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CM defect and Hilbert functions of monomial curves

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ABSTRACT

In this article we consider a semigroup ring $R = K[\![\Gamma]\!]$ of a numerical semigroup Γ and study the Cohen–Macaulayness of the associated graded ring $\mathrm{G}(\Gamma) := \mathrm{gr}_{\mathfrak{m}}(R) := \oplus_{n\in\mathbb{N}} \mathfrak{m}^n/\mathfrak{m}^{n+1}$ and the behaviour of the Hilbert function H_R of R. We define a certain (finite) subset $\mathrm{B}(\Gamma) \subseteq \Gamma$ and prove that $\mathrm{G}(\Gamma)$ is Cohen–Macaulay if and only if $\mathrm{B}(\Gamma) = \emptyset$. Therefore the subset $\mathrm{B}(\Gamma)$ is called the *Cohen–Macaulay defect* of $\mathrm{G}(\Gamma)$. Further, we prove that if the degree sequence of elements of the standard basis of Γ is non-decreasing, then $\mathrm{B}(\Gamma) = \emptyset$ and hence $\mathrm{G}(\Gamma)$ is Cohen–Macaulay. We consider a class of numerical semigroups $\Gamma = \sum_{i=0}^{3} \mathbb{N}m_i$ generated by 4 elements m_0, m_1, m_2, m_3 such that $m_1 + m_2 = m_0 + m_3$—so called "balanced semigroups". We study the structure of the Cohen–Macaulay defect $\mathrm{B}(\Gamma)$ of Γ and particularly we give an estimate on the cardinality $|\mathrm{B}(\Gamma, r)|$ for every $r \in \mathbb{N}$. We use these estimates to prove that the Hilbert function of R is non-decreasing. Further, we prove that every balanced "unitary" semigroup Γ is "2-good" and is not "1-good", in particular, in this case, $\mathrm{G}(\Gamma)$ is not Cohen–Macaulay. We consider a certain special subclass of balanced semigroups Γ. For this subclass we try to determine the Cohen–Macaulay defect $\mathrm{B}(\Gamma)$ using the explicit description of the standard basis of Γ; in particular, we prove that these balanced semigroups are 2-good and determine when exactly $\mathrm{G}(\Gamma)$ is Cohen–Macaulay.

0. Introduction

In this article we study the Cohen–Macaulayness of the tangent cone and the behaviour of the Hilbert function of monomial curves. These problems are studied by several authors and many results are obtained for 1-dimensional local Cohen–Macaulay rings. More precisely, let $(R, \mathfrak{m})$ be a 1-dimensional Cohen–Macaulay local ring of multiplicity e and embedding dimension $\nu \geq 2$. If either $\nu \leq 3$ or $\nu \leq \mathrm{e} \leq \nu + 2$, then (see [4,5]) the Hilbert function H_R of R is non-decreasing. However, if either $\nu \geq 4$ or $\mathrm{e} \geq \nu + 3$, then H_R can be locally decreasing (see for example, [16,22,6,13]). Moreover, if R is the semigroup ring $K[\![\Gamma]\!]$ of a numerical semigroup $\Gamma \subseteq \mathbb{N}$ generated by an arithmetic sequence, then the associated graded ring $\mathrm{G}(\Gamma) := \mathrm{gr}_{\mathfrak{m}}(R) := \oplus_{n\in\mathbb{N}} \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is always Cohen–Macaulay and hence the Hilbert function of R is non-decreasing (see [13]). Furthermore, if Γ is generated by an almost arithmetic sequence, then a characterization (in most cases) for the Cohen–Macaulayness of $\mathrm{G}(\Gamma)$ is given in [12] and the Hilbert function of R is non-decreasing (see [26]). More recently, Arslan and Mete in [1] and Shibuta in [25] have proved that $\mathrm{G}(\Gamma)$ is Cohen–Macaulay for a particular class of semigroups generated by 4 elements.

In Section 1, for the semigroup ring $R = K[\![\Gamma]\!]$ of a numerical semigroup Γ, we study the Cohen–Macaulayness of $\mathrm{G}(\Gamma)$. In this case we define a certain (finite) subset $\mathrm{B}(\Gamma) \subseteq \Gamma$ (see 1.4 and 1.5) and prove that (see Theorem 1.6) $\mathrm{G}(\Gamma)$ is Cohen–Macaulay if and only if $\mathrm{B}(\Gamma) = \emptyset$. Therefore the subset $\mathrm{B}(\Gamma)$ is called the *Cohen–Macaulay defect* of $\mathrm{G}(\Gamma)$. Finally, we prove

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that (see Theorem 1.12) if the degree sequence of elements of the standard basis of Γ is non-decreasing, then $\mathrm{B}(\Gamma) = \emptyset$ and hence $\mathrm{G}(\Gamma)$ is Cohen–Macaulay. This theorem allows us to check whether $\mathrm{G}(\Gamma)$ is Cohen–Macaulay rather easily if one knows the description of standard basis of Γ explicitly, for example, if Γ is generated by an arithmetic progression.

In Section 2, we consider a class of numerical semigroups $\Gamma = \sum_{i=0}^{3} \mathbb{N}m_i$ generated by 4 elements m_0, m_1, m_2, m_3 such that $m_1 + m_2 = m_0 + m_3$; these semigroups are called *balanced semigroups*. For this class we study the structure of the Cohen–Macaulay defect $\mathrm{B}(\Gamma)$ and particularly we give (see Corollary 2.10) an estimate on the cardinality $|\mathrm{B}(\Gamma, r)|$ for every $r \in \mathbb{N}$. We use these estimates to prove that (see Theorem 2.11) the Hilbert function of $R = K[\![\Gamma]\!]$ is non-decreasing. Further, we prove that (see Example 2.12) every balanced "unitary" semigroup Γ is "2-good" and is not "1-good" (see the definitions in 2.3), in particular, in this case, $\mathrm{G}(\Gamma)$ is not Cohen–Macaulay.

In Section 3 we consider a certain special subclass of balanced semigroups Γ. For this subclass we try to determine (see Proposition 3.3) the Cohen–Macaulay defect $\mathrm{B}(\Gamma)$ using the explicit description (see Proposition 3.2) of the standard basis of Γ; in particular, we prove that these balanced semigroups are 2-good and determine (see Theorem 3.4) when exactly $\mathrm{G}(\Gamma)$ is Cohen–Macaulay. It is interesting to note that our subclass contains the special classes of semigroups considered by Kraft (see [8], [9]); Bresinsky considered a similar class of examples in [2]. Bresinsky and Kraft constructed these special classes to show that the defining ideals $\mathfrak{P}(m_0, m_1, m_2, m_3)$ of the monomial curves $\mathcal{C}(m_0, m_1, m_2, m_3) \subseteq \mathbb{A}^4_K$ with parametrization $X_0 = T^{m_0}, X_1 = T^{m_1}, X_2 = T^{m_2}, X_3 = T^{m_3}$ and the type of R can be arbitrarily large, respectively (unless the semigroup is symmetric, see [3]). A more general class of examples was considered by the first author in [17] and he gave a systematic method using the explicit description of the standard basis of Γ to construct a minimal set of generators for $\mathfrak{P}(m_0, m_1, m_2, m_3)$ and the derivation module $\mathrm{Der}_K(R)$ of R. This method of construction explains to some extent why both the numbers $\mu(\mathfrak{P}(m_0, m_1, m_2, m_3))$ and $\mu(\mathrm{Der}_K(R))$ can be arbitrary large. Therefore this class of examples contains many counterexamples. However, our result (Theorem 2.11) supports an informal conjecture made by Rossi which says that: *The Hilbert function of a* 1*-dimensional Gorenstein local ring is always non-decreasing.*

1. Cohen–Macaulay defect

Let $(R, \mathfrak{m})$ be a d-dimensional noetherian local ring of multiplicity e and embedding dimension $\nu := \mathrm{Dim}_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) = \ell(\mathfrak{m}/\mathfrak{m}^2) = \mu(\mathfrak{m}) :=$ the minimal number of generators of $\mathfrak{m}$. Let $\mathrm{gr}_{\mathfrak{m}}(R) := \oplus_{n\in\mathbb{N}}\, \mathfrak{m}^n/\mathfrak{m}^{n+1}$ denote the associated graded ring of R with respect to the maximal ideal $\mathfrak{m}$ of R. The numerical function $\mathrm{H}_R : \mathbb{N} \to \mathbb{N}$ defined by $n \mapsto \mathrm{Dim}_{R/\mathfrak{m}}(\mathfrak{m}^n/\mathfrak{m}^{n+1})$ is called the Hilbert function of $\mathrm{gr}_{\mathfrak{m}}(R)$ or just that of R. The generating function of the numerical function H_R is the Poincaré series $\mathrm{P}_R(Z) := \sum_{n\in\mathbb{N}} \mathrm{H}_R(n) \cdot Z^n$. It is well known that there exists a polynomial $\mathrm{h}_R(Z) \in \mathbb{Z}[Z]$ such that $\mathrm{P}_R(Z) = \frac{\mathrm{h}_R(Z)}{(1-Z)^d}$. The polynomial $\mathrm{h}_R(Z) = \mathrm{h}_0 + \mathrm{h}_1 Z + \cdots + \mathrm{h}_{n_0} Z^{n_0}$ is called the h-polynomial of R; $\mathrm{h}_0 = \mathrm{H}_R(0) = 1$, $\mathrm{h}_1 = \mathrm{H}_R(1) - d = \nu - d$.

For convenience, some definitions and some well-known results for 1-dimensional noetherian local rings are collected in 1.1 and 1.2:

Some Results 1.1. Let $(R, \mathfrak{m})$ be a 1-dimensional Cohen–Macaulay local ring of multiplicity e and embedding dimension $\nu \geq 2$. Further, let $\mathrm{h}_R := \mathrm{h}_R(Z) = \mathrm{h}_0 + \mathrm{h}_1 Z + \cdots + \mathrm{h}_{n_0} Z^{n_0}$, $n_0 := \deg \mathrm{h}_R$ denote the h-polynomial of R. Then $\mathrm{h}_0 = 1$, $\mathrm{h}_i = \mathrm{H}_R(i) - \mathrm{H}_R(i-1)$ for every $i \geq 1$ and $n_0 = \min\{n \in \mathbb{N} \mid \mathrm{H}_R(i) = e \text{ for all } i \geq n\} = \min\{n \in \mathbb{N} \mid \mathrm{H}_R(n) = e\}$. Further, we note the following results:

(1) We may assume that (if necessary replace the ring R by $R[X]_{\mathfrak{m}[X]}$ and assume that the residue field $R/\mathfrak{m}$ of R is infinite) there exists a superficial element $x \in \mathfrak{m}$ of degree 1, i.e. $(\mathfrak{m}^{n+1} : x) = \mathfrak{m}^n$ for large $n >> 0$; or equivalently, $x\mathfrak{m}^n = \mathfrak{m}^{n+1}$ for large $n >> 0$, i.e. the principal ideal Rx is a minimal reduction of $\mathfrak{m}$ (see [15, Theorem 1] and [10]). Let $x \in \mathfrak{m}$ be a superficial element of degree 1 and $\overline{R} := R/Rx$. Then x is a non-zero divisor in R, $x \notin \mathfrak{m}^2$ and $\mathrm{P}_{\overline{R}}(Z) = \sum_{n\geq 0} \mathrm{Dim}_{R/\mathfrak{m}}\left(\overline{\mathfrak{m}}^n/\overline{\mathfrak{m}}^{n+1}\right) \cdot Z^n = \mathrm{h}_{\overline{R}}(Z)$, where $\overline{\mathfrak{m}} := \mathfrak{m}/Rx$ is the maximal ideal of $\overline{R}$.

(2) (See [23] and [27, Theorems (6.1), (1.5) and (1.3)]) Let $x \in \mathfrak{m}$ be a superficial element of degree 1 and let $\mathrm{G} := \mathrm{gr}_{\mathfrak{m}}(R)$ be the associated graded ring of R with respect to the maximal ideal $\mathfrak{m}$. Then: G *is Cohen–Macaulay* $\iff$ *the image* x^* *of* x *in* G *is a non-zero divisor in* G $\iff$ $\mathrm{h}_R = \mathrm{h}_{\overline{R}}$. *In particular, if* G *is Cohen–Macaulay, then* $\mathrm{h}_r \geq 0$ *for every* $r \geq 0$, i.e. *the Hilbert function* $\mathrm{H}_R : \mathbb{N} \to \mathbb{N}$ *of* R *is non-decreasing.*

(3) (See [10, Prop. 8 and Prop. 9]) $n_0 = \deg(\mathrm{h}_R) = \min\{n \in \mathbb{N} \mid \mathrm{H}_R(n) = e\} = \min\{n \in \mathbb{N} \mid x \cdot \mathfrak{m}^n = \mathfrak{m}^{n+1}\} =: r_0$. The natural number r_0 is called the *reduction exponent* of the minimal reduction Rx of $\mathfrak{m}$. This equality also shows that r_0 is independent of the choice of x, i.e. if Ry is another minimal reduction of $\mathfrak{m}$, then $r_0 = \min\{n \in \mathbb{N} \mid y \cdot \mathfrak{m}^n = \mathfrak{m}^{n+1}\}$. Therefore r_0 is called the *reduction exponent* of the local ring R.

Notation and Definitions 1.2. Throughout this article we make the following assumptions and fix the following notation:

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{N}^+ = \mathbb{Z}^+$ and $\mathbb{Z}^-$ denote the sets of natural numbers, integers, positive integers and negative integers, respectively.

Let $R := K[\![\Gamma]\!] = K[\![T^g \mid g \in \Gamma]\!] (\subseteq K[\![T]\!])$ be the semigroup ring of a numerical semigroup $\Gamma \subseteq \mathbb{N}$ over a field K. The integral closure of R is the formal power series ring $K[\![T]\!]$ over K. Let $v := \mathrm{ord}_T : K(\!(T)\!) \to \mathbb{Z} \cup \{\infty\}$ be the discrete valuation of $K[\![T]\!]$. Let $\mathrm{G}(\Gamma)$ denote the associated graded ring $\mathrm{gr}_{\mathfrak{m}}(R)$ of R with respect to the maximal ideal $\mathfrak{m}$.

(1) Let $m_0, \ldots, m_{\nu-1}$ be a minimal generating set for Γ with $m_0 < m_1 < \cdots < m_{\nu-1}$. Then $m_0 = \mathrm{e}$ is the multiplicity of R and the element $x = T^{m_0} \in R$ is a superficial element of degree 1 (see for example [21, Lemma (1.3)]). Further, since R is a semigroup ring of $\Gamma = v(R)$, we have $T^{v(F)} \in R$ for every $F \in R$.
(2) (Standard basis) Let $\mathrm{S}_{m_0} := \{z \in \Gamma \mid z - m_0 \notin \Gamma\}$ be the standard basis (or Apéry set) of Γ with respect to m_0. It is clear that S_{m_0} depends on Γ and m_0, but for simplicity we put $\mathrm{S} := \mathrm{S}_{m_0}$. Further it is easy to see that $\mathrm{S} = \{s_0 = 0, s_1, \ldots, s_{m_0-1}\} \subseteq \Gamma$, where $0 = s_0 < s_1 < \cdots < s_{m_0-1} \in \Gamma$ are non-negative integers with the following properties:
 (i) $s_i \not\equiv s_j \pmod{m_0}$ for all $i, j \in [0, m_0 - 1], i \neq j$.
 (ii) If $z \in \Gamma$, then $z = am_0 + s_i$ for unique $i \in [0, m_0 - 1]$ and $a \in \mathbb{N}$.
(3) (Maximum degree) For each $g \in \Gamma$, we put

$$\text{max-deg}(g) := \max\left\{\sum_{i=0}^{\nu-1} a_i \mid g = \sum_{i=0}^{\nu-1} a_i m_i \text{ with } a_0, \ldots, a_{\nu-1} \in \mathbb{N}\right\}.$$

The natural number max-deg(g) is called the maximum degree of g. For $g \in \Gamma$, let $[T^g]$ denote the natural image of T^g in $\mathrm{G}(\Gamma)$. Then it is easy to see that max-deg$(g) = \deg([T^g]) = \max\{r \in \mathbb{N} \mid T^g \in \mathfrak{m}^r\}$.
(4) For $r \in \mathbb{N}^+$, we put:

$$\Gamma(r) := \{g \in \Gamma \mid \text{max-deg}(g) = r\} \quad \text{and} \quad \mathrm{S}(r) := \Gamma(r) \cap \mathrm{S}.$$

For completeness we collect some observations on H_R and $\mathrm{H}_{\overline{R}}$ in the following lemma.

Lemma 1.3. *Let Γ and R be as in 1.2. Then for every $r \in \mathbb{N}^+$, we have:*

(1) $\mathrm{H}_R(r) = |\Gamma(r)|$.
(2) *Let $g \in \Gamma(r)$. Then $g \in \mathrm{S}(r)$ if and only if $T^g \notin Rx$.*
(3) *For $s \in \mathrm{S}(r)$, let $\overline{T^s}$ denote the image of T^s in $\overline{\mathfrak{m}}^r/\overline{\mathfrak{m}}^{r+1}$. Then $\{\overline{T^s} \mid s \in \mathrm{S}(r)\}$ is a basis of the $R/\mathfrak{m}$-vector space $\overline{\mathfrak{m}}^r/\overline{\mathfrak{m}}^{r+1}$. In particular, $\mathrm{H}_{\overline{R}}(r) = |\mathrm{S}(r)|$.*

Proof. (1) $\mathrm{H}_R(r) = \mathrm{Dim}_K(\mathfrak{m}^r/\mathfrak{m}^{r+1}) = \ell(\mathfrak{m}^r/\mathfrak{m}^{r+1}) = |v(\mathfrak{m}^r) \setminus v(\mathfrak{m}^{r+1})|$ by [11, Section 2], where $v = \mathrm{ord}_T$ (see 1.2). Therefore by 1.2 (parts (1), (3) and (4)) we have the equalities $v(\mathfrak{m}^r) \setminus v(\mathfrak{m}^{r+1}) = \{v(f) \mid f \in \mathfrak{m}^r \setminus \mathfrak{m}^{r+1}\} = \{g \in \Gamma \mid T^g \in \mathfrak{m}^r \setminus \mathfrak{m}^{r+1}\} = \{g \in \Gamma \mid \text{max-deg}(g) = r\} = \Gamma(r)$.
(2) If $g \notin \mathrm{S}(r)$, then $g = m_0 + g'$ for some $g' \in \Gamma$ and so $T^g = T^{m_0}T^{g'} \in Rx$. Conversely, if $T^g \in Rx$, then $T^g = Fx$ with $F \in R$ and so $T^g = T^{v(F)}T^{m_0} = T^{v(F)+m_0}$, i.e. $g = v(T^g) = v(F) + m_0 \notin \mathrm{S}$, since $v(F) \in v(R) = \Gamma$.
(3) Since $T^s \in \mathfrak{m}^r \setminus \mathfrak{m}^{r+1}$ for every $s \in \mathrm{S}(r)$, it follows that $\{\overline{T^s} \mid s \in \mathrm{S}(r)\}$ generates the $R/\mathfrak{m}$-vector space $\overline{\mathfrak{m}}^r/\overline{\mathfrak{m}}^{r+1}$. If there is a non-trivial linear dependence relation $\overline{T^{s_1}} = \sum_{i=2}^{k} c_i\overline{T^{s_i}}$ with distinct $s_1, s_2, \ldots, s_k \in \mathrm{S}(r)$ and $c_2, \ldots, c_k \in R/\mathfrak{m}$, then $T^{s_1} = \sum_{i=2}^{k} c_i T^{s_i} + FT^{m_0} + F'$ with $F, F' \in R$ and $F' \in \mathfrak{m}^{r+1}$. Therefore, since $s_1 \neq s_i$ for all $i = 2, \ldots, k$ and $s_1 \notin \Gamma(r+1)$, by 1.2 (parts (3) and (1)) we must have $T^{s_1} = T^{v(F)}T^{m_0} \in Rx$, i.e. $s_1 = v(F) + m_0 \notin \mathrm{S}(r)$, a contradiction. □

Definition 1.4. Let Γ be as in 1.2. For $r \in \mathbb{N}$, we put

$$\mathrm{B}(\Gamma, r) := \{g \in \Gamma(r) \mid \text{max-deg}(g + m_0) \geq r + 2\} \quad \text{and} \quad \mathrm{B}(\Gamma) = \bigcup_{r\in\mathbb{N}} \mathrm{B}(\Gamma, r).$$

It is clear that $\mathrm{B}(\Gamma, 0) = \emptyset$ and each $\mathrm{B}(\Gamma, r)$ is a finite set. We shall call the set $\mathrm{B}(\Gamma)$ Cohen–Macaulay defect of $\mathrm{G}(\Gamma)$, since we shall prove below in Theorem 1.6 that $\mathrm{B}(\Gamma) = \emptyset$ if and only if $\mathrm{G}(\Gamma)$ is Cohen–Macaulay. First we prove that $\mathrm{B}(\Gamma)$ is a finite set.

Lemma 1.5. *Let Γ and R be as in 1.2 and let r_0 be the reduction exponent of R. Then:*

(1) $\mathrm{S}(r) = \emptyset$ *for every $r \geq r_0 + 1$.*
(2) $\mathrm{B}(\Gamma, r) = \emptyset$ *for every $r \geq r_0 - 1$. In particular,* $\mathrm{B}(\Gamma)$ *is a finite set.*

Proof. Since $r_0 = \min\{n \in \mathbb{N} \mid x \cdot \mathfrak{m}^n = \mathfrak{m}^{n+1}\}$ (see 1.1(3)), we have $\mathfrak{m}^{r_0+t} = x^t \cdot \mathfrak{m}^{r_0}$ for every $t \in \mathbb{N}$. Therefore (1) is immediate from 1.3(2).

(2) If $r = r_0 + t$ with $t \geq -1$ and if $g \in \mathrm{B}(\Gamma, r)$, then $g' = g + m_0 \in \Gamma(r')$ with $r' \geq r + 2 = r_0 + t + 2$ and so $T^g \cdot x = T^{g+m_0} = T^{g'} \in \mathfrak{m}^{r_0+t+2} = x^{t+2} \cdot \mathfrak{m}^{r_0}$. Therefore $T^g \in x^{t+1} \cdot \mathfrak{m}^{r_0} = \mathfrak{m}^{r_0+t+1}$, since x is a non-zero divisor in R. This contradicts that fact that $g \in \Gamma(r_0 + t)$ (see 1.3(3)). □

Theorem 1.6. *Let Γ be a numerical semigroup, $R = K[\![\Gamma]\!]$ be a semigroup ring of Γ over a field K and let $\mathrm{G}(\Gamma) = \mathrm{gr}_{\mathfrak{m}}(R)$ be the associated graded ring of R. Then $\mathrm{G}(\Gamma)$ is Cohen–Macaulay if and only if $\mathrm{B}(\Gamma) = \emptyset$.*

Proof. For $t \in \mathbb{N}^+$, let $(*_t)$ be the assertion:

$(*_t)$ The equality max-deg$(g + tm_0) =$ max-deg$(g) + t$ holds for every $g \in \Gamma$.

In view of [12, Proposition 2.2(3)] and the definition of $\mathrm{B}(\Gamma)$, it is enough to prove the implication: $(*_1) \Rightarrow (*_t)$. We prove this by induction on t. Let $t \geq 2$, $g \in \Gamma$ and $g' := g + (t-1)m_0$. Then max-deg$(g + tm_0) =$ max-deg$(g' + m_0) =$ max-deg$(g') + 1 = ($max-deg$(g) + (t-1)) + 1 =$ max-deg$(g) + t$ by $(*_1)$ and by induction on t. □

For the computation of the coefficients h_i of the h-polynomial of R, we introduce the following subsets:

Definition 1.7. Let Γ be as in 1.2. For $r \in \mathbb{N}$, we define

$$\mathrm{C}(\Gamma, r) := \{g + m_0 \in \Gamma(r) \mid g \in \mathrm{B}(\Gamma, r') \text{ for some } r' \le r - 2\}.$$

Clearly, for $r \le 2$, $\mathrm{C}(\Gamma, r) = \emptyset$. Further, if $\mathrm{B}(\Gamma) = \emptyset$, then $\mathrm{C}(\Gamma, r) = \emptyset$ for every $r \in \mathbb{N}$. Moreover, we have:

Lemma 1.8. *Let Γ and R be as in 1.2 and let r_0 be the reduction exponent of R. Then:*

(1) $\mathrm{C}(\Gamma, r) = \emptyset$ *for every* $r \ge r_0 + 1$.
(2) $\mathrm{S}(r_0) \cup \mathrm{C}(\Gamma, r_0) \ne \emptyset$.

Proof. If $r = r_0 + t$ with $t \in \mathbb{N}^+$ and if $g + m_0 \in \mathrm{C}(\Gamma, r)$, then $g \in \mathrm{B}(\Gamma)$ and $T^g \cdot x = T^{g+m_0} \in \mathfrak{m}^r = x^t \cdot \mathfrak{m}^{r_0}$. Therefore $T^g \in x^{t-1} \cdot \mathfrak{m}^{r_0} \subseteq \mathfrak{m}^{r_0}$, since x is a non-zero divisor in R and so $g \in \Gamma(r') \cap \mathrm{B}(\Gamma) = \mathrm{B}(\Gamma, r')$ with $r' \ge r_0$ which is absurd by Lemma 1.5(2). This proves (1).

(2) Suppose that $\mathrm{S}(r_0) = \emptyset$. Since $\mathfrak{m}^{r_0+1} = x \cdot \mathfrak{m}^{r_0} \subseteq x \cdot \mathfrak{m}^{r_0-1} \subsetneq \mathfrak{m}^{r_0}$ by definition of r_0, there exists $T^g \in \mathfrak{m}^{r_0} \setminus x \cdot \mathfrak{m}^{r_0-1}$. Then $g \in \Gamma(r_0)$ and $g = m_0 + g'$ for some $g' \in \Gamma(r')$ with $r' \le r_0 - 2$, since $\mathrm{S}(r_0) = \emptyset$. Therefore $g = g' + m_0 \in \mathrm{C}(\Gamma, r_0)$. □

Proposition 1.9. *Let Γ and R be as in 1.2 and let $r \in \mathbb{N}$. Then:*

(1) *The sets* $\mathrm{S}(r)$, $\mathrm{B}(\Gamma, r-1)$ *and* $\mathrm{C}(\Gamma, r)$ *are pairwise disjoint.*
(2) *The subsets* $\mathrm{S}(r)$, $\mathrm{D}(\Gamma, r) := \{g + m_0 \in \Gamma(r) \mid g \in \Gamma(r-1) \setminus \mathrm{B}(\Gamma, r-1)\}$, $\mathrm{C}(\Gamma, r)$ *form a partition of* $\Gamma(r)$.
(3) $\mathrm{h}_r = |\mathrm{S}(r)| + |\mathrm{C}(\Gamma, r)| - |\mathrm{B}(\Gamma, r-1)|$.

Proof. (1) Clearly $\mathrm{S}(r) \cap \mathrm{C}(\Gamma, r) = \emptyset$ by definition of the standard basis S of Γ. Further, since $\Gamma(r) \cap \Gamma(r-1) = \emptyset$, $\mathrm{S}(r) \subseteq \Gamma(r)$, $\mathrm{B}(\Gamma, r-1) \subseteq \Gamma(r-1)$ and $\mathrm{C}(\Gamma, r) \subseteq \Gamma(r)$ by definitions, it follows that $\mathrm{S}(r) \cap \mathrm{B}(\Gamma, r-1) = \emptyset$ and $\mathrm{B}(\Gamma, r-1) \cap \mathrm{C}(\Gamma, r) = \emptyset$.

(2) In view of the definition of the standard basis S of Γ and (1) it is enough to prove that $\mathrm{D}(\Gamma, r) \cap \mathrm{C}(\Gamma, r) = \emptyset$. For this it is enough to note that: if $g \in \Gamma$, then $g + m_0 \in \Gamma(r)$ if and only if either $g + m_0 \in \mathrm{C}(\Gamma, r)$, or $g \in \Gamma(r-1) \setminus \mathrm{B}(\Gamma, r-1)$.

(3) Since $|\mathrm{D}(\Gamma, r)| = |\Gamma(r-1) \setminus \mathrm{B}(\Gamma, r-1)|$, we have $\mathrm{h}_r = \mathrm{H}_R(r) - \mathrm{H}_R(r-1) = |\Gamma(r)| - |\Gamma(r-1)| = |\mathrm{S}(r)| + |\mathrm{C}(\Gamma, r)| - |\mathrm{B}(\Gamma, r-1)|$ by 1.3(1) and (2). □

Remark 1.10. From Lemmas 1.9(3), 1.8 and 1.5 it follows that $\mathrm{h}_r = 0$ for every $r > r_0$ and $\mathrm{h}_{r_0} > 0$ (see also [14, Corollary 1.11]). This proves once again 1.1(3) for a semigroup ring R: The degree $\deg(\mathrm{h}_R)$ of the h-polynomial is the reduction exponent r_0, of R. Further, if $\mathrm{G}(\Gamma)$ is Cohen–Macaulay, then $\mathrm{h}_R(Z) = \sum_{r=0}^{r_0} |\mathrm{S}(r)| \cdot Z^r$ by Theorem 1.6 and 1.9(3). If Γ is generated by an arithmetic progression, then $\mathrm{G}(\Gamma)$ is always Cohen–Macaulay and the explicit computation of the h-polynomial is done in [13]. If Γ is generated by an almost arithmetic progression, then a characterization for the Cohen–Macaulayness of $\mathrm{G}(\Gamma)$ is given (in most cases) in [12] and the explicit computation of the h-polynomial is done in [20].

The following general lemma will be used in the proof of Theorem 1.12 and many other propositions in Section 2.

Lemma 1.11. *Let Γ be as in 1.2, $r \in \mathbb{N}$ and let $g \in \Gamma(r)$. Then*

(1) *If* $g = \sum_{i=0}^{\nu-1} a_i m_i$ *with* $\sum_{i=0}^{\nu-1} a_i = r$, *then* $\sum_{i=0}^{\nu-1} a'_i m_i \in \Gamma(\sum_{i=0}^{\nu-1} a'_i)$ *for every* $(a'_0, \ldots, a'_{\nu-1}) \in \mathbb{N}^\nu$ *with* $a'_i \le a_i$, $i = 0, \ldots, \nu - 1$.
(2) *If* $g + m_0 = \sum_{i=0}^{\nu-1} b_i m_i$ *with* $\tilde{r} = \sum_{i=0}^{\nu-1} b_i = \text{max-deg}(g + m_0) \ge r + 2$, *then* $\sum_{i=0}^{\nu-1} b'_i m_i \in \mathrm{S}(r') \uplus \mathrm{C}(\Gamma, r')$ *for every* $(b'_0, \ldots, b'_{\nu-1}) \in \mathbb{N}^\nu$ *with* $b'_i \le b_i$, $i = 0, \ldots, \nu - 1$ *and* $r' = \sum_{i=0}^{\nu-1} b'_i$.

Proof. (1) Since $g = \sum_{i=0}^{\nu-1} a'_i m_i + \sum_{i=0}^{\nu-1} (a_i - a'_i) m_i$, the assertion is immediate from the inequality $r = \sum_{i=0}^{\nu-1} a_i = \text{max-deg}(g) \ge \text{max-deg}(\sum_{i=0}^{\nu-1} a'_i m_i) + \sum_{i=0}^{\nu-1} (a_i - a'_i)$.

(2) First note that $g \in \mathrm{B}(\Gamma, r)$. Put $g' := \sum_{i=0}^{\nu-1} b'_i m_i$. Then $g' \in \Gamma(r')$ with $r' := \sum_{i=0}^{\nu-1} b'_i \le \tilde{r}$ by (1). If $r' = \tilde{r}$, then $b'_i = b_i$ for every $0 \le i \le \nu - 1$, i.e. $g' = g + m_0$ and so $g' \in \mathrm{C}(\Gamma, r')$, since $g \in \mathrm{B}(\Gamma, r)$ with $r \le \tilde{r} - 2 = r' - 2$. If $r' \le \tilde{r} - 1$ and if $g' \notin \mathrm{S}(r')$, then $g' = m_0 + h$ with $h \in \Gamma$ and so $g = h + \sum_{i=0}^{\nu-1} (b_i - b'_i) m_i$ and $r = \text{max-deg}(g) \ge \text{max-deg}(h) + \tilde{r} - r'$. Therefore, since $\tilde{r} \ge r + 2$, we have $\text{max-deg}(h) \le r' + r - \tilde{r} \le r' - 2$. This proves that $\text{max-deg}(m_0 + h) = \text{max-deg}(g') = r' \ge \text{max-deg}(h) + 2$, i.e. $h \in \mathrm{B}(\Gamma, \text{max-deg}(h))$ and hence $g' = m_0 + h \in \mathrm{C}(\Gamma, r')$. □

Finally we give an application of Theorem 1.6 and Lemma 1.11.

Theorem 1.12. *Let Γ be a numerical semigroup generated by $m_0, m_1, \ldots, m_{\nu-1}$ with $0 < m_0 < m_1 < \cdots < m_{\nu-1}$ and let $\mathrm{S} = \{0 = s_0, s_1, \ldots, s_{m_0-1}\}$ with $0 = s_0 < s_1 < \cdots < s_{m_0-1}$ be the standard basis of Γ with respect to m_0. If the sequence $d_i := \text{max-deg}(s_i)$ $i = 0, \ldots, m_0 - 1$ of maximal degrees of elements of S is not decreasing, then $\mathrm{B}(\Gamma) = \emptyset$. In particular, the associated graded ring $\mathrm{G}(\Gamma)$ of $R = K[\![\Gamma]\!]$ with respect to the maximal ideal $\mathfrak{m}$ is Cohen–Macaulay.*

Proof. Suppose that $\mathrm{B}(\Gamma) \neq \emptyset$. Let $r := \min\{s \in \mathbb{N} \mid \mathrm{B}(\Gamma, s) \neq \emptyset\}$ and let $g \in \mathrm{B}(\Gamma, r)$. Then by definition $r \geq 1$, $\mathrm{C}(\Gamma, r') = \emptyset$ for every $r' \leq r+1$ and $k := \text{max-deg}(g + m_0) \geq r+2$, i.e. $g + m_0 \in \Gamma(k)$. Further, by the definition of the standard basis $g = s_i + am_0$ for some $1 \leq i \leq m_0 - 1$ and $a \in \mathbb{N}^+$. On the other hand choose a maximal expression for $g + m_0$, i.e.

$$g + m_0 = \sum_{j\in J} b_j m_j \quad \text{with } J \subseteq \{0, 1, \ldots, \nu - 1\}, b_j \in \mathbb{N}^+ \text{ for all } j \in J \text{ and } k = \text{max-deg}(g + m_0) = \sum_{j\in J} b_j.$$

Note that since $\text{max-deg}(g) = r \leq k - 2$, by the minimality of r we have $0 \notin J$. Furthermore, since $g = s_i + am_0$, we have $r = \text{max-deg}(g) \geq \text{max-deg}(s_i) + a = d_i + a$, i.e.

$$k \geq r + 2 \geq d_i + a + 2 \geq d_i + 2. \tag{1.12.1}$$

Now, for every J-tuple $(b'_j)_{j\in J}$ with $0 \leq b'_j \leq b_j$ for all $j \in J$ and $\sum_{j\in J} b'_j = d_i + 1$, by Lemma 1.11(2) (with $\tilde{r} = k$ and $r' = d_i + 1$) we have $g' := \sum_{j\in J} b'_j m_j \in \mathrm{S}(d_i + 1) \cup \mathrm{C}(d_i + 1)$. But, since $d_i \leq r$, we have $\mathrm{C}(\Gamma, d_i + 1) = \emptyset$ as noted above. Therefore $g' \in \mathrm{S}(d_i + 1)$ and hence $g' = s_\ell \in \mathrm{S}$ for exactly one $\ell \in \{1, \ldots, m_0 - 1\}$. Now, from $g + m_0 = s_i + (a+1)m_0 = s_\ell + \sum_{j\in J}(b_j - b'_j)m_j$ and $\sum_{j\in J}(b_j - b'_j) = k - (d_i + 1) \geq a + 1$ (by (1.12.1) and since $0 \notin J$), we get

$$s_i + (a+1)m_0 > s_\ell + \sum_{j\in J}(b_j - b'_j)m_0 \geq s_\ell + (a+1)m_0.$$

Therefore $s_\ell < s_i$. This contradicts the assumption, since $d_\ell = \text{max-deg}(s_\ell) = d_i + 1 > d_i$. □

Remark 1.13. Note that since the explicit description of the standard basis of the numerical semigroup Γ generated by an almost arithmetic progression in known (for example, see [17,18]), one can use the above theorem to obtain the proofs of the results proved in [13,12] much easily than their original proofs.

Examples 1.14. The converse of Theorem 1.12 is not true in general, even if R is Gorenstein or if $\mathrm{G}(\Gamma)$ is Gorenstein. For example:

(a) If Γ is generated by $7, 9, 17, 19$, then $R = K[\![\Gamma]\!]$ is Gorenstein, $\mathrm{G}(\Gamma)$ is Cohen–Macaulay (see [24]) and $\{0, 9, 17, 18, 19, 27, 36\}$ is the standard basis of Γ with respect to 7, but the sequence $d_0, d_1, \ldots, d_6$ is $0, 1, 1, 2, 1, 3, 4$ is not non-decreasing.
(b) If Γ is generated by $15, 18, 70$, then $R = K[\![\Gamma]\!]$ and $\mathrm{G}(\Gamma)$ are Gorenstein (see [19, 4.2 and 4.4]) and $\{0, 18, 36, 54, 70, 72, 88, 106, 124, 140, 142, 158, 176, 194, 212\}$ is the standard basis of Γ with respect to 15, but the sequence $d_0, d_1, \ldots, d_{15}$ is $0, 1, 2, 3, 1, 4, 2, 3, 4, 2, 5, 3, 4, 5, 6$ is not non-decreasing.

2. Balanced semigroups

In this section we consider a class of "balanced semigroups" and study their Cohen–Macaulay defect. Let us first recall:

Balanced semigroups 2.1 (*See also [7]*). Let $\Gamma \subseteq \mathbb{N}$ be a numerical semigroup $\sum_{i=0}^{3} \mathbb{N}m_i$ with $m_0 < m_1 < m_2 < m_3$, $\gcd(m_0, m_1, m_2, m_3) = 1$ and $m_i \nmid m_j$ for all $i \neq j$. We say that Γ is balanced if $m_1 + m_2 = m_0 + m_3$. In this section we assume that $R = K[\![\Gamma]\!]$ is the semigroup ring of a balanced semigroup Γ. We put $D := \gcd(m_0, m_3)$ and $E := \gcd(m_1, m_2)$. Then the following equations are easy to verify:

(a) $\gcd(D, E) = 1$, $m_0 = q_0 \cdot D$, $m_3 = q_3 \cdot D$ and $m_1 = q_1 \cdot E$, $m_2 = q_2 \cdot E$ with $1 < q_0 < q_3$, $\gcd(q_0, q_3) = 1$ and $1 < q_1 < q_2$, $\gcd(q_1, q_2) = 1$.
(b) $q_0 \cdot m_3 = q_3 \cdot m_0$ and $q_1 \cdot m_2 = q_2 \cdot m_1$.
(c) $q_1 + q_2 = u \cdot D$ and $q_0 + q_3 = u \cdot E$ for some $u \in \mathbb{N}^+$.
(d) $(q_2 + 1) \cdot m_1 = m_0 + (q_1 - 1) \cdot m_2 + m_3$ and $(q_1 + 1) \cdot m_2 = m_0 + (q_2 - 1) \cdot m_1 + m_3$.

Throughout this section let Γ denote a balanced semigroup. We use the notation introduced in 2.1 without any reference to it.

Lemma 2.2. *Let $r \in \mathbb{N}$, $g \in \mathrm{B}(\Gamma, r)$ and let $\lambda_0, \lambda_1, \lambda_2, \lambda_3 \in \mathbb{N}$ be such that $g + m_0 = \sum_{i=0}^{3} \lambda_i m_i$ and* $\text{max-deg}(g + m_0) = \lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 \geq r + 2$. *Then:*

(1) $\lambda_0 = 0$.
(2) $\lambda_1 = 0$ *or* $\lambda_2 = 0$.
(3) $\lambda_3 \leq r$. *Moreover, if $\lambda_3 \neq 0$, then exactly one of λ_1 or λ_2 is non-zero.*

Proof. If either $\lambda_0 > 0$, or both $\lambda_1 > 0$ and $\lambda_2 > 0$, then using $m_1 + m_2 = m_0 + m_3$ it follows that $r = \text{max-deg}(g) \geq r + 1$ which is absurd. This proves (1) and (2). Since $\text{max-deg}(g) = r$, $g \leq rm_3$, we have $g + m_0 < (r+1)m_3$ and so $\lambda_3 \leq r$. If both $\lambda_1 = \lambda_2 = 0$, then $r \geq \lambda_3 \geq r + 2$ which is absurd. This proves (3). □

For convenience, by using Lemma 2.2, we divide the set $\mathrm{B}(\Gamma, r)$ into the following 4 subsets and make the following assumptions:

Definitions 2.3. Let Γ be a balanced semigroup. For $r \in \mathbb{N}$, we define subsets

$$\mathrm{B}_{1,0}(\Gamma, r) := \left\{g \in \mathrm{B}(\Gamma, r) \mid g + m_0 = \lambda_1 m_1 \text{ with } \lambda_1 \in \mathbb{N}^+ \text{ and } \lambda_1 = \text{max-deg}(g + m_0)\right\},$$
$$\mathrm{B}_{1,3}(\Gamma, r) := \left\{g \in \mathrm{B}(\Gamma, r) \mid g + m_0 = \lambda_1 m_1 + \lambda_3 m_3 \text{ with } \lambda_1, \lambda_3 \in \mathbb{N}^+ \text{ and } \lambda_1 + \lambda_3 = \text{max-deg}(g + m_0)\right\},$$
$$\mathrm{B}_{2,0}(\Gamma, r) := \left\{g \in \mathrm{B}(\Gamma, r) \mid g + m_0 = \lambda_2 m_2 \text{ with } \lambda_2 \in \mathbb{N}^+ \text{ and } \lambda_2 = \text{max-deg}(g + m_0)\right\},$$
$$\mathrm{B}_{2,3}(\Gamma, r) := \left\{g \in \mathrm{B}(\Gamma, r) \mid g + m_0 = \lambda_2 m_2 + \lambda_3 m_3 \text{ with } \lambda_2, \lambda_3 \in \mathbb{N}^+ \text{ and } \lambda_2 + \lambda_3 = \text{max-deg}(g + m_0)\right\}.$$

First note that $\mathrm{B}_{i,0}(\Gamma, r) \cap \mathrm{B}_{i,3}(\Gamma, r) = \emptyset$ for both $i = 1, 2$ and $\mathrm{B}_{1,0}(\Gamma, r) \cap \mathrm{B}_{2,0}(\Gamma, r) = \emptyset$. We put $\mathrm{B}_1(\Gamma, r) := \mathrm{B}_{1,0}(\Gamma, r) \cup \mathrm{B}_{1,3}(\Gamma, r)$, $\mathrm{B}_2(\Gamma, r) := \mathrm{B}_{2,0}(\Gamma, r) \cup \mathrm{B}_{2,3}(\Gamma, r)$ and $\mathrm{B}_1(\Gamma) := \bigcup_{r\in\mathbb{N}} \mathrm{B}_1(\Gamma, r)$, $\mathrm{B}_2(\Gamma) := \bigcup_{r\in\mathbb{N}} \mathrm{B}_2(\Gamma, r)$.

With this notation by Lemma 2.2 we have: $\mathrm{B}(\Gamma, r) = \mathrm{B}_1(\Gamma, r) \cup \mathrm{B}_2(\Gamma, r)$ for every $r \in \mathbb{N}$. In particular, $\mathrm{B}(\Gamma) = \mathrm{B}_1(\Gamma) \cup \mathrm{B}_2(\Gamma)$.

For the proof of Theorem 2.11, in view of Proposition 1.9(3) we need to compare the cardinalities $|\mathrm{B}(\Gamma, r)|$ and $|\mathrm{S}(r + 1) \cup \mathrm{C}(\Gamma, r + 1)|$ for every $r \in \mathbb{N}$. For this we study different representations (in m_0, m_1, m_2, m_3 with coefficients in $\mathbb{N}$) of elements of the subsets $\mathrm{B}_{i,0}(\Gamma, r)$, $\mathrm{B}_{i,3}(\Gamma, r)$ for every $i = 1, 2$ and every $r \in \mathbb{N}$, and estimate the cardinalities of these subsets. This will be done in Propositions 2.4 and 2.7–2.9. With this view in mind it therefore useful to make the following definitions:

A balanced semigroup Γ is called 1-good (respectively, 2-good) if $\mathrm{B}_1(\Gamma) = \emptyset$ (respectively, $\mathrm{B}_2(\Gamma) = \emptyset$). We further say that Γ is good if it is both 1-good and 2-good, i.e. if the Cohen–Macaulay defect of Γ is empty, see Theorem 1.6. In this section and in the next section we give examples of good (see Proposition 3.3(1), (2) and also Examples 3.2(1), 3.2(2), 3.2(3) (with $p \le 2$) and 3.2(4) (with $p = -1$)) and not good (see Proposition 3.3(3) and also Examples 3.2(3) (with $p \ge 3$), 3.2(4) (with $p \le -2$), 3.2(5) and 2.12) balanced semigroups.

For a better arrangement of the proofs of Propositions 2.4 and 2.7, we further make the following definitions:

(1) If $\mathrm{B}_{1,3}(\Gamma) = \bigcup_{r\in\mathbb{N}} \mathrm{B}_{1,3}(\Gamma, r) \neq \emptyset$, then we put $r_1 := \min\{r \in \mathbb{N} \mid \mathrm{B}_{1,3}(\Gamma, r) \neq \emptyset\}$. An element $g \in \mathrm{B}_{1,3}(\Gamma, r_1)$ is called $(1, 3)$-good if there exist $a_0, a_2 \in \mathbb{N}$, $a_3 \in \mathbb{N}^+$ such that $g = a_0m_0 + a_2m_2 + a_3m_3$ and $\text{max-deg}(g) = a_0 + a_2 + a_3$. The set of $(1, 3)$-good elements of Γ is denoted by $\mathrm{B}'_{1,3}(\Gamma)$. Note that $\mathrm{B}'_{1,3}(\Gamma) \subseteq \mathrm{B}_{1,3}(\Gamma, r_1) \subseteq \mathrm{B}_1(\Gamma)$.

(2) If $\mathrm{B}_{2,3}(\Gamma) = \bigcup_{r\in\mathbb{N}} \mathrm{B}_{2,3}(\Gamma, r) \neq \emptyset$, then we put $r_2 := \min\{r \in \mathbb{N} \mid \mathrm{B}_{2,3}(\Gamma, r) \neq \emptyset\}$. An element $g \in \mathrm{B}_{2,3}(\Gamma, r_2)$ is called $(2, 3)$-good if there exist $a_0, a_1 \in \mathbb{N}$, $a_3 \in \mathbb{N}^+$ such that $g = a_0m_0 + a_1m_1 + a_3m_3$ and $\text{max-deg}(g) = a_0 + a_1 + a_3$. The set of $(2, 3)$-good elements of Γ is denoted by $\mathrm{B}'_{2,3}(\Gamma)$. Note that $\mathrm{B}'_{2,3}(\Gamma) \subseteq \mathrm{B}_{2,3}(\Gamma, r_2) \subseteq \mathrm{B}_2(\Gamma)$.

Proposition 2.4. *Let Γ be a balanced semigroup. Then*

(1) $\mathrm{B}'_{2,3}(\Gamma) = \mathrm{B}_{2,3}(\Gamma, r_2)$.
(2) *If $g \in \mathrm{B}_{2,3}(\Gamma, r_2)$, then $g - m_3 \in \mathrm{B}_{2,0}(\Gamma, r_2 - 1)$.*
(3) *Γ is 2-good if and only if $\mathrm{B}_{2,0}(\Gamma, r) = \emptyset$ for every $r \in \mathbb{N}$.*
(4) *Suppose that $\mathrm{B}_{2,0}(\Gamma) := \bigcup_{r\in\mathbb{N}} \mathrm{B}_{2,0}(\Gamma, r) \neq \emptyset$ and $s_2 := \min\{s \in \mathbb{N} \mid \mathrm{B}_{2,0}(\Gamma, s) \neq \emptyset\}$. Then:*
   (a) *If $\mathrm{B}_{2,3}(\Gamma) \neq \emptyset$, i.e. if r_2 is defined, then $r_2 \ge s_2 + 1$.*
   (b) *If $g = a_0m_0 + a_1m_1 + a_2m_2 + a_3m_3 \in \mathrm{B}_{2,0}(\Gamma, s_2)$, $a_0, a_1, a_2, a_3 \in \mathbb{N}$ and if $s_2 = \text{max-deg}(g) = a_0 + a_1 + a_2 + a_3$, then $a_2 = 0$, $3 \le a_3 < q_0$ and $g + m_0 = \lambda_2 m_2$ with $1 \le \lambda_2 \le q_1 - 1$. In particular, $(q_1 - 1)m_2 \notin \mathrm{S}$.*
(5) *If $(q_1 - 1)m_2 \in \mathrm{S}$, then $\mathrm{B}_2(\Gamma) = \emptyset$.*

Proof. (1) Let $g \in \mathrm{B}_{2,3}(\Gamma, r_2)$. Then by definition $g + m_0 = \lambda_2 m_2 + \lambda_3 m_3$ with $\lambda_2, \lambda_3 \in \mathbb{N}^+$ and $\lambda_2 + \lambda_3 = \text{max-deg}(g + m_0) \ge r_2 + 2$. Let $a_0, a_1, a_2, a_3 \in \mathbb{N}$ be such that $g = a_0m_0 + a_1m_1 + a_2m_2 + a_3m_3$ and $r_2 = \text{max-deg}(g) = a_0 + a_1 + a_2 + a_3$. First note that $a_2 = 0$, otherwise $g' = g - m_2 = a_0m_0 + a_1m_1 + (a_2 - 1)m_2 + a_3m_3$, $\text{max-deg}(g') = r_2 - 1$ by Lemma 1.11(1) and $g' + m_0 = (\lambda_2 - 1)m_2 + \lambda_3 m_3$ (note that $\lambda_2 \neq 1$ by 2.2(3)) and hence $g' \in \mathrm{B}_{2,3}(\Gamma, r_2 - 1) = \emptyset$ which is absurd. Next, $a_3 > 0$, otherwise $(r_2 + 2)m_2 \le (\lambda_2 + \lambda_3)m_2 \le \lambda_2 m_2 + \lambda_3 m_3 = g + m_0 = (a_0 + 1)m_0 + a_1m_1 < m_0 + (a_0 + a_1)m_1 = m_0 + r_2m_1 < (r_2 + 1)m_2$ which is absurd. Therefore $g \in \mathrm{B}'_{2,3}(\Gamma)$ by Definition 2.3(2). This proves the inclusion $\mathrm{B}_{2,3}(\Gamma, r_2) \subseteq \mathrm{B}'_{2,3}(\Gamma)$ and hence the equality.

(2) Suppose that $g \in \mathrm{B}_{2,3}(\Gamma, r_2)$. Then $g \in \mathrm{B}'_{2,3}(\Gamma)$ by (1) and so by Definition 2.3(2) there exist $a_0, a_1 \in \mathbb{N}$, $a_3 \in \mathbb{N}^+$ such that $g = a_0m_0 + a_1m_1 + a_3m_3$, $r_2 = \text{max-deg}(g) = a_0 + a_1 + a_3$ and $g + m_0 = \lambda_2 m_2 + \lambda_3 m_3$ with $\lambda_2, \lambda_3 \in \mathbb{N}^+$, $\lambda_2 + \lambda_3 \ge r_2 + 2$. But then

$$g - m_3 + m_0 = (a_0 + 1)m_0 + a_1m_1 + (a_3 - 1)m_3 = \begin{cases}\lambda_2 m_2, & \text{if } \lambda_3 = 1,\\ \lambda_2 m_2 + (\lambda_3 - 1)m_3, & \text{if } \lambda_3 \ge 2,\end{cases}$$

and $\text{max-deg}(g - m_3 + m_0) = \lambda_2 + \lambda_3 - 1 \ge (r_2 - 1) + 2$ by Lemma 1.11(1). Therefore

$$g - m_3 \in \begin{cases}\mathrm{B}_{2,0}(\Gamma, r_2 - 1), & \text{if } \lambda_3 = 1,\\ \mathrm{B}_{2,3}(\Gamma, r_2 - 1), & \text{if } \lambda_3 \ge 2.\end{cases}$$

Therefore $\lambda_3 = 1$, since $\mathrm{B}_{2,3}(\Gamma, r_2 - 1) = \emptyset$ by the definition of r_2. This proves that $g - m_3 \in \mathrm{B}_{2,0}(\Gamma, r_2 - 1)$.

(3) Since $\mathrm{B}_2(\Gamma, r) = \mathrm{B}_{2,0}(\Gamma, r) \cup \mathrm{B}_{2,3}(\Gamma, r)$ for every $r \in \mathbb{N}$, the assertion follows from (2) and the definition of 2-good.

(4) Part (a) is immediate from (2). To prove part (b), let $g = a_0m_0 + a_1m_1 + a_2m_2 + a_3m_3 \in \mathrm{B}_{2,0}(\Gamma, s_2)$, $a_0, a_1, a_2, a_3 \in \mathbb{N}$ with $s_2 = \text{max-deg}(g) = a_0 + a_1 + a_2 + a_3$. Then $g + m_0 = \lambda_2 m_2$ with $\lambda_2 \in \mathbb{N}^+$, $\lambda_2 = \text{max-deg}(g + m_0) \geq s_2 + 2$. Clearly $a_2 = 0$ by the definition of s_2 and hence $s_2 = a_0 + a_1 + a_3$. Further, $a_3 \geq 3$, since otherwise (using $m_1 + m_2 = m_0 + m_3$ it is easy to verify the inequalities)

$$g + m_0 = \lambda_2 m_2 \geq (s_2 + 2)m_2 = (a_0 + a_1)m_2 + (a_3 + 2)m_2 > \begin{cases} g + 2m_0 & \text{if } a_3 = 0, \\ g + m_2 & \text{if } a_3 = 1, \\ g + 2m_0 & \text{if } a_3 = 2. \end{cases}$$

In each case this is absurd. Furthermore, we must have $a_3 < q_0$, otherwise we will get $g = a_0m_0 + a_1m_1 + q_0m_3 + (a_3 - q_0)m_3 = (a_0 + q_3)m_0 + a_1m_1 + (a_3 - q_0)m_3$, since $q_0m_3 = q_3m_0$ by 2.1(b) and so $s_2 = \text{max-deg}(g) \geq a_0 + a_1 + a_3 + q_3 - q_0 \geq a_0 + a_1 + a_3 + 1 = s_2 + 1$ which is absurd. Finally, if $\lambda_2 \geq q_1$, then by 2.1(d) and (b), we have

$$g + m_0 = \lambda_2 m_2 = \begin{cases} (\lambda_2 - q_1 - 1)m_2 + m_0 + (q_2 - 1)m_1 + m_3, & \text{if } \lambda_2 > q_1, \\ q_2 m_1, & \text{if } \lambda_2 = q_1, \end{cases}$$

and hence (since $q_2 \geq q_1 + 1$)

$$\begin{cases} g = (\lambda_2 - q_1 - 1)m_2 + (q_2 - 1)m_1 + m_3 \text{ and } \lambda_2 - 2 \geq \text{max-deg}(g) \geq \lambda_2, & \text{if } \lambda_2 > q_1, \\ \lambda_2 = \text{max-deg}(g + m_0) \geq q_2 \geq q_1 + 1 = \lambda_2 + 1, & \text{if } \lambda_2 = q_1, \end{cases}$$

which is impossible in both cases. Therefore $\lambda_2 \leq q_1 - 1$.

(5) If $\mathrm{B}_2(\Gamma) \neq \emptyset$, then by part (3) above $\mathrm{B}_{2,0}(\Gamma) \neq \emptyset$ and so $(q_1 - 1)m_2 \notin \mathrm{S}$ by part (4). □

Example 2.5. Let $\Gamma = \sum_{i=0}^{3} \mathbb{N}m_i$ be a balanced numerical semigroup. Then in general it is not true that $(q_1 - 1)m_2 \in \mathrm{S}$. For example, if Γ is generated by 12, 35, 55 and 78, then $D := \gcd(12, 78) = 6$, $E := \gcd(35, 55) = 5$ and hence Γ is balanced with $q_0 = 2$, $q_1 = 7$, $q_2 = 11$, $q_3 = 13$ and $(q_1 - 1)m_2 = 6m_2 = 330 = 10m_0 + 6m_1 \notin \mathrm{S}$. However, in this case we have $\mathrm{B}_{2,0} = \emptyset$ trivially, since the set $\{a_3 \mid 3 \leq a_3 < q_0\} = \emptyset$ (see 2.4(4)(b)) and hence $\mathrm{B}_2(\Gamma) = \emptyset$ by 2.4(3).

Remark 2.6. We have no examples of balanced semigroups Γ with $\mathrm{B}_2(\Gamma) \neq \emptyset$. It is likely that $\mathrm{B}_2(\Gamma) = \emptyset$ for every balanced semigroup Γ but we are not able to prove this in general.

The following analogous proposition to that of Proposition 2.4 will be used in the proof of Proposition 3.3.

Proposition 2.7. *Let Γ be a balanced semigroup.*

(1) *If $g \in \mathrm{B}'_{1,3}(\Gamma)$, then $g' = g - m_3 \in \mathrm{B}_{1,0}(\Gamma, r_1 - 1)$.*
(2) *Suppose that $\mathrm{B}'_{1,3}(\Gamma) = \mathrm{B}_{1,3}(\Gamma, r_1)$. Then Γ is 1-good if and only if $\mathrm{B}_{1,0}(\Gamma, r) = \emptyset$ for every $r \in \mathbb{N}$.*

Proof. (1) Since $g \in \mathrm{B}'_{1,3}(\Gamma) \subseteq \mathrm{B}_{1,3}(\Gamma, r_1)$, by Definition 2.3(1) there exists $a_0, a_2 \in \mathbb{N}$, $a_3 \in \mathbb{N}^+$ such that $g = a_0m_0 + a_2m_2 + a_3m_3$, $r_1 = \text{max-deg}(g) = a_0 + a_2 + a_3$, $g + m_0 = \lambda_1 m_1 + \lambda_3 m_3$ with $\lambda_1, \lambda_3 \in \mathbb{N}^+$ and $\text{max-deg}(g + m_0) = \lambda_1 + \lambda_3 \geq r_1 + 2$. Then

$$g - m_3 + m_0 = (a_0 + 1)m_0 + a_2m_2 + (a_3 - 1)m_3 = \begin{cases} \lambda_1 m_1, & \text{if } \lambda_3 = 1, \\ \lambda_1 m_1 + (\lambda_3 - 1)m_3, & \text{if } \lambda_3 \geq 2, \end{cases}$$

and $\text{max-deg}(g - m_3 + m_0) = \lambda_1 + \lambda_3 - 1 \geq (r_1 - 1) + 2$ by Lemma 1.11(1). Therefore

$$g - m_3 \in \begin{cases} \mathrm{B}_{1,0}(\Gamma, r_1 - 1), & \text{if } \lambda_3 = 1, \\ \mathrm{B}_{1,3}(\Gamma, r_1 - 1), & \text{if } \lambda_3 \geq 2. \end{cases}$$

This shows that $\lambda_3 = 1$, since $\mathrm{B}_{1,3}(\Gamma, r_1 - 1) = \emptyset$. Then $g - m_3 \in \mathrm{B}_{1,0}(\Gamma, r_1 - 1)$.

(2) Suppose that $\mathrm{B}_{1,0}(\Gamma, r) = \emptyset$ for every $r \in \mathbb{N}$. If $g \in \mathrm{B}_{1,3}(\Gamma, r_1)$, then $g \in \mathrm{B}'_{1,3}(\Gamma)$ by assumption and hence $g - m_3 \in \mathrm{B}_{1,0}(\Gamma, r_1 - 1)$ by (1). This is absurd, since $\mathrm{B}_{1,0}(\Gamma, r_1 - 1) = \emptyset$. This proves that $\mathrm{B}_{1,3}(\Gamma, r) = \emptyset$ for every $r \in \mathbb{N}$ and hence $\mathrm{B}_1(\Gamma) = \emptyset$, i.e. Γ is 1-good. The converse is clear from definitions. □

Finally, for the proof of Theorem 2.11 we need the following two propositions and Corollary 2.10.

Proposition 2.8. *Let Γ be a balanced semigroup, $i \in \{1, 2\}$ and let $r \in \mathbb{N}$. Then:*

(1) *$|\mathrm{B}_{i,0}(\Gamma, r)| \leq 1$. Moreover, if $\mathrm{B}_{i,0}(\Gamma, r) \neq \emptyset$, then $(r + 1)m_i \in \mathrm{S}(r + 1) \cup \mathrm{C}(\Gamma, r + 1)$.*
(2) *For every $g \in \mathrm{B}_{i,3}(\Gamma, r)$ with $g + m_0 = \lambda_i m_i + \lambda_3 m_3$ and $\lambda_i + \lambda_3 \geq r + 2$, we have $(r + 1 - \lambda_3)m_i + \lambda_3 m_3 \in \mathrm{S}(r + 1) \cup \mathrm{C}(\Gamma, r + 1)$.*

Proof. First note that the sets $\mathrm{S}(r + 1)$ and $\mathrm{C}(\Gamma, r + 1)$ are disjoint by their definitions and their (disjoint) union is denoted by $\mathrm{S}(r + 1) \uplus \mathrm{C}(\Gamma, r + 1)$.

(1) Suppose that $g, h \in \mathrm{B}_{i,0}(\Gamma, r)$ with $g \geq h$, then by definition $g + m_0 = \lambda_i m_i$ and $h + m_0 = \mu_i m_i$ with $\lambda_i \geq \mu_i \geq r + 2$ and so $g = h + (\lambda_i - \mu_i)m_i$. But then $r = \text{max-deg}(g) \geq \text{max-deg}(h) + (\lambda_i - \mu_i) = r + (\lambda_i - \mu_i)$ and hence $\lambda_i = \mu_i$ and $g = h$. This proves the first part. For the last part, let $g \in \mathrm{B}_{i,0}(\Gamma, r)$. Then by definition $g + m_0 = \lambda_i m_i$ with $\lambda_i \geq r + 2$ and so $(r + 1)m_i \in \mathrm{S}(r + 1) \uplus \mathrm{C}(\Gamma, r + 1)$ by Lemma 1.11(2) (applied to $\nu = 4, b_0 = 0 = b_3, b_i = \lambda_i = \tilde{r}, b'_0 = b'_3 = 0, b'_i = r + 1 = r'$).

(2) First note that $\lambda_3 \leq r$ by Lemma 2.2(3). The element $(r + 1 - \lambda_3)m_i + \lambda_3 m_3 \in \mathrm{S}(r + 1) \uplus \mathrm{C}(\Gamma, r + 1)$ by Lemma 1.11(2) (applied to $\nu = 4, b_0 = 0 = b_2, b_i = \lambda_i, b_3 = \lambda_3, \tilde{r} = \lambda_i + \lambda_3, b'_0 = b'_2 = 0, b'_i = r + 1 - \lambda_3 (\geq 1), b'_3 = \lambda_3$ and $r + 1 = r'$). □

Proposition 2.9. *Let Γ be a balanced semigroup and let $r \in \mathbb{N}$. Suppose that $\mathrm{B}_{1,3}(\Gamma, r) = \{g_1, \ldots, g_k\}$ with $g_i + m_0 = \lambda_{1i}m_1 + \lambda_{3i}m_3$ and $\lambda_{1i} + \lambda_{3i} \geq r + 2$ for every $i = 1, \ldots, k$; $\mathrm{B}_{2,3}(\Gamma, r) = \{g'_1, \ldots, g'_{k'}\}$ with $g'_j + m_0 = \lambda'_{2j}m_2 + \lambda'_{3j}m_3$ and $\lambda'_{2j} + \lambda'_{3j} \geq r + 2$ for every $j = 1, \ldots, k'$. Further, suppose that $g_0 \in \mathrm{B}_{1,0}(\Gamma, r)$ with $g_0 + m_0 = \lambda m_1$ and $\lambda \geq r + 2$; $g'_0 \in \mathrm{B}_{2,0}(\Gamma, r)$ with $g'_0 + m_0 = \lambda' m_2$ and $\lambda' \geq r + 2$. Then:*

(1) *$(r+1-\lambda_{3i})m_1+\lambda_{3i}m_3 \neq (r+1-\lambda_{3\ell})m_1+\lambda_{3\ell}m_3$ for all $1 \leq i, \ell \leq k, i \neq \ell$. Further, $(r+1)m_1 \neq (r+1-\lambda_{3i})m_1+\lambda_{3i}m_3$ for every $i = 1, \ldots, k$.*

(2) *$(r+1-\lambda'_{3j})m_2+\lambda'_{3j}m_3 \neq (r+1-\lambda'_{3\ell})m_2+\lambda'_{3\ell}m_3$ for all $1 \leq j, \ell \leq k', j \neq \ell$. Further, $(r+1)m_2 \neq (r+1-\lambda'_{3j})m_2+\lambda'_{3j}m_3$ for every $j = 1, \ldots, k'$.*

(3) *$(r + 1 - \lambda_{3i})m_1 + \lambda_{3i}m_3 \neq (r + 1 - \lambda'_{3j})m_2 + \lambda'_{3j}m_3$ for all $1 \leq i \leq k$ and for all $1 \leq j \leq k'$.*

(4) *$(r + 1)m_1 \neq (r + 1 - \lambda'_{3j})m_2 + \lambda'_{3j}m_3$ for every $j = 1, \ldots, k'$.*

(5) *$(r + 1)m_2 \neq (r + 1 - \lambda_{3i})m_1 + \lambda_{3i}m_3$ for every $i = 1, \ldots, k$.*

Proof. Note that by Lemma 2.2(3) we have $r \geq \lambda_{3i}$ for every $i = 1, \ldots, k$ and $r \geq \lambda'_{3j}$ for every $j = 1, \ldots, k'$.

(1) For $1 \leq i, \ell \leq k$, we have $(r+1-\lambda_{3i})m_1+\lambda_{3i}m_3 = (r+1-\lambda_{3\ell})m_1+\lambda_{3\ell}m_3 \iff (\lambda_{3\ell}-\lambda_{3i})m_3 = (\lambda_{3\ell}-\lambda_{3i})m_1 \iff \lambda_{3i} = \lambda_{3\ell}$ (and hence $\lambda_{1i} = \lambda_{1\ell}$. If $\lambda_{1i} \neq \lambda_{1\ell}$, then without loss of generality we may assume that $\lambda_{1i} > \lambda_{1\ell}$. Now, subtracting the equation $g_\ell + m_0 = \lambda_{1\ell}\, m_1 + \lambda_{3\ell}\, m_3$ from the equation $g_i + m_0 = \lambda_{1i}\, m_1 + \lambda_{3i}\, m_3$ and using the equality $\lambda_{3i} = \lambda_{3\ell}$, it follows that $g_i = g_\ell + (\lambda_{1i} - \lambda_{1\ell})m_1$. In particular, $r = \text{max-deg}(g_i) \geq \text{max-deg}(g_\ell) + (\lambda_{1i} - \lambda_{1\ell}) > \text{max-deg}(g_\ell) = r$ which is absurd.) $\iff g_i = g_\ell \iff i = \ell$. The last assertion is immediate, since $\lambda_{3i}m_1 \neq \lambda_{3i}m_3$ for every $i = 1, \ldots, k$.

(2) For $1 \leq j, \ell \leq k'$, we have $(r+1-\lambda'_{3j})m_2+\lambda'_{3j}m_3 = (r+1-\lambda'_{3\ell})m_2+\lambda'_{3\ell}m_3 \iff (\lambda'_{3\ell}-\lambda'_{3j})m_3 = (\lambda'_{3\ell}-\lambda'_{3j})m_2 \iff \lambda'_{3j} = \lambda'_{3\ell}$ (and hence $\lambda'_{2j} = \lambda'_{2\ell}$. If $\lambda'_{2j} \neq \lambda'_{2\ell}$, then without loss of generality we may assume that $\lambda'_{2j} > \lambda'_{2\ell}$. Now, subtracting the equation $g'_\ell + m_0 = \lambda'_{2\ell}\, m_2 + \lambda'_{3\ell}\, m_3$ from the equation $g'_j + m_0 = \lambda'_{2j}\, m_2 + \lambda'_{3j}\, m_3$ and using the equality $\lambda'_{3j} = \lambda'_{3\ell}$, it follows that $g'_j = g'_\ell + (\lambda'_{2j} - \lambda'_{2\ell})m_2$. In particular, $r = \text{max-deg}(g'_j) \geq \text{max-deg}(g'_\ell) + (\lambda'_{2j} - \lambda'_{2\ell}) > \text{max-deg}(g'_\ell) = r$ which is absurd.) $\iff g'_j = g'_\ell \iff j = \ell$. The last assertion is immediate, since $\lambda'_{3j}m_2 \neq \lambda'_{3j}m_3$ for every $j = 1, \ldots, k'$.

(3) Let $i \in \{1, \ldots, k\}$ and let $j \in \{1, \ldots, k'\}$. Therefore, if $(r + 1 - \lambda_{3i})m_1 + \lambda_{3i}m_3 = (r + 1 - \lambda'_{3j})m_2 + \lambda'_{3j}m_3$, then (since $m_1 + m_2 = m_0 + m_3$), we get

$$\begin{aligned} g_i + m_0 = \lambda_{1i}m_1 + \lambda_{3i}m_3 &= (r + 1 - \lambda_{3i})m_1 + (\lambda_{1i} + \lambda_{3i} - r - 1)m_1 + \lambda_{3i}m_3 \\ &= (r + 1 - \lambda'_{3j})m_2 + \lambda'_{3j}m_3 + (\lambda_{1i} + \lambda_{3i} - r - 1)m_1 \\ &= (r - \lambda'_{3j})m_2 + m_2 + \lambda'_{3j}m_3 + (\lambda_{1i} + \lambda_{3i} - r - 2)m_1 + m_1 \\ &= (r - \lambda'_{3j})m_2 + \lambda'_{3j}m_3 + (\lambda_{1i} + \lambda_{3i} - r - 2)m_1 + m_0 + m_3. \end{aligned}$$

This shows that $g_i = (r - \lambda'_{3j})m_2 + \lambda'_{3j}m_3 + (\lambda_{1i} + \lambda_{3i} - r - 2)m_1 + m_3$, in particular, $r = \text{max-deg}(g_i) \geq (r - \lambda'_{3j}) + \lambda'_{3j} + (\lambda_{1i} + \lambda_{3i} - r - 2) + 1 = (\lambda_{1i} + \lambda_{3i} - 1) \geq r + 2 - 1 = r + 1$ which is absurd.

(4) For $j \in \{1, 2, \ldots, k'\}$, we have $(r + 1 - \lambda'_{3j})m_2 + \lambda'_{3j}m_3 > (r + 1 - \lambda'_{3j})m_2 + \lambda'_{3j}m_2 = (r + 1)m_2 > (r + 1)m_1$ (since $\lambda'_{3j} \in \mathbb{N}^+$ by Definition 2.3).

(5) Let $i \in \{1, \ldots, k\}$. Therefore, if $(r + 1)m_2 = (r + 1 - \lambda_{3i})m_1 + \lambda_{3i}m_3$, then (since $m_1 + m_2 = m_0 + m_3$), we get

$$\begin{aligned} g'_0 + m_0 = \lambda' m_2 &= (r + 1)m_2 + (\lambda' - r - 1)m_2 \\ &= (r + 1 - \lambda_{3i})m_1 + \lambda_{3i}m_3 + (\lambda' - r - 1)m_2 \\ &= (r - \lambda_{3i})m_1 + m_1 + \lambda_{3i}m_3 + (\lambda' - r - 2)m_2 + m_2 \\ &= (r - \lambda_{3i})m_1 + \lambda_{3i}m_3 + (\lambda' - r - 2)m_1 + m_0 + m_3. \end{aligned}$$

This shows that $g'_0 = (r - \lambda_{3i})m_1 + (\lambda_{3i} + 1)m_3 + (\lambda' - r - 2)m_2$, in particular, $r = \text{max-deg}(g'_0) \geq (r - \lambda_{3i}) + (\lambda_{3i} + 1) + (\lambda' - r - 2) = \lambda' - 1 \geq r + 2 - 1 = r + 1$ which is absurd. □

Corollary 2.10. *Let Γ be a balanced semigroup and let $r \in \mathbb{N}$. Then:*

$$|\mathrm{B}(\Gamma, r)| \leq |\mathrm{S}(r + 1)| + |\mathrm{C}(\Gamma, r + 1)|.$$

Proof. Use Proposition 2.8 to define maps $\varphi_{i,0} : \mathrm{B}_{i,0}(\Gamma, r) \to \mathrm{S}(r+1) \cup \mathrm{C}(\Gamma, r+1)$ and the maps $\varphi_{i,3} : \mathrm{B}_{i,3}(\Gamma, r) \to \mathrm{S}(r+1) \cup \mathrm{C}(\Gamma, r+1)$, for $i = 1, 2$. Moreover, by Proposition 2.9 these maps are injective and their images $\mathrm{Im}(\varphi_{i,0})$ and $\mathrm{Im}(\varphi_{i,3})$, $i = 1, 2$ are pairwise disjoint subsets in $\mathrm{S}(r+1) \cup \mathrm{C}(\Gamma, r+1)$ and hence $|\mathrm{B}(\Gamma, r)| = |\bigcup_{i=1}^{2} \mathrm{B}_{i,0}(\Gamma, r) \cup \bigcup_{i=1}^{2} \mathrm{B}_{i,3}(\Gamma, r)| \le \sum_{i=1}^{2} |\mathrm{B}_{i,0}(\Gamma, r)| + \sum_{i=1}^{2} |\mathrm{B}_{i,3}(\Gamma, r)| = \sum_{i=1}^{2} |\mathrm{Im}(\varphi_{i,0})| + \sum_{i=1}^{2} |\mathrm{Im}(\varphi_{i,3})| \le |\mathrm{S}(r+1) \cup \mathrm{C}(\Gamma, r+1)| \le |\mathrm{S}(r+1)| + |\mathrm{C}(\Gamma, r+1)|$. □

Theorem 2.11. *Let Γ be a balanced semigroup and let $R = K[\![\Gamma]\!]$ be a semigroup ring over a field K. Then the Hilbert function $\mathrm{H}_R : \mathbb{N} \to \mathbb{N}$ of R is non-decreasing.*

Proof. By Corollary 2.10 and the formula in 1.9(3), we have $\mathrm{h}_r \ge 0$ for every $r \in \mathbb{N}$. □

Example 2.12 (*Balanced Unitary Semigroups—See [7]*). Let $\Gamma = \sum_{i=0}^{3} \mathbb{N}m_i$ with $m_0 < m_1 < m_2 < m_3$ with $\gcd(m_0, m_1, m_2, m_3) = 1$ and $m_i \nmid m_j$ for all $i \ne j$ be a balanced semigroup. Let $D = \gcd(m_0, m_3)$ and $E = \gcd(m_1, m_2)$ be as in 2.1. We now use the notation as in 2.1 and the equations (a), (b), (c) and (d) in 2.1, first note that by 2.1(c) there exists $u \in \mathbb{N}^+$ such that $q_1+q_2 = u \cdot D$ and $q_0+q_3 = u \cdot E$. The semigroup Γ is called unitary if the common sum $m_0+m_3 = m_1+m_2$ is the product of $\gcd(m_0, m_3) \cdot \gcd(m_1, m_2)$ (see [7, Definition (1.3)]), or equivalently, $u = 1$. For example, the numerical semigroup generated by $m_0 = 14, m_1 = 15, m_2 = 20, m_3 = 21$ is balanced and unitary.

Now, let $\Gamma = \sum_{i=0}^{3} \mathbb{N}m_i$ be a balanced unitary semigroup. Then:

(1) Standard basis of Γ is: $\mathrm{S} = \mathrm{S}(1, 3) \uplus \mathrm{S}(2, 3) \uplus \mathrm{S}(0, 3)$, where

$$\mathrm{S}(1,3) := \{(a_1 \cdot m_1 + a_3 \cdot m_3 \mid (a_1, a_3) \in [1, q_2] \times [0, q_0 - 1]\},$$
$$\mathrm{S}(2,3) := \{b_2 \cdot m_2 + b_3 \cdot m_3 \mid (b_2, b_3) \in [1, q_1 - 1] \times [0, q_0 - 1]\} \quad \text{and}$$
$$\mathrm{S}(0,3) := \{c \cdot m_3 \mid c \in [0, q_0 - 1]\}.$$

Proof (*See also [7, Proposition (1.9)]*). From the definition of S and the right-hand sides of the equations in 2.1(b) and 2.1(d) it follows that $\mathrm{S} \subseteq \mathrm{S}(1, 3) \uplus \mathrm{S}(2, 3) \uplus \mathrm{S}(0, 3)$. Further, since $|\mathrm{S}| = m_0 = q_0 \cdot q_2 + q_0 \cdot (q_1 - 1) + q_0 = |\mathrm{S}(1, 3)| + |\mathrm{S}(2, 3)| + |\mathrm{S}(0, 3)|$ by equations 2.1(a), 2.1(c), the equality holds. □

(2) It is proved in [7, 2.6] that Γ is symmetric (this can also be checked easily from the description of the standard basis of Γ given in part (1)).

(3) For $k = 1, \dots, q_0 - 1$, let $g_k := (q_1 - 1) \cdot m_2 + k \cdot m_3$. Then $\text{max-deg}(g_k) = q_1 + k - 1$ for every $k = 1, \dots, q_0 - 1$.

Proof. Since $g_k \in \mathrm{S}$ for every $k \in [1, q_0 - 1]$, the assertion easily follows from part (1) and the equations in 2.1. □

(4) $g_1 \in \mathrm{B}_{1,0}(q_1)$ and $g_k \in \mathrm{B}_{1,3}(q_1 + k - 1)$ for every $k \in [2, q_0 - 1]$. In particular, Γ is not 1-good by Definition 2.3 and so Γ is not good.

Proof. $g_k + m_0 = (q_1 - 1) \cdot m_2 + k \cdot m_3 + m_0 = (q_2 + 1) \cdot m_1 + (k - 1) \cdot m_3$ and so $\text{max-deg}(g_k + m_0) \ge q_2 + k \ge q_1 + k + 1 = \text{max-deg}(g_k) + 2$ by 2.1(d) and part (3). Therefore the assertion is immediate from the definitions. □

(5) The semigroup $\Gamma = \mathbb{N}15 + \mathbb{N}22 + \mathbb{N}33 + \mathbb{N}40$ generated by 15, 22, 33, 40 is balanced with $D = \gcd(m_0, m_3) = \gcd(15, 40) = 5$, $E = \gcd(m_1, m_2) = \gcd(22, 33) = 11$, $q_0 = 3$, $q_1 = 2$, $q_2 = 3$, $q_3 = 8$ and $u = 1$. Therefore it is unitary and by part (4) $\mathrm{B}_{1,3}(\Gamma) \ne \emptyset$, since $q_0 > 2$.

(6) Since $(q - 1)m_2 \in \mathrm{S}$ by part (1), Γ is 2-good by Proposition 2.4(5).

(7) The associated graded ring $\mathrm{G}(\Gamma)$ is not Cohen–Macaulay.

Proof. Immediate from 1.6 and (4). □

(8) Let $\mathfrak{P}$ and $\mathrm{A} := K[X_0, X_1, X_2, X_3]/\mathfrak{P}$ be the defining ideal and the coordinate ring of the monomial curve in K^4 defined by $X_0 = t^{m_0}, X_1 = t^{m_1}, X_2 = t^{m_2}, X_3 = t^{m_3}$. Then $\mathfrak{P}$ is generated by the three binomials $F := X_0X_3 - X_1X_2$, $F_0 := X_0^{q_3} - X_3^{q_0}$ and $F_1 := X_1^{q_2} - X_2^{q_1}$. In particular, $\mathfrak{P}$ is a complete intersection and hence A is Gorenstein.

3. A class of balanced semigroups

In this section we study a certain special class of balanced numerical semigroups and study the structure of their Cohen–Macaulay defects. First let us fix the following notation.

Notation 3.1. Let p, q be fixed integers, n be a variable positive integer with $n \gg p$ (e.g. $n \ge 4|p| + 1$). Let $m := m_0 = n^2 + pn + q$, $m_1 = m_0 + n$, $m_2 = m_0 + 2n + 1$, $m_3 = m_0 + 3n + 1$ and let $\Gamma(n; p, q) = \sum_{i=0}^{3} \mathbb{N}m_i$. Then $\Gamma(n; p, q)$ is a balanced semigroup. Further, let $\mathrm{R}(n; p, q) := K[\![\Gamma(n; p, q)]\!]$, where K is an arbitrary field and let $\mathfrak{P}(n; p, q)$ be the defining ideal of the (algebroid) monomial curve in K^4 with parametrization $X_0 = t^{m_0}, X_1 = t^{m_1}, X_2 = t^{m_2}, X_3 = t^{m_3}$. Finally, let $\mathrm{G}(n; p, q)$ denote the associated graded ring of the local ring $\mathrm{R}(n; p, q)$.

For $(p, q) \in (\mathbb{Z}^- \times \{0\}) \cup (\mathbb{Z}^+ \times \{0\}) \cup \Delta_{\mathbb{Z}}$, where $\Delta_{\mathbb{Z}} := \{(m, m) \mid m \in \mathbb{Z}\}$ in $\mathbb{Z} \times \mathbb{Z}$, we use the results of Sections 1 and 2 to give a characterization for the Cohen–Macaulayness of $\mathrm{G}(n; p, q)$ (see Theorem 3.4).

The examples $(p, q) = (-2, 0)$ and $(p, q) = (-1, -1)\}$ were considered by Kraft in [8,9] to prove that the minimal number $\mu(\mathrm{Der}_K(\mathrm{R}(n; p, q)))$ of generators of the derivation modules $\mathrm{Der}_K(\mathrm{R}(n; p, q))$ can be arbitrarily large. In [2] Bresinsky considered a similar class of examples to prove that the minimal number $\mu(\mathfrak{P}(n; p, q))$ of $\mathfrak{P}(n; p, q)$ can be arbitrary large.

Now our aim is to study the structure of the Cohen–Macaulay defect using the explicit description of the standard bases of the numerical semigroups $\Gamma(n; p, q)$ which is given in Proposition 3.2. For this study we use similar arguments as in [17, Chapter 3, Section 4].

Proposition 3.2. *With the assumptions and notation as in 3.1 we have:*

(1) *Suppose that* $(p, q) = (0, 0)$.
  (i) *Equations for* $\Gamma(n; 0, 0)$*: The following equations are easy to verify.*
    (a) $m_1 + m_2 = m_0 + m_3$.
    (b) $i \cdot m_1 + (n - i) \cdot m_3 = (i + 1) \cdot m_0 + (n - i) \cdot m_2$ *for all* $i = 0, \ldots, n$.
    (c) $j \cdot m_2 + (n - j) \cdot m_3 = (j + 1) \cdot m_0 + (n - j + 1) \cdot m_1$ *for all* $j = 0, \ldots, n$.
  (ii) *The standard basis of* $\Gamma(n; 0, 0)$ *is:*
    $\mathrm{S} = \mathrm{S}(1, 3) \uplus \mathrm{S}(2, 3) \uplus \mathrm{S}(0, 3)$, *where*
    $\mathrm{S}(1, 3) := \{a_1 \cdot m_1 + a_3 \cdot m_3 \mid a_1 \in [1, n - 1], a_3 \in [0, n - 2], a_1 + a_3 \le n - 1\}$,
    $\mathrm{S}(2, 3) := \{b_2 \cdot m_2 + b_3 \cdot m_3 \mid b_2 \in [1, n-1], b_3 \in [0, n-2], b_2 + b_3 \le n-1\}$ *and* $\mathrm{S}(0, 3) := \{c \cdot m_3 \mid c \in [0, n-1]\}$.

(2) *Suppose that* $p < 0$ *and* $q = 0$ *and* $n \ge -2p$.
  (i) *Equations for* $\Gamma(n; p < 0, 0)$*: The following equations are easy to verify.*
    (a) $m_1 + m_2 = m_0 + m_3$.
    (b) $i \cdot m_1 + (n + p - i) \cdot m_3 = (i + 1) \cdot m_0 + (n + p - i) \cdot m_2$ *for all* $i = 0, \ldots, n + p$.
    (c) $j \cdot m_2 + (n - j) \cdot m_3 = (j + 2p + 1) \cdot m_0 + (n - j - 2p + 1) \cdot m_1$ *for all* $j = -2p, \ldots, n$.
    (d) $\ell \cdot m_2 + (n - \ell) \cdot m_3 = (n + 3p + 2 + \ell) \cdot m_0 + (1 - 3p - \ell) \cdot m_1$ *for all* $\ell = -p + 1, \ldots, -2p - 1$.
  (ii) *The standard basis of* $\Gamma(n; p < 0, 0)$ *is:*
    $\mathrm{S} = \mathrm{S}(1, 3) \uplus \mathrm{S}(2, 3) \uplus \mathrm{S}(0, 3)$, *where*
    $\mathrm{S}(1, 3) := \{a_1 \cdot m_1 + a_3 \cdot m_3 \mid a_1 \in [1, n + p - 1], a_3 \in [0, n + p - 2], a_1 + a_3 \le n + p - 1\}$,
    $\mathrm{S}(2, 3) := \{b_2 \cdot m_2 + b_3 \cdot m_3 \mid b_2 \in [1, n - 1], b_3 \in [0, n + p - 1], b_2 + b_3 \le n - 1\}$ *and*
    $\mathrm{S}(0, 3) := \{c \cdot m_3 \mid c \in [0, n + p - 1]\}$.

(3) *Suppose that* $p > 0$ *and* $q = 0$ *and* $n \ge 2p - 1$.
  (i) *Equations for* $\Gamma(n; p > 0, 0)$*: The following equations are easy to verify.*
    (a) $m_1 + m_2 = m_0 + m_3$.
    (b) $(n + 2) \cdot m_1 = m_0 + (n - p - 1) \cdot m_2 + (p + 1)m_3$.
    (c) $i \cdot m_1 + (n + p - i) \cdot m_3 = (i + 1) \cdot m_0 + (n + p - i) \cdot m_2$ *for all* $i = 0, \ldots, n + p$.
    (d) $j \cdot m_2 + (n - j) \cdot m_3 = (j + 2p + 1) \cdot m_0 + (n - j - 2p + 1) \cdot m_1$ *for all* $j = 0, \ldots, n - 2p + 1$.
    (e) $j \cdot m_2 + (n - j) \cdot m_3 = (j + p - n) \cdot m_0 + (2n - p - j + 1)m_1$ *for all* $j = n - p + 1, \ldots, n$.
    (f) $(n - p) \cdot m_2 + (p + \ell) \cdot m_3 = (n + 1) \cdot m_1 + \ell \cdot m_3$ *for all* $\ell = 0, \ldots, p - 1$.
  (ii) *The standard basis of* $\Gamma(n; p > 0, 0)$ *is:*
    $\mathrm{S} = \mathrm{S}(1, 3) \uplus \mathrm{S}(2, 3) \uplus \widetilde{\mathrm{S}}(2, 3) \uplus \mathrm{S}(0, 3)$, *where*
    $\mathrm{S}(1, 3) := \{(a_1 \cdot m_1 + a_3 \cdot m_3 \mid a_1 \in [1, n + 1], a_3 \in [0, n - 1], a_1 + a_3 \le n + p - 1\}$,
    $\mathrm{S}(2, 3) := \{(b_2 \cdot m_2 + b_3 \cdot m_3 \mid b_2 \in [1, n - 1], b_3 \in [0, n - 1], b_2 + b_3 \le n - 1\}$,
    $\widetilde{\mathrm{S}}(2, 3) := \{b_2 \cdot m_2 + b_3 \cdot m_3 \mid b_2 \in [n - 2p + 2, n - p - 1], b_3 \in [p + 1, 2p - 2], b_2 + b_3 \ge n\}$ *and*
    $\mathrm{S}(0, 3) := \{c \cdot m_3 \mid c \in [0, n - 1]\}$.

(4) *Suppose that* $p = q < 0$ *and* $n \ge -4p - 1$.
  (i) *Equations for* $\Gamma(n; p < 0, p)$*: The following equations are easy to verify.*
    (a) $m_1 + m_2 = m_0 + m_3$.
    (b) $i \cdot m_1 + (n - p - i) \cdot m_3 = (p + i + 1) \cdot m_0 + (n - 2p - i) \cdot m_2$ *for all* $i = -3p + 1, \ldots, n - p$.
    (c) $(n + p + k + 1) \cdot m_1 = k \cdot m_0 + (n + p - k) \cdot m_2 + k \cdot m_3$ *for all* $k = 0, \ldots, -2p - 1$.
    (d) $j \cdot m_2 + (n + 2p - j) \cdot m_3 = (j - 2p + 1) \cdot m_0 + (n + 4p + 1 - j) \cdot m_1$ *for all* $j = 0, \ldots, n + 4p + 1$.
    (e) $(n + p + 1) \cdot m_2 + \ell \cdot m_3 = m_0 + (n + p) \cdot m_1 + (\ell + 1) \cdot m_3$ *for all* $\ell = 0, \ldots, -2p - 2$.
  (ii) *The standard basis of* $\Gamma(n; p < 0, p)$ *is:*
    $\mathrm{S} = \mathrm{S}(1, 3) \uplus \mathrm{S}(2, 3) \uplus \widetilde{\mathrm{S}}(2, 3) \uplus \mathrm{S}(0, 3)$, *where*
    $\mathrm{S}(1, 3) := \{(a_1 \cdot m_1 + a_3 \cdot m_3 \mid a_1 \in [1, n + p + 1], a_3 \in [0, n + 2p - 1], a_1 + a_3 \le n - p - 1\}$,
    $\mathrm{S}(2, 3) := \{(b_2 \cdot m_2 + b_3 \cdot m_3 \mid b_2 \in [1, n + 2p - 1], b_3 \in [0, n + 2p - 1], b_2 + b_3 \le n + 2p - 1\}$,
    $\widetilde{\mathrm{S}}(2, 3) := \{b_2 \cdot m_2 + b_3 \cdot m_3 \mid b_2 \in [n + 4p + 2, n + p - 1], b_3 \in [0, -2p - 2], b_2 + b_3 \ge n + 2p\}$ *and*
    $\mathrm{S}(0, 3) := \{c \cdot m_3 \mid c \in [0, n + 2p - 1]\}$.

(5) *Suppose that* $p = q > 0$ *and* $n \ge 2p$.
  (i) *Equations for* $\Gamma(n; p > 0, p)$*: The following equations are easy to verify.*
    (a) $m_1 + m_2 = m_0 + m_3$.
    (b) $i \cdot m_1 + (n - p - i) \cdot m_3 = (p + i + 1) \cdot m_0 + (n - 2p - i) \cdot m_2$ *for all* $i = 0, \ldots, n - 2p$.
    (c) $(n + p + 2) \cdot m_1 = m_0 + (n + p - 1) \cdot m_2 + m_3$.
    (d) $j \cdot m_2 + (n + 2p - j) \cdot m_3 = (j - 2p + 1) \cdot m_0 + (n + 4p + 1 - j) \cdot m_1$ *for all* $j = 3p, \ldots, n + p$.
    (e) $(n + p) \cdot m_2 = (n + p + 1) \cdot m_1$.
  (ii) *The standard basis of* $\Gamma(n; p > 0, p)$ *is:*
    $\mathrm{S} = \mathrm{S}(1, 3) \uplus \widetilde{\mathrm{S}}(1, 3) \uplus \mathrm{S}(2, 3) \uplus \mathrm{S}(0, 3)$, *where*
    $\mathrm{S}(1, 3) := \{(a_1 \cdot m_1 + a_3 \cdot m_3 \mid a_1 \in [1, n - p - 1], a_3 \in [0, n - p - 1], a_1 + a_3 \le n - p - 1\}$,

$\widetilde{\mathrm{S}}(1,3) := \{a_1 \cdot m_1 + a_3 \cdot m_3 \mid a_1 \in [n-2p+1, n+p+1], a_3 \in [0, p-1], a_1 + a_3 \geq n-p\}$
$\mathrm{S}(2,3) := \{(b_2 \cdot m_2 + b_3 \cdot m_3 \mid b_2 \in [1, n+p-1], b_3 \in [0, n-p-1], b_2 + b_3 \leq n+2p-1\}$, *and*
$\mathrm{S}(0,3) := \{c \cdot m_3 \mid c \in [0, n-p-1]\}$.

Proof. In each case the equations in part (i) are easy to verify. For the proof of part (ii) use equations in part (i) to show that S is contained in the required (disjoint) union of the subsets defined in each case in part (ii). Now, since the cardinality of S is m_0 which is also the cardinality of the required (disjoint) union, the equality is immediate. □

Now we use Proposition 3.2 to give a description of the Cohen–Macaulay defect $\mathrm{B}(\Gamma)$.

Proposition 3.3. *Suppose that* $(p,q) \in (\mathbb{Z}^- \times \{0\}) \cup (\mathbb{Z}^+ \times \{0\}) \cup \Delta_{\mathbb{Z}}$. *For a variable positive integer n with $n \gg p$ (e.g. $n \geq 4|p|+1$), put $\Gamma(n; p, q) = \Gamma$. Then:*

(1) $\mathrm{B}_{2,0}(\Gamma, r) = \emptyset$ *for every $r \in \mathbb{N}$ and Γ is 2-good.*
(2) *Assume that* $(p,q) \in (\mathbb{Z}^- \times \{0\}) \cup \{(-1,-1), (0,0), (1,0), (2,0)\}$. *Let $g = a_0 m_0 + a_2 m_2 \in \Gamma$ with $r_1 = \text{max-deg}(g) = a_0 + a_2$ be such that $g + m_0 = \lambda_1 m_1 + \lambda_3 m_3$ with $\lambda_1, \lambda_3 \in \mathbb{N}^+$ and $\text{max-deg}(g+m_0) = \lambda_1 + \lambda_3$. Then $r_1 \geq \lambda_1 + \lambda_3$. (For the definition of r_1 see 2.3(1).) In particular, $g \notin \mathrm{B}_{1,3}(\Gamma, r_1)$. Furthermore:*
   (a) $\mathrm{B}'_{1,3}(\Gamma) = \mathrm{B}_{1,3}(\Gamma, r_1)$.
   (b) $\mathrm{B}_{1,0}(\Gamma, r) = \emptyset$ *for every $r \in \mathbb{N}$ and Γ is 1-good. Moreover, Γ is good.*
(3) *Assume that* $(p,q) \notin (\mathbb{Z}^- \times \{0\}) \cup \{(-1,-1), (0,0), (1,0), (2,0)\}$. *Then Γ is not 1-good. More precisely:*
   (a) *If $(p,q) \in \mathbb{Z}^+ \times \{0\}$ with $p \geq 3$, then $(n-p-1)m_2 + (p+1)m_3 \in \mathrm{B}_{1,0}(\Gamma, n)$.*
   (b) *If $(p,q) \in \Delta_{\mathbb{Z}}$ with $p \leq -2$, then $(n+p-1)m_2 + m_3 \in \mathrm{B}_{1,0}(\Gamma, n+p)$.*
   (c) *If $(p,q) \in \Delta_{\mathbb{Z}}$ with $p \geq 1$, then $(n+p-1)m_2 + m_3 \in \mathrm{B}_{1,0}(\Gamma, n+p)$.*

Proof. (1) Suppose that $g \in \mathrm{B}_{2,0}(\Gamma, r)$ for some $r \in \mathbb{N}$. Then by the definition $g + m_0 = \lambda_2 m_2$ with $r = \text{max-deg}(g)$ and $\lambda_2 = \text{max-deg}(g + m_0) \geq r + 2$. Therefore $\lambda_2 m_2 \notin \mathrm{S}$ and hence by the explicit description of S given in each case (see Proposition 3.2) we have

$$\lambda_2 \geq \begin{cases} n, & \text{if } (p,q) = (0,0), \\ n, & \text{if } (p,q) \in \mathbb{Z}^- \times \{0\}, \\ n, & \text{if } (p,q) \in \mathbb{Z}^+ \times \{0\}, \\ n+p+1, & \text{if } (p,q) \in \Delta_{\mathbb{Z}^-}, \text{ since } (n+p)m_2 = (n+p+1)m_1 \in \mathrm{S}(1,3) \subseteq \mathrm{S} \text{ by 3.2(4)(ii)}, \\ & \quad \text{and } (n+p+1)m_2 = m_0 + (n+p)m_1 + m_3 \notin \mathrm{S}, \\ n+p+1, & \text{if } (p,q) \in \Delta_{\mathbb{Z}^+}, \text{ since } (n+p)m_2 = (n+p+1)m_1 \in \tilde{\mathrm{S}}(1,3) \subseteq \mathrm{S} \text{ by 3.2(5)(ii)}, \\ & \quad \text{and } (n+p+1)m_2 = m_0 + (n+p)m_1 + m_3 \notin \mathrm{S} \end{cases}$$

and hence from equations 3.2(1)(i)(c) ($j = n$), 3.2(2)(i)(c), 3.2(3)(i)(e), 3.2(4)(i)(e) ($\ell = 0$) and 3.2(5)(i)(e) and 3.2(5)(i)(a), respectively, we have

$$g + m_0 = \lambda_2 m_2 = \begin{cases} (n+1)m_0 + m_1 + (\lambda_2 - n)m_2, & \text{if } (p,q) = (0,0), \\ (n+2p+1)m_0 + (1-2p)m_1 + (\lambda_2 - n)m_2, & \text{if } (p,q) \in \mathbb{Z}^- \times \{0\}, \\ pm_0 + (n-p+1)m_1 + (\lambda_2 - n)m_2, & \text{if } (p,q) \in \mathbb{Z}^+ \times \{0\}, \\ m_0 + (n+p)m_1 + (\lambda_2 - n - p - 1)m_2 + m_3, & \text{if } (p,q) \in \Delta_{\mathbb{Z}^-}, \\ m_0 + (n+p)m_1 + (\lambda_2 - n - p - 1)m_2 + m_3, & \text{if } (p,q) \in \Delta_{\mathbb{Z}^+}. \end{cases}$$

But then we get $r = \text{max-deg}(g) \geq \lambda_2 \geq r + 2$, a contradiction. This proves that $\mathrm{B}_{2,0}(\Gamma, r) = \emptyset$ for every $r \in \mathbb{N}$ and hence Γ is 2-good by Proposition 2.4.

(2) Since $\lambda_1 m_1 + \lambda_3 m_3 = g + m_0 \notin \mathrm{S}$, by the explicit description of S given in each case (see Proposition 3.2) we have

(i) $\begin{cases} \lambda_1 + \lambda_3 \geq n, & \text{if } (p,q) = (0,0), \\ \lambda_1 + \lambda_3 \geq n+p, & \text{if } (p,q) \in \mathbb{N}^- \times \{0\}. \end{cases}$
(ii) If $(p,q) \in \{(1,0), (2,0)\}$, then exactly one of the following cases holds:
   (ii.a) $\lambda_1 \geq n+2$.
   (ii.b) $p \leq \lambda_1 \leq n+1$ and $\lambda_1 + \lambda_3 \geq n+p$.
   (ii.c) $0 < \lambda_1 < p$ and $\lambda_3 \geq n$.
(iii) If $(p,q) = (-1,-1)$, then exactly one of the following cases holds
   (iii.a) $\lambda_1 \geq n+p+2 = n+1$.
   (iii.b) $4 = -3p+1 \leq \lambda_1 \leq n+p+1 = n$ and $\lambda_1 + \lambda_3 \geq n-p = n+1$.
   (iii.c) $0 < \lambda_1 < 4$ and $\lambda_3 \geq n+2p = n-2$.

Now, conclude that $r_1 = \text{max-deg}(g) \geq \lambda_1 + \lambda_3$ by using the equations:

– if $(p,q) = (0,0)$, then use $\begin{cases} \text{3.2(1)(i)(b) } (i = \lambda_1), & \text{if } \lambda_1 < n, \\ \text{3.2(1)(i)(b) } (i = n), & \text{if } \lambda_1 \geq n, \end{cases}$

– if $(p,q)\in\mathbb{N}^-\times\{0\}$, then use $\begin{cases}3.2(2)(\mathrm{i})(\mathrm{b})\ (i=n+p), & \text{if } \lambda_1\ge n+p,\\ 3.2(2)(\mathrm{i})(\mathrm{b})\ (i=\lambda_1), & \text{if } \lambda_1< n+p,\end{cases}$

– if $(p,q)\in\{(1,0),(2,0)\}$, then use $\begin{cases}3.2(3)(\mathrm{i})(\mathrm{c})\ (i=n+p), & \text{if (ii.a) holds},\\ 3.2(3)(\mathrm{i})(\mathrm{c})\ (i=\lambda_1), & \text{if (ii.b) holds},\\ 3.2(3)(\mathrm{i})(\mathrm{d})\ (j=0), & \text{if (ii.c) holds},\end{cases}$

– if $(p,q)=(-1,-1)$, then use $\begin{cases}3.2(4)(\mathrm{i})(\mathrm{b})\ (i=n-p) & \text{if (iii.a) holds},\\ 3.2(4)(\mathrm{i})(\mathrm{b})\ (i=\lambda_1) & \text{if (iii.b) holds},\\ 3.2(4)(\mathrm{i})(\mathrm{d})\ (j=0) & \text{if (iii.c) holds}.\end{cases}$

Therefore in all the cases $g\notin \mathrm{B}_{1,3}(\Gamma,r_1)$ by definition and the assertion (a) is immediate from Definition 2.3(1).

(b) Suppose that $g\in\mathrm{B}_{1,0}(\Gamma,r)$ for some $r\in\mathbb{N}$. Then by definition $g+m_0=\lambda_1m_1$ with $\lambda_1\in\mathbb{N}^+$ and $\lambda_1=\text{max-deg}(g+m_0)\ge r+2$. Therefore $\lambda_1m_1\notin \mathrm{S}$ and hence by the explicit description of S given in each case (see Proposition 3.2) we have

$$\lambda_1\ge\begin{cases}n, & \text{if } (p,q)=(0,0),\\ n+p, & \text{if } (p,q)\in\mathbb{N}^-\times\{0\},\\ n+p, & \text{if } (p,q)\in\{(1,0),(2,0)\},\\ n-p=n+1, & \text{if } (p,q)=(-1,-1),\end{cases}$$

and hence from equations 3.2(1)(i)(b) $(i=n)$, 3.2(2)(i)(b) $(i=n+p)$, 3.2(3)(i)(c) $(i=n+p)$ and 3.2(4)(i)(b) $(i=n-p)$, respectively, we have

$$g+m_0=\lambda_1m_1=\begin{cases}(n+1)m_0+(\lambda_1-n)m_1, & \text{if } (p,q)=(0,0),\\ (n+p+1)m_0+(\lambda_1-n-p)m_1, & \text{if } (p,q)\in\mathbb{N}^-\times\{0\},\\ (n+p+1)m_0+(\lambda_1-n-p)m_1, & \text{if } (p,q)\in\{(1,0),(2,0)\},\\ (n+1)m_0+(\lambda_1-n-1)m_1+m_2, & \text{if } (p,q)=(-1-1).\end{cases}$$

But then we get $r=\text{max-deg}(g)\ge\lambda_1\ge r+2$, a contradiction. This proves that $\mathrm{B}_{1,0}(\Gamma,r)=\emptyset$ for every $r\in\mathbb{N}$ and hence Γ is 1-good by part (a) and Proposition 2.7(2). Therefore Γ is good by part (1) and part (b).

(3) (a) Note that by 3.2(3)(ii) $\tilde{\mathrm{S}}(2,3)\ne\emptyset$, since $n\ge 2p-1$ and $p\ge 3$ by assumptions. Further, since $(n-p-1)m_2+(p+1)m_3\in\tilde{\mathrm{S}}(2,3)\subseteq\mathrm{S}$, $(n+1)m_1\in\mathrm{S}$ by 3.2(3)(ii) and $(n-p-1)m_2+(p+1)m_3+m_0=(n+2)m_1$ by 3.2(3)(i)(b), it is easy to see that $\text{max-deg}((n-p-1)m_2+(p+1)m_3)=n$ and $\text{max-deg}((n-p-1)m_2+(p+1)m_3+m_0)=n+2$. Therefore $(n-p-1)m_2+(p+1)m_3\in\mathrm{B}_{1,0}(\Gamma,n)$.

(b) Note that by 3.2(4)(ii) $\tilde{\mathrm{S}}(2,3)\ne\emptyset$, since $n\ge-4p-1$ and $p\le-2$ by assumptions. Further, since $(n+p-1)m_2+m_3\in\tilde{\mathrm{S}}(2,3)\subseteq\mathrm{S}$, $(n+p+1)m_1\in\mathrm{S}$ by 3.2(4)(ii) and $(n+p-1)m_2+m_3+m_0=(n+p+2)m_1$ by 3.2(4)(i)(c) $(k=1)$, it is easy to see that $\text{max-deg}((n+p-1)m_2+m_3)=n+p$ and $\text{max-deg}((n+p-1)m_2+m_3+m_0)=n+p+2$. Therefore $(n+p-1)m_2+m_3\in\mathrm{B}_{1,0}(\Gamma,n+p)$.

(c) Note that by 3.2(5)(ii) $\tilde{\mathrm{S}}(1,3)\ne\emptyset$, since $n\ge 2p$ and $p\ge 1$ by assumptions. Further, since $(n+p-1)m_2+m_3\in\mathrm{S}(2,3)\subseteq\mathrm{S}$, $(n+p+1)m_1\in\tilde{\mathrm{S}}(1,3)$ by 3.2(4)(ii) and $(n+p-1)m_2+m_3+m_0=(n+p+2)m_1$ by 3.2(5)(i)(c), it is easy to see that $\text{max-deg}((n+p-1)m_2+m_3)=n+p$ and $\text{max-deg}((n+p-1)m_2+m_3+m_0)=n+p+2$. Therefore $(n+p-1)m_2+m_3\in\mathrm{B}_{1,0}(\Gamma,n+p)$. □

Theorem 3.4. *Let p,q be fixed integers, n be a variable positive integer with $n\gg p$ (e.g. $n\ge 4|p|+1$), $m:=m_0=n^2+pn+q$, $m_1=m_0+n$, $m_2=m_0+2n+1$, $m_3=m_0+3n+1$, $\Gamma(n;p,q)=\mathbb{N}m_0+\mathbb{N}m_1+\mathbb{N}m_2+\mathbb{N}m_3$ and let $\mathrm{R}(n;p,q)=K[\![\Gamma(n;p,q)]\!]$ be the semigroup ring of $\Gamma(n;p,q)$ over a field K. Suppose that $(p,q)\in(\mathbb{Z}^-\times\{0\})\cup(\mathbb{Z}^+\times\{0\})\cup\Delta_{\mathbb{Z}}$. Then $\mathrm{G}(n;p,q)$ is Cohen–Macaulay if and only if $(p,q)\in(\mathbb{Z}^-\times\{0\})\cup\{(-1,-1),(0,0),(1,0),(2,0)\}$.*

Proof. Immediate from Proposition 3.3(2), (3) and Theorem 1.6. □

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