Available online at www.sciencedirect.com
ScienceDirect
Linear Algebra and its Applications 425 (2007) 486–517
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

# On the Nevanlinna-Pick interpolation problem: Analysis of the McMillan degree of the solutions

A. Gombani [a,*], György Michaletzky [b]

[a] *LADSEB-CNR, Corso Stati Uniti 4, 35127 Padova, Italy*
[b] *Eötvös Loránd University, H-1111 Pázmány Péter sétány 1/C, Computer and Automation Institute of HAS, H-1111 Kende u. 13-17, Budapest, Hungary*

Received 12 February 2006; accepted 12 February 2006
Available online 24 March 2007
Submitted by J. Rosenthal

Dedicated to Professor Paul Fuhrmann on the occasion of his 70th birthday

**Abstract**

In this paper we investigate some aspect of the Nevanlinna-Pick and Schur interpolation problem formulated for Schur-functions considered on the right-half plane of $\mathbb{C}$. We consider the well established parametrization of the solution $Q = T_\Theta(S) := (S\Theta_{12} + \Theta_{22})^{-1}(S\Theta_{11} + \Theta_{21})$ (see e.g. [J.A. Ball, I. Gohberg, L. Rodman, Realization and interpolation of rational matrix functions, Operator Theory: Adv. Appl. 33 (1988) 1–72; H. Dym, *J*-contractive matrix functions, reproducing kernel Hilbert spaces and interpolation, CBMS Regional Conference Series in Mathematics, No. 71, Amer. Math. Soc., Providence, RI, 1989]), where the *J*-inner function $\Theta$ is completely determined by the interpolation data and *S* is an arbitrary Schur function. We then compare the relations between the realizations of *Q* and *S* induced by $\Theta$. We show in particular that *S* generates a solution with a low McMillan degree if and only if *S* satisfies some interpolation conditions formulated on the left-half plane of $\mathbb{C}$. This analysis can be considered to be partially complementary to the results of A. Lindquist, C. Byrnes et al. on Carathéodory functions, [C. Byrnes, A. Lindquist, On the partial stochastic realization problem, IEEE Trans. Automatic Control 42 (1997) 1049–1070; C. Byrnes, A. Lindquist, S.V. Gusev, A.S. Mateev, A complete parametrization of all positive rational extension of a covariance sequence, IEEE Trans. Automatic Control 40 (1995) 1841–1857; C. Byrnes, A. Lindquist, S.V. Gusev, A convex optimization approach to the rational covariance extension problem, TRIA/MAT, 1997].


* Corresponding author.
*E-mail addresses:* gombani@ladseb.pd.cnr.it (A. Gombani), michgy@ludens.elte.hu (G. Michaletzky).

0024-3795/$ - see front matter 
doi:10.1016/j.laa.2006.02.045

*AMS classification:* 30E05; 93B29; 93B30

*Keywords:* Schur parameters; Interpolation; Schur functions; Positive real functions

## 1. Introduction

We consider here the problem of parametrizing the interpolating Schur functions $Q$ for a given set of Schur data by means of a linear fractional transformation with a bound on the degree. The general problem (i.e. without bound) has been studied extensively (see [2,6] and references therein). The problem with bounds has been first considered in [9] (see also [10,11]). The technique which is most widespread in the literature (and which will also use) starts from the interpolating conditions to build a linear fractional transformation (LFT), and the whole set of solutions is parametrized by means of the LFT of another Schur function $S$, which is chosen freely in the unit ball of $H^{\infty}$.

Generically, the degree of $Q$ is the sum of the number of interpolating conditions and the degree of the parameter function $S$; that is, if $n$ is the number of interpolating conditions, we have $\deg(Q) = n + \deg(S)$. In Section 4 we characterize those Schur functions $S$ for which there is degree drop in the corresponding $Q$. This is an important problem which has been studied with different methods (see [3–5,9–11,13,14]). Here we use state space realizations for the LFT of $S$, denoted by $T_{\Theta}(S)$, and we analyze the controllability and observability subspace of this realization. It turns out that, while the realization is always controllable, it is not necessarily observable. A study of the unobservability subspace is thus what is developed in Section 4. These results together with similar statements obtained from the realization of $S$ arising using the inverse of the LFT are used to characterize the Schur-functions $S$ which yield a lower degree interpolant $Q$ proving that a drop in the degree can be achieved only by imposing interpolation conditions on $S$ in the left-half plane. At the same time if $\deg(Q) < n + \deg(S)$, then the interpolant $Q$ satisfies additional interpolation conditions prescribed in the left-half plane at the mirror images (with respect to the imaginary axis) of the original interpolation nodes. In general, we show in Section 4 that the total number of additional interpolation conditions satisfied by the functions $S$ and $Q$ is at least $n$. The nodes of these interpolation problems are in the left-half plane, while the interpolation directions and values are determined by those of the original interpolation problem.

In Section 5 instead of analysing the difference in the McMillan-degrees of the functions $S$ and $Q$ connected by the linear fractional transformation, a special form of low degree interpolants is considered similar to those developed in [9,14]. We prove that these interpolants can be characterized by special solutions of a (matrix-valued) quadratic equation.

Although the main focus of the paper is on Nevanlinna-Pick interpolants, we often obtain, as a byproduct, results on interpolating function which are stable but not Schur, or even unstable. Since we feel that these results are of independent interest, we have included them when possible.

## 2. Preliminaries and notation

Let $F$ be a rational $p \times m$ matrix of McMillan degree $N$, analytic in the right half-plane $\mathbb{C}^{+}$. We shall denote by $\mathscr{H}_{+}^{\infty}$ the Hardy space of essentially bounded vector or matrix valued functions (the proper dimension will be understood from the context). If $M$ is a complex matrix, Tr shall denote its trace, $M^{\mathrm{T}}$ its transpose and $M^{*}$ its transpose conjugate. For a function $Q \in \mathscr{H}_{+}^{\infty}$, we set $Q^{*}(s) := \overline{Q(-\bar{s})^{\mathrm{T}}}$. A function $Q$ is said to be Schur if it is in $\mathscr{H}_{+}^{\infty}$ and $\|Q\|_{\infty} \leqslant 1$. It is *inner* if it is

Schur and $Q(\mathrm{i}\omega)Q^*(\mathrm{i}\omega) = I$ for a.e. on $\mathbb{R}$. For $F \in \mathscr{H}_+^\infty$, we say that the factorization $F = \tilde{F}^*M$ is a Douglas–Shapiro–Shields (DSS) factorization (see Fuhrmann [8]) if $\tilde{F} \in \mathscr{H}_+^\infty$, $M$ is inner and it has minimal McMillan degree. Suppose $F$ is right invertible in $\mathscr{H}_+^\infty$. The observable pair $(C, A)$ is said to be a *right null pair* for $F$ (see [2]) if $FC(sI - A)^{-1}$ is analytic in the spectrum of $A$.

We assume that we are given a set of interpolation points $s_1, \dots, s_n$ in the right half-plane $\mathbb{C}^+$ and interpolating conditions

$$U = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} V = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \tag{2.1}$$

with $u_i, v_i$ row vectors in $\mathbb{C}^p$, $\|u_i\| = 1$ and $\|v_i\| < 1$ for $i = 1, \dots, n$ and we want to find the solutions $Q$ to the problem

$$u_i Q(s_i)^* = v_i \quad i = 1, \dots, n \tag{2.2}$$

which are Schur-functions.

It is well-known (see e.g. [2]) that all solutions of this problem can be given using a rational fractional representation defined by a $J$-inner function. This representation is even valid for a more general form of the interpolation problem (2.2) which allows for multiplicities of the interpolation nodes. This can be defined in the following way.

**Problem 2.1** (*Schur*, *Nevanlinna-Pick*). Given the matrix $\mathscr{A}$ of size $n \times n$ with eigenvalues in the left half plane $\mathbb{C}^-$, and matrices $U$, $V$ of size $n \times p$ and a contractive matrix $D$ of size $p \times p$ parameterize all functions $Q$ for which

(i) $Q$ is a Schur-function;
(ii)

$$\begin{cases} (Q(s)U^* - V^*)(sI + \mathscr{A}^*)^{-1} & \text{is analytic on } \mathbb{C}^+, \\ Q(\infty) = D. \end{cases} \tag{2.3}$$

In this case the eigenvalues of $-\mathscr{A}^*$ determine the interpolation nodes. It is well known that Problem 2.1 has a solution if and only if

$$\mathscr{A}\mathscr{P} + \mathscr{P}\mathscr{A}^* + UU^* - VV^* = 0. \tag{2.4}$$

has a positive semi-definite solution.

Among the solutions of the Problem 2.1 we would like to consider solutions with "low complexity". In the scalar case – i.e. when $p = 1$ – this can be formulated as solutions with McMillan-degree exactly $n$ or in other words as solutions in the form of ratio of two coprime polynomials with degree $n$ (see [14,3]). In this case exactly $n$ parameters are needed to express all the solutions The exact constraint in this context on the degree was first imposed by Kimura [14], and it seems to be the only known way to make the problem tractable (if the degree drops, less parameters than $n$ are needed and this leads to a very thorny problem, since it is not known how to make this reduction and still preserve the Schur property). This type of formulation can be applied in the multivariate case as well, although some care is needed. In fact, in this case, saying that a function has degree $n$ is not enough. To see this, assume that $Q_1$ is an interpolant of degree $k$ strictly less than $n$. Then it can be shown that, under suitable conditions, there exist functions $Q_2$ analytic in $\mathbb{C}^+$ of degree $n-k$ such $Q_2(s)U^*(sI+\mathscr{A}^*)^{-1}$ is still analytic in $\mathbb{C}^+$ and $Q = Q_1 + Q_2$ is Schur. By construction it is of degree $n$ and it is also interpolating. On the other hand, it poses the same parametrization problems as interpolants of lower degree in the scalar case. An example of how to construct such functions is given in Section 5.

In conclusion, the functions we want to exclude from our parametrization are not only the functions of degree strictly less than $n$, as in the scalar case, but also those which can be reduced to a lower degree one by subtraction of a suitable function having the same DSS factor and vanishing on the interpolation nodes.

To this end, we make the following definition. Let $Q$ be a solution of degree $n$ to Problem 2.1 and let $Q = \tilde{Q}^*M$ be its DSS factorization: then we say that $Q$ is *irreducible* if, for any other solution, the condition that $Q_1M^*$ is coanalytic implies $Q_1 = Q$. We will therefore restrict ourselves to the set of interpolants which are irreducible; this is not a very restrictive assumption. In fact, in Section 5 it will be seen that the excluded functions are an algebraic set.

To simplify the calculations, we will assume in the following that the matrix $D$ is *strictly* contractive.

**Problem 2.2.** Given the matrix $\mathscr{A}$ of size $n \times n$ with eigenvalues in the left half plane $\mathbb{C}^-$, matrices $U$, $V$ of size $n \times p$ and a constant strictly contractive matrix $D$ of size $p \times p$, parametrize all functions $Q$, for which

(i) $Q$ is a Schur-function;
(ii)

$$\begin{cases} (Q(s)U^* - V^*)(sI+\mathscr{A}^*)^{-1} \quad \text{is analytic on } \mathbb{C}^+, \\ Q(\infty) = D. \end{cases} \tag{2.5}$$

(iii) $Q$ is rational of McMillan degree exactly $n$,
(iv) $Q$ is irreducible.

It is well known that solutions of degree strictly less than $n$ might exist; nevertheless, the set of solutions of degree $n$ can be shown to be a smooth manifold (see [13]) and therefore it is of particular interest in many problems involving optimization, like some identification algorithms. Moreover, it has also been studied in [9,14].

## 3. Interpolation conditions as matrix equations

Our first result is to express the interpolation conditions in Problem 2.1 in terms of matrix equations for a given realization for $Q$. Although quite simple, the following conditions do not seem to have appeared in the literature (the first formulation presented here was derived independently for the discrete time case by Marmorat and Olivi, see [15]). Although we will mainly use the result in connection with Schur function, we state the interpolation conditions in a more general setting.

**Proposition 3.1.** *Let $Q$ be a rational function analytic in $\sigma(-\mathscr{A}^*)$ admitting an observable realization*

$$Q = \left(\begin{array}{c|c} A_Q & B_Q \\ \hline C_Q & D \end{array}\right).$$

*Then* $(Q(s)U^* - V^*)(sI + \mathscr{A}^*)^{-1}$ *is analytic in* $\sigma(-\mathscr{A}^*)$ *if and only if there exists a matrix* $Y$ *such that*

$$DU^* - V^* = C_Q Y \tag{3.1}$$

*and*

$$B_Q U^* = Y\mathscr{A}^* + A_Q Y. \tag{3.2}$$

**Proof.** First suppose that $(Q(s)U^* - V^*)(sI + \mathscr{A}^*)^{-1}$ is analytic in $\sigma(-\mathscr{A}^*)$; then, it can only have poles at the eigenvalues of $A_Q$, and it vanishes at $\infty$; thus the observability of $(C_Q, A_Q)$ implies the existence of the matrix $Y$ in

$$(Q(s)U^* - V^*)(sI + \mathscr{A}^*)^{-1} = C_Q(sI - A_Q)^{-1}Y. \tag{3.3}$$

We can thus evaluate the identity

$$Q(s)U^* - V^* = DU^* - V^* + C_Q(sI - A_Q)^{-1}B_Q U^* = C_Q(sI - A_Q)^{-1}Y(sI + \mathscr{A}^*)$$

at $\infty$ obtaining the equation

$$DU^* - V^* = C_Q Y.$$

Subtraction from the previous equation yields:

$$\begin{aligned} C_Q(sI - A_Q)^{-1}B_Q U^* &= C_Q(sI - A_Q)^{-1}(Y(sI + \mathscr{A}^*) - (sI - A_Q)Y) \\ &= C_Q(sI - A_Q)^{-1}(Y\mathscr{A}^* + A_Q Y). \end{aligned}$$

Observability of $(C_Q, A_Q)$ implies that

$$B_Q U^* = Y\mathscr{A}^* + A_Q Y.$$

Conversely, if an observable realization of $Q$ satisfies (3.1) and (3.2), by backtracking the above argument, we see that $(Q(s)U^* - V^*)(sI + \mathscr{A}^*)^{-1}$ is analytic in $\sigma(-\mathscr{A}^*)$. □

Remark that, if we assume $Q$ to be analytic in $\mathbb{C}^+$, the above conditions (3.1) and (3.2) are satisfied if and only if $(Q(s)U^* - V^*)(sI + \mathscr{A}^*)^{-1}$ is also analytic in $\mathbb{C}^+$, that is, if condition (ii) in Problem 2.1 holds.

Notice also that, to obtain the Eqs. (3.1) and (3.2), we have exploited the fact that the function $Q$ cannot have poles at the interpolation nodes. In fact this assumption is quite unnatural and therefore removing it will be the main purpose of the next theorem.

Although it is not apparent from the formulation of Problem 2.1, since interpolation nodes and poles of $Q$ lie in disjoint parts of the plane, there is a compelling reason for resolving this issue. As it will be seen in Sections 4.2.2 and 4.2.3, there is a natural way to define an inverse problem, which leads to a situation where interpolation nodes and poles might coincide. Therefore this result will be needed.

**Theorem 3.1.** *Let* $F(s) = D + C(sI - A)^{-1}B$ *be a rational function.*

(i) *Assume that there exists a – possibly matrix-valued – function*

$$\phi(s) = \mathscr{C}_\phi(sI - \mathscr{A}_\phi)^{-1}\mathscr{B}_\phi + \psi(s),$$

*where* $\psi$ *is a matrix-valued polynomial and* $(\mathscr{C}_\phi, \mathscr{A}_\phi)$ *is an observable pair such that*

*$F\phi^*$ is analytic at the eigenvalues of $-\mathscr{A}_\phi^*$*

*and assume, moreover, that the pair $(C, A)$ is observable. Then there exists a matrix $\Pi$ such that*

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \Pi \\ \mathscr{B}_\phi^* \end{bmatrix} = \begin{bmatrix} -\Pi \mathscr{A}_\phi^* \\ 0 \end{bmatrix}. \tag{3.4}$$

(ii) *Assume that the matrices $\mathscr{A}_\phi, \mathscr{B}_\phi, \Pi$ satisfy Eq. (3.4), where $(\mathscr{A}_\phi, \mathscr{B}_\phi)$ and $(A, B)$ are controllable pairs. Then there exists a matrix polynomial $\psi$ such that for*

$$\phi(s) = (sI - \mathscr{A}_\phi)^{-1}\mathscr{B}_\phi + \psi(s) \tag{3.5}$$

*the function $F\phi^*$ is analytic at the eigenvalues of $-\mathscr{A}_\phi^*$.*
*If $A$ and $-\mathscr{A}_\phi^*$ have no common eigenvalues then Eq. (3.4) implies that*

$$F(s)\mathscr{B}_\phi^*(sI + \mathscr{A}_\phi^*)^{-1} = C(sI - A)^{-1}\Pi,$$

*which is analytic at the eigenvalues of $-\mathscr{A}_\phi^*$.*

**Remark 3.1.** If for a function $\phi$ defined in (3.5) the product $F\phi^*$ is analytic at the eigenvalues of $-\mathscr{A}_\phi^*$ then we are going to say for short that the pair $(\mathscr{B}_\phi^*, -\mathscr{A}_\phi^*)$ forms a right-zero pair of the function $F$.

**Proof.** See the Appendix. □

The above conditions should be compared with the **zero module** definition of Wyman and Sain (see [16]). In fact, assuming that the *transfer* function $F$ is of size $p \times m$, and denoting by $\mathbb{C}^p[s]$ (and $\mathbb{C}^m[s]$) the space of $p$-vectors ($m$-vectors) of polynomials, the finite zero module $Z_{\text{fin}}(F)$ is defined as follows:

$$Z_{\text{fin}}(F) = \frac{F^{-1}(\mathbb{C}^p[z]) + \mathbb{C}^m[z]}{\ker F + \mathbb{C}^m[z]},$$

where the factorization is meant in vector space sense.

In words, the numerator contains those rational $m$-dimensional "input" functions for which there exists a polynomial vector of size $m$ such that applying the transfer function $F$ to the sum a polynomial vector is obtained. But – due to the factorization – two "input" functions $\phi_1, \phi_2$ are considered to be equivalent if there exists a polynomial $\psi$ such that $F(z)(\phi_1(z) - \phi_2(z) + \psi(z)) = 0$.

A straightforward modification of the proof presented in the appendix shows that if Eq. (3.4) holds, then the function $\psi$ in (3.5) can be chosen in such a way that $F\phi^*$ be a polynomial.

The connection between interpolation and zeros is quite natural, since interpolation is, after all, a prescription on the zeros of some function.

Theorem 3.1 is formulated for right zero functions but obviously a similar theorem is valid for left zero functions, as well.

Note that writing Problem 2.1 in the form

$$[Q, I] \begin{bmatrix} U^* \\ -V^* \end{bmatrix} (sI + \mathscr{A}^*)^{-1}$$

is analytic at the eigenvalues of $-\mathscr{A}^*$ Eqs. (3.1), (3.2) – in the following form:

$$\begin{bmatrix} A_Q & B_Q & 0 \\ C_Q & D & I \end{bmatrix} \begin{bmatrix} -Y \\ U^* \\ -V^* \end{bmatrix} = \begin{bmatrix} Y\mathscr{A}^* \\ 0 \end{bmatrix}$$

follow from Theorem 3.1(i) using Eq. (3.4).

## 4. Analysis of the McMillan-degree using state-space realizations of the solutions

It is well known that all solutions of the interpolation Problem 2.1 without constraints on the McMillan-degree can be generated by a linear fractional transformation (a detailed definition will be given in Section 4.1 below) of a Schur function $S$ defined by a $J$-inner function $\Theta$, which will be denoted by $T_\Theta(S)$. (See Ball, Gohberg and Rodman [2]). We discuss here (Section 4.2.1) the properties of this linear fractional transformation $T_\Theta(S)$ in terms of minimal realizations of $\Theta$ and $S$. In Section 4.2.2 we discuss the properties of the inverse transform; notice that, although the formulas in Sections 4.2.1 and 4.2.2 might look similar, the direct problem (studying $Q$ in terms of $S$) and the inverse problem (studying $S$ in terms of $Q$) are intrinsically different because, in the construction of a realization of $S$ from $Q$, we use the interpolating property of $Q$; moreover, while in the first case the interpolating conditions are in the domain of analyticity of $Q$, in the second case can coincide with some pole of $S$. In Section 4.2.3 we connect the results of the previous two sections.

### 4.1. LFT in terms of realizations

We define here the linear fractional transformation connected to the interpolation conditions of Problem 2.1. This transformation is given in terms of a $J$-inner function $\Theta$ (see [2]) with the following realization

$$\Theta = \left(\begin{array}{c|cc} \mathscr{A} & U & V \\ \hline -U^*\mathscr{P}^{-1} & I & 0 \\ V^*\mathscr{P}^{-1} & 0 & I \end{array}\right), \tag{4.1}$$

where $\mathscr{P} = \mathscr{P}^* > 0$ satisfies the Lyapunov-equation

$$\mathscr{A}\mathscr{P} + \mathscr{P}\mathscr{A}^* + UU^* - VV^* = 0. \tag{4.2}$$

Then the Schur-function $Q$ is a solution of the interpolation Problem 2.1 – without any constraint on the MacMillan-degree – if and only if

$$Q := (S\Theta_{12} + \Theta_{22})^{-1}(S\Theta_{11} + \Theta_{21}),$$

where

$$\Theta = \begin{pmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{pmatrix} \tag{4.3}$$

and $S$ is a Schur-function with $S(\infty) = D$. We shall write for short

$$Q = T_\Theta(S). \tag{4.4}$$

Especially, if $u_i$ and $v_i$ are the rows of $U$ and $V$ and

$$\mathscr{A} = \operatorname{diag}\{-\bar{s}_1, \ldots, -\bar{s}_n\},$$

then the Schur function $Q$ is a solution to the Nevanlinna-Pick problem

$$Q(s_i)u_i^* = v_i^*.$$

Similarly, if we take $\mathscr{A}$ lower triangular and $\mathscr{P}$ diagonal, we obtain a Potapov factorization of $\Theta$ and therefore a Schur-problem.

The following two lemmata play an important although essentially technical role expressing the connection between $S$ and $Q$ provided by the linear fractional transformation (4.4) in terms of their realizations. Their proofs are postponed to the Appendix.

**Lemma 4.1.** *Let $\Theta$ be a $J$-inner function as in* (4.1), *and let*

$$S = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right). \tag{4.5}$$

*Then $Q = T_\Theta(S)$ has the following* (*possibly non-minimal*) *realization*:

$$Q = \left(\begin{array}{cc|c} \mathscr{A} + V(DU^* - V^*)\mathscr{P}^{-1} & -VC & -U + VD \\ -BU^*\mathscr{P}^{-1} & A & -B \\ \hline (DU^* - V^*)\mathscr{P}^{-1} & -C & D \end{array}\right). \tag{4.6}$$

Observe that the feedthrough term $D$ of $S$ goes through the linear fractional transformation without any change.

In order to find some connection between the McMillan-degree of $S$ and $Q$ we need a state-space description of the inverse linear fractional transformation as well.

**Lemma 4.2.** *Let $\Theta$ be a $J$-inner function as in* (4.1) *and let*

$$Q = \left(\begin{array}{c|c} A_Q & B_Q \\ \hline C_Q & D \end{array}\right). \tag{4.7}$$

*Then $S = T_\Theta^{-1}(Q)$ has the following* (*possibly non-minimal*) *realization*:

$$S = \left(\begin{array}{cc|c} A_Q & -B_Q U^*\mathscr{P}^{-1} & B_Q \\ VC_Q & \mathscr{A} + (U - VD)U^*\mathscr{P}^{-1} & VD - U \\ \hline C_Q & (V^* - DU^*)\mathscr{P}^{-1} & D \end{array}\right). \tag{4.8}$$

### *4.2. Controllability and observability analysis*

Due to the fact that these realizations above are not necessarily minimal to get a detailed picture concerning the McMillan-degree of the Schur-functions $S$ and $Q$ the controllability and unobservability subspaces of the realizations (4.6) and (4.8) should be analysed. We will show that, while the realization (4.6) $Q$ is always controllable if the realization of $S$ is minimal, its unobservability subspace might not be trivial and it has a particular structure. Symmetrically, the realization (4.8) of $S$ is always observable, but its controllability subspace has a particular structure in terms of the data. These structures are explained in terms of some additional interpolating conditions.

#### *4.2.1. Direct transformation*

We start with the analysis of the controllability.

**Proposition 4.1.** *If the pair $(A, B)$ in the realization* (4.5) *is controllable then the realization of the function $Q$ defined in* (4.6) *is controllable*, *as well*.

**Proof.** The identity

$$\mathscr{A} + V(DU^* - V^*)\mathscr{P}^{-1} = -\mathscr{P}\mathscr{A}^*\mathscr{P}^{-1} + (VD - U)U^*\mathscr{P}^{-1}$$

implies that the controllability subspace defined by the pair

$$\left(\begin{bmatrix} \mathscr{A} + V(DU^* - V^*)\mathscr{P}^{-1} & -VC \\ -BU^*\mathscr{P}^{-1} & A \end{bmatrix}, \begin{bmatrix} -U + VD \\ -B \end{bmatrix}\right)$$

coincide with that determined by

$$\left(\begin{bmatrix} -\mathscr{P}\mathscr{A}^*\mathscr{P}^{-1} & -VC \\ 0 & A \end{bmatrix}, \begin{bmatrix} -U + VD \\ -B \end{bmatrix}\right).$$

To prove controllability the celebrated P-B-H test will be applied. The orthogonal complement of the controllability subspace is invariant under the adjoint of the state-transition matrix. Thus we can consider an eigenvector belonging to that subspace.

$$[\alpha^*, \beta^*]\begin{bmatrix} U - VD \\ B \end{bmatrix} = 0, \tag{4.9}$$

$$[\alpha^*, \beta^*]\begin{bmatrix} -\mathscr{P}\mathscr{A}^*\mathscr{P}^{-1} & -VC \\ 0 & A \end{bmatrix} = \lambda\left[\alpha^*, \beta^*\right]. \tag{4.10}$$

Eq. (4.10) gives that

$$\lambda\alpha^* = -\alpha^*\mathscr{P}\mathscr{A}^*\mathscr{P}^{-1}, \tag{4.11}$$

$$\lambda\beta^* = -\alpha^*VC + \beta^*A. \tag{4.12}$$

If $\alpha \neq 0$ then $\lambda$ should be an eigenvalues of $-\mathscr{A}^*$, i.e. $\lambda$ coincides with one of the interpolation nodes. In particular, $\operatorname{Re}\lambda > 0$. Consequently, $\lambda$ is not an eigenvalue of $A$, thus the matrix $\lambda I - A$ is nonsingular implying that

$$\beta^* = -\alpha^*VC(\lambda I - A)^{-1}. \tag{4.13}$$

Substituting this back to the equation in (4.9)

$$\alpha^*[U - V(D + C(\lambda I - A)^{-1}B)] = 0. \tag{4.14}$$

Shortly,

$$\alpha^*[U - VS(\lambda)] = 0. \tag{4.15}$$

Observe that since $\lambda$ is in the right half plane the inequality

$$S(\lambda)S(\lambda)^* \leqslant I \tag{4.16}$$

holds true.

Multiplying the Lyapunov-equation (4.2) by $\alpha^*$ and $\alpha$ from the left and right, respectively, we get that

$$-2\operatorname{Re}\lambda\alpha^*\mathscr{P}\alpha + \alpha^*V[S(\lambda)S(\lambda)^* - I]V^*\alpha = 0.$$

If $\alpha$ is nonzero then the first term is strictly negative while the second term is non positive leading to a contradiction.

If $\alpha = 0$ then equations in (4.9) and (4.10) give that

$$\beta^* B = 0,$$
$$\beta^* A = \lambda\beta^*.$$

Now the controllability of the pair $A, B$ implies that $\beta = 0$, concluding our proof. $\square$

Let us continue with the analysis of the observability.

**Theorem 4.1.** *Assume that the realization* (4.5) *of $S$ is observable. Consider a matrix* $\begin{bmatrix}\alpha_S\\ \beta_S\end{bmatrix}$ *such that its columns are linearly independent and span the unobservability subspace of the realization* (4.6) *of $Q$.*

*Then the range of $\alpha_S$ is an invariant subspace of $\mathscr{A}$, i.e. there exists a matrix $\Gamma_S$ such that*

$$\mathscr{A}\alpha_S = \alpha_S\Gamma_S \tag{4.17}$$

*and $\beta_S$ is determined by the following equation*:

$$\begin{aligned}(S(s)U^* - V^*)\mathscr{P}^{-1}(sI - \mathscr{A})^{-1}\alpha_S &= (S(s)U^* - V^*)\mathscr{P}^{-1}\alpha_S(sI - \Gamma_S)^{-1}\\ &= C(sI - A)^{-1}\beta_S.\end{aligned} \tag{4.18}$$

*Especially, in this case the dimension of the unobservability subspace in the realization of $Q$ given by* (4.6) *is at most $n$.*

*Conversely, if the matrices $\alpha$ and $\beta$ satisfy Eqs.* (4.17) *and* (4.18) *for some matrix $\Gamma$ then the range of the matrix* $\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$ *is a subspace of the unobservability subspace of the realization* (4.6).

**Proof.** Obviously the unobservability subspace of the realization given in (4.6) is determined by the pair

$$\left([(DU^* - V^*)\mathscr{P}^{-1}, -C], \begin{bmatrix}\mathscr{A} & 0\\ -BU^*\mathscr{P}^{-1} & A\end{bmatrix}\right).$$

Now let us assume that the column vectors of a matrix $\begin{bmatrix}\alpha_S\\ \beta_S\end{bmatrix}q$ determine a basis in the unobservability subspace. Then

$$(DU^* - V^*)\mathscr{P}^{-1}\alpha_S - C\beta_S = 0, \tag{4.19}$$

$$\begin{bmatrix}\mathscr{A} & 0\\ -BU^*\mathscr{P}^{-1} & A\end{bmatrix}\begin{bmatrix}\alpha_S\\ \beta_S\end{bmatrix} = \begin{bmatrix}\alpha_S\\ \beta_S\end{bmatrix}\Gamma_S \tag{4.20}$$

for some matrix $\Gamma_S$.

First we show that the column vectors of $\alpha_S$ are linearly independent. In fact, if a vector $\gamma$ is in the kernel space of $\alpha_S$, i.e. $\alpha_S\gamma = 0$, then the equation in (4.20) implies that

$$\alpha_S\Gamma_S\gamma = \mathscr{A}\alpha_S\gamma = 0,$$

thus $\Gamma_S\gamma \in \operatorname{Ker}(\alpha_S)$, so the kernel subspace of $\alpha_S$ is $\Gamma_S$ invariant. Especially, it contains an eigenvector of $\Gamma_S$. In other words there exists a vector $\gamma$, for which the identities

$$\alpha_S\gamma = 0,$$
$$\Gamma_S\gamma = \mu\gamma$$

for some $\mu$ hold. Multiplying the Eqs. (4.19) and (4.20) from the right by $\gamma$ we obtain that

$$C\beta_S\gamma = 0,$$
$$A\beta_S\gamma = \mu\beta_S\gamma.$$

Thus the vector $\beta\gamma$ is in the unobservability subspace determined by the pair $(C, A)$. Using the assumed observability of the realization $S$ we get that $\beta_S\gamma = 0$. Consequently

$$\begin{bmatrix}\alpha_S\\ \beta_S\end{bmatrix}\gamma = 0.$$

But the column vectors of the matrix $\begin{bmatrix}\alpha_S\\ \beta_S\end{bmatrix}$ are linearly independent, thus $\gamma = 0$, proving that the column vectors of $\alpha_S$ are linearly independent.

Now Eq. (4.20) gives that

$$\mathscr{A}\alpha_S = \alpha_S\Gamma_S,$$
$$-BU^*\mathscr{P}^{-1}\alpha_S + A\beta_S = \beta_S\Gamma_S.$$

The first relation proves (4.17). Adding $-s\beta_S$ to both sides of the second equation and multiplying on the left by $-(sI - A)^{-1}$ we get

$$(sI - A)^{-1}BU^*\mathscr{P}^{-1}\alpha_S + \beta_S = (sI - A)^{-1}\beta_S(sI - \Gamma_S).$$

Multiplying from the left by $C$ and using Eq. (4.19) we obtain that

$$(D + C(sI - A)^{-1}B)U^*\mathscr{P}^{-1}\alpha_S - V^*\mathscr{P}^{-1}\alpha_S = C(sI - A)^{-1}\beta_S(sI - \Gamma_S).$$

In other words

$$(S(s)U^* - V^*)\mathscr{P}^{-1}\alpha_S(sI - \Gamma_S)^{-1} = C(sI - A)^{-1}\beta_S. \tag{4.21}$$

Since from (4.17) we have $\alpha_S(sI - \Gamma_S)^{-1} = (sI - \mathscr{A})^{-1}\alpha_S$, formula (4.18) follows immediately.

At the same time (4.17) together with the fact that the column vectors of the matrix $\alpha_S$ are linearly independent implies that the dimension of the unobservability subspace in the realization of $Q$ given by (4.6) is at most $n$.

Conversely, assume that $\alpha$ and $\beta$ satisfy (4.17) and (4.18). Then the first equation in (4.20) is immediately satisfied. Now using equation $\alpha(sI - \Gamma)^{-1} = (sI - \mathscr{A})^{-1}\alpha$ in Eq. (4.18) then multiplying by $(sI - \Gamma)^{-1}$ and evaluating it at infinity we obtain that

$$(DU^* - V^*)\mathscr{P}^{-1}\alpha - C\beta = 0$$

which is Eq. (4.19).

Now subtracting this value we get that

$$C(sI - A)^{-1}BU^*\mathscr{P}^{-1}\alpha + C\beta = C(sI - A)^{-1}\beta(sI - \Gamma).$$

I.e.

$$C(sI - A)^{-1}(BU^*\mathscr{P}^{-1}\alpha - A\beta + \beta\Gamma) = 0.$$

Using the assumed observability of the pair $(C, A)$ the second equation in (4.18) is obtained. Eqs. (4.19) and (4.20) together imply that the range of $\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$ is in the unobservability subspace, concluding the proof. □

To formulate a consequence of Theorem 4.1 and Proposition 4.1 as a connection between the McMillan-degrees of $S$ and $Q$ let us introduce the notation:

$$d_{o,Q} = \text{dimension of the unobservability subspace of the realization (4.6).}$$

and

$$\begin{aligned} n &= \text{McMillan-degree of } \Theta, \\ n_S &= \text{McMillan-degree of } S, \\ n_Q &= \text{McMillan-degree of } Q. \end{aligned}$$

**Corollary 4.1.** *Assume that the realization of the function S is minimal. Then*

$$n_Q = n + n_S - d_{o,Q}. \tag{4.22}$$

*Moreover*

$$d_{o,Q} \leqslant n \quad i.e. \quad n_Q \geqslant n_S.$$

*4.2.2. Inverse transformation*

Now let us analyze the controllability and unobservability subspaces of the realization (4.8) of $S$ obtained from that of $Q$ using the inverse linear fractional transformation.

**Proposition 4.2.** *Let Q be a Schur-function with an observable realization*

$$Q = \left(\begin{array}{c|c} A_Q & B_Q \\ \hline C_Q & D \end{array}\right),$$

*satisfying condition (2.3). Then the function* $S = T_\Theta^{-1}(Q)$ *has the observable realization*

$$S = \left(\begin{array}{c|c} A_Q - Y\mathscr{P}^{-1}VC_Q & B_Q - Y\mathscr{P}^{-1}(VD - U) \\ \hline C_Q & D \end{array}\right), \tag{4.23}$$

*where Y is defined by the equation*

$$(Q(s)U^* - V^*)(sI + \mathscr{A}^*)^{-1} = C_Q(sI - A_Q)^{-1}Y. \tag{4.24}$$

**Proof.** Let us apply the state transformation defined by the matrix

$$T = \begin{bmatrix} I & Y \\ 0 & \mathscr{P} \end{bmatrix}$$

to the realization (4.8). Using Eqs. (3.2), (3.1) we get that

$$[C_Q, (V^* - DU^*)\mathscr{P}^{-1}] \begin{bmatrix} I & Y \\ 0 & \mathscr{P} \end{bmatrix} = [C_Q, 0],$$

$$\begin{bmatrix} I & Y \\ 0 & \mathscr{P} \end{bmatrix}^{-1} \begin{bmatrix} B_Q \\ VD - U \end{bmatrix} = \begin{bmatrix} B_Q - Y\mathscr{P}^{-1}(VD - U) \\ \mathscr{P}^{-1}(VD - U) \end{bmatrix},$$

$$\begin{bmatrix} A_Q & -B_QU^*\mathscr{P}^{-1} \\ VC_Q & \mathscr{A} + (U - VD)U^*\mathscr{P}^{-1} \end{bmatrix} \begin{bmatrix} I & Y \\ 0 & \mathscr{P} \end{bmatrix} = \begin{bmatrix} I & Y \\ 0 & \mathscr{P} \end{bmatrix} \begin{bmatrix} A_Q - Y\mathscr{P}^{-1}VC_Q & 0 \\ \mathscr{P}^{-1}VC_Q & -\mathscr{A}^* \end{bmatrix}.$$

Thus the realization (4.23) is obtained. It is obviously observable if $(C_Q, A_Q)$ is an observable pair. This concludes the proof of the proposition. □

Let us start the analysis of the controllability subspace of the realization defined in (4.23) with a special case: it will provide a sufficient condition for the minimality of (4.23) and ensure that the McMillan degrees of $S$ and $Q$ coincide.

**Theorem 4.2.** *Assume that the realization of $Q$ given in* (4.7) *is minimal and $Q$ satisfies* (2.3). *If the functions $VQ(s) - U$ and $sI - \mathscr{A}$ have no common left zero-functions then the realization of $S$ in* (4.23) *is minimal, especially $n_S = n_Q$.*

**Proof.** Observability follows from Proposition 4.2. To check controllability, let us apply the P-H-B test. Assume that there exist a nonzero column vector $\alpha_Q$ and $\mu \in \mathbb{C}$ such that

$$\alpha_Q^*(B_Q - Y\mathscr{P}^{-1}(VD - U)) = 0, \tag{4.25}$$

$$\alpha_Q^*(A_Q - Y\mathscr{P}^{-1}VC_Q) = \mu\alpha_Q^*. \tag{4.26}$$

It is immediately obtained that $\alpha_Q^* Y \neq 0$. Otherwise Eqs. $\alpha_Q^* B_Q = 0$, $\alpha_Q^* A_Q = \mu\alpha^*$ would hold, contradicting to the controllability of $(A_Q, B_Q)$.

Computing (4.25)$U^*$ − (4.26)$Y$, we obtain that

$$\alpha_Q^*(B_Q U^* - Y\mathscr{P}^{-1}VDU^* + Y\mathscr{P}^{-1}UU^* - A_Q Y + Y\mathscr{P}^{-1}VC_Q Y) = -\mu\alpha_Q^* Y.$$

Using (3.1) and (3.2) we get that

$$\alpha_Q^*(Y\mathscr{A}^* - Y\mathscr{P}^{-1}VV^* + Y\mathscr{P}^{-1}UU^*) = -\mu\alpha_Q^* Y.$$

Using (2.4),

$$(\alpha_Q^* Y\mathscr{P}^{-1})(\mu I - \mathscr{A}) = 0. \tag{4.27}$$

On the other hand equations in (4.25) and (4.26) can be arranged into the following form

$$[\alpha_Q^*, -\alpha_Q^* Y\mathscr{P}^{-1}]\begin{bmatrix} A_Q - \mu I & B_Q \\ VC_Q & VD - U \end{bmatrix} = [0, 0]. \tag{4.28}$$

Eqs. (4.27) and (4.28) indicate that the functions $sI - \mathscr{A}$ and $VQ(s) - U$ have a common zero direction at $s = \mu$, contradicting our assumption. Thus the realization of $S$ given in (4.23) is controllable, concluding the proof. □

Introduce the notation

$$d_{c,S} = \text{dimension of the controllability subspace of the realization (4.23).}$$

According to Proposition 4.2 the realization (4.23) is observable. Consequently

$$n_S = d_{c,S}.$$

The argument applied in the proof of Theorem 4.2 can be modified to give a more precise description between the McMillan-degree of $S$ and some extra "interpolation-type" properties of $Q$. This leads to the following theorem.

**Theorem 4.3.** *Assume that the realization of $Q$ given in* (4.7) *is minimal and $Q$ satisfies* (2.3).

*Using the matrices in the realization of $S$ given in* (4.23) *consider the following set of equations*:

$$\beta^* \mathscr{A} = \Gamma \beta^*, \tag{4.29}$$

$$[\gamma^*, \beta^*] \begin{bmatrix} A_Q & B_Q \\ VC_Q & VD - U \end{bmatrix} = [\Gamma \gamma^* \quad 0]. \tag{4.30}$$

*Then,*

(i) *if the linearly independent column vectors of the* $\alpha_Q$ *span the orthogonal complement of the controllability subspace of the realization* (4.23), *then there exists a matrix* $\Gamma_Q$, *for which*

$$\gamma^* = \alpha_Q^*, \quad \Gamma = \Gamma_Q, \quad \text{and} \quad \beta^* = \alpha_Q^* Y \mathscr{P}^{-1},$$

*solve Eqs.* (4.29) *and* (4.30).

(ii) *Conversely, if the matrices* $\beta, \gamma, \Gamma$ *solve these equations, where the column vectors of the matrix* $\beta$ *are linearly independent, then* $\beta^* = \gamma^* Y \mathscr{P}^{-1}$, *and the column vectors of* $\gamma$ *are orthogonal to the controllability subspace.*

**Proof.** See the Appendix. □

Comparing Eq. (4.30) to (3.4) in Theorem 3.1 we see that according to the previous theorem the uncontrollability of the realization of $S$ given in (4.23) is connected to the fact that the function $Q$ satisfies extra interpolation conditions. This connection will be analysed in more details in the next section.

Note that Theorem 4.3 implies again that the inequalities $n_S \leqslant n_Q \leqslant n_S + n$ hold.

*4.2.3. Additionally prescribed interpolation conditions*

Let us recall Eq. (4.18):

$$(S(s)U^* - V^*)\mathscr{P}^{-1}\alpha_S(sI - \Gamma_S)^{-1} = C(sI - A)^{-1}\beta_S.$$

Observe that the poles of the function on the right hand side are among the eigenvalues of the matrix $A$, while the function standing on the left hand side has formally poles at the eigenvalues of $A$ and $\Gamma_S$. Thus the eigenvalues of this latter one should be cancelled. This is again an interpolation type condition where the interpolation nodes are defined by the eigenvalues of $\Gamma_S$ forming a subset of the eigenvalues of $\mathscr{A}$. These interpolation nodes are in the left half plane $\mathbb{C}^-$. This kind of interpolation problems is analyzed in [7].

Theorem 3.1 gives a possibility of formulating this interpolation problem in a more precise way. To this aim observe that Eqs. (4.19), (4.20), characterizing the unobservability subspace of the realization (4.6), can be written in the following form:

$$\mathscr{A}\alpha_S = \Gamma_S \alpha_S, \tag{4.31}$$

$$\begin{bmatrix} A & BU^*\mathscr{P}^{-1} \\ C & (DU^* - V^*)\mathscr{P}^{-1} \end{bmatrix} \begin{bmatrix} \beta_S \\ -\alpha_S \end{bmatrix} = \begin{bmatrix} \beta_S \Gamma_S \\ 0 \end{bmatrix}. \tag{4.32}$$

Thus Theorem 3.1 implies the following corollary using the notations introduced in Theorem 4.1.

**Corollary 4.2.** *There exists a matrix-valued polynomial* $\psi$ *such that the function*

$$(S(s)U^* - V^*)\mathscr{P}^{-1}(\alpha_S(sI - \Gamma_S)^{-1} + \psi(s)) \tag{4.33}$$

*is analytic at the eigenvalues of* $\Gamma_S$.

*Conversely, if for an observable pair* $(\alpha_S, \Gamma_S)$ *satisfying Eq.* (4.31) *there exists a matrix-valued polynomial* $\psi$ *such that the function* (4.33) *is analytic at the eigenvalues of* $\Gamma_S$ *then there exists a matrix* $\beta_S$ *such that the range of* $\begin{bmatrix}\alpha_S\\ \beta_S\end{bmatrix}$ *is a subspace of the unobservability subspace of the realization* (4.6) *of* $Q$.

Note that in the special case when $\mathscr{A} = \operatorname{diag}(-\bar{s}_1, \ldots, -\bar{s}_n)$, $n_Q = n_S$ and $S$ is analytic at the eigenvalues of $\mathscr{A}$ then these interpolation conditions can be expressed as

$$S(-\bar{s}_j)U^*\mathscr{P}^{-1}e_j = V^*\mathscr{P}^{-1}e_j, \quad i = 1, \ldots, n, \tag{4.34}$$

where $e_j$ denotes the $j$th unit vector.

Let us emphasize that although the analysis carried out in Theorem 4.1 based on a special realization of $Q$, Corollary 4.2 is formulated for the function $S$, roughly stating that the number of interpolation conditions satisfied by the function $S$ is at least

$$n + n_S - n_Q.$$

Now let us recall Eqs. (B.4) and (B.3) from the Appendix.

$$[\alpha_Q^*, -\alpha_Q^* Y\mathscr{P}^{-1}]\begin{bmatrix} A_Q & B_Q \\ VC_Q & VD-U \end{bmatrix} = [\Gamma_Q\alpha_Q^*, 0], \tag{4.35}$$

$$(\alpha_Q^* Y\mathscr{P}^{-1})\mathscr{A} = \Gamma_Q(\alpha_Q^* Y\mathscr{P}^{-1}). \tag{4.36}$$

Eq. (4.35) can be formulated as the pair $(\alpha_Q^* Y\mathscr{P}^{-1}, \Gamma_Q)$ forms a left zero pair of the function $VQ(s) - U$. Furthermore Eq. (4.36) implies – using again that the row vectors of $\alpha_Q^* Y\mathscr{P}^{-1}$ are linearly independent – that the eigenvalues of $\Gamma_Q$ form a subset of that of $\mathscr{A}$.

In particular, if $\mathscr{A} = \operatorname{diag}(-\bar{s}_1, \ldots, -\bar{s}_n)$ and assuming – for the sake of simplicity – that $A_Q$ and $\mathscr{A}$ has no common eigenvalues, then for some subset $N_c \subset \{1, \ldots, n\}$ of size $|N_c| = n_Q - n_S$ the function $Q$ should satisfy the following set of *extra* interpolation conditions:

$$e_j^* VQ(-\bar{s}_j) = e_j^* U, \quad j \in N_c, \tag{4.37}$$

where $e_j$ denotes the $j$th unit vector.

Theorem 3.1 gives again a possibility of expressing these additional interpolation conditions to be satisfied now by the function $Q$ in a more precise way. Namely, the following corollary holds – using the notations introduced in Theorem 4.3.

**Corollary 4.3.** *There exists a matrix-valued polynomial* $\psi$ *such that*

$$((sI - \Gamma_Q)^{-1}\alpha_Q^* Y\mathscr{P}^{-1} + \psi)(VQ(s) - U)$$

*is analytic at the eigenvalues of* $\Gamma_Q$.

*Conversely, if for a pair of matrices* $(\beta_Q, \Gamma_Q)$ *satisfying Eq.* (4.29) *assuming that the column vectors of* $\beta_Q$ *are linearly independent there exists a matrix-valued polynomial* $\psi$ *such that the function*

$$((sI - \Gamma_Q)^{-1}\beta_Q^* + \psi)(VQ(s) - U)$$

*is analytic at the eigenvalues of* $\Gamma_Q$ *then there exists a matrix* $\alpha_Q$ *such that*

$$\beta_Q^* = \alpha_Q^* Y\mathscr{P}^{-1},$$

*and the column vectors of* $\alpha_Q$ *are orthogonal to the controllability subspace of the realization* (4.23).

Let us emphasize in this case again that although the analysis carried out in the proof of Theorem 4.3 is based on the special realization of $S$ taken in (4.23), Corollary 4.3 is formulated for the function $Q$, roughly stating that the number of additional interpolation condition satisfied by $Q$ is equal to

$$n_Q - n_S.$$

Comparing Corollary 4.3 with Corollary 4.2 we obtain that the total number of additional interpolation conditions formulated for the functions $Q$ and $S$, with interpolation nodes in $\mathbb{C}^-$, is $(n + n_S - n_Q) + (n_Q - n_S) = n$. For both functions the additional interpolation nodes form subsets of the eigenvalues of $\mathscr{A}$. The following theorem shows that these sets of prescribed interpolation condition are in a certain sense "complementary" to each other.

**Theorem 4.4.** *Assume that $Q = T_\Theta(S)$ is a solution of the interpolation Problem 2.1. Then there exists a nonsingular transformation $R$ such that*

$$R\mathscr{A}R^{-1} = \begin{bmatrix} \Gamma_Q & 0 \\ \Gamma_0 & \Gamma_S \end{bmatrix}, \tag{4.38}$$

*and matrices $\alpha_Q, \alpha_S$ of full column rank such that*

*$(\Gamma_S, \mathscr{P}^{-1}\alpha_S)$ form a right-zero pair of the function $(SU^* - V^*)$,*

*while*

*$(\Gamma_Q, \alpha_Q^* Y\mathscr{P}^{-1})$ form a left-zero pair of the function $VQ - U$.*

**Proof.** Consider a minimal realization of $Q$ in the form (4.6) and the corresponding – possibly non minimal – realization of $S$ given in (4.23). Define $\alpha_Q$ as in Theorem 4.3, i.e. its linearly independent column vectors span the orthogonal complement of the controllability subspace of the realization (4.23). Extend these vectors to a basis and form this way the nonsingular matrix $T^* = [\delta, \alpha_Q]$, and consider the corresponding partition of its inverse $T^{-1} = [\mu, \nu]$.

Then

$$S = \left(\begin{array}{c|c} \delta^*(A_Q - Y\mathscr{P}^{-1}VC_Q)\mu & \delta^*(B_Q - Y\mathscr{P}^{-1}(VD - U)) \\ \hline C_Q\mu & D \end{array}\right) \tag{4.39}$$

provides a minimal realization for $S$. Let us introduce the following notations:

$$A_S = \delta^*(A_Q - Y\mathscr{P}^{-1}VC_Q)\mu,$$
$$B_S = \delta^*(B_Q - Y\mathscr{P}^{-1}(VD - U)),$$
$$C_S = C_Q\mu.$$

As we know from the proof of Theorem 4.3 the row vectors of $\alpha_Q^* Y\mathscr{P}^{-1}$ are linearly independent. Thus we can define $R$ as follows:

$$R = \begin{bmatrix} \alpha_Q^* Y\mathscr{P}^{-1} \\ \kappa^* \end{bmatrix}, \tag{4.40}$$

where the row vectors of $\kappa^*$ provide an arbitrary extension of the row vectors of $\alpha_Q^* Y\mathscr{P}^{-1}$ to a basis. Consider the induced partition of $R^{-1}$:

$$R^{-1} = [\rho, \sigma].$$

Then part (i) of Theorem 4.3 and Eq. (4.29) together with the identities $\alpha_Q^* Y\mathscr{P}^{-1}\rho = I$, $\alpha_Q^* Y\mathscr{P}^{-1}\sigma = 0$ give that

$$R\mathscr{A}R^{-1} = \begin{bmatrix} \Gamma_Q & 0 \\ \kappa^*\mathscr{A}\rho & \kappa^*\mathscr{A}\sigma \end{bmatrix}, \tag{4.41}$$

and according to Theorem 4.3 and Corollary 4.3 the pair $(\Gamma_Q, \alpha_Q^* Y \mathscr{P}^{-1})$ forms a left-zero pair of $VQ - U$.

Define the matrices $\Gamma_S, \alpha_S, \beta_S$ as follows:

$$\Gamma_S = \kappa^* \mathscr{A} \sigma, \Gamma_0 = \kappa^* \mathscr{A} \rho,$$
$$\alpha_S = \sigma,$$
$$\beta_S = \delta^* Y \mathscr{P}^{-1} \sigma.$$

According to Theorem 3.1 to prove that $(\Gamma_S, \mathscr{P}^{-1}\alpha_S)$ provide a right-zero pair of the function $SU^* - V^*$ it is enough to check that Eqs. (4.31) and (4.32) hold, where the (minimal) realization of $S$ is given by (4.39).

The (2, 2) equation in (4.41) gives that

$$\mathscr{A}\sigma = \sigma\kappa^* \mathscr{A}\sigma,$$

i.e.

$$\mathscr{A}\alpha_S = \alpha_S \Gamma_S \tag{4.42}$$

proving that Eq. (4.31) holds.

To prove that Eq. (4.32) we need the following identity:

$$\begin{aligned}
\mu\beta_S &= \mu\delta^* Y \mathscr{P}^{-1}\sigma \\
&= (I - \nu\alpha_Q^*) Y \mathscr{P}^{-1}\sigma \\
&= Y\mathscr{P}^{-1}\sigma = Y\mathscr{P}^{-1}\alpha_S,
\end{aligned}$$

using that $\alpha_Q^* Y \mathscr{P}^{-1}\sigma = 0$. Consequently,

$$\begin{aligned}
C_S\beta_S - (DU^* - V^*)\mathscr{P}^{-1}\alpha_S &= C_Q\mu\beta_S - (DU^* - V^*)\mathscr{P}^{-1}\alpha_S \\
&= C_Q(\mu\beta_S - Y\mathscr{P}^{-1}\sigma) \\
&= 0,
\end{aligned}$$

using Eq. (3.1), i.e. $DU^* - V^* = C_Q Y$, giving that the second equation in (4.32) holds.

To prove the second equation of (4.32) the quantity

$$\begin{aligned}
&A_S\beta_S - B_S U^* \mathscr{P}^{-1}\alpha_S \\
&\quad = \delta^*(A_Q - Y\mathscr{P}^{-1}VC_Q)\mu\beta_S - \delta^*(B_Q - Y\mathscr{P}^{-1}(VD - U))U^*\mathscr{P}^{-1}\alpha_S
\end{aligned}$$

should be computed. Using the Eqs. $\mu\beta_S = Y\mathscr{P}^{-1}\alpha_S$ obtained above, furthermore that $B_Q U^* - A_Q Y = Y\mathscr{A}^*$ (Eq. (3.2)), $C_Q Y = DU^* - V^*$ (Eq. (3.1)) and also that $-\mathscr{A}^* + \mathscr{P}^{-1}VV^* - \mathscr{P}^{-1}UU^* = \mathscr{P}^{-1}\mathscr{A}\mathscr{P}$ (Eq. (4.2)) we get that

$$\begin{aligned}
&\delta^*(A_Q - Y\mathscr{P}^{-1}VC_Q)\mu\beta_S - \delta^*(B_Q - Y\mathscr{P}^{-1}(VD - U))U^*\mathscr{P}^{-1}\alpha_S \\
&\quad = \delta^*(A_Q - Y\mathscr{P}^{-1}VC_Q)Y\mathscr{P}^{-1}\sigma - \delta^*(B_Q - Y\mathscr{P}^{-1}(VD - U))U^*\mathscr{P}^{-1}\alpha_S \\
&\quad = \delta^*(-Y\mathscr{A}^* - Y\mathscr{P}^{-1}(VC_QY - (VD - U)U^*))\mathscr{P}^{-1}\alpha_S \\
&\quad = \delta^*Y(-\mathscr{A}^* + \mathscr{P}^{-1}VV^* - \mathscr{P}^{-1}UU^*)\mathscr{P}^{-1}\alpha_S \\
&\quad = \delta^*Y\mathscr{P}^{-1}\mathscr{A}\alpha_S \\
&\quad = \delta^*Y\mathscr{P}^{-1}\alpha_S\Gamma_S \\
&\quad = \beta_S\Gamma_S,
\end{aligned}$$

proving that the second equation on (4.32) holds, as well. Now Theorem 3.1 gives that $(\Gamma_S, \mathscr{P}^{-1}\alpha_S)$ form a right-zero pair of $SU^* - V^*$, concluding the proof of the theorem. □

## 5. Solutions of the interpolation problem with "low complexity"

We have considered, so far, the situation where the degree of the interpolant $Q$ could differ from the number $n$ of interpolating conditions. The main motivation for this analysis was to reduce the McMillan degree of the interpolant as much as possible. That meant that, while the degree of the interpolant was in general greater than $n$, it could possibly drop below that value. If we now *impose* the degree of the interpolant to be exactly $n$, then we can say something more about the structure of the interpolants. This will provide explicit state space conditions for Schur interpolants. We will show that, if we relax the condition of $Q$ being Schur, we in fact obtain a parametrization for stable interpolating functions; if we relax also the stability condition, we obtain a parametrization for interpolating proper rational functions.

Consider again the interpolation Problem 2.2:

**Problem 2.2.** Given the matrix $\mathscr{A}$ of size $n \times n$ with eigenvalues in the left half plane $\mathbb{C}^-$, matrices $U$, $V$ of size $n \times p$ and a constant strictly contractive matrix $D$ of size $p \times p$, parametrize all functions $Q$, for which

(i) $Q$ is a Schur-function;
(ii)
$$\begin{cases} (Q(s)U^* - V^*)(sI + \mathscr{A}^*)^{-1} \text{ is analytic on } \mathbb{C}^+, \\ Q(\infty) = D. \end{cases} \tag{5.1}$$
(iii) $Q$ is rational of McMillan degree exactly $n$,
(iv) $Q$ is irreducible.

Let us recall that irreducibility of an interpolating Schur function means that it is uniquely determined by its DSS inner factor.

For any given matrix $B$ of size $n \times p$, denote by

$$F_B^* = \left(\begin{array}{c|c} -\mathscr{A}^* & B \\ \hline -U^* & I \end{array}\right),$$
$$G_B^* = \left(\begin{array}{c|c} -\mathscr{A}^* & B \\ \hline -V^* & D \end{array}\right).$$

**Theorem 5.1.** *The Schur-function $Q$ is a solution of the interpolation Problem 2.2 if and only if it has the realization*

$$Q = \left(\begin{array}{c|c} -\mathscr{A}^* + BU^* & B \\ \hline DU^* - V^* & D \end{array}\right). \tag{5.2}$$

*Especially, in this case*

$$Q = G_B^*(F_B^*)^{-1}.$$

**Proof.** Assume that $Q$ is an interpolating irreducible function of degree $n$; then, in view of Proposition 3.1, any realization $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ of $Q$ satisfies Eqs. (3.1) and (3.2) for some matrix $Y$. A check on dimensions show that $Y$ is square. We claim that $Y$ is invertible. If not, there exists a non identically zero matrix $C_1$ of the same size as $C$ such that $C_1Y = 0$; but then also $Q_1 := \left[\begin{array}{c|c} A & B \\ \hline C + C_1 & D \end{array}\right]$ satisfies the interpolation conditions (2.3); but $Q$ and $Q_1$ have the same $(A, B)$ pair and thus the same DSS factor, which leads to a contradiction. Therefore, Eqs. (3.1) and (3.2) can be rewritten as:

$$DU^* - V^* = C_QY \tag{5.3}$$

and

$$Y^{-1}A_QY = -\mathscr{A}^* + Y^{-1}B_QU^* \tag{5.4}$$

that is, $Q = \left[\begin{array}{c|c} -\mathscr{A}^* + Y^{-1}B_QU^* & Y^{-1}B_Q \\ \hline DU^* - V^* & D \end{array}\right]$; setting $B := Y^{-1}B_Q$, we immediately get (5.2).

Conversely, if $Q$ has realization (5.2), then it is an interpolating function (in view of Proposition 3.1); it is irreducible because $Y = I$ and thus has 0 kernel.

Finally, the calculation

$$\begin{aligned} G^*(F^*)^{-1} &= \left(\begin{array}{c|c} -\mathscr{A}^* & B \\ \hline -V^* & D \end{array}\right)\left(\begin{array}{c|c} -\mathscr{A}^* + BU^* & B \\ \hline U^* & I \end{array}\right) \\ &= \left(\begin{array}{cc|c} -\mathscr{A}^* + BU^* & 0 & B \\ BU^* & -\mathscr{A}^* & B \\ \hline DU^* & -V^* & D \end{array}\right) \\ &= \left(\begin{array}{c|c} -\mathscr{A}^* + BU^* & B \\ \hline DU^* - V^* & D \end{array}\right) = Q \end{aligned}$$

concludes the proof. □

Notice that the fact that the function $Q$ in (5.2) is Schur implies that $(-\mathscr{A}^*, B)$ is controllable, because $-\mathscr{A}^*$ is antistable and $-\mathscr{A}^* + BU^*$ is stable.

It is now easy to see how to construct an example of an interpolating function of lower degree. Let $Y$ be a flat full row-rank matrix of dimension $n_1 \times n$ (which is thus right invertible), and $B_1$ a matrix of dimension $n_1 \times p$. Denote by $Y^{-R}$ the right inverse of $Y$. Then, setting $A_{11} := -Y\mathscr{A}^*Y^{-R} + B_1U^*Y^{-R}$ and $C_1 := (DU^* - V^*)Y^{-R}$, we immediately see that $Q_1 = \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array}\right]$ is an interpolating function of degree at most $n_1$.

On the other hand to construct an interpolant of degree $n$ which is not irreducible let us start with two controllable pairs of matrices $(A_Q, B_Q)$ and $(\mathscr{A}, U)$ where $A$ and $\mathscr{A}$ are stable matrices of size $n \times n$ for which the solution $Y$ of the Sylvester-equation

$$A_QY - B_QU^* = Y\mathscr{A}^*$$

is *singular* (it is not hard to produce such matrices, provided $U$ and $B_Q$ have at least two columns; in fact, to find conditions to ensure the invertibility of the solution of a Sylvester-equation in the multivariable case is an interesting open problem). Now consider any matrix $C_Q$ (of the appropriate size) for which the pair $(C_Q, A_Q)$ is observable such that there exists a matrix $D$ making the function $Q(s) = D + C_Q(sI - A_Q)^{-1}B_Q$ a Schur-function. Finally define $V$ as follows:

$$V^* = DU^* - C_Q Y.$$

Then the equations in Proposition 3.1 are satisfied and thus $Q$ is a solution of Problem 2.1 defined by the triplet $\mathscr{A}, U, V$ constructed above. $Q$ is not irreducible but it is of McMillan-degree $n$.

In Theorem 5.2 and Proposition 5.1 we provide conditions assuring that the function $Q$ with realization (5.2) becomes a Schur or stable function, respectively.

**Theorem 5.2.** *Let $Q$ be as in (5.2), and suppose it has McMillan-degree $n$. Then $Q$ is a Schur-function if and only if*

$$B = -R^{-1}\left(U - VD - \tilde{V}(I - D^*D)^{\frac{1}{2}}\right),$$

*where $\tilde{V} \in \mathbb{C}^{n\times p}$ and $R \in \mathbb{C}^{n\times n}$, $R$ is positive definite and the equation*

$$\mathscr{A}R + R\mathscr{A}^* + UU^* - VV^* - \tilde{V}\tilde{V}^* = 0 \tag{5.5}$$

*holds.*

**Proof.** According to the bounded real lemma the function $Q$ is a Schur-function if and only if the equation

$$\begin{aligned} &R(A_Q + B_Q D^*(I - DD^*)^{-1}C_Q) + (A_Q + B_Q D^*(I - DD^*)^{-1}C_Q)^* R \\ &\quad + C_Q^*(I - DD^*)^{-1}C_Q + RB_Q(I - D^*D)^{-1}B_Q^* R = 0 \end{aligned} \tag{5.6}$$

has a positive definite solution $R$, where the matrices $(A_Q, B_Q, C_Q, D)$ provide a minimal realization of $Q$.

The assumption on the McMillan degree of $Q$ implies that (5.2) is also a minimal realization. Expressing the matrices $A_Q$ and $C_Q$ using (5.2) straightforward calculation gives that

$$\begin{aligned} &R\mathscr{A}^* + \mathscr{A}R + (UU^* - VV^*) \\ &\quad - (RB + U - VD)(I - D^*D)^{-1}(B^*R + U^* - D^*V^*) = 0 \end{aligned}$$

holds. Introducing the notation

$$\tilde{V} = (RB + U - VD)(I - D^*D)^{-\frac{1}{2}}$$

Eq. (5.5) is obtained. □

Based on this a parameterization of Schur solutions to the interpolation problem can be obtained using the results in [1,12].

Notice that (5.2) yields an interpolant even if we drop the requirement to be Schur, but require, for instance just stability. This is analysed in the next proposition.

**Proposition 5.1.** *Let $Q$ be as in (5.2), and suppose it has McMillan-degree $n$. Then $Q$ is stable if and only if*

$$B = \mathscr{P}_W^{-1}(U - W),$$

*where $W \in \mathbb{C}^{n\times p}$ satisfies the generalized Pick condition, i.e. the Lyapunov equation*

$$\mathscr{A}\mathscr{P}_W + \mathscr{P}_W\mathscr{A}^* + UU^* - WW^* = 0 \tag{5.7}$$

*has a unique positive definite solution.*

**Proof.** Assume that $Q$ with the realization given in (5.2) is a stable function. The assumption concerning its McMillan-degree implies that this realization is minimal, i.e. there exists a matrix $\mathscr{P}_Q > 0$ for which the equation

$$(-\mathscr{A}^* + BU^*)\mathscr{P}_Q + \mathscr{P}_Q(-\mathscr{A} + UB^*) + BB^* = 0$$

holds. This can be arranged to

$$\mathscr{A}\mathscr{P}_Q^{-1} + \mathscr{P}_Q^{-1}\mathscr{A}^* + UU^* - (U + \mathscr{P}_Q^{-1}B)(U + \mathscr{P}_Q^{-1}B)^* = 0.$$

Introducing the notations

$$\mathscr{P}_W = \mathscr{P}_Q^{-1},$$
$$W = U - \mathscr{P}_Q^{-1}B.$$

Eq. (5.7) is obtained.

Conversely assume that (5.7) holds. A straightforward calculation gives that

$$\mathscr{P}_W(-\mathscr{A}^* - \mathscr{P}_W^{-1}(U - W)U^*)\mathscr{P}_W^{-1} = \mathscr{A} + W(U^* - W^*)\mathscr{P}_W^{-1},$$
$$(\mathscr{A} + W(U^* - W^*)\mathscr{P}_W^{-1})\mathscr{P}_W + \mathscr{P}_W(\mathscr{A} + W(U^* - W^*)\mathscr{P}_W^{-1})^* + (U - W)(U^* - W^*) = 0.$$

Now the controllability of the pair

$$(\mathscr{A}^* + \mathscr{P}_W^{-1}(U - W)W^*, \mathscr{P}_W^{-1}(U - W))$$

implies the stability of $(\mathscr{A} + W(U^* - W^*)\mathscr{P}_W^{-1})$. $\square$

Notice that, if $Q$ has realization (5.2), then the matrix $Y$ in (3.2) is equal to the identity, and therefore, the representation (4.23) of $S$ in Proposition 4.2 takes on the form:

$$S = \left(\begin{array}{c|c} -\mathscr{A}^* + BU^* - \mathscr{P}^{-1}V(DU^* - V^*) & B - \mathscr{P}^{-1}(VD - U) \\ \hline DU^* - V^* & D \end{array}\right)$$
$$= \left(\begin{array}{c|c} \mathscr{P}^{-1}(\mathscr{A}\mathscr{P} + \mathscr{P}BU^* - VDU^* + UU^*) & B + \mathscr{P}^{-1}(U - VD) \\ \hline DU^* - V^* & D \end{array}\right)$$
$$= \left(\begin{array}{c|c} \mathscr{A} + \mathscr{P}BU^*\mathscr{P}^{-1} + (U - VD)U^*\mathscr{P}^{-1} & \mathscr{P}B + (U - VD) \\ \hline (DU^* - V^*)\mathscr{P}^{-1} & D \end{array}\right).$$

Setting eventually

$$B_S := \mathscr{P}B + U - VD,$$

we get

$$S = \left(\begin{array}{c|c} \mathscr{A} + B_S U^*\mathscr{P}^{-1} & B_S \\ \hline (DU^* - V^*)\mathscr{P}^{-1} & D \end{array}\right), \tag{5.8}$$

Finally note that the right null-pair corresponding to the zeros of the function $I - QQ^*$ inside $\mathbb{C}^-$ is given by the matrices

$$\left(-\mathscr{A}^* + B(I - D^*D)^{-\frac{1}{2}}\tilde{V}^*,\ V^* - (I - DD^*)^{-1}D(I - D^*D)^{\frac{1}{2}}\tilde{V}^*\right).$$

**Remark 5.1.** For any $B$, the transfer function $G_B^*(F_B^*)^{-1}$ has the realization given in (5.2) but it is not necessarily a Schur-function (or not even a stable one), therefore it does not necessarily produce a solution of the interpolation Problem 2.2. Still, if the spectra of $-\mathscr{A}^*$ and $-\mathscr{A}^* + BU^*$ are disjoint, it will provide a function which interpolates the data $\mathscr{A}, U, V, D$ without further constraints. In this sense, the above is a generalization of the Kimura–Georgiou parametrization.

## Acknowledgments

Part of this research took place while the authors were visiting the Mittag–Leffler Institute in February 2003. Their warm hospitality and support is gratefully acknowledged.

## Appendix A

### A.1. *Proof of Theorem 3.1*

(i) Assume that for $\phi(s) = \mathscr{C}_\phi(sI - \mathscr{A}_\phi)^{-1}\mathscr{B}_\phi + \psi(s)$, where $\psi$ is a matrix-valued polynomial, the function $F\phi^*$ is analytic at the eigenvalues of $-\mathscr{A}_\phi^*$. First we show that there exists a function $\overline{\phi}(s) = (sI - \mathscr{A}_\phi)^{-1}\mathscr{B}_\phi + \overline{\psi}(s)$, where $\overline{\psi}$ is a matrix-valued polynomial, such that $F\overline{\phi}^*$ is still analytic at the eigenvalues of $-\mathscr{A}_\phi^*$. In fact, assume that $\mathscr{A}_\phi$ is a $k \times k$ matrix. Then using that

$$s^j\phi(s) = \mathscr{C}_\phi\mathscr{A}_\phi^j(sI - \mathscr{A}_\phi)^{-1}\mathscr{B}_\phi + \psi_j(s),$$

for $j = 1, \dots, k-1$ where

$$\psi_j(s) := \sum_{l=0}^{j-1}\mathscr{C}_\phi\mathscr{A}_\phi^{j-l-1}\mathscr{B}_\phi s^l + s^j\psi(s) = \mathscr{C}_\phi\mathscr{A}_\phi^{j-1}\mathscr{B}_\phi + s\psi_{j-1}$$

is a matrix-valued polynomial, we get that the function

$$\begin{aligned} &F(s)[\phi^*(s), -s\phi^*(s), s^2\phi^*(s), \dots, (-s)^{k-1}\phi^*(s)] \\ &\quad = -F(s)\{\mathscr{B}_\phi^*(sI + \mathscr{A}_\phi^*)^{-1}[\mathscr{C}_\phi^*, \mathscr{A}_\phi^*\mathscr{C}_\phi^*, (\mathscr{A}_\phi^*)^2\mathscr{C}_\phi^*, \dots, (\mathscr{A}_\phi^*)^{k-1}\mathscr{C}_\phi^*] \\ &\qquad - [\psi^*(s), \psi_1^*(s), \psi_2^*(s), \dots, \psi_{k-1}^*(s)]\} \end{aligned}$$

should be analytic at the eigenvalues of $-\mathscr{A}_\phi^*$. Since, according to our assumption, the pair $(\mathscr{C}_\phi, \mathscr{A}_\phi)$ is observable, the matrix

$$M := \begin{bmatrix} \mathscr{C}_\phi \\ \mathscr{C}_\phi\mathscr{A}_\phi \\ \mathscr{C}_\phi\mathscr{A}_\phi^2 \\ \vdots \\ \mathscr{C}_\phi\mathscr{A}_\phi^{k-1} \end{bmatrix}$$

is left-invertible. Multiplying by $(M^{-L})^*$ we obtain that

$$F(s)(\mathscr{B}_\phi^*(sI + \mathscr{A}_\phi^*)^{-1} - \overline{\psi}(s)^*)$$

is analytic at the eigenvalues of $-\mathscr{A}_\phi^*$, where

$$\overline{\psi}(s) = M^{-L}\begin{bmatrix} \psi(s) \\ \psi_1(s) \\ \psi_2(s) \\ \vdots \\ \psi_{k-1}(s) \end{bmatrix}.$$

Now consider a closed contour $\Gamma$ separating the eigenvalues of $-\mathscr{A}_\phi^*$ from the (remaining) poles of $F$. Then

$$\int_\Gamma F(s)[\mathscr{B}_\phi^*(sI + \mathscr{A}_\phi^*)^{-1} - \overline{\psi}^*(s)]\,\mathrm{d}s = 0,$$

because the integrand is analytic at the eigenvalues of $-\mathscr{A}_\phi^*$. Multiplying term by term we obtain that

$$\begin{aligned}&\int_\Gamma D\mathscr{B}_\phi^*(sI + \mathscr{A}_\phi^*)^{-1}\,\mathrm{d}s - \int_\Gamma D\overline{\psi}^*(s)\,\mathrm{d}s\\ &\quad + \int_\Gamma C(sI - A)^{-1}B[\mathscr{B}_\phi^*(sI + \mathscr{A}_\phi^*)^{-1} - \overline{\psi}^*(s)]\,\mathrm{d}s = 0.\end{aligned}$$

The first term is well known to be equal to $D\mathscr{B}_\phi^*$; the second term is zero because $\overline{\psi}$ is a polynomial. Thus, introducing the notation

$$\Pi = \int_\Gamma (sI - A)^{-1}B[\mathscr{B}_\phi^*(sI + \mathscr{A}_\phi^*)^{-1} - \overline{\psi}^*(s)]\,\mathrm{d}s,$$

we obtain that

$$D\mathscr{B}_\phi^* + C\Pi = 0. \tag{A.1}$$

Let us compute the value of $A\Pi + \Pi\mathscr{A}_\phi^*$; using the identity $(sI - A)^{-1}A = -I + s(sI - A)^{-1}$ (and a similar one for $\mathscr{A}^*$), we get

$$\begin{aligned}A\Pi + \Pi\mathscr{A}_\phi^* &= \int_\Gamma A(sI - A)^{-1}B[\mathscr{B}_\phi^*(sI + \mathscr{A}_\phi^*)^{-1} - \overline{\psi}^*(s)]\,\mathrm{d}s\\ &\quad + \int_\Gamma (sI - A)^{-1}B[\mathscr{B}_\phi^*(sI + \mathscr{A}_\phi^*)^{-1} - \overline{\psi}^*(s)]\mathscr{A}_\phi^*\,\mathrm{d}s\\ &= -\int_\Gamma B[\mathscr{B}_\phi^*(sI + \mathscr{A}_\phi^*)^{-1} - \overline{\psi}^*(s)]\,\mathrm{d}s\\ &\quad + \int_\Gamma s(sI - A)^{-1}B[\mathscr{B}_\phi^*(sI + \mathscr{A}_\phi^*)^{-1} - \overline{\psi}^*(s)]\,\mathrm{d}s\\ &\quad + \int_\Gamma (sI - A)^{-1}B[\mathscr{B}_\phi^* - \overline{\psi}^*(s)(sI + \mathscr{A}_\phi^*)]\,\mathrm{d}s\\ &\quad - \int_\Gamma (sI - A)^{-1}B[\mathscr{B}_\phi^*(sI + \mathscr{A}_\phi^*)^{-1} - \overline{\psi}^*(s)]s\,\mathrm{d}s\\ &= -B\mathscr{B}_\phi^*\\ &\quad + \int_\Gamma (sI - A)^{-1}B[\mathscr{B}_\phi^* - \overline{\psi}^*(s)(sI + \mathscr{A}_\phi^*)]\,\mathrm{d}s.\end{aligned}$$

We claim that the integrand in the last term is analytic in $\Gamma$ and thus the integral vanishes. In fact introducing the notation

$$\eta(s) = (sI - A)^{-1}B[\mathscr{B}_\phi^* - \overline{\psi}^*(s)(sI + \mathscr{A}_\phi^*)]$$

the identity

$$F(s)[\mathscr{B}_\phi^*(sI + \mathscr{A}_\phi^*)^{-1} - \overline{\psi}^*(s)](sI + \mathscr{A}_\phi^*) = D[\mathscr{B}_\phi^* - \overline{\psi}^*(s)(sI + \mathscr{A}_\phi^*)] + C\eta(s)$$

gives that $C\eta$ is analytic inside $\Gamma$. At the same time

$$s\eta(s) - A\eta(s) = B[\mathscr{B}_\phi^* - \overline{\psi}^*(s)(sI + \mathscr{A}_\phi^*)]$$

gives that $CA\eta$ is again analytic inside $\Gamma$. Continuing in a similar way we obtain that the functions $CA^j\eta$, $j \geqslant 0$ are analytic inside $\Gamma$. Using the assumption about the observability of the pair $(C, A)$ the equation

$$A\Pi + \Pi\mathscr{A}_\phi^* = -B\mathscr{B}_\phi^* \tag{A.2}$$

is obtained proving together with (A.1) the first part of the theorem.

(ii) First consider the case when $A$ and $-\mathscr{A}_\phi^*$ have no common eigenvalues. Then equation $A\Pi + B\mathscr{B}_\phi^* = -\Pi\mathscr{A}_\phi^*$ yields that

$$(sI - A)\Pi - B\mathscr{B}_\phi^* = \Pi(sI + \mathscr{A}_\phi^*).$$

Multiplying by $(sI + \mathscr{A}_\phi^*)^{-1}$ from the right and by $C(sI - A)^{-1}$ from the left and rearranging the terms we obtain that

$$C(sI - A)^{-1}B\mathscr{B}_\phi^*(sI + \mathscr{A}_\phi^*)^{-1} = C(sI - A)^{-1}\Pi + C\Pi(sI - \mathscr{A}_\phi^*)^{-1}.$$

Using that $C\Pi = -D\mathscr{B}_\phi^*$ we get that

$$F(s)\mathscr{B}_\phi^*(sI + \mathscr{A}_\phi^*)^{-1} = C(sI - A)^{-1}\Pi.$$

If $A$ and $-\mathscr{A}_\phi^*$ have no common eigenvalues then the right hand side is obviously analytic at the eigenvalues of $-\mathscr{A}_\phi^*$ proving the last part of (ii). (Note that controllability of $(A, B)$ and $(\mathscr{A}_\phi, \mathscr{B}_\phi)$ was not needed in these computations. They will play a role in the general case.)

In the case when $A$ and $-\mathscr{A}_\phi^*$ have some common eigenvalues essentially the same type of argument can be applied with a small modification. Namely, assume that for some matrices – we shall specify them later – the extended form of the first equation in (3.4) holds:

$$A[\Pi, \overline{\Pi}] + B[\mathscr{B}_\phi^*, \overline{\mathscr{B}}_\phi^*] = [\Pi, \overline{\Pi}]\begin{bmatrix} -\mathscr{A}_\phi^* & A_0 \\ 0 & -\overline{\mathscr{A}}_\phi^* \end{bmatrix}. \tag{A.3}$$

Applying the same argument as before we can arrive at the equation

$$\begin{aligned}(D + C(sI - A)^{-1}B)[\mathscr{B}_\phi^*, \overline{\mathscr{B}}_\phi^*]\left(sI - \begin{bmatrix} -\mathscr{A}_\phi^* & A_0 \\ 0 & -\overline{\mathscr{A}}_\phi^* \end{bmatrix}\right)^{-1} \\ = -C(sI - A)^{-1}[\Pi, \overline{\Pi}] + [0, C\overline{\Pi} + D\overline{\mathscr{B}}_\phi^*]\left(sI - \begin{bmatrix} -\mathscr{A}_\phi^* & A_0 \\ 0 & -\overline{\mathscr{A}}_\phi^* \end{bmatrix}\right)^{-1}.\end{aligned}$$

Calculating explicitly the inverse of a block upper triangular matrix the second block equation gives that

$$\begin{aligned}F(s)[\mathscr{B}_\phi^*(sI + \mathscr{A}_\phi^*)^{-1}A_0(sI + \overline{\mathscr{A}}_\phi^*)^{-1} + \overline{\mathscr{B}}_\phi^*(sI + \overline{\mathscr{A}}_\phi^*)^{-1}] \\ = -C(sI - A)^{-1}\overline{\Pi} + (C\overline{\Pi} + D\overline{\mathscr{B}}_\phi^*)(sI + \overline{\mathscr{A}}_\phi^*)^{-1}.\end{aligned}$$

Now multiply both sides by $\det(sI+\overline{\mathscr{A}}_\phi^*)$. If this scalar-valued function cancels the poles of the function $C(sI-A)^{-1}\overline{\Pi}$ falling into the set of eigenvalues of $-\mathscr{A}_\phi^*$ then the right hand side will be analytic at the eigenvalues of $-\mathscr{A}_\phi^*$. Consequently for the function

$$\phi^*(s)=\mathscr{B}_\phi^*(sI+\mathscr{A}_\phi^*)^{-1}A_0\,\mathrm{adj}(sI+\overline{\mathscr{A}}_\phi^*)+\overline{\mathscr{B}}_\phi^*\mathrm{adj}(sI+\overline{\mathscr{A}}_\phi^*).$$

where adj denotes the adjoint of a matrix, the product $F\phi^*$ is analytic at the eigenvalues of $-\mathscr{A}_\phi^*$.

Thus it remains to prove that Eq. (A.3) has a solution and the realization above of the rational part of $\phi^*$ is controllable. Concerning the first statement we are going to prove that for any fixed matrices $A_0,\overline{\mathscr{A}}_\phi^*$ the equation

$$A\overline{\Pi}+B\overline{\mathscr{B}}_\phi^*+\overline{\Pi}\overline{\mathscr{A}}_\phi^*=\Pi A_0$$

has a solution in $\overline{\Pi},\overline{\mathscr{B}}_\phi^*$. In fact, the adjoint of the linear mapping

$$[\overline{\Pi},\overline{\mathscr{B}}_\phi^*]\ \mapsto\ A\overline{\Pi}+B\overline{\mathscr{B}}_\phi^*+\overline{\Pi}\overline{\mathscr{A}}_\phi^*$$

can be written as

$$L\ \mapsto\ [A^{\mathrm{T}}L+L(\overline{\mathscr{A}}_\phi^*)^{\mathrm{T}},B^{\mathrm{T}}L].$$

If $L$ is in the kernel of this linear mapping then equations

$$A^{\mathrm{T}}L+L(\overline{\mathscr{A}}^*)^{\mathrm{T}}=0,\quad B^{\mathrm{T}}L=0$$

hold, implying that the range of $L$ is an $A^{\mathrm{T}}$-invariant subspace in the kernel of the matrix $B^{\mathrm{T}}$. The controllability of the pair $(A,B)$ implies that $L=0$. Now since the kernel of the adjoint mapping is trivial the original mapping should be onto, proving that Eq. (A.3) has a solution even in the case when we fix the matrices $A_0,\overline{\mathscr{A}}_\phi^*$ in advance.

Now we are going to address the problem of choosing the matrices $A_0,\overline{\mathscr{A}}_\phi^*$. As we have pointed out the multiplication by $\det(sI+\overline{\mathscr{A}}_\phi^*)$ should cancel the poles of $(sI-A)^{-1}$ arising at the common eigenvalues of $A$ and $-\mathscr{A}_\phi^*$. Without loss of generality we might assume that the matrix $\mathscr{A}_\phi^*$ is in Jordan form, i.e.

$$-\mathscr{A}_\phi^*=J_{n_1,z_1}\oplus J_{n_2,z_2}\oplus\cdots\oplus J_{n_m,z_m},$$

where $z_1,\dots,z_m$ denote the eigenvalues of $\mathscr{A}_\phi^*$, $n_1,\dots,n_m$ are the corresponding multiplicities and $J_{l,z}$ is an $l\times l$ upper triangular Jordan-block matrix with eigenvalue at $z$. Then set

$$-\overline{\mathscr{A}}_\phi^*=J_{n,z_1}\oplus J_{n,z_2}\oplus\cdots\oplus J_{n,z_m},$$
$$A_0=M_1\oplus\cdots\oplus M_k,$$

where $M_j$ is a matrix of size $n_j\times n$ with the only nonzero element at the south-west corner. Note that the matrix $-\overline{\mathscr{A}}_\phi^*$ repeats the eigenvalues of $-\mathscr{A}_\phi^*$ but with multiplicities equal the size of $A$.

Thus $\det(sI+\overline{\mathscr{A}}_\phi^*)(sI-A)^{-1}$ has no pole at the eigenvalues of $-\mathscr{A}_\phi^*$, consequently $F\phi^*$ is analytic at the eigenvalues of $-\mathscr{A}_\phi^*$.

On the other hand the strictly proper part of $(sI+\mathscr{A}_\phi^*)^{-1}A_0\,\mathrm{adj}(sI+\overline{\mathscr{A}}_\phi^*)$ is

$$\sum_j(sI+\mathscr{A}_\phi^*)^{-1}(-\mathscr{A}_\phi^*)^jA_0R_j,$$

where the matrices $R_j$, $j=0,\dots,$ are defined by $\mathrm{adj}(sI+\overline{A}_\phi^*)=\sum_j s^jR_j$. To check the controllability of the pair $(\mathscr{A}_\phi^*,\sum_j(-\mathscr{A}_\phi^*)^jA_0R_j)$ the P-B-H test is applied. If $\xi$ is a left eigenvector of unit length of $-\mathscr{A}_\phi^*$ with eigenvalue $z_l$ (thus in block structure it has only one non-zero block, in which the last element is 1 all the others are zero), then

$$\begin{aligned}\sum_j \xi(-\mathscr{A}_\phi^*)^j A_0 R_j = &\sum_j z_l^j \xi A_0 R_j \\ &= \xi A_0 \mathrm{adj}(z_l I + \overline{\mathscr{A}}_\phi^*) \neq 0.\end{aligned}$$

(In fact, in block form it has only one non-zero block, and in this block its last element is 1, all the others are zero.) The last identity follows is obtained from direct calculations using the Jordan-form of $-\overline{\mathscr{A}}_\phi^*$, and the block-diagonal form of $A_0$.

This proves the controllability of the pair thus applying the argument used at the beginning of the proof of (i) we can conclude that there exist a function

$$\phi^*(s) = \mathscr{B}_\phi^*(sI + \mathscr{A}_\phi^*)^{-1} + \psi^*(s),$$

($\psi$ is a polynomial) such that $F\phi^*$ is analytic at the eigenvalues of $-\mathscr{A}_\phi^*$ concluding the proof of (ii) in the general case, as well. □

### A.2. *Example*

The following example shows that the polynomial part in Theorem 3.1 is indeed needed. Suppose we take $F = \left(\begin{array}{c|c} A & B \\ C & D\end{array}\right)$ with

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}.$$

Then $F$ is easily seen to be

$$\begin{bmatrix} 1 + \frac{1}{s-1} & 1 \\ -\frac{1}{(s-1)^2} + \frac{1}{s-1} & 2 - \frac{1}{s-1} \end{bmatrix}.$$

If we take

$$\mathscr{A}_\phi := -A^*, \quad \mathscr{B}_\phi = I_2,$$

we see that

$$A\Pi + \Pi\mathscr{A}_\phi^* + B\mathscr{B}_\phi^* = 0$$

is satisfied by $\Pi = \begin{bmatrix} -1 & -1 \\ 0 & -2 \end{bmatrix}$.

In fact,

$$A\Pi + \Pi\mathscr{A}_\phi^* + B\mathscr{B}_\phi^* = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 0 & -2 \end{bmatrix},$$
$$-\begin{bmatrix} -1 & -1 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0.$$

Thus

$$F(s)\mathscr{B}_\phi^*(sI + \mathscr{A}_\phi^*)^{-1}$$

$$
\begin{aligned}
&= (D + C(sI - A)^{-1}B)\mathscr{B}_\phi^*(sI + \mathscr{A}_\phi^*)^{-1} \\
&= \left(\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} s-1 & 0 \\ 1 & s-1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s-1 & 0 \\ 1 & s-1 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} 1 + \frac{1}{s-1} & 1 \\ -\frac{1}{(s-1)^2} + \frac{1}{s-1} & 2 - \frac{1}{s-1} \end{bmatrix} \begin{bmatrix} \frac{1}{s-1} & 0 \\ -\frac{1}{(s-1)^2} & \frac{1}{s-1} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{s-1} & \frac{1}{s-1} \\ -\frac{1}{(s-1)^3} + \frac{1}{(s-1)^2} - 2\frac{1}{(s-1)^2} + \frac{1}{(s-1)^3} & \frac{2}{s-1} - \frac{1}{(s-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{s-1} & \frac{1}{s-1} \\ -\frac{1}{(s-1)^2} & \frac{2}{s-1} - \frac{1}{(s-1)^2} \end{bmatrix}.
\end{aligned}
$$

We are now looking for a polynomial matrix $\psi(s)$ such that

$$F(s)(\mathscr{B}_\phi^*(sI + \mathscr{A}_\phi^*)^{-1} + \psi(s))$$

is analytic at 1. Notice that this can be done because, for each row, the negative degree of $F$ is greater that or equal to that of the product $F(s)\mathscr{B}_\phi^*(sI + \mathscr{A}_\phi^*)^{-1}$. It easily seen that, for example, the polynomial

$$\psi(s) := \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}(s-1)$$

will do. In fact,

$$
\begin{aligned}
&F(s)(\mathscr{B}_\phi^*(sI + \mathscr{A}_\phi^*)^{-1} + \psi(s)) \\
&\quad= \begin{bmatrix} \frac{1}{s-1} & \frac{1}{s-1} \\ -\frac{1}{(s-1)^2} & \frac{2}{s-1} - \frac{1}{(s-1)^2} \end{bmatrix} \\
&\qquad+ \begin{bmatrix} 1 + \frac{1}{s-1} & 1 \\ -\frac{1}{(s-1)^2} + \frac{1}{s-1} & 2 - \frac{1}{s-1} \end{bmatrix} \left(\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}(s-1)\right) \\
&\quad= \begin{bmatrix} \frac{1}{s-1} & \frac{1}{s-1} \\ -\frac{1}{(s-1)^2} & \frac{2}{s-1} - \frac{1}{(s-1)^2} \end{bmatrix} \\
&\qquad+ \begin{bmatrix} -\frac{1}{s-1} & -\frac{1}{s-1} \\ \frac{1}{(s-1)^2} - \frac{2}{s-1} + 2 & \frac{1}{(s-1)^2} - \frac{2}{s-1} + 2 \end{bmatrix} + \begin{bmatrix} -2(s-1) - 2 & 0 \\ \frac{2}{s-1} - 2 & 0 \end{bmatrix} \\
&\quad= \begin{bmatrix} -2s & 0 \\ 0 & 2 \end{bmatrix}.
\end{aligned}
$$

## Appendix B. Proof of Lemma 4.1

We can write

$$
\begin{aligned}
&S\Theta_{11} + \Theta_{21} \\
&\quad= \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \left(\begin{array}{c|c} \mathscr{A} & U \\ \hline -U^*\mathscr{P}^{-1} & I \end{array}\right) + \left(\begin{array}{c|c} \mathscr{A} & U \\ \hline V^*\mathscr{P}^{-1} & 0 \end{array}\right)
\end{aligned}
$$

$$= \left(\begin{array}{cc|c} \mathscr{A} & 0 & U \\ -BU^*\mathscr{P}^{-1} & A & B \\ \hline -DU^*\mathscr{P}^{-1} & C & D \end{array}\right) + \left(\begin{array}{cc|c} \mathscr{A} & 0 & U \\ -BU^*\mathscr{P}^{-1} & A & B \\ \hline V^*\mathscr{P}^{-1} & 0 & 0 \end{array}\right)$$

$$= \left(\begin{array}{cc|c} \mathscr{A} & 0 & U \\ -BU^*\mathscr{P}^{-1} & A & B \\ \hline -(DU^* - V^*)\mathscr{P}^{-1} & C & D \end{array}\right).$$

Similarly,

$$S\Theta_{12} + \Theta_{22}$$

$$= \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \left(\begin{array}{c|c} \mathscr{A} & V \\ \hline -U^*\mathscr{P}^{-1} & 0 \end{array}\right) + \left(\begin{array}{c|c} \mathscr{A} & V \\ \hline V^*\mathscr{P}^{-1} & I \end{array}\right)$$

$$= \left(\begin{array}{cc|c} \mathscr{A} & 0 & V \\ -BU^*\mathscr{P}^{-1} & A & 0 \\ \hline -DU^*\mathscr{P}^{-1} & C & 0 \end{array}\right) + \left(\begin{array}{cc|c} \mathscr{A} & 0 & V \\ -BU^*\mathscr{P}^{-1} & A & 0 \\ \hline V^*\mathscr{P}^{-1} & 0 & I \end{array}\right)$$

$$= \left(\begin{array}{cc|c} \mathscr{A} & 0 & V \\ -BU^*\mathscr{P}^{-1} & A & 0 \\ \hline -(DU^* - V^*)\mathscr{P}^{-1} & C & I \end{array}\right)$$

So the inverse of the above matrix is easily seen to be

$$(S\Theta_{12} + \Theta_{22})^{-1} = \left(\begin{array}{cc|c} \mathscr{A} + V(DU^* - V^*)\mathscr{P}^{-1} & -VC & V \\ -BU^*\mathscr{P}^{-1} & A & 0 \\ \hline (DU^* - V^*)\mathscr{P}^{-1} & -C & I \end{array}\right).$$

Therefore,

$$Q = (S\Theta_{12} + \Theta_{22})^{-1}(S\Theta_{11} + \Theta_{21})$$

$$= \left(\begin{array}{cc|c} \mathscr{A} + V(DU^* - V^*)\mathscr{P}^{-1} & -VC & V \\ -BU^*\mathscr{P}^{-1} & A & 0 \\ \hline (DU^* - V^*)\mathscr{P}^{-1} & -C & I \end{array}\right) \left(\begin{array}{cc|c} \mathscr{A} & 0 & U \\ -BU^*\mathscr{P}^{-1} & A & B \\ \hline -(DU^* - V^*)\mathscr{P}^{-1} & C & D \end{array}\right)$$

$$= \left(\begin{array}{cccc|c} \mathscr{A} & 0 & 0 & 0 & U \\ -BU^*\mathscr{P}^{-1} & A & 0 & 0 & B \\ -V(DU^* - V^*)\mathscr{P}^{-1} & VC & \mathscr{A} + V(DU^* - V^*)\mathscr{P}^{-1} & -VC & VD \\ 0 & 0 & -BU^*\mathscr{P}^{-1} & A & 0 \\ \hline -(DU^* - V^*)\mathscr{P}^{-1} & C & (DU^* - V^*)\mathscr{P}^{-1} & -C & D \end{array}\right)$$

$$=: \left(\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & D \end{array}\right)$$

Using the change of basis $T = \begin{pmatrix} I & 0 \\ -I & I \end{pmatrix}$ we get

$$\mathbf{A}' = T\mathbf{A}T^{-1}, \quad \mathbf{B}' = T\mathbf{B}, \quad \mathbf{C}' = \mathbf{C}T^{-1},$$

which applied to our system yields:

$$\left(\begin{array}{c|c}\mathbf{A}' & \mathbf{B}' \\ \hline \mathbf{C}' & D\end{array}\right) = \left(\begin{array}{cccc|c} \mathscr{A} & 0 & 0 & 0 & U \\ -BU^*\mathscr{P}^{-1} & A & 0 & 0 & B \\ 0 & 0 & \mathscr{A} + V(DU^* - V^*)\mathscr{P}^{-1} & -VC & -U + VD \\ 0 & 0 & -BU^*\mathscr{P}^{-1} & A & -B \\ \hline 0 & 0 & (DU^* - V^*)\mathscr{P}^{-1} & -C & D \end{array}\right)$$

$$= \left(\begin{array}{cc|c} \mathscr{A} + V(DU^* - V^*)\mathscr{P}^{-1} & -VC & -U + VD \\ -BU^*\mathscr{P}^{-1} & A & -B \\ \hline (DU^* - V^*)\mathscr{P}^{-1} & -C & D \end{array}\right).$$

**Proof of Lemma 4.2.** Inverting the relation $Q = T_\Theta(S)$ we get

$$\begin{aligned} &Q = T_\Theta(S) = (S\Theta_{12} + \Theta_{22})^{-1}(S\Theta_{11} + \Theta_{21}), \\ &(S\Theta_{12} + \Theta_{22})Q = (S\Theta_{11} + \Theta_{21}), \\ &S(\Theta_{12}Q - \Theta_{11}) = -(\Theta_{22}Q - \Theta_{21}), \\ &S = (\Theta_{22}Q - \Theta_{21})(-\Theta_{12}Q + \Theta_{11})^{-1} \end{aligned}$$

and so we can write

$$\begin{aligned} &\Theta_{22}Q - \Theta_{21} \\ &\quad = \left(\begin{array}{c|c} \mathscr{A} & V \\ \hline V^*\mathscr{P}^{-1} & I \end{array}\right)\left(\begin{array}{c|c} A_Q & B_Q \\ \hline C_Q & D \end{array}\right) - \left(\begin{array}{c|c} \mathscr{A} & U \\ \hline V^*\mathscr{P}^{-1} & 0 \end{array}\right) \\ &\quad = \left(\begin{array}{cc|c} A_Q & 0 & B_Q \\ VC_Q & \mathscr{A} & VD \\ \hline C_Q & V^*\mathscr{P}^{-1} & D \end{array}\right) - \left(\begin{array}{cc|c} A_Q & 0 & 0 \\ VC_Q & \mathscr{A} & U \\ \hline C_Q & V^*\mathscr{P}^{-1} & 0 \end{array}\right) \\ &\quad = \left(\begin{array}{cc|c} A_Q & 0 & B_Q \\ VC_Q & \mathscr{A} & VD - U \\ \hline C_Q & V^*\mathscr{P}^{-1} & D \end{array}\right). \end{aligned}$$

Similarly,

$$\begin{aligned} &-\Theta_{12}Q + \Theta_{11} \\ &\quad = -\left(\begin{array}{c|c} \mathscr{A} & V \\ \hline -U^*\mathscr{P}^{-1} & 0 \end{array}\right)\left(\begin{array}{c|c} A_Q & B_Q \\ \hline C_Q & D \end{array}\right) + \left(\begin{array}{c|c} \mathscr{A} & U \\ \hline -U^*\mathscr{P}^{-1} & I \end{array}\right) \\ &\quad = -\left(\begin{array}{cc|c} A_Q & 0 & B_Q \\ VC_Q & \mathscr{A} & VD \\ \hline 0 & -U^*\mathscr{P}^{-1} & 0 \end{array}\right) + \left(\begin{array}{cc|c} A_Q & 0 & 0 \\ VC_Q & \mathscr{A} & U \\ \hline 0 & -U^*\mathscr{P}^{-1} & I \end{array}\right) \\ &\quad = \left(\begin{array}{cc|c} A_Q & 0 & -B_Q \\ VC_Q & \mathscr{A} & -VD + U \\ \hline 0 & -U^*\mathscr{P}^{-1} & I \end{array}\right). \end{aligned}$$

So the inverse of the above matrix is easily seen to be

$$(-\Theta_{12}Q+\Theta_{11})^{-1}=\left(\begin{array}{cc|c} A_Q & -B_QU^*\mathscr{P}^{-1} & -B_Q \\ VC_Q & \mathscr{A}+(-VD+U)U^*\mathscr{P}^{-1} & -VD+U \\ \hline 0 & U^*\mathscr{P}^{-1} & I \end{array}\right).$$

Therefore,

$$\begin{aligned} S &= (\Theta_{22}Q-\Theta_{21})(-\Theta_{12}Q+\Theta_{11})^{-1} \\ &= \left(\begin{array}{cc|c} A_Q & 0 & B_Q \\ VC_Q & \mathscr{A} & VD-U \\ \hline C_Q & V^*\mathscr{P}^{-1} & D \end{array}\right)\left(\begin{array}{cc|c} A_Q & -B_QU^*\mathscr{P}^{-1} & -B_Q \\ VC_Q & \mathscr{A}+(-VD+U)U^*\mathscr{P}^{-1} & -VD+U \\ \hline 0 & U^*\mathscr{P}^{-1} & I \end{array}\right) \\ &= \left(\begin{array}{cccc|c} A_Q & -B_QU^*\mathscr{P}^{-1} & 0 & 0 & -B_Q \\ VC_Q & \mathscr{A}+(-VD+U)U^*\mathscr{P}^{-1} & 0 & 0 & -VD+U \\ 0 & B_QU^*\mathscr{P}^{-1} & A_Q & 0 & B_Q \\ 0 & (VD-U)U^*\mathscr{P}^{-1} & VC_Q & \mathscr{A} & VD-U \\ \hline 0 & DU^*\mathscr{P}^{-1} & C_Q & V^*\mathscr{P}^{-1} & D \end{array}\right) \\ &=: \left(\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & D \end{array}\right) \end{aligned}$$

Using the change of basis $T=\begin{pmatrix} I & 0 \\ -I & I \end{pmatrix}$ we get

$$\mathbf{A}'=T\mathbf{A}T^{-1}, \quad \mathbf{B}'=T\mathbf{B}, \quad \mathbf{C}'=\mathbf{C}T^{-1}$$

which applied to our system yields

$$\begin{aligned} \left(\begin{array}{cc} \mathbf{A}' & \mathbf{B}' \\ \hline \mathbf{C}' & D \end{array}\right) &= \left(\begin{array}{cccc|c} A_Q & -B_QU^*\mathscr{P}^{-1} & 0 & 0 & -B_Q \\ VC_Q & \mathscr{A}+(-VD+U)U^*\mathscr{P}^{-1} & 0 & 0 & -VD+U \\ 0 & 0 & A_Q & 0 & 0 \\ 0 & 0 & VC_Q & \mathscr{A} & 0 \\ \hline -C_Q & (DU^*-V^*)\mathscr{P}^{-1} & C_Q & V^*\mathscr{P}^{-1} & D \end{array}\right) \\ &= \left(\begin{array}{cc|c} A_Q & -B_QU^*\mathscr{P}^{-1} & B_Q \\ VC_Q & \mathscr{A}+(U-VD)U^*\mathscr{P}^{-1} & VD-U \\ \hline C_Q & (V^*-DU^*)\mathscr{P}^{-1} & D \end{array}\right) \end{aligned}$$

concluding the proof of the lemma.

**Proof of Theorem 4.3.** Consider first the controllability subspace of the realization (4.23) and assume that the linearly independent columns of a matrix $\alpha_Q$ span the orthogonal complement of this controllability subspace. Then due to the fact that this orthogonal complement is $(A_Q - Y\mathscr{P}^{-1}VC_Q)^*$ invariant there exists a matrix $\Gamma_Q$ such that the following equations hold:

$$\alpha_Q^*(B_Q - Y\mathscr{P}^{-1}(VD-U)) = 0, \tag{B.1}$$

$$\alpha_Q^*(A_Q - Y\mathscr{P}^{-1}VC_Q) = \Gamma_Q\alpha_Q^*. \tag{B.2}$$

Computing (B.1)$U^*$ − (B.2)$Y$, we get that

$$\alpha_Q^*(B_Q U^* - Y\mathscr{P}^{-1}VDU^* + Y\mathscr{P}^{-1}UU^* - A_Q Y + Y\mathscr{P}^{-1}VC_Q Y) = -\Gamma_Q\alpha_Q^* Y.$$

Now using (3.1) and (3.2) we obtain that

$$\alpha_Q^*(Y\mathscr{A}^* - Y\mathscr{P}^{-1}VV^* + Y\mathscr{P}^{-1}UU^*) = -\Gamma_Q\alpha_Q^* Y.$$

I.e.

$$(\alpha_Q^* Y\mathscr{P}^{-1})\mathscr{A} = \Gamma_Q(\alpha_Q^* Y\mathscr{P}^{-1}). \tag{B.3}$$

An immediate consequence of this equation and the controllability of the pair $(A_Q, B_Q)$ that the row vectors of $\alpha_Q^* Y\mathscr{P}^{-1}$ are linearly independent. In fact, from Eq. (B.3) it follows that $\mathrm{Ker}[\mathscr{P}^{-1}Y^*\alpha_Q]$ is $\Gamma_Q^*$ invariant. So, let $\kappa$ be an eigenvector of $\Gamma_Q^*$, $\Gamma_Q^*\kappa = \mu\kappa$, for which

$$\mathscr{P}^{-1}Y^*\alpha_Q\kappa = 0.$$

Then multiplying (B.1) and (B.2) from the right by $\kappa^*$ we obtain that equations

$$\begin{aligned}&\kappa^*\alpha_Q^* B_Q = 0,\\ &\kappa^*\alpha_Q^* A_Q = \bar{\mu}\kappa^*\alpha_Q^*\end{aligned}$$

hold, implying – due to the controllability of the pair $(A_Q, B_Q)$ – that $\alpha_Q\kappa = 0$. Since the columns are $\alpha_Q$ are linearly independent, $\kappa$ should be the zero vector.

On the other hand equations in (B.1) and (B.2) can now be arranged into the following form

$$[\alpha_Q^*, -\alpha_Q^* Y\mathscr{P}^{-1}]\begin{bmatrix} A_Q & B_Q \\ VC_Q & VD - U\end{bmatrix} = [\Gamma_Q\alpha_Q^*, 0], \tag{B.4}$$

proving (i).

Conversely, if for some matrices $\beta, \gamma, \Gamma$, assuming that the column vectors of $\beta$ are linearly independent, the equations

$$\beta^*\mathscr{A} = \Gamma\beta^*, \tag{B.5}$$

$$[\gamma^*, \beta^*]\begin{bmatrix} A_Q & B_Q \\ VC_Q & VD - U\end{bmatrix} = [\Gamma\gamma^*, 0], \tag{B.6}$$

hold, then – using that the row vectors of $\beta^*$ are linearly independent – Eq. (B.5) implies that the eigenvalues of $\Gamma$ form a subset of that of $\mathscr{A}$, and furthermore multiplying (B.6) from the right by $\begin{bmatrix} -Y \\ U^*\end{bmatrix}$ we obtain that

$$-\Gamma\gamma^* Y = \gamma^* Y\mathscr{A}^* + \beta^*\mathscr{A}\mathscr{P} + \beta^*\mathscr{P}\mathscr{A}^* \tag{B.7}$$

holds. Invoking Eq. (B.5) we get that

$$(\gamma^* Y + \beta^*\mathscr{P})\mathscr{A} = -\Gamma(\gamma^* Y + \beta^*\mathscr{P}).$$

Since the matrices $\Gamma$ and $-\mathscr{A}$ have no common eigenvalues we get that

$$\gamma^* Y + \beta^*\mathscr{P} = 0,$$

i.e.

$$\beta^* = -\gamma^* Y\mathscr{P}^{-1},$$

Substituting this into (B.6) the obtained equations

$$\gamma^*(B_Q - Y\mathscr{P}^{-1}(VD - U)) = 0,$$
$$\gamma^*(A_Q - Y\mathscr{P}^{-1}VC_Q) = \Gamma\gamma^*$$

imply that the column vectors of $\gamma$ are orthogonal to the controllability subspace of the realization given in (4.23), concluding the proof of (ii).

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