions $F : \mathcal{G}_n \to \mathbb{R}$. Note that the characteristic functions $\{\chi_{\{G\}} : G \in \mathcal{G}_n\}$ form a basis in $L^2(\mathcal{G}_n)$ over the reals. The action of σ_n on $\mathcal{G}_n$ induces an action of σ_n on $L^2(\mathcal{G}_n)$ given by

$$(\sigma \cdot F)(G) = F(\sigma^{-1} \cdot G), \quad G \in \mathcal{G}_n, \quad \sigma \in \sigma_n.$$

Then

$$\sigma \cdot \chi_{\{G\}} = \chi_{\{\sigma \cdot G\}}. \tag{22}$$

Of course the relation (22) (followed by $\mathbb{R}$-linear extension) could be taken as the definition of the action of σ_n on $\mathcal{G}_n$. We shall need the *unlabelling operator* $U : L^2(\mathcal{G}_n) \to L^2(\mathcal{G}_n)$ given by

$$U(F) = \sum_{\sigma \in \sigma_n} \sigma \cdot F, \quad F \in L^2(\mathcal{G}_n).$$

Also $U(\chi_{\{G\}})$ is called the *unlabelling* of the graph $G \in \mathcal{G}_n$. The scope of this section is to establish the following:

Theorem 2. *Let $G, H \in \mathcal{G}_n$ such that $G(x_i) \approx H(x_i)$ (a graph isomorphism) for any $1 \leq i \leq n$. Then*

$$U(\chi_{\{H\}}) = U(\chi_{\{G\}}) + F_{GH} \tag{23}$$

where $F_{GH} : \mathcal{G}_n \to \mathbb{R}$ is given by

$$F_{GH}(g) = 2^{-n(n-1)/2} \sum_{\hat{\chi}_S(X)=0} (-1)^{X \cdot k(g)} \sum_{\sigma \in \sigma_n} \left[(-1)^{X \cdot k(\sigma \cdot H)} - (-1)^{X \cdot k(\sigma \cdot G)} \right]$$

for any $g \in \mathcal{G}_n$. Here $S = k(\Gamma) \subset \mathbb{Z}_2^{n/(n-1)/2}$ and Γ is the set of all claws $K_{1,n-1}$ on the vertices $\{x_1, \cdots, x_n\}$.

The explicit expression of F_{GH} in (23) is new and Theorem 2 may be considered as a generalization of R. P. Stanley's result (Corollary 1 in the Introduction).

To prove Theorem 2 we need some preparation (based on the ideas in [17]). The *Stanley morphism* is the map given by

$$\Phi : L^2(\mathcal{G}_n) \to L^2(\mathcal{G}_n), \quad \Phi(\chi_{\{G\}}) = \sum_{i=1}^{n} \chi_{\{G(x_i)\}},$$

for any $G \in \mathcal{G}_n$ (followed by $\mathbb{R}$-linear extension). The bijection $k : \mathcal{G}_n \to V(G_N)$ ($N = n(n-1)/2$) defined in the Introduction induces the $\mathbb{R}$-linear isomorphism

$$k_* : L^2(\mathcal{G}_n) \to \mathcal{D}(G_N), \quad k_*(F) = F \circ k^{-1}, \quad F \in L^2(\mathcal{G}_n).$$

Next let

$$E_i = \{e \in \{e_1, \cdots, e_N\} : x_i \in e\}, \quad 1 \leq i \leq n,$$

i.e. E_i consists of all edges in K^n which are incident with the vertex x_i. Now let $G_i \in \mathcal{G}_n$ be the graph on $\{x_1, \cdots, x_n\}$ whose edge set is $E(G_i) = E_i$

$$G_i = k^{-1}(\chi_{E_i}(e_1), \cdots, \chi_{E_i}(e_N)).$$

Here χ_{E_i} is thought of as $\mathbb{Z}_2$-valued. Then G_i is a claw $K_{1,n-1} = \overline{K^1} * \overline{K^{n-1}}$ on $\{x_1, \cdots, x_n\}$. As in Theorem 2 we set

$$\Gamma = \{G_1, \cdots, G_n\} \subset \mathcal{G}_n.$$

We need the following:

Lemma 4. *Let $S = k(\Gamma) \subset \mathbb{Z}_2^N$. Then the diagram*

$$\begin{array}{ccc} L^2(\mathcal{G}_n) & \xrightarrow{\Phi} & L^2(\mathcal{G}_n) \\ {\scriptstyle k_*}\downarrow & & \downarrow{\scriptstyle k_*} \\ \mathcal{D}(G_N) & \xrightarrow{R_S} & \mathcal{D}(G_N) \end{array}$$

is commutative.

So the Stanley morphism is, up to an isomorphism, the Radon transform (see Appendix A) based on translates of $S \subset \mathbb{Z}_2^N$ (the set of all codewords corresponding to all possible claws in K^n).

Proof of Lemma 4. For any $\zeta \in \mathbb{Z}_2^N$

$$(k_* \, \Phi \, F)(\zeta) = (\Phi F)(k^{-1}\zeta) = \Phi\left(\sum_{G \in \mathcal{G}_n} F(G) \, \chi_{\{G\}}\right)(k^{-1}\zeta)$$

that is

$$(k_* \, \Phi \, F)(\zeta) = \sum_{G \in \mathcal{G}_n} F(G) \sum_{i=1}^{n} \chi_{G(x_i)}(k^{-1}\zeta). \tag{24}$$

On the other hand

$$\begin{aligned}(R_S \, k_* \, F)(\zeta) &= \sum_{z \in S+\zeta} (k_* F)(z) = \sum_{z \in S} (k_* F)(z + \zeta) \\ &= \sum_{i=1}^{n} (k_* F)(k(G_i) + \zeta)\end{aligned}$$

that is

$$(R_S \, k_* \, F)(\zeta) = \sum_{G \in \mathcal{G}_n} F(G) \sum_{i=1}^{n} \chi_{\{G\}}\big(k^{-1}(k(G_i) + \zeta)\big). \tag{25}$$

We adopt the notations

$$G_\zeta = k^{-1}(\zeta) \in \mathcal{G}_n, \quad G_\zeta^i = k^{-1}(k(G_i) + \zeta) \in \mathcal{G}_n,$$

i.e. G_ζ is the graph on $\{x_1, \cdots, x_n\}$ corresponding to the codewords ζ and G_ζ^i is obtained from G_i by deleting all edges in common with G_ζ and by inserting all edges of G_ζ initially not in G_i. Then

$$E(G_\zeta^i) = E(G_i) \Delta E(G_\zeta)$$

(symmetric difference). By (24)–(25) the diagram in Lemma 4 is commutative if only if

$$\sum_{G \in \mathcal{G}_n} F(G) \sum_{i=1}^{n} \big\{ \chi_{\{G\}}(G_\zeta^i) - \chi_{\{G(x_i)\}}(G_\zeta) \big\} = 0.$$

This follows by observing that $G = G_\zeta^i$ if and only if $G(x_i) = G_\zeta$. For instance let us check the sufficiency. The hypothesis is equivalent to

$$\chi_{E(G(x_i))}(e_\alpha) = \zeta_\alpha, \quad 1 \le \alpha \le N,$$

while the conclusion reads

$$\chi_{E(G)}(e_\alpha) = \chi_{E_i}(e_\alpha) + \zeta_\alpha, \quad 1 \le \alpha \le N.$$

This may be checked as follows

$$\begin{aligned}\chi_{E_i}(e_\alpha) + \zeta_\alpha &= \chi_{E_i}(e_\alpha) + \chi_{E(G(x_i))}(e_\alpha) \\ &= \begin{cases} 1 + \chi_{E(G(x_i))}(e_\alpha), & x_i \in e_\alpha \\ \chi_{E(G(x_i))}(e_\alpha), & x_i \notin e_\alpha \end{cases} \\ &= \begin{cases} 1, & x_i \in e_\alpha \text{ and } e_\alpha \in N_G(e_i) \\ 0, & x_i \in e_\alpha \text{ and } e_\alpha \notin N_G(x_i) \\ 1, & x_i \notin e_\alpha \text{ and } e_\alpha \in E(G) \\ 0, & x_i \notin e_\alpha \text{ and } e_\alpha \notin E(G) \end{cases} = \chi_{E(G)}(e_\alpha).\end{aligned}$$

Q.e.d.

Proof of Theorem 2. By (iii) in Lemma 7 in Appendix B

$$\begin{aligned}(\Phi U)(\chi_{\{G\}}) = (U\Phi)(\chi_{\{G\}}) = \sum_{i=1}^{n} U(\chi_{\{G(x_i)\}}) = \quad &\text{(by (i) in Lemma 7)} \\ = \sum_{i=1}^{n} U(\chi_{\{H(x_i)\}}) = (U\Phi)(\chi_{\{H\}}) = (\Phi U)(\chi_{\{H\}})&\end{aligned}$$

so that

$$U(\chi_{\{H\}}) - U(\chi_{\{G\}}) \in \operatorname{Ker}(\Phi). \tag{26}$$

Then $U(\chi_{\{H\}}) = U(\chi_{\{G\}}) + F$ for some $F \in \operatorname{Ker}(\Phi)$ which we need to compute. Let us apply k_* to this identity. Yet $k_* F \in \operatorname{Ker}(R_S)$, $S = k(\Gamma)$ (by Lemma 4) and then (by the Diaconis-Graham lemma, see Appendix A)

$$k_* U(\chi_{\{H\}}) = k_* U(\chi_{\{G\}}) + 2^{-N} \sum_{\hat{\chi}_S(X)=0} \lambda_X E_X$$

for some $\lambda_X \in \mathbb{R}$. Applying the Fourier transform we get (by (36))

$$\mathcal{F}(k_* U(\chi_{\{H\}})) = \mathcal{F}(k_* U(\chi_{\{G\}})) + \sum_{\hat{\chi}_S(X)=0} \lambda_X \chi_{\{X\}}. \tag{27}$$

On the other hand

$$\begin{aligned}\mathcal{F}(k_* U(\chi_{\{G\}}))(X) &= \sum_{\zeta \in V(G_N)} (-1)^{X\cdot\zeta} (k_* U(\chi_{\{G\}}))(\zeta) \\ &= \sum_{\zeta} (-1)^{X\cdot\zeta} U(\chi_{\{G\}})(k^{-1}\zeta) = \sum_{\zeta} (-1)^{X\cdot\zeta} \sum_{\sigma\in\sigma_n} (\sigma\cdot\chi_{\{G\}})(k^{-1}\zeta)\end{aligned}$$

i.e.

$$\mathcal{F}(k_* U(\chi_{\{G\}}))(X) = \sum_{\zeta \in V(G_N)} (-1)^{X\cdot\zeta} \sum_{\sigma\in\sigma_n} \chi_{\{\sigma\cdot G\}}(G_\zeta). \tag{28}$$

Let us apply (27) to a zero X of $\hat{\chi}_S$ and use (28). We obtain

$$\lambda_X = \sum_{\zeta} (-1)^{X\cdot\zeta} \sum_{\sigma} \left[\chi_{\{\sigma\cdot H\}}(G_\zeta) - \chi_{\{\sigma\cdot G\}}(G_\zeta)\right]$$

hence

$$\begin{aligned} U(\chi_{\{H\}}) = U(\chi_{\{G\}}) + \\ + 2^{-N} \sum_{\hat{\chi}_S(X)=0} \sum_{\zeta} (-1)^{X\cdot\zeta} \sum_{\sigma} \left[\chi_{\{\sigma\cdot H\}}(G_\zeta) - \chi_{\{\sigma\cdot G\}}(G_\zeta)\right] E_X \circ k.\end{aligned}$$

Assume that $G \neq H$. Finally for any $g \in \mathcal{G}_n$

$$\begin{aligned} F(g) &= 2^{-N} \sum_{\hat{\chi}_S(X)=0} \sum_{\sigma} \sum_{\zeta} (-1)^{X\cdot(\zeta + k(g))} \left[\chi_{\{\sigma\cdot H\}}(G_\zeta) - \chi_{\{\sigma\cdot G\}}(G_\zeta)\right] \\ &= 2^{-N} \sum_{\hat{\chi}_S(X)=0} \sum_{\sigma} \sum_{\zeta} (-1)^{X\cdot(\zeta + k(g))} \begin{cases} 1, & G_\zeta = \sigma\cdot H \\ -1, & G_\zeta = \sigma\cdot G \\ 0, & \text{otherwise} \end{cases} \\ &= 2^{-N} \sum_{\hat{\chi}_S(X)=0} \sum_{\sigma} \left[(-1)^{X\cdot(k(\sigma\cdot H) + k(g))} - (-1)^{X\cdot(k(\sigma\cdot G)+k(g))}\right].\end{aligned}$$

Q.e.d.

At this point we may prove Corollary 2. To this end we compute $\|F_{GH}\|^2_{L^2(\mathcal{G}_n)} = \sum_{g\in\mathcal{G}_n} F_{GH}(g)^2$. Let

$$f(X) := \sum_{\sigma\in\sigma_n} \{(-1)^{X\cdot k(\sigma\cdot H)} - (-1)^{X\cdot k(\sigma\cdot G)}\}, \quad X \in V(G_{n(n-1)/2}).$$

If $\{X_1, \cdots, X_m\}$ is an enumeration of the zeros of $\mathcal{F}(\chi_S)$ then

$$\left[\sum_{\hat{\chi}_S(X)=0} (-1)^{X\cdot Y} f(X)\right]^2 = \sum_{i=1}^{m} f(X_i)^2 + 2\sum_{i<j} (-1)^{(X_i+X_j)\cdot Y} f(X_i) f(X_j)$$

so that (as $i < j$ implies $X_i + X_j \neq 0$ and then $\sum_Y (-1)^{(X_i+X_j)\cdot Y} = 0$)

$$\begin{aligned} \|F_{GH}\|^2 &= 2^{-2N} \sum_{g\in\mathcal{G}_n} \left[\sum_{\hat{\chi}_S(X)=0} (-1)^{X\cdot k(g)} f(X)\right]^2 \\ &= 2^{-2N} \sum_{Y\in V(G_N)} \sum_{\hat{\chi}_S(X)=0} f(X)^2 \end{aligned}$$

i.e.

$$\|F_{GH}\|^2_{L^2(\mathcal{G}_n)} = 2^{-N} \sum_{\hat{\chi}_S(X)=0} f(X)^2$$

hence $F_{GH} = 0$ if and only if $f = 0$ on $\mathcal{Z}(\hat{\chi}_S)$. Finally

$$f(X) = |I_H| \sum_{g\in\mathrm{Orb}(H)} (-1)^{X\cdot k(g)} - |I_G| \sum_{g\in\mathrm{Orb}(G)} (-1)^{X\cdot k(g)}$$

and one may use the description of $\mathcal{Z}(\hat{\chi}_S)$ at the end of Appendix A.

4.2 The vertex-switching spectral reconstruction problem.

As two isomorphic graphs have the same spectrum a natural weakening of R. P. Stanley's problem is the following *vertex-switching spectral reconstruction problem.* Let $n \geq 4$ and let $\mathcal{G}_n(\Delta)$ be the set of all graphs on the vertices $\{x_1, \cdots, x_n\}$ such that for any $G \in \mathcal{G}_n(\Delta)$ the degree of each vertex x of G satisfies $2 \leq m_G(x) < n-1$. Given $G, H \in \mathcal{G}_n(\Delta)$ let us assume that $\sigma(G(x)) = \sigma(H(x))$ for any $x \in V$. Then is it true that $\sigma(G) = \sigma(H)$? As well as in the case of the vertex-switching reconstruction up to graph isomorphisms the problem is easily seen to possess no solution for $n = 4$. Indeed

$$\sigma(\overline{K^4}) = \{0\}, \quad \sigma(C^4) = \{0, 2, 4\},$$

while the graphs got by switching at some vertex have the spectrum of a claw $\sigma(K_{1,3}) = \{0, 1, 4\}$. The combinatorial Laplacians of G and $G(x_i)$ are related by

Proposition 6. *For any $G \in \mathcal{G}_n(\Delta)$ and for any vertex $x \neq x_i$ one has*

$$(\Delta_{G(x_i)} f)(x) = \begin{cases} \dfrac{1}{m(x)-1}\big\{m(x)\ \Delta f(x) - Q_i f(x)\big\}, & x \in N(x_i) \\ \dfrac{1}{m(x)+1}\big\{m(x)\ \Delta f(x) + Q_i f(x)\big\}, & x \notin N(x_i), \end{cases}$$

$$(\Delta_{G(x_i)} f)(x_i) = \frac{n-1}{n-m(x_i)-1}\,(\Delta_{K^n} f)(x_i) - \frac{m(x_i)}{n-m(x_i)-1}\,(\Delta f)(x_i),$$

where $Q_i f(x) = f(x) - f(x_i)$ for any $f : \{x_1, \cdots, x_n\} \to \mathbb{R}$ and Δ_{K^n} is the combinatorial Laplacian of the complete graph on $\{x_1, \cdots, x_n\}$.

The proof of Proposition 6 follows from

$$N_{G(x_i)}(x) = \begin{cases} N(x) \setminus \{x_i\}, & x \in N(x_i) \\ N(x) \cup \{x_i\}, & x \notin N(x_i), \end{cases} \quad x \neq x_i,$$

$$N_{G(x_i)}(x_i) = \{x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n\} \setminus N(x_i).$$

The problem of relating $\sigma(G(x_i))$ to $\sigma(G)$ is however open.

4.3 On the range of the unlabelling operator.

Let E be a nonempty set and $\mathcal{F}(E)$ the space of all real valued functions on E. Let $\mathcal{H} \subseteq \mathcal{F}(E)$ be a Hilbert space with the inner product $(\ ,\ )_{\mathcal{H}}$. A *reproducing kernel* for $\mathcal{H}$ (cf. e.g. N. Aronszajn, [2]) is a function $K : E \times E \to \mathbb{R}$ such that

$$F(q) = (F, K(\cdot, q))_{\mathcal{H}}, \quad q \in E,$$

for any $F \in \mathcal{H}$. Given a function $\mathbf{h} : E \to \mathcal{H}$ let us consider the linear map

$$L : \mathcal{H} \to \mathcal{F}(E), \quad (LF)(p) = (F, \mathbf{h}(p))_{\mathcal{H}}, \quad F \in \mathcal{H}, \ \ p \in E.$$

S. Saitoh, [12], devised a method for organizing the range of L as a Hilbert space with a reproducing kernel. Precisely one may set

(29) $$K(p, q) = (\mathbf{h}(q), \mathbf{h}(p))_{\mathcal{H}}, \quad p, q \in E,$$

and then (by Theorem 2.1 in [14], p. 51) the range $\mathcal{R}(L)$ admits a natural structure of a Hilbert space such that the induced norm is

$$\|f\|_{\mathcal{R}(L)} = \inf\{\|F\|_{\mathcal{H}} : F \in L^{-1}(f)\}, \quad f \in \mathcal{R}(L),$$

and $K(p, q)$ given by (29) is a reproducing kernel for $\mathcal{R}(L)$. Although L is not invertible in general (its fibers are in general non empty) one may identify an element of minimal norm in each fibre $L^{-1}(f)$ (cf. Theorem 2.2 in [14], p. 52) so as to produce useful generalized inversion formulas (cf. e.g. [13] and [6] for applications to analysis).

Let us endow $L^2(\mathcal{G}_n)$ with the inner product

$$(F, F')_{L^2(\mathcal{G}_n)} = \sum_{G \in \mathcal{G}_n} F(G) F'(G), \quad F, F' \in L^2(\mathcal{G}_n).$$

The general theory recalled above applies to $E = \mathcal{G}_n$ and the (finite dimensional) Hilbert space $\mathcal{H} = L^2(\mathcal{G}_n)$ as follows. Let $\mathbf{h} : \mathcal{G}_n \to L^2(\mathcal{G}_n)$ be given by

$$\mathbf{h}(G) = |I_G|\ \chi_{\mathrm{Orb}(G)}, \quad G \in \mathcal{G}_n.$$

Here $I_G = \{\sigma \in \sigma_n : \sigma \cdot G = G\}$. Let $G \in \mathcal{G}_n$. For each $H \in \mathrm{Orb}(G)$ we set

$$A_G(H) := \{\sigma \in \sigma_n : \sigma \cdot G = H\}.$$

As $H = \tau \cdot G$ for some $\tau \in \sigma_n$ it follows that

$$A_G(H) = \{\sigma \in \sigma_n : \tau^{-1}\sigma \in I_G\} = \tau I_G.$$

Then for any $F \in L^2(\mathcal{G}_n)$

$$\begin{aligned} U(F)(G) &= \sum_{\sigma\in\sigma_n} F(\sigma^{-1}\cdot G) = \sum_{H\in \mathrm{Orb}(G)} |A_G(H)|\, F(H) \\ &= |I_G| \sum_{H\in\mathrm{Orb}(G)} F(H) = \sum_{H\in\mathcal{G}_n} F(H)\mathbf{h}(G)(H) = (F, \mathbf{h}(G))_{L^2(\mathcal{G}_n)}. \end{aligned}$$

Then Theorems 2.1 and 2.2 in [14], p. 51–52, do apply to $L = U$ and we may state the following:

Proposition 7. *The range of the unlabelling operator $U : L^2(\mathcal{G}_n) \to L^2(\mathcal{G}_n)$ is a (finitely dimensional) Hilbert space $\mathcal{H}_K$ with the reproducing kernel*

$$K(G,H) = |I_G|\,|I_H|\,|\mathrm{Orb}(G)\cap \mathrm{Orb}(H)|, \quad G,H \in \mathcal{G}_n.$$

Moreover

$$\left\{ \left(\sum_{G\in O} |I_G| \right) \chi_O : O \in \mathcal{G}_n/\sigma_n \right\}$$

is a basis in $\mathcal{H}_K$.

Proof. The inner product in $\mathcal{H}_K = \mathcal{R}(U)$ is given by

$$(f,f')_{\mathcal{H}_K} = (PF, PF')_{L^2(\mathcal{G}_n)}, \quad f, f' \in \mathcal{H}_K,$$

where $F \in U^{-1}(f)$ and $F' \in U^{-1}(f')$. Also $P : L^2(\mathcal{G}_n) \to L^2(\mathcal{G}_n) \ominus \mathrm{Ker}(U)$ is the natural projection. The definition doesn't depend upon the choice of elements in the fibres of U over f and f'. The reproducing kernel is

$$\begin{aligned} K(G,H) &= (\mathbf{h}(H), \mathbf{h}(G))_{L^2(\mathcal{G}_n)} \\ &= |I_G|\,|I_H| \sum_{g\in\mathcal{G}_n} \chi_{\mathrm{Orb}(G)}(g)\, \chi_{\mathrm{Orb}(H)}(g) = |I_G|\,|I_H|\,|\mathrm{Orb}(G)\cap\mathrm{Orb}(H)| \end{aligned}$$

with the obvious consequence

Corollary 5. *If $K(G,H) > 0$ then $G \approx H$.*

Proposition 8. *Let $G_1, \cdots, G_\ell \in \mathcal{G}_n$ be a fixed choice of graphs on the vertices $\{x_1, \cdots, x_n\}$ such that the quotient space of $\mathcal{G}_n$ by σ_n is*

$$\mathcal{G}_n/\sigma_n = \{\mathrm{Orb}(G_1), \cdots, \mathrm{Orb}(G_\ell)\}.$$

Then the system of functions

$$\{\chi_{\{g\}} - \chi_{\{G_a\}} : g \in \mathrm{Orb}(G_a) \setminus \{G_a\}, \quad 1 \le a \le \ell\}$$

is a basis in Ker(U) *over* $\mathbb{R}$ *while*

$$\left\{ \sum_{g\in\mathrm{Orb}(G_a)} \chi_{\{g\}} : 1 \le a \le \ell \right\}$$

is a basis in $L^2(\mathcal{G}_n) \ominus \mathrm{Ker}(U)$ over $\mathbb{R}$. In particular $\dim_{\mathbb{R}} \mathrm{Ker}(U) = 2^{n(n-1)/2} - \ell$.

Proof. The kernel of U consists of all $F \in L^2(\mathcal{G}_n)$ such that

$$\sum_{H\in\mathrm{Orb}(G)} F(H) = 0$$

for any $G \in \mathcal{G}_n$. As $\{\chi_{\{g\}} : g \in \mathcal{G}_n\}$ is an orthonormal basis in $L^2(\mathcal{G}_n)$ for any $F = \sum_{g\in\mathcal{G}_n} \lambda_g \chi_{\{g\}} \in \mathrm{Ker}(U)$ ($\lambda_g \in \mathbb{R}$)

$$0 = \sum_{H\in\mathrm{Orb}(G)} \sum_{g\in\mathcal{G}_n} \lambda_g\, \chi_{\{g\}}(H) = \sum_{H\in\mathrm{Orb}(G)} \lambda_H$$

hence $\lambda_G = -\sum_{H\in\mathrm{Orb}(G)\setminus\{G\}} \lambda_H$. Then

$$F = \sum_{a=1}^{\ell} \sum_{g\in\mathrm{Orb}(G_a)} \lambda_g\, \chi_{\{g\}} = \sum_a \sum_{g\in\mathrm{Orb}(G_a)\setminus\{G_a\}} \lambda_g(\chi_{\{g\}} - \chi_{\{G_a\}})$$

and the first statement in Proposition 8 is proved. In particular

$$\dim_{\mathbb{R}} \operatorname{Ker}(U) = \sum_{a=1}^{\ell} n_a - \ell = 2^N - \ell,$$

where $n_a = |\operatorname{Orb}(G_a)|$ and $N = n(n-1)/2$. Next for any $F \in L^2(\mathcal{G}_n) \ominus \operatorname{Ker}(U)$ and any $H \in \operatorname{Orb}(G_a) \setminus \{G_a\}$, $1 \le a \le \ell$

$$\begin{aligned} 0 &= \big(F, \chi_{\{H\}} - \chi_{\{G_a\}}\big)_{L^2(\mathcal{G}_n)} \\ &= \sum_b \sum_{g \in \operatorname{Orb}(G_b)} \lambda_g \big(\chi_{\{g\}}, \chi_{\{H\}} - \chi_{\{G_a\}}\big)_{L^2(\mathcal{G}_n)} = \lambda_H - \lambda_{G_a} \end{aligned}$$

hence $\lambda_H = \lambda_{G_a}$ for any $H \in \operatorname{Orb}(G_a) \setminus \{G_a\}$ which proves the second statement in Proposition 8.

Corollary 6. *The projection $P : L^2(\mathcal{G}_n) \to L^2(\mathcal{G}_n) \ominus \operatorname{Ker}(U)$ is given by*

$$P\chi_{\{G_a\}} = \frac{1}{2-n_a} \sum_{g \in \operatorname{Orb}(G_a)} \chi_{\{g\}}, \quad 1 \le a \le \ell, \tag{30}$$

while for any $H \in \mathcal{G}_n \setminus \{G_1, \cdots, G_\ell\}$

$$P\chi_{\{H\}} = \frac{n_{a(H)} - 1}{n_{a(H)}} \sum_{g \in \operatorname{Orb}(G_{a(H)})} \chi_{\{g\}} \tag{31}$$

where $a(H) \in \{1, \cdots, \ell\}$ is the unique index such that $H \in \operatorname{Orb}(G_{a(H)})$.

Proof. Let $H \in \mathcal{G}_n$. Let us look for λ_g, $\mu_a \in \mathbb{R}$ such that

$$\chi_{\{H\}} = \sum_a \sum_{g \in \operatorname{Orb}(G_a) \setminus \{G_a\}} \lambda_g (\chi_{\{g\}} - \chi_{\{G_a\}}) + \sum_a \mu_a \sum_{g \in \operatorname{Orb}(G_a)} \chi_{\{g\}}.$$

We distinguish two cases as I) $H = G_b$ for some $1 \le b \le \ell$ or II) $H \in \mathcal{G}_n \setminus \{G_1, \cdots, G_\ell\}$. In the first case

$$\mu_a = \sum_{g \in \operatorname{Orb}(G_a) \setminus \{G_a\}} \lambda_g, \quad a \in \{1, \cdots, \ell\} \setminus \{b\}, \tag{32}$$

$$\mu_b = 1 + \sum_{g \in \operatorname{Orb}(G_b) \setminus \{G_b\}} \lambda_g, \tag{33}$$

$$\mu_a + \lambda_g = 0, \quad g \in \operatorname{Orb}(G_a) \setminus \{G_a\}, \ \ 1 \le a \le \ell. \tag{34}$$

Then (32) and (34) yield $(2 - n_a)\mu_a = 0$ hence $\mu_a = 0$ for any $a \neq b$ due to the following:

Lemma 5. *Let $n \ge 2$. For any $1 \le a \le \ell$ either $n_a = 1$ or $n_a \ge 3$.*

Proof. Clearly $\operatorname{Orb}(\overline{K^n}) = \{\overline{K^n}\}$ and $\operatorname{Orb}(K^n) = \{K^n\}$. If $n = 2$ then $\mathcal{G}_2 = \{\overline{K^2}, K^2\}$ hence $n_a = 1$ for $a \in \{1, 2\}$. Let $n \ge 3$ and $G \in \mathcal{G}_n \setminus \{\overline{K^n}, K^n\}$. If $|E(G)| = 1$ then $|E(K^n) \setminus E(G)| \ge 2$. Let then $E(G) = \{e\}$ and let us consider $e', e'' \in E(K^n) \setminus E(G)$, $e' \neq e''$. As $n \ge 3$ there exist $\tau_i \in \sigma_n \setminus \{1\}$, $i \in \{1, 2\}$, such that $\tau_1 \cdot e = e'$ and $\tau_2 \cdot e = e''$. It follows that $|\operatorname{Orb}(G)| \ge 3$. A similar argument may be given when $|E(G)| \ge 2$.

Then (33)–(34) imply $\lambda_g = 0$ for any $g \in \operatorname{Orb}(G_a) \setminus \{G_a\}$, $a \neq b$, and $\mu_b = 1/(2 - n_b)$ yielding (30). Similarly in the second case

$$\mu_a = \sum_{g \in \operatorname{Orb}(G_a) \setminus \{G_a\}} \lambda_g, \quad 1 \le a \le \ell,$$

$$\mu_a + \lambda_g = 0, \quad g \in \operatorname{Orb}(G_a) \setminus \{G_a\}, \ \ a \in \{1, \cdots, \ell\} \setminus \{a(H)\},$$

$$\mu_{a(H)} + \lambda_g = 1, \quad g \in \operatorname{Orb}(G_{a(H)}) \setminus \{G_{a(H)}\},$$

hence $\mu_{a(H)} = (n_{a(H)} - 1)/n_{a(H)}$ yielding (31). Q.e.d.

At this point we may complete the proof of Proposition 7. Let $f_a = U(\sum_{g \in \operatorname{Orb}(G_a)} \chi_{\{g\}})$. By the second statement in Proposition 8 it follows that $\{f_a : 1 \le a \le \ell\}$ is a basis of $\mathcal{H}_K$ over $\mathbb{R}$. On the other hand each f_a may be explicitly computed as

$$\begin{aligned} f_a(H) &= \sum_{g \in \operatorname{Orb}(G_a)} \sum_{\sigma \in \sigma_n} \chi_{\{g\}}(\sigma^{-1} \cdot H) \\ &= \sum_{g \in \operatorname{Orb}(G_a)} |A_g(H)| = \left(\sum_{g \in \operatorname{Orb}(G_a)} |I_g| \right) \chi_{\operatorname{Orb}(G_a)}. \end{aligned}$$

Appendix A: The Diaconis-Graham Lemma

The *Radon transform* based on translates of a set $S \subseteq V(G_N)$ is the map

$$R_S : \mathcal{D}(G_N) \to \mathcal{D}(G_N), \quad (R_S f)(X) = \sum_{Y \in X+S} f(Y),$$

where $X + S = \{X + Z : Z \in S\}$. P. Diaconis & R. L. Graham have studied (cf. [7]) the Radon transform on $\mathbb{Z}_2^N$ and proved an inversion formula for $R_{\overline{B}_1(0)}$ (respectively for $R_{S_1(0)}$) when N is even (respectively odd). The Fourier and Radon transforms are related as shown by the following:

Lemma 6 ([7], p. 325–326). *Let $S \subseteq V(G_N)$ be a nonempty subset. Then $\{2^{-N} E_X : \hat{\chi}_S(X) = 0\}$ is a basis of* $\mathrm{Ker}(R_S)$ *hence*

$$\dim_{\mathbb{R}} \mathrm{Ker}(R_S) = |\{X \in V(G_N) : \hat{\chi}_S(X) = 0\}|.$$

In particular the Radon transform R_S is injective if and only if $\hat{\chi}_S \neq 0$ everywhere in $V(G_N)$.

For any $f \in \mathcal{D}(G_N)$

$$(f * \chi_S)(X) = \sum_{Y \in V(G_N)} \chi_S(Y) f(X - Y)$$
$$= \sum_{Y \in S} f(X - Y) = \sum_{Z \in X+S} f(Z)$$

i.e. $f * \chi_S = R_S f$. Taking Fourier transforms

$$\mathcal{F}(R_S f) = \hat{f}\ \hat{\chi}_S. \tag{35}$$

Let us compute the Fourier transform of E_Z

$$\hat{E}_Z(X) = \sum_{Y \in V(G_N)} (-1)^{X \cdot Y} E_Z(Y)$$
$$= \sum_{Y} (-1)^{(X+Z) \cdot Y} = \begin{cases} 2^N & X = Z, \\ 0 & X \neq Z, \end{cases}$$

as $X \neq 0$ implies $\sum_{Y \in V(G_N)} (-1)^{X \cdot Y} = 0$ (cf. [4], p. 15–16). We may conclude that

$$\mathcal{F}(2^{-N} E_Z) = \chi_{\{Z\}}. \tag{36}$$

Proof of Lemma 6. Given $f \in \mathcal{D}(G_N)$ we let $\mathcal{Z}(f) = f^{-1}(0)$ be the set of all zeros of f. Let us observe that the system $\{\mathcal{F}(2^{-N} E_Z) : \hat{\chi}_S(Z) = 0\} \subset \mathcal{D}(G_N)$ is free over $\mathbb{R}$. Indeed let us consider a null linear combination

$$0 = \sum_{Z \in \mathcal{Z}(\hat{\chi}_S)} \lambda_Z\, \mathcal{F}(2^{-N} E_Z)$$

with $\lambda_Z \in \mathbb{R}$. Then (by (36)) $0 = \sum_{Z \in \mathcal{Z}(\hat{\chi}_S)} \lambda_Z\, \chi_{\{Z\}}(X) = \lambda_X$ for any $X \in \mathcal{Z}(\hat{\chi}_S)$. Yet $\mathcal{F}$ is an isometry of $\mathcal{D}(G_N)$ into itself hence the system $\{2^{-N} E_X : X \in \mathcal{Z}(\hat{\chi}_S)\} \subset \mathcal{D}(G_N)$ is free as well.

To end the proof of Lemma 6 we ought to show that $\{2^{-N} E_X : X \in \mathcal{Z}(\hat{\chi}_S)\}$ is also a system of generators in $\mathrm{Ker}(R_S)$. Let $X \in \mathcal{Z}(\hat{\chi}_S)$. Then (by (35)–(36))

$$\mathcal{F}(R_S E_X) = \hat{E}_X\, \hat{\chi}_S = 2^N \chi_{\{X\}}\, \hat{\chi}_S = 0$$

hence $2^{-N} E_X \in \mathrm{Ker}(R_S)$. Finally let us show that any $f \in \mathrm{Ker}(R_S)$ may be written as the linear combination $2^{-N} \sum_{X \in \mathcal{Z}(\hat{\chi}_S)} \hat{f}(X) E_X$. Indeed (by (35)) $0 = \mathcal{F}(R_S f) = \hat{f}\ \hat{\chi}_S$ implies that $\hat{f} = 0$ on $V(G_N) \setminus \mathcal{Z}(\hat{\chi}_S)$ hence

$$\left(\sum_{X \in \mathcal{Z}(\hat{\chi}_S)} \hat{f}(X) E_X \right)(Y) = \sum_{X \in \mathcal{Z}(\hat{\chi}_S)} \hat{f}(X) (-1)^{X \cdot Y}$$
$$= \sum_{X \in V(G_N)} (-1)^{X \cdot Y} \hat{f}(X) - \sum_{V(G_N) \setminus \mathcal{Z}(\hat{\chi}_S)} (-1)^{X \cdot Y} \hat{f}(X) = 2^N f(Y)$$

as $\mathcal{F} = 2^N \mathcal{F}^{-1}$. Q.e.d.

By a result in [17] the zero set $\mathcal{Z}(\hat{\chi}_{k(\Gamma)})$ consists (up to the bijection k) of all the graphs g on $\{x_1, \cdots, x_n\}$ such that $\sum_{i=1}^{n} (-1)^{m_i} = 0$, where m_i is the degree of x_i as a vertex of g. Indeed (following the arguments in [17])

$$(\mathcal{F} \chi_{k(\Gamma)})(\zeta) = \sum_{z \in k(\Gamma)} (-1)^{\zeta \cdot z} = \sum_{i=1}^{n} (-1)^{\zeta \cdot k(G_i)}$$

and

$$\zeta \cdot k(G_i) = \sum_{\alpha=1}^{N} \zeta_\alpha \chi_{E_i}(e_\alpha) = \sum_{\alpha=1}^{N} \chi_{E(G_\zeta)}(e_\alpha)\chi_{E_i}(e_\alpha) = \sum_{e \in E(G_\zeta)} \chi_{E_i}(e)$$

$$= \sum_{e \in E(G_\zeta)} \begin{cases} 1, & x_i \in e \\ 0, & x_i \notin e \end{cases} = m_{G_\zeta}(x_i),$$

the degree of x_i as a vertex of G_ζ. To describe the components of $X \in \mathbb{Z}_2^{n(n-1)/2}$ we may use the index set $\{(i,j) : 1 \le i < j \le n\}$. If $\{x_i, x_j\} \in E(K^n)$ $(i < j)$ then $\{x_i, x_j\} \in E(G_X)$ if and only if $X_{ij} = 1 \in \mathbb{Z}_2$. Let $[X_{ij}] \in \mathcal{M}_n(\mathbb{Z}_2)$ be the $n \times n$ symmetric matrix obtained from X such that $X_{ii} = 0$, $1 \le i \le n$. Then the degree of the vertex x_i in G_X is

$$m_{G_X}(x_i) = \sum_{j=1}^{n} \iota(X_{ij}), \quad 1 \le i \le n,$$

where $\iota : \mathbb{Z}_2 \to \mathbb{R}$ is the natural injection, so that $\mathcal{Z}(\hat{\chi}_S)$ may be identified with the set A_n given by (3) in the Introduction.

Appendix B: Stanley's Lemma

The scope of this appendix is to gather a few results spread through [17] as the following:

Lemma 7 (R. P. Stanley, [17]).
i) The unlabellings of two graphs on $\{x_1, \cdots, x_n\}$ coincide if and only if the two graphs are isomorphic i.e. given $G, H \in \mathcal{G}_n$ one has $U(\chi_{\{G\}}) = U(\chi_{\{H\}})$ if and only if $G \approx H$.
ii) For any graph $G \in \mathcal{G}_n$, any permutation $\sigma \in \sigma_n$, and any index $1 \le i \le n$

$$(\sigma \cdot G)(x_i) = \sigma \cdot G(x_{\sigma^{-1}(i)}).$$

iii) The Stanley morphism Φ and the unlabelling operator U commute i.e. $\Phi \circ U = U \circ \Phi$.

Proof. Clearly the unlabellings of two isomorphic graphs coincide. Conversely, let us assume that G and H have the same unlabellings and let us apply the identity $U(\chi_{\{G\}}) = U(\chi_{\{H\}})$ (an equality of functions on $\mathcal{G}_n$) to H. We obtain

$$\sum_{\sigma \in \sigma_n} \chi_{\{G\}}(\sigma^{-1} \cdot H) = \sum_{\sigma \in \sigma_n} \chi_{\{H\}}(\sigma^{-1} \cdot H)$$

or

$$|\{\sigma \in \sigma_n : H = \sigma \cdot G\}| = |I_H| \ge 1 \tag{37}$$

where I_H is the isotropy group of H with respect to the action of σ_n on $\mathcal{G}_n$ (of course $|I_H| \ge 1$ as I_H contains at least the identical permutation). Statement (i) is proved. The proof of (ii) is immediate. To prove (iii) we conduct the calculation

$$(\Phi U)(\chi_{\{G\}}) = \sum_{\sigma \in \sigma_n} \Phi(\sigma \cdot \chi_{\{G\}}) = \quad \text{(by (22))}$$

$$= \sum_{\sigma \in \sigma_n} \Phi(\chi_{\{\sigma \cdot G\}}) = \sum_{\sigma \in \sigma_n} \sum_{i=1}^{n} \chi_{\{(\sigma \cdot G)(x_i)\}} = \quad \text{(by (ii) in Lemma 7)}$$

$$= \sum_{\sigma \in \sigma_n} \sum_{i=1}^{n} \chi_{\{\sigma \cdot G(x_{\sigma^{-1}(i)})\}} = \sum_{\sigma \in \sigma_n} \sum_{i=1}^{n} \chi_{\{G(x_i)\}} = (U\Phi)(\chi_{\{G\}}).$$

Q.e.d.

REFERENCES

[1] Alon, N., "Eigenvalues and expanders," *Combinatorica*, **6**: 83–96 (1986).
[2] Aronszajn, N., "Theory of reproducing kernels," *Trans. A.M.S.*, **68**: 337–404 (1950).
[3] Barletta, E., and Dragomir, S., "A new upper bound on the Cheeger number of a graph," *J. Combin. Theory, Ser. B*, **82**: 167–174 (2001).
[4] Barletta, E., and Dragomir, S., "Combinatorial PDEs on Hamming graphs," *Discrete Mathematics*, **254**: 1–18 (2002).
[5] Bondy, J. A., and Hemminger, R. L., "Graph reconstruction—A survey," *J. Graph. Theory*, **1**: 227–268 (1977).
[6] Byun, D.-W., and Saitoh, S., "A real inversion formula for the Laplace transform," *Zeitschrift für Analysis und ihre Anwendungen*, **12**: 597–603 (1993).
[7] Diaconis, P., and Graham, R., "The Radon transform on $\mathbb{Z}_2^N$," *Pacific J. Math.*, **118(2)**: 323–345 (1985).
[8] Diestel, R., *Graph theory*, Graduate Texts in Mathematics, 173, Springer, New York, 1997.
[9] Kanai, M., "Rough isometries and the parabolicity of Riemannian manifolds," *J. Math. Soc. Japan*, **38(2)**: 227–238 (1986).
[10] Kecs, W., *The convolution product and its applications*, Editura Academiei Republicii Socialiste România, Bucureşti, 1978.

[11] Ohno, Y., and Urakawa, H., "On the first eigenvalue of the combinatorial Laplacian for a graph," *Interdisciplinary Informations Sciences*, **1(1)**: 33–46 (1994).
[12] Saitoh, S., "Hilbert spaces induced by Hilbert space valued functions," *Proc. A.M.S.*, **89**: 74–78 (1983).
[13] Saitoh, S., "Representations of the norms in Bergman-Selberg spaces on strips and half planes," *Complex Variables*, **19**: 231–241 (1992).
[14] Saitoh, S., "One approach to some general integral transforms and its applications," *Integral transforms and special functions*, **3(1)**: 49–84 (1995).
[15] Soardi, P. M., "Rough isometries and Dirichlet finite harmonic functions on graphs," *Proc. A.M.S.*, **119(4)**: 1239–1248 (1993).
[16] Stanley, R. P., "Quotients of Peck posets," *Order*, **1**: 29–34 (1984).
[17] Stanley, R. P., "Reconstruction from vertex-switching," *J. Combin. Theory, Ser. B*, **38**: 132–137 (1985).
[18] Urakawa, H., "A discrete analogue of the harmonic morphism and Green kernel comparison theorems," *Glasgow Math. J.*, **42**: 319–344 (2000).
[19] Urakawa, H., "The spectrum of an infinite graph," *Canadian J. Math.*, **52**: 1057–1084 (2000).