

Sequent and Hypersequent Calculi for Abelian and Łukasiewicz Logics

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Abstract

We present two embeddings of infinite-valued Łukasiewicz logic \mathbf{L} into Meyer and Slaney's abelian logic \mathbf{A} , the logic of lattice-ordered abelian groups. We give new analytic proof systems for \mathbf{A} and use the embeddings to derive corresponding systems for \mathbf{L} . These include: hypersequent calculi for \mathbf{A} and \mathbf{L} and terminating versions of these calculi; labelled single sequent calculi for \mathbf{A} and \mathbf{L} of complexity co-NP; unlabelled single sequent calculi for \mathbf{A} and \mathbf{L} .

1 Introduction

In [24] Hájek classifies *truth-functional fuzzy logics* as logics whose conjunction and implication functions are interpreted via *continuous t-norms* and their residua. Since each continuous t-norm may be constructed from the Łukasiewicz, Gödel and Product t-norms, the resulting logics, Łukasiewicz logic \mathbf{L} , Gödel logic \mathbf{G} and Product logic \mathbf{P} respectively, are fundamental for this classification. Also important is Hájek's Basic Logic \mathbf{BL} [24] which was proved in [16] to characterise validity in logics based on continuous t-norms. In a similar vein Godo and Esteva [20] have defined the logics \mathbf{MTL} (Monoidal T-norm based Logic) and \mathbf{IMTL} (Involutive Monoidal T-norm based Logic) which turn out to characterise validity in *left-continuous t-norm based logics* and *left-continuous t-norm with involutive negation based logics* respectively. Underlying all the above systems is Höhle's monoidal logic \mathbf{ML} [25] which provides a common basis for both t-norm logics and logics based on Heyting algebras. A diagram of the relationships between these logics is given in Figure 1 with arrows signifying inclusions between logics.

Systematic accounts of Hilbert-style axiomatisations and algebraic semantics for fuzzy logics have been presented eg in [22, 24] but no comprehensive Gentzen-style sequent calculus approach such as that given for substructural logics has been forthcoming. Progress for each logic has varied. Łukasiewicz

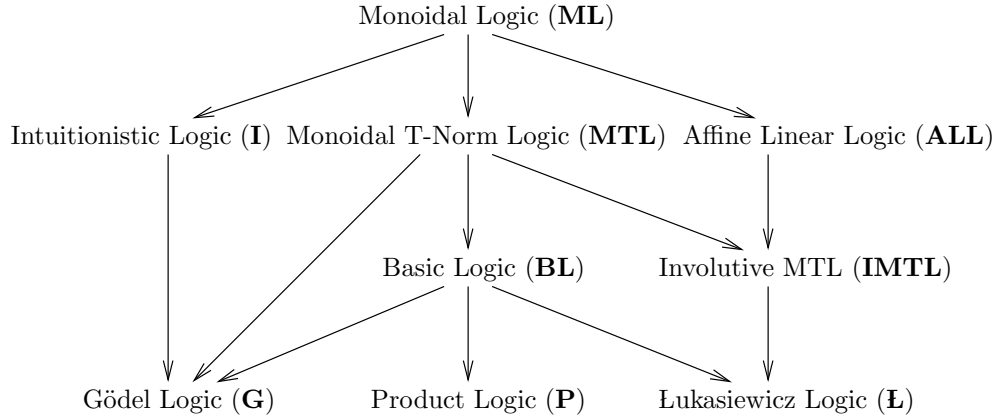


Figure 1: Relationships between fuzzy logics

logic \mathbf{L} , despite having been explored thoroughly from a semantic perspective (see eg [17]), has been given only calculi that either fail to be analytic (ie cut-free) [39, 15] or internal (ie avoiding non-logical calculations) [23, 35, 36]. A notable exception is the work of Aguzzoli, Ciabattoni and Gerla [1, 2] who give cut-free internal calculi for \mathbf{L} , \mathbf{G} and \mathbf{P} ; their approach however is based on validity in *finite-valued* logics with sequents consisting of components representing sets of truth values, and therefore does not integrate well with more standard Gentzen presentations. For Gödel logic \mathbf{G} the proof-theoretic picture is more healthy; there are several (rather complicated) single sequent calculi [42, 4, 19, 8], a calculus employing *sequents of relations* [10], and also a very natural *hypersequent* formulation [7]. Hypersequent calculi, introduced independently by Avron in [5] and Pottinger in [38] as a generalisation of the usual sequent calculi, have also been given for \mathbf{MTL} and other variants of Urquhart’s \mathbf{C} logic [43] in [9, 14]. A first semantics-based calculus for Hájek’s *Basic Logic* \mathbf{BL} has been presented in [33].

The approach taken in this work towards developing a proof theory of fuzzy logics is new. It results from the identification of fuzzy logics with *fragments of comparative logics*. Intuitively, a fuzzy logic with truth values between 0 and 1 may be viewed as being part of an “extended” logic with truth values between $-\infty$ and $+\infty$. These extended logics are called comparative logics¹ here to distinguish them from fuzzy logics with truth values in $[0, 1]$. The idea proposed in this paper is that (at least in some cases) comparative logics are more natural to work with for proof-theoretic purposes. Developing proof systems for comparative logics then allows us to exploit our translations to obtain proof systems for their fragments, the fuzzy logics. In fact this mirrors the strategy of systems with extra relation symbols or labels; calculations are performed in an extended language to find answers for queries in the original language. The novelty here is that this approach is followed at the *logical* level

rather than the *metalogical* level.

We apply this approach here to Łukasiewicz infinite-valued logic \mathbf{L} . For \mathbf{L} the appropriate comparative logic is *abelian* logic \mathbf{A} , the logic of lattice-ordered abelian groups with characteristic models in the integers, rationals and reals. \mathbf{A} was introduced and motivated independently by Meyer and Slaney [31] and Casari [11] as a relevance and comparative logic respectively. In [31] the so-called *enthymematic* fragment of \mathbf{A} was proved to coincide exactly with \mathbf{L}^+ , the positive part of \mathbf{L} and this result is extended here to identify the *material* fragment of \mathbf{A} with the whole of \mathbf{L} . A further, perhaps more natural, translation from \mathbf{L} into \mathbf{A} is also provided. By introducing analytic sequent and hypersequent calculi for \mathbf{A} (the first proof systems for this logic) we are then able to derive the first Gentzen-style analytic sequent and hypersequent calculi for \mathbf{L} .

We proceed as follows. In section 2 we introduce Łukasiewicz logic \mathbf{L} and Abelian logic \mathbf{A} presenting some important and relevant results. In section 3 we relate the logics via two embeddings of \mathbf{L} into \mathbf{A} . We then turn our attention to proof theory. In section 4 we present *hypersequent calculi* for \mathbf{A} and \mathbf{L} and in section 5 we introduce *terminating* versions of these calculi. In section 6 we introduce *labelled calculi* for \mathbf{A} and \mathbf{L} which we show to be co-NP. Finally in section 7 we present *unlabelled single sequent calculi* for \mathbf{A} and \mathbf{L} .²

2 Background

In this section we first present some algebraic preliminaries and then introduce Łukasiewicz infinite-valued logic \mathbf{L} and abelian logic \mathbf{A} .

2.1 Preliminaries

The work reported here is concerned with *propositional* logics; the language used is built inductively from a denumerable set of propositional variables p_1, p_2, \dots and a finite set of connectives.

Definition 1 (Con) $Con = \{\wedge, \vee, \oplus, +, \supset, \rightarrow, \Rightarrow, \leftrightarrow, \neg, \sim, t, \perp\}$

Definition 2 (For) *For* is built inductively as follows: (1) $p_1, p_2, \dots \in For$, (2) if $\phi_1, \dots, \phi_m \in For$ then $*(\phi_1, \dots, \phi_m) \in For$ where $* \in Con$, $arity(*) = m$.

Definition 3 (A-valuation) Given an algebra $A = \langle L, O \rangle$ with carrier $L \neq \emptyset$ and operations $\{t\} \subseteq O \subseteq Con$, an *A-valuation* is a function $v : For \rightarrow L$ such that for $* \in Con$, $arity(*) = m$, $v(*(\phi_1, \dots, \phi_m)) = *(v(\phi_1), \dots, v(\phi_m))$.

¹The expression *comparative* logic was first used by Casari in [11, 12] to characterise a set of systems formalising ideas expressed by Aristotle about comparisons of majority, minority and equality. There is considerable overlap between Casari's comparative logics and those presented here but whereas Casari regards fuzzy logics such as Łukasiewicz infinite-valued logic as particular comparative logics, this work instead presents fuzzy logics as *fragments* of comparative logics.

²Some of the results presented in sections 6 and 7 have already appeared in [29].

Definition 4 (Validity) $\phi \in \text{For}$ is valid in an algebra A iff $v(\phi) \geq t$ for all A -valuations v . ϕ is X -valid (written $\models_X \phi$) iff ϕ is valid in all algebras of the class X .

2.2 Łukasiewicz Logic

Łukasiewicz infinite-valued logic \mathbf{L} was introduced by Łukasiewicz in 1930 [27]. Good references for historical details and an overview of the main results are [28] and [43]. An in-depth algebraic treatment of \mathbf{L} is provided by [17] and the fuzzy logic perspective is described in [24].

Definition 5 (Łukasiewicz Infinite-Valued Logic, \mathbf{L}) \supset and \perp are primitive. \mathbf{L} has the rule (mp) and the following definitions and axioms:

$$\begin{array}{ll}
D\sim & \sim A = A \supset \perp \\
Dt & t = \sim \perp \\
L1 & A \supset (B \supset A) \\
L2 & (A \supset B) \supset ((B \supset C) \supset (A \supset C)) \\
D\vee & A \vee B = (A \supset B) \supset B \\
D\oplus & A \oplus B = \sim A \supset B \\
L3 & ((A \supset B) \supset B) \supset ((B \supset A) \supset A) \\
L4 & ((A \supset \perp) \supset (B \supset \perp)) \supset (B \supset A)
\end{array}$$

$$(mp) \quad \frac{A \supset B, A}{B}$$

The following axiomatisation of *positive* Łukasiewicz logic was given by Rose and Rosser in [40].

Definition 6 (Positive Łukasiewicz Infinite-Valued Logic, \mathbf{L}^+) \supseteq, \vee, \wedge and t are primitive. \mathbf{L}^+ has the rule (mp) and the following axioms:

$$\begin{array}{ll}
L^{+1} & (D \supseteq B) \supseteq ((D \supseteq C) \supseteq (D \supseteq (B \wedge C))) \\
L^{+2} & B \supseteq (C \supseteq B) \\
L^{+3} & (B \supseteq C) \supseteq ((C \supseteq D) \supseteq (B \supseteq D)) \\
L^{+4} & ((B \supseteq C) \supseteq C) \supseteq (B \vee C) \\
L^{+5} & (B \vee C) \supseteq (C \vee B) \\
L^{+6} & (B \supseteq C) \vee (C \supseteq B) \\
L^{+7} & (B \wedge C) \supseteq B \\
L^{+8} & (B \wedge C) \supseteq C \\
L^{+9} & t
\end{array}$$

Algebraic structures for \mathbf{L} were introduced by Chang in 1958 [13].

Definition 7 (MV-algebra) An MV-algebra is an algebra $\langle A, \oplus, \sim, \perp \rangle$ with a binary operation \oplus , a unary operation \sim and a constant \perp , satisfying the following equations:

$$\begin{array}{ll}
mv1 & \perp \oplus a = a \\
mv2 & a \oplus b = b \oplus a \\
mv3 & (a \oplus b) \oplus c = a \oplus (b \oplus c) \\
mv4 & \sim \sim a = a \\
mv5 & a \oplus \sim \perp = \sim \perp \\
mv6 & \sim(\sim a \oplus b) \oplus b = \sim(\sim b \oplus a) \oplus a
\end{array}$$

We also define: $a \supset b = \sim a \oplus b$, $t = \sim \perp$, $a \vee b = \sim(\sim a \wedge \sim b)$.

Let $[0, 1]_{\mathbb{R}}$ be the real unit interval and define $a \oplus b = \min(1, a + b)$, $\sim a = 1 - a$ and $\perp = 0$. We have that $[0, 1]_{\mathbf{L}} = \langle [0, 1], \oplus, \sim, \perp \rangle$ is an MV-algebra with $a \supset b = \min(1, 1 + b - a)$. In fact $[0, 1]_{\mathbf{L}}$ is characteristic for MV-algebras.

Theorem 8 (Chang 1958 [13]) *The following are equivalent: (1) ϕ is a theorem of \mathbf{L} . (2) ϕ is valid in all MV-algebras. (3) ϕ is valid in $[0, 1]_{\mathbf{L}}$.*

It will be convenient for us to consider the (also characteristic) MV-algebra $[-1, 0]_{\mathbf{L}} = \langle [-1, 0]_{\mathbb{R}}, \oplus, \sim, \perp \rangle$ where $a \oplus b = \min(0, a + b + 1)$, $\sim a = -1 - a$ and $\perp = -1$ with $a \supset b = \min(0, b - a)$.

Finally we mention the following complexity result for \mathbf{L} .

Theorem 9 (Mundici 1987 [34]) *The tautology problem for \mathbf{L} is co-NP-complete.*

2.3 Abelian Logic

Abelian logic \mathbf{A} was introduced and investigated by Meyer and Slaney [31, 32] as a logic of *relevance*, obtained from Anderson and Belnap's relevant logic \mathbf{R} (see [3] for details) by rejecting the contraction axiom $((A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B))$ and liberalising the negation axiom $((A \rightarrow \perp) \rightarrow \perp) \rightarrow A$ to $((A \rightarrow B) \rightarrow B) \rightarrow A$. \mathbf{A} has also been motivated independently by Casari [12] as a logic of *comparison* formalising comparisons of majority, minority and equality in natural language. More recently Galli et al. [21] have derived \mathbf{A} as a logic of equilibrium for arguments. Finally we note that a sequent calculus for the *intensional* fragment of \mathbf{A} (the logic of abelian groups) has been provided by Paoli in [37].

We begin here by giving Meyer and Slaney's axiomatisation of \mathbf{A} .

Definition 10 (Abelian Logic, \mathbf{A}) $\rightarrow, +, \wedge, \vee$ and t are primitive. \mathbf{A} has the following definitions, axioms and rules:

$$\begin{array}{ll}
D \leftrightarrow & A \leftrightarrow B = (A \rightarrow B) \wedge (B \rightarrow A) & D \neg & \neg A = A \rightarrow t \\
A1 & ((A \vee B) \rightarrow C) \leftrightarrow ((A \rightarrow C) \wedge (B \rightarrow C)) & A6 & A \leftrightarrow (t \rightarrow A) \\
A2 & ((A + B) \rightarrow C) \leftrightarrow (A \rightarrow (B \rightarrow C)) & A7 & (A \wedge B) \rightarrow A \\
A3 & (A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C)) & A8 & (A \wedge B) \rightarrow B \\
A4 & ((A \rightarrow B) \wedge (A \rightarrow C)) \rightarrow (A \rightarrow (B \wedge C)) & A9 & A \rightarrow ((A \rightarrow B) \rightarrow B) \\
A5 & (A \wedge (B \vee C)) \rightarrow ((A \wedge B) \vee (A \wedge C)) & A10 & ((A \rightarrow B) \rightarrow B) \rightarrow A
\end{array}$$

$$\begin{array}{ll}
(mp) & \frac{A \rightarrow B, A}{B} & (\wedge I) & \frac{A, B}{A \wedge B}
\end{array}$$

The appropriate class of algebras for \mathbf{A} are *lattice-ordered abelian groups*:

Definition 11 (Lattice-Ordered Abelian Group (Abelian L-Group)) *An abelian l-group is an algebra $\langle G, +, \vee, \neg, t \rangle$ with binary operations $+$ and \vee , a unary operation \neg and a constant t , satisfying the following equations:*

$$\begin{array}{ll}
a1 & t + a = a & a5 & a \vee b = b \vee a \\
a2 & a + b = b + a & a6 & (a \vee b) \vee c = a \vee (b \vee c) \\
a3 & (a + b) + c = a + (b + c) & a7 & a = a \vee a \\
a4 & a + \neg a = t & a8 & a + (b \vee c) = (a + b) \vee (a + c)
\end{array}$$

In addition, we define: $a \wedge b = \neg(\neg a \vee \neg b)$, $a \rightarrow b = \neg a + b$, $a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a)$ and $a \leq b$ iff $a \vee b = b$.

Well known examples of abelian l-groups are the integers $\mathbb{Z} = \langle \mathbb{Z}, +, \max, -, 0 \rangle$, the rationals $\mathbb{Q} = \langle \mathbb{Q}, +, \max, -, 0 \rangle$ and the reals $\mathbb{R} = \langle \mathbb{R}, +, \max, -, 0 \rangle$. In fact any of these serves as a characteristic model for \mathbf{A} .

Theorem 12 (Characterisation Theorem for \mathbf{A}) *The following are equivalent: (1) ϕ is a theorem of \mathbf{A} . (2) ϕ is valid in all abelian l-groups. (3) ϕ is valid in \mathbb{Z} . (4) ϕ is valid in \mathbb{Q} . (5) ϕ is valid in \mathbb{R} .*

Proof. The equivalence of (2)-(5) is well known from pure mathematics. The equivalence of (1) and (2) is proved by Meyer and Slaney in [31]; also given there is a more logic minded proof of the equivalence of (2) and (3). \square

Notice that in \mathbf{A} we have $a + b = \neg(-a + \neg b)$ ie intensional *disjunction* and intensional *conjunction* are exactly the same thing. Similarly we have that $t = \neg t$ so truth and canonical falsity are also identical.

The complexity of the tautology problem for \mathbf{A} follows directly from the following theorem.

Theorem 13 (Weispfenning [44]) *The word problem for abelian l-groups is co-NP-complete.*

Corollary 14 *The tautology problem for \mathbf{A} is co-NP-complete.*

3 Relating \mathbf{A} and \mathbf{L}

The use of abelian l-groups in the theory of Łukasiewicz infinite-valued logic is not new. Chang's original completeness proof for \mathbf{L} involved extending MV-algebras to abelian l-groups and more recently Cignoli et al. have proved that MV-algebras and abelian l-groups are *categorically equivalent* [17]. Ordered abelian groups are also fundamental to the Kripke-style semantics for \mathbf{L} provided independently by Scott [41] and Urquhart [43]. It is not surprising therefore that the *logic* of abelian l-groups and \mathbf{L} should be related in some way. Here we give both an embedding of \mathbf{L} into \mathbf{A} which mirrors relationships between other logics investigated by Meyer and co-workers, and also a translation from \mathbf{L} into \mathbf{A} which will prove useful later in deriving calculi for \mathbf{L} .

We begin by defining two new implications for \mathbf{A} .

Definition 15 (Enthymematic Implication (\supseteq)) $A \supseteq B = (t \wedge A) \rightarrow B$

Definition 16 (Material Implication (\supset)) $A \supset B = (t \wedge A) \rightarrow (\perp \vee B)$

In [30] Meyer shows that fragments obtained using these new implications frequently correspond to other well-known logics; for example the material and enthymematic fragments of \mathbf{R} are classical logic and positive intuitionistic logic respectively. Of particular interest here is the fact, demonstrated by Dunn and Meyer in [18], that the material fragment of Gödel logic \mathbf{G} is the relevance logic \mathbf{RM} . We now investigate the corresponding fragments for \mathbf{A} , noting that since \mathbf{A} has no falsity constant, \perp is treated as a *propositional variable*.

Definition 17 (Enthymematic Fragment of \mathbf{A} , \mathbf{A}_E) \mathbf{A}_E consists of the theorems of \mathbf{A} in the language built up from propositional variables, t , \wedge , \vee and \supseteq .

Definition 18 (Material Fragment of \mathbf{A} , \mathbf{A}_M) \mathbf{A}_M consists of the theorems of \mathbf{A} in the language built up from propositional variables, \perp and \supseteq .

Meyer and Slaney in [31] show that the enthymematic fragment of \mathbf{A} is \mathbf{L}^+ , the positive part of Lukasiewicz logic. We generalise this here to identify the material fragment of \mathbf{A} with the whole of \mathbf{L} .

Theorem 19 (Meyer and Slaney [31]) $\mathbf{L}^+ = \mathbf{A}_E$

Theorem 20 $\mathbf{L} = \mathbf{A}_M$

Proof.

$\mathbf{L} \subseteq \mathbf{A}_M$. It is straightforward if laborious to check that all the (translated) axioms of \mathbf{L} are valid in \mathbb{Q} so it remains to show that (mp) is admissible. Consider \mathbf{A}_M formulae, ϕ and ψ where $\models_A \phi \supseteq \psi$ and $\models_A \phi$. Since $\models_A \phi$ we have $\phi \geq t$, whence $\phi \supseteq \psi = (\phi \wedge t) \rightarrow (\psi \vee \perp) = \psi \vee \perp \geq t$. If ψ is a propositional variable then by taking a valuation $v(\psi) = v(\perp) = -1$ in \mathbb{Q} we have $v(\phi \supseteq \psi) < t$, a contradiction. So either $\psi = t$ or $\psi = \psi_1 \supseteq \psi_2$. If the former then clearly $\models_A \psi$, if the latter then we have $\psi = (\psi_1 \wedge t) \rightarrow (\psi_2 \vee \perp) \geq \psi_2 \vee \perp \geq \perp$ whence $\psi \vee \perp = \psi \geq t$ and $\models_A \psi$ as required.

$\mathbf{A}_M \subseteq \mathbf{L}$. We show that if $\not\models_{\mathbf{L}} \phi$ then $\not\models_A \phi$. Given a valuation v for $[-1, 0]_{\mathbf{L}}$ such that $v(\phi) < 0$, define a valuation v' in \mathbb{Q} as follows: $v'(\perp) = -1$, $v'(p) = v(p)$ for all propositional variables p . We claim that for all formulae ψ : (1) if $v(\psi) < 0$ then $v'(\psi) = v(\psi)$, (2) if $v(\psi) = 0$ then $v'(\psi) \geq 0$. We would then have $v'(\phi) = v(\phi) < 0$ as required. We prove the claim by induction on the complexity of ψ . The base case holds by stipulation. If $\psi = \psi_1 \supseteq \psi_2 = (t \wedge \psi_1) \rightarrow (\perp \vee \psi_2)$, then if $v(\psi_1) > v(\psi_2)$ we have $v(\psi) = v(\psi_2) - v(\psi_1)$. $v(\psi_2) < 0$ so $v'(\perp \vee \psi_2) = v(\psi_2)$. Also $v'(t \wedge \psi_1) = v(\psi_1)$, whence $v'(\psi) = v(\psi_2) - v(\psi_1)$. If $v(\psi_1) \leq v(\psi_2)$ then if $v(\psi_2) = 0$ we have $v'(\perp \vee \psi_2) \geq 0$ by the induction hypothesis and $v'(t \wedge \psi_1) \leq 0$ by definition, whence we have $v'(\psi) \geq 0$; if $v(\psi_2) < 0$ then $v(\psi_1) < 0$ and we have $v'(\psi_1) = v(\psi_1)$ and $v'(\psi_2) = v(\psi_2)$ whence $v'(\psi) = v(\psi_2) - v(\psi_1) \leq 0$ as required. \square

We now give a less general but possibly more intuitive translation of \mathbf{L} into \mathbf{A} . The idea is to restrict the valuations of atoms in \mathbf{A} to lie between t and an arbitrary fixed propositional variable q^\perp that acts as falsity. Łukasiewicz implication (replacing \supseteq by \Rightarrow from now on) is defined in \mathbf{A} as follows:

Definition 21 (Positive Implication \Rightarrow) $A \Rightarrow B = (A \rightarrow B) \wedge t$

Before continuing we note that \wedge and \vee can be interpreted using only \Rightarrow and the intensional connectives:

Proposition 22 (i) $a \wedge b = a + (a \Rightarrow b)$ (ii) $a \vee b = (b \Rightarrow a) \rightarrow a$

Proof. (i) $a + (a \Rightarrow b) = a + (t \wedge (a \rightarrow b)) = (a + t) \wedge (a + (a \rightarrow b)) = a \wedge b$
(ii) $(b \Rightarrow a) \rightarrow a = \neg((t \wedge (b \rightarrow a)) + \neg a) = \neg((t + \neg a) \wedge ((b \rightarrow a) + \neg a)) = \neg(\neg a \wedge (\neg(b + \neg a) + \neg a)) = \neg(\neg a \wedge (\neg b + (a + \neg a))) = \neg(\neg a \wedge \neg b) = a \vee b. \quad \square$

Our translation of \mathbf{L} into \mathbf{A} is given below:

Definition 23 (Translation *) $p^* = (p \vee q^\perp) \wedge t$, $\perp^* = q^\perp \wedge t$, $(A \Rightarrow B)^* = t \wedge (A^* \rightarrow B^*)$ where q^\perp is an arbitrary fixed propositional variable of \mathbf{A} .

Theorem 24 $\models_{\mathbf{L}} \phi$ iff $\models_{\mathbf{A}} \phi^*$.

Proof.

For the left-to-right direction suppose that $v(\phi^*) < 0$ in \mathbb{Q} . If $v(q_\perp) \geq 0$ then it is easy to prove that $v(\phi^*) = 0$ which is a contradiction. Hence we have that $v(q_\perp) < 0$ and in fact we can assume WLOG that $v(q_\perp) = -1$ (multiplying our valuation v if necessary). We now define a valuation v' for $[-1, 0]_{\mathbf{L}}$ where $v'(p) = v(p^*)$. We prove inductively that for all \mathbf{L} -formulae ψ , $v'(\psi) = v(\psi^*)$ which gives us $v'(\phi) = v(\phi^*) < 0$ in $[-1, 0]_{\mathbf{L}}$ as required. The base cases for propositional variables and \perp follow by definition, for \Rightarrow we have that $v'(\psi_1 \Rightarrow \psi_2) = \min(0, v'(\psi_2) - v'(\psi_1)) = \min(0, v(\psi_2^*) - v(\psi_1^*)) = v(t \wedge (\psi_1^* \rightarrow \psi_2^*)) = v((\psi_1 \Rightarrow \psi_2)^*)$ and we are done.

For the right-to-left direction suppose that $v(\phi) < 0$ in $[-1, 0]_{\mathbf{L}}$, we define a valuation v' in \mathbb{Q} such that $v'(p) = v(p)$ and $v'(q^\perp) = -1$ and prove inductively that $v'(\psi^*) = v(\psi)$ for all \mathbf{L} -formulae ψ , whence $v'(\phi) < 0$ in \mathbb{Q} as required. The base cases follow immediately by definition and for \Rightarrow we have $v'((\psi_1 \Rightarrow \psi_2)^*) = v'(t \wedge (\psi_1^* \rightarrow \psi_2^*)) = \min(0, v'(\psi_2^*) - v'(\psi_1^*)) = \min(0, v(\psi_2) - v(\psi_1)) = v(\psi_1 \Rightarrow \psi_2)$ and we are done. \square

4 Hypersequent Calculi

Hypersequent calculi were introduced independently by Avron [5] and Pottinger [38] for logics with a reasonably simple semantics lacking a correspondingly simple sequent calculus. Hypersequents consist of multiple components (sequents) interpreted disjunctively and hypersequent rules include, in addition to single sequent rules, external structural rules that can operate on more than one component at a time. Hypersequent calculi are particularly suitable for logics obeying *prelinearity* (ie $(A \rightarrow B) \vee (B \rightarrow A)$) since the multiple components of the hypersequent allow several alternative hypotheses to be processed in parallel. All t-norm based logics have prelinearity and this suggests that the hypersequent formalism could be a more suitable framework for a systematic proof theory of fuzzy logics than the usual Gentzen single-sided sequents; indeed natural and intuitive hypersequent calculi have already been provided for Gödel logic \mathbf{G} [6] and \mathbf{MTL} [9, 14].

Definition 25 (Hypersequent) *A component is an ordered pair of multisets of formulae written $\Gamma \vdash \Delta$ ie a sequent; a hypersequent is a multiset of components written $\Gamma_1 \vdash \Delta_1 | \dots | \Gamma_n \vdash \Delta_n$.*

Note that we choose components to consist of pairs of *multisets* of formulae and hypersequents to consist of *multisets* of components and therefore avoid the need in our calculi for internal and external exchange rules. In what follows *all* set terminology $\cup, \cap, \{\}$ and so on will refer to multisets. We will also make use of the following definitions.

Definition 26 (Formula Complexity $\text{cp}(\mathbf{A})$) $\text{cp}(q) = 0$ for q atomic, $\text{cp}(* (A_1, \dots, A_m)) = 1 + \sum_{i=1}^m \text{cp}(A_i)$ where $*$ \in *Con*, $\text{arity}(*) = m$.

Definition 27 (Hypersequent Multiset Complexity $\text{mc}(\mathbf{G})$) $\text{mc}(\Gamma_1 \vdash \Delta_1 | \dots | \Gamma_n \vdash \Delta_n) = \{\text{cp}(A) : A \in \cup_{i=1}^n \Gamma_i \cup \Delta_i\}$

Definition 28 (Integer Multiset Ordering $<_m$) For Γ and Δ multisets of integers, $\Gamma <_m \Delta$ if either (1) $\Gamma \subset \Delta$ or (2) $\Gamma <_m \Delta'$ where $\Delta' = \Delta - \{j\} \cup \{i, \dots, i\}$ and $i < j$.

Note that $<_m$ is a *well-order* on multisets of integers.

4.1 A Hypersequent Calculus for \mathbf{A}

Hypersequents for \mathbf{A} are interpreted in the standard way, recalling that intensional conjunction and intensional disjunction are equivalent in this logic:

Definition 29 (Interpretation of Hypersequents for \mathbf{A}) Let the interpretation of a sequent $S = A_1, \dots, A_n \vdash B_1, \dots, B_m$ be $\phi^S = (A_1 + \dots + A_n) \rightarrow (B_1 + \dots + B_m)$ (where $C_1 + \dots + C_k = t$ if $k = 0$) so that $\models S$ iff $\models \phi^S$. Let the interpretation of a hypersequent $G = S_1 | \dots | S_n$ be $\phi^G = \phi^{S_1} \vee \dots \vee \phi^{S_n}$ so that $\models G$ iff $\models \phi^G$.

We introduce the following hypersequent calculus for \mathbf{A} .

Definition 30 (GA) *GA* has the following rules:

Axioms

$$(ID) \quad A \vdash A \qquad (\wedge) \quad \vdash$$

Structural rules

$$\begin{array}{ll} (EW) & \frac{G|\Gamma \vdash \Delta}{G|\Gamma \vdash \Delta | \Gamma' \vdash \Delta'} \qquad (EC) \quad \frac{G|\Gamma \vdash \Delta | \Gamma \vdash \Delta}{G|\Gamma \vdash \Delta} \\ (S) & \frac{G|\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}{G|\Gamma_1 \vdash \Delta_2 | \Gamma_2 \vdash \Delta_2} \qquad (M) \quad \frac{G|\Gamma_1 \vdash \Delta_1 \quad G|\Gamma_2 \vdash \Delta_2}{G|\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2} \end{array}$$

Logical rules

$$\begin{array}{ll}
(t, l) & \frac{G|\Gamma \vdash \Delta}{G|\Gamma, t \vdash \Delta} & (t, r) & \frac{G|\Gamma \vdash \Delta}{G|\Gamma \vdash t, \Delta} \\
(\neg, l) & \frac{G|\Gamma \vdash A, \Delta}{G|\Gamma, \neg A \vdash \Delta} & (\neg, r) & \frac{G|\Gamma, A \vdash \Delta}{G|\Gamma \vdash \neg A, \Delta} \\
(\rightarrow, l) & \frac{G|\Gamma, B \vdash A, \Delta}{G|\Gamma, A \rightarrow B \vdash \Delta} & (\rightarrow, r) & \frac{G|\Gamma, A \vdash B, \Delta}{G|\Gamma \vdash A \rightarrow B, \Delta} \\
(+, l) & \frac{G|\Gamma, A, B \vdash \Delta}{G|\Gamma, A + B \vdash \Delta} & (+, r) & \frac{G|\Gamma \vdash A, B, \Delta}{G|\Gamma \vdash A + B, \Delta} \\
(\wedge, l) & \frac{G|\Gamma, A \vdash \Delta | \Gamma, B \vdash \Delta}{G|\Gamma, A \wedge B \vdash \Delta} & (\wedge, r) & \frac{G|\Gamma \vdash A, \Delta \quad G|\Gamma \vdash B, \Delta}{G|\Gamma \vdash A \wedge B, \Delta} \\
(\vee, l) & \frac{G|\Gamma, A \vdash \Delta \quad G|\Gamma, B \vdash \Delta}{G|\Gamma, A \vee B \vdash \Delta} & (\vee, r) & \frac{G|\Gamma \vdash A, \Delta | \Gamma \vdash B, \Delta}{G|\Gamma \vdash A \vee B, \Delta}
\end{array}$$

Alternative (non-invertible) rules to replace (\wedge, l) and (\vee, r) without losing completeness are:

$$(\wedge'_i, l) \quad \frac{G|\Gamma, A_i \vdash \Delta}{G|\Gamma, A_1 \wedge A_2 \vdash \Delta} \quad i = 1, 2 \quad (\vee'_i, r) \quad \frac{G|\Gamma \vdash A_i, \Delta}{G|\Gamma \vdash A_1 \vee A_2 \vdash \Delta} \quad i = 1, 2$$

We could also replace the rules (M) , (EW) and the axioms (ID) , (Λ) , (t, l) , (t, r) with the single axiom:

$$(AX) \quad G|\Gamma, t, \dots, t \vdash \Gamma, t, \dots, t$$

Prelinearity for \mathbf{A} is proved as follows.

Example 31 A proof of $(A \rightarrow B) \vee (B \rightarrow A)$ in \mathbf{GA} :

$$\frac{\frac{\frac{A \vdash A \quad B \vdash B}{A, B \vdash A, B}}{A \vdash B | B \vdash A}}{A \vdash B | B \rightarrow A}}{\vdash A \rightarrow B | \vdash B \rightarrow A}}{\vdash (A \rightarrow B) \vee (B \rightarrow A)}$$

We now turn our attention to the soundness and completeness of \mathbf{GA} . Note that we use the convention in what follows of writing Γ for both the multiset Γ and the sum of elements of Γ . We also write $\lambda\Gamma$ for $\underbrace{\Gamma \cup \dots \cup \Gamma}_{\lambda}$.

Theorem 32 (Soundness of \mathbf{GA}) If G succeeds in \mathbf{GA} then $\models_{\mathbf{A}} G$.

Proof. We reason inductively on the height of a proof in \mathbf{GA} showing that the axioms are sound and that the rules of \mathbf{GA} preserve validity in \mathbb{Q} :

- Axioms. Clearly $A - A \geq 0$ and $0 \geq 0$
- Structural Rules. For (S) , if $\models_{\mathbf{A}} G | \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2$ then either $G \geq 0$ or $(\Delta_1 + \Delta_2) - (\Gamma_1 + \Gamma_2) \geq 0$. So we have either $G \geq 0$ or $\Delta_1 - \Gamma_1 \geq 0$ or $\Delta_2 - \Gamma_2 \geq 0$ whence we have $\models_{\mathbf{A}} G | \Gamma_1 \vdash \Delta_1 | \Gamma_2 \vdash \Delta_2$. The other structural rules are similar.
- Logical Rules. For (\rightarrow, r) , if $\models_{\mathbf{A}} G | \Gamma, A \vdash \Delta, B$ then either $G \geq 0$ or $(\Delta + B) - (\Gamma + A) \geq 0$ so either $G \geq 0$ or $(\Delta + B - A) - \Gamma \geq 0$ ie $\models_{\mathbf{A}} G | \Gamma \vdash \Delta, A \rightarrow B$. The other logical rules are similar. \square

We prepare for our completeness proof by proving some simple properties of \mathbf{GA} .

Proposition 33 $\Pi \vdash \Pi$ *succeeds in* \mathbf{GA}

Proof. By induction on the size of Π . If $\Pi = \emptyset$ then we apply (Λ) . Otherwise $\Pi = \{A\} \cup \Pi'$ and we apply (M) to obtain $A \vdash A$ which succeeds by (ID) , and $\Pi' \vdash \Pi'$ which succeeds by the induction hypothesis. \square

Proposition 34 *The logical rules of* \mathbf{GA} *are terminating.*

Proof. It is easy to see that all the logical rules strictly reduce the multiset complexity of the hypersequent. \square

Definition 35 (Invertible) *A rule is invertible if whenever the conclusion of the rule is valid then all its premises are valid.*

Proposition 36 *The logical rules of* \mathbf{GA} *are invertible.*

Proof. We reason in \mathbb{Q} . As an example consider (\wedge, l) ; if $\models_{\mathbf{A}} G | \Gamma, A \vdash \Delta | \Gamma, B \vdash \Delta$ then either $G \geq 0$, $\Delta - (\Gamma + B) \geq 0$ or $\Delta - (\Gamma + A) \geq 0$. So if $G < 0$ then we have $\Delta - (\Gamma + \min(A, B)) \geq 0$ ie $\models_{\mathbf{A}} G | \Gamma, A \wedge B \vdash \Delta$. The other logical rules are similar. \square

We now show that proving an atomic hypersequent in \mathbf{GA} is equivalent to solving a linear programming problem over \mathbb{Q} .

Proposition 37 *Given* Γ_i, Δ_i *containing only atoms for* $i = 1 \dots n$ *then* $\models_{\mathbf{A}} \Gamma_1 \vdash \Delta_1 | \dots | \Gamma_n \vdash \Delta_n$ *iff there exist* $\lambda_1, \dots, \lambda_n \in \mathbb{Z}^+$ *such that* $\lambda_i > 0$ *for some* i , $1 \leq i \leq n$ *and* $\cup_{i=1}^n \lambda_i \Gamma_i = \cup_{i=1}^n \lambda_i \Delta_i$.

Proof. Reasoning in \mathbb{Q} we have $\models_{\mathbf{A}} \Gamma_1 \vdash \Delta_1 | \dots | \Gamma_n \vdash \Delta_n$ iff $\Gamma_1 \leq \Delta_1$ or \dots or $\Gamma_n \leq \Delta_n$. This is equivalent to the set $\{\Gamma_1 > \Delta_1, \dots, \Gamma_n > \Delta_n\}$ being inconsistent over \mathbb{Q} which holds iff there exist $\lambda_1, \dots, \lambda_n \in \mathbb{Z}^+$ such that $\lambda_i > 0$ for some i , $1 \leq i \leq n$ and $\sum_{i=1}^n \lambda_i \Gamma_i = \sum_{i=1}^n \lambda_i \Delta_i$ ie iff $\cup_{i=1}^n \lambda_i \Gamma_i = \cup_{i=1}^n \lambda_i \Delta_i$. \square

Proposition 38 *If there exist* $\lambda_1, \dots, \lambda_n \in \mathbb{Z}^+$ *such that* $\cup_{i=1}^n \lambda_i \Gamma_i = \cup_{i=1}^n \lambda_i \Delta_i$ *then* $\Gamma_1 \vdash \Delta_1 | \dots | \Gamma_n \vdash \Delta_n$ *succeeds in* \mathbf{GA} .

Proof. For each i , if $\lambda_i = 0$ then we apply (EW) to remove $\Gamma_i \vdash \Delta_i$ from the hypersequent, otherwise we apply (EC) $\lambda_i - 1$ times to obtain λ_i copies of $\Gamma_i \vdash \Delta_i$. We then apply (S) repeatedly to get a component of the form $\Pi \vdash \Pi$ (since $\cup_{i=1}^n \lambda_i \Gamma_i = \cup_{i=1}^n \lambda_i \Delta_i$) which succeeds by Proposition 33. \square

We are now ready to prove the completeness of \mathbf{GA} .

Theorem 39 (Completeness of \mathbf{GA}) *If $\models_{\mathbf{A}} G$ then G succeeds in \mathbf{GA}*

Proof. We show that given a hypersequent G valid in \mathbf{A} we are able to find a proof of G in \mathbf{GA} . Our first step in the proof is to apply the logical rules exhaustively to G ; since by Propositions 34 and 36 these rules are terminating and invertible respectively we obtain atomic hypersequents valid in \mathbf{A} . It therefore suffices to show that each *atomic* hypersequent $G' = \Gamma_1 \vdash \Delta_1 \mid \dots \mid \Gamma_n \vdash \Delta_n$ valid in \mathbf{A} is provable in \mathbf{GA} . By Proposition 37 we have that G' is valid iff there exist $\lambda_1, \dots, \lambda_n \in \mathbb{Z}^+$ such that $\lambda_i > 0$ for some i , $1 \leq i \leq n$ and $\cup_{i=1}^n \lambda_i \Gamma_i = \cup_{i=1}^n \lambda_i \Delta_i$ is a linear combination of the components of G' (taking λ_i copies of each $\Gamma_i \vdash \Delta_i$ and adding them all together) gives a component of the form $\Pi \vdash \Pi$. So now by Proposition 38, which says that we can apply (EC) , (EW) and (S) to G' to obtain the required combination, we have that G' succeeds. \square

We note that Theorem 39 tells us that the following *cut rules* are admissible in \mathbf{A} :

$$\frac{\Gamma, A \vdash \Delta, A}{\Gamma \vdash \Delta} \quad \frac{\Gamma, A \vdash \Delta \quad \Pi \vdash \Sigma, A}{\Gamma, \Pi \vdash \Delta, \Sigma}$$

In fact these are interderivable in \mathbf{GA} :

$$\frac{\frac{\Gamma, A \vdash \Delta \quad \Pi \vdash \Sigma, A}{\Gamma, \Pi, A \vdash \Delta, \Sigma, A}}{\Gamma, \Pi \vdash \Delta, \Sigma} \quad \frac{\frac{\Gamma, A \vdash \Delta, A}{\Gamma \vdash \Delta, A \rightarrow A} \quad \frac{A \vdash A}{A \rightarrow A \vdash}}{\Gamma \vdash \Delta}$$

4.2 A Hypersequent Calculus for \mathbf{L}

We derive a hypersequent calculus for \mathbf{L} from \mathbf{GA} , our hypersequent calculus for \mathbf{A} , in several stages. First we give an interpretation for hypersequents for \mathbf{L} in terms of the characteristic model $[-1, 0]_{\mathbf{L}}$. This allows us to give a very natural translation from hypersequents for \mathbf{L} to hypersequents for \mathbf{A} . We then immediately have a hypersequent calculus for \mathbf{L} using \mathbf{GA} which we refine to obtain a more direct and natural calculus for \mathbf{L} .

We start by interpreting hypersequents for \mathbf{L} . We extend the standard notion of validity for \mathbf{L} from $\models_{\mathbf{L}}$ to $\models_{\mathbf{L}}^*$ using the characteristic model $[-1, 0]_{\mathbf{L}}$. A formula is interpreted as usual for this model but multisets of formulae are interpreted (as for \mathbf{A}) as *sums* of elements in \mathbb{Q} (as opposed to bounded sums using the Łukasiewicz conjunction \oplus). Interpretations are extended to sequents and hypersequents in the usual way.

Definition 40 (Interpretation of Hypersequents for \mathbf{L}) $\models_{\mathbf{L}}^* \Gamma_1 \vdash \Delta_1 \mid \dots \mid \Gamma_n \vdash \Delta_n$ iff for all $v : \text{For} \rightarrow [-1, 0]_{\mathbf{L}}$ there exists i such that $\sum_{A \in \Gamma_i} v(A) \leq \sum_{B \in \Delta_i} v(B)$.

We emphasise that for *formulae* this notion coincides with the standard notion of validity for \mathbf{L} .

Proposition 41 $\models_{\mathbf{L}}^* \phi$ iff $\models_{\mathbf{L}} \phi$.

Proof. Immediate from Definition 40. \square

Clearly our hypersequents for \mathbf{L} can also be interpreted in \mathbf{A} . The following translation, extending that given by Definition 23 makes this relationship precise.

Definition 42 (Translation *) $p^* = (p \vee q^\perp) \wedge t$, $\perp^* = q^\perp \wedge t$, $(A \Rightarrow B)^* = t \wedge (A^* \rightarrow B^*)$, $(\Gamma \vdash \Delta)^* = \Gamma^* \vdash \Delta^*$, $(S_1 | \dots | S_n)^* = S_1^* | \dots | S_n^*$ where q^\perp is an arbitrary fixed propositional variable.

Theorem 43 $\models_{\mathbf{L}}^* G$ iff $\models_{\mathbf{A}} G^*$.

Proof. The proof of Theorem 24 is easily extended from formulae to hypersequents. \square

Theorem 43 means that we have a calculus for \mathbf{L} simply by using \mathbf{GA} applied to formulae of the form A^* . We refine this calculus in several steps. We first simplify the rules for formulae of the form A^* to obtain the following calculus:

Definition 44 (\mathbf{GA}^*) \mathbf{GA}^* consists of the axioms and structural rules of \mathbf{GA} together with the following logical rules:

$$\begin{array}{ll}
(\perp^*, l) & \frac{G|\Gamma \vdash \Delta | \Gamma, q^\perp \vdash \Delta}{G|\Gamma, \perp^* \vdash \Delta} & (\perp^*, r) & \frac{G|\Gamma \vdash \Delta \quad G|\Gamma \vdash q^\perp, \Delta}{G|\Gamma \vdash \perp^*, \Delta} \\
(q^*, l) & \frac{G|\Gamma \vdash \Delta | \Gamma, q^\perp \vdash \Delta \quad G|\Gamma \vdash \Delta | \Gamma, q \vdash \Delta}{G|\Gamma, q^* \vdash \Delta} & (q^*, r) & \frac{G|\Gamma \vdash \Delta \quad G|\Gamma \vdash \Delta, q|\Gamma \vdash q^\perp, \Delta}{G|\Gamma \vdash q^*, \Delta} \\
(\Rightarrow^*, l) & \frac{G|\Gamma, B^* \vdash A^*, \Delta | \Gamma \vdash \Delta}{G|\Gamma, (A \Rightarrow B)^* \vdash \Delta} & (\Rightarrow^*, r) & \frac{G|\Gamma, A^* \vdash B^*, \Delta \quad G|\Gamma \vdash \Delta}{G|\Gamma \vdash (A \Rightarrow B)^*, \Delta}
\end{array}$$

Proposition 45 If G^* succeeds in \mathbf{GA} then G^* succeeds in \mathbf{GA}^* .

Proof. It is easy to check that the logical rules of \mathbf{GA}^* are derivable using the logical rules of \mathbf{GA} and are hence invertible. Since they also reduce a hypersequent G^* to atomic hypersequents and the two calculi share the same structural rules and axioms we are done. \square

Our next move is to define a calculus that is *stronger* than \mathbf{GA}^* and operates directly on formulae of \mathbf{L} rather than their translations:

Definition 46 (\mathbf{GL}_i) \mathbf{GL}_i consists of the axioms and structural rules of \mathbf{GA}

and the following logical rules:

$$\begin{array}{l}
(\perp, l) \quad \frac{G|\Gamma \vdash \Delta|\Gamma, \perp \vdash \Delta}{G|\Gamma, \perp \vdash \Delta} \\
(q, l) \quad \frac{G|\Gamma \vdash \Delta|\Gamma, q \vdash \Delta}{G|\Gamma, q \vdash \Delta} \quad (q, r) \quad \frac{G|\Gamma \vdash q, \Delta|\Gamma \vdash \perp, \Delta}{G|\Gamma \vdash q, \Delta} \\
(\Rightarrow, l) \quad \frac{G|\Gamma, B \vdash A, \Delta|\Gamma \vdash \Delta}{G|\Gamma, A \Rightarrow B \vdash \Delta} \quad (\Rightarrow, r) \quad \frac{G|\Gamma, A \vdash B, \Delta \quad G|\Gamma \vdash \Delta}{G|\Gamma \vdash A \Rightarrow B, \Delta}
\end{array}$$

Proposition 47 *If G^* succeeds in \mathbf{GA}^* then G succeeds in \mathbf{GL}_i .*

Proof. First we observe that removing the left premises from (\perp^*, r) and (q^*, l) and (q^*, r) gives a calculus \mathbf{GA}_1^* such that if G^* succeeds in \mathbf{GA}^* then G^* succeeds in \mathbf{GA}_1^* . We now replace (for the calculus \mathbf{GA}_1^*) q by q^* in the rules (q^*, l) and (q^*, r) and q^\perp by \perp^* in the rules (\perp^*, l) and (\perp^*, r) to obtain a calculus \mathbf{GA}_2^* . Since no rules have conclusions with formulae that must be atomic we have that if G^* succeeds in \mathbf{GA}_1^* then G^* succeeds in \mathbf{GA}_2^* . But this is just the translation of the calculus \mathbf{GL}_i (removing (\perp^*, r) as it is just the trivial rule) so we have that if G^* succeeds in \mathbf{GA}_2^* then G succeeds in \mathbf{GL}_i and we are done. \square

We now present our final calculus for \mathbf{L} :

Definition 48 (\mathbf{GL}) *\mathbf{GL} has the following rules:*

Axioms

$$(ID) \quad A \vdash A \quad (\wedge) \quad \vdash \quad (\perp) \quad \perp \vdash A$$

Structural rules

$$\begin{array}{l}
(IW) \quad \frac{G|\Gamma \vdash \Delta}{G|\Gamma, A \vdash \Delta} \\
(EW) \quad \frac{G|\Gamma \vdash \Delta}{G|\Gamma \vdash \Delta|\Gamma' \vdash \Delta'} \quad (EC) \quad \frac{G|\Gamma \vdash \Delta|\Gamma \vdash \Delta}{G|\Gamma \vdash \Delta} \\
(S) \quad \frac{G|\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}{G|\Gamma_1 \vdash \Delta_2|\Gamma_2 \vdash \Delta_2} \quad (M) \quad \frac{G|\Gamma_1 \vdash \Delta_1 \quad G|\Gamma_2 \vdash \Delta_2}{G|\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}
\end{array}$$

Logical rules

$$(\Rightarrow, l) \quad \frac{G|\Gamma, B \vdash A, \Delta|\Gamma \vdash \Delta}{G|\Gamma, A \Rightarrow B \vdash \Delta} \quad (\Rightarrow, r) \quad \frac{G|\Gamma, A \vdash B, \Delta \quad G|\Gamma \vdash \Delta}{G|\Gamma \vdash A \Rightarrow B, \Delta}$$

Below we give a proof of the characteristic axiom of \mathbf{L} .

Example 49 A proof of $((A \Rightarrow B) \Rightarrow B) \Rightarrow ((B \Rightarrow A) \Rightarrow A)$ in **GL**:

$$\begin{array}{c}
\frac{\frac{\frac{B \vdash B \quad A \vdash A}{B, A \vdash A, B}}{B, A \vdash A, B | B \vdash A}}{B, B \Rightarrow A \vdash A} \quad \frac{\frac{B \vdash B \quad A \vdash A}{B, A \vdash A, B}}{B, B \Rightarrow A, A \vdash A, B} \\
\frac{B, B \Rightarrow A \vdash A, A \Rightarrow B}{B, B \Rightarrow A \vdash A, A \Rightarrow B | B \Rightarrow A \vdash A} \\
\frac{\frac{\vdash}{(A \Rightarrow B) \Rightarrow B \vdash} \quad (A \Rightarrow B) \Rightarrow B, B \Rightarrow A \vdash A}{(A \Rightarrow B) \Rightarrow B \vdash (B \Rightarrow A) \Rightarrow A} \\
\vdash \frac{(A \Rightarrow B) \Rightarrow B \vdash (B \Rightarrow A) \Rightarrow A}{\vdash ((A \Rightarrow B) \Rightarrow B) \Rightarrow ((B \Rightarrow A) \Rightarrow A)}
\end{array}$$

We prove soundness and completeness proof-theoretically using the translation of Definition 42.

Proposition 50 The translations of the rules (IW) , (\perp) , (\Rightarrow, l) and (\Rightarrow, r) are derivable in **GA** for hypersequents containing only formulae of the form A^* .

Proof. For (IW) we note that all formulae A^* are of the form $X \wedge t$ and proceed as follows:

$$\frac{\frac{G | \Gamma \vdash \Delta}{G | \Gamma, t \vdash \Delta}}{G | \Gamma, X \vdash \Delta | \Gamma, t \vdash \Delta} \\
\frac{G | \Gamma, X \wedge t \vdash \Delta}$$

For (\perp) we proceed by induction on the complexity of A . If A is a propositional variable p then we have:

$$\frac{\frac{\frac{q^\perp \vdash q^\perp}{q^\perp \vdash p | q^\perp \vdash q^\perp}}{q^\perp \vdash p \vee q^\perp} \quad \frac{t \vdash t}{t \vdash t | q^\perp \vdash t}}{\frac{t \vdash p \vee q^\perp | q^\perp \vdash p \vee q^\perp}{t \wedge q^\perp \vdash p \vee q^\perp} \quad \frac{t \wedge q^\perp \vdash t}{t \wedge q^\perp \vdash (p \vee q^\perp) \wedge t}}$$

If $A = \perp$ then we succeed immediately by (ID) . For $A = B \Rightarrow C$ we have the following situation:

$$\frac{\frac{\frac{t \vdash t}{t \vdash t | q^\perp \vdash t} \quad \frac{t \wedge q^\perp, B^* \vdash C^*}{t \wedge q^\perp \vdash B^* \rightarrow C^*}}{t \wedge q^\perp \vdash t \wedge (B^* \rightarrow C^*)}$$

But now since (IW) is derivable in **GA** we can step to $t \wedge q^\perp \vdash C^*$ for the right branch which succeeds by the induction hypothesis.

(\Rightarrow, l) and (\Rightarrow, r) are derived as follows:

$$\frac{\frac{\frac{G|\Gamma \vdash \Delta|\Gamma, B^* \vdash \Delta, A^*}{G|\Gamma \vdash \Delta|\Gamma, A^* \rightarrow B^* \vdash \Delta}}{G|\Gamma, t \vdash \Delta|\Gamma, A^* \rightarrow B^* \vdash \Delta}}{G|\Gamma, t \wedge (A^* \rightarrow B^*) \vdash \Delta}$$

$$\frac{\frac{G|\Gamma \vdash \Delta}{G|\Gamma \vdash \Delta, t} \quad \frac{G|\Gamma, A^* \vdash \Delta, B^*}{G|\Gamma \vdash \Delta, A^* \rightarrow B^*}}{G|\Gamma \vdash \Delta, t \wedge (A^* \rightarrow B^*)} \quad \square$$

Proposition 51 *The following rule is admissible in \mathbf{GL} :*

$$\frac{G|\Gamma_1 \vdash \perp, \Delta_1| \dots |\Gamma_n \vdash \perp, \Delta_n}{G|\Gamma_1 \vdash A, \Delta_1| \dots |\Gamma_n \vdash A, \Delta_n}$$

Proof. By induction on the height h of a proof of $G|\Gamma_1 \vdash \perp, \Delta_1| \dots |\Gamma_n \vdash \perp, \Delta_n$. If $h = 0$ then we must have $\perp \vdash \perp$ and we get $\perp \vdash A$ by (\perp) . The cases for $h > 0$ follow easily from the induction hypothesis. \square

Proposition 52 *The rules (\perp, l) , (q, l) and (q, r) of \mathbf{GL}_i are admissible in \mathbf{GL} .*

Proof. (q, l) is derivable using (EC) and (IW) . (q, r) and (\perp, l) are admissible using (EC) and Proposition 51.

Proposition 53 *G succeeds in \mathbf{GL} iff G^* succeeds in \mathbf{GA} .*

Proof.

For the left-to-right direction we simply note that by Proposition 50 we have that the translations of all the rules of \mathbf{GL} are either rules of \mathbf{GA} or derivable in \mathbf{GA} for hypersequents containing only formulae of the form A^* .

For the right-to-left direction by Propositions 47 and 45 we only have to show that if G succeeds in \mathbf{GL}_i then G succeeds in \mathbf{GL} . We simply note that all the extra rules of \mathbf{GL}_i are either derivable or admissible in \mathbf{GL} by Proposition 52. \square

Theorem 54 (Soundness and Completeness of \mathbf{GL}) *G succeeds in \mathbf{GL} iff $\models_{\mathbf{L}}^* G$.*

Proof. By Proposition 53, Theorems 39 and 32, and Theorem 43 respectively we have that G succeeds in \mathbf{GL} iff G^* succeeds in \mathbf{GA} iff $\models_A G^*$ iff $\models_{\mathbf{L}}^* G$. \square

Corollary 55 *$\vdash \phi$ succeeds in \mathbf{GL} iff $\models_{\mathbf{L}} \phi$.*

Finally we note that by disregarding the rule (\perp) we obtain a calculus for the positive part of Łukasiewicz logic \mathbf{L}^+ . Moreover unlike \mathbf{L} we can interpret hypersequents for \mathbf{L}^+ within \mathbf{L}^+ in the standard way using \oplus instead of $+$.

5 Terminating Hypersequent Calculi

Although **GA** and **GL** are natural and elegant calculi they are not particularly suitable for proof search. Most obviously they are *non-terminating* due to the presence of the external contraction rule (*EC*). One method of obtaining terminating calculi is to replace (*EC*) and (*S*) with new rules that apply external contraction and splitting more carefully. In particular propositional variables can be removed from hypersequents *one at a time* by adding a particular propositional variable as a *focus* to each hypersequent; the focus is changed only when all occurrences of the marked propositional variable have been removed from the hypersequent.

Definition 56 (Focussed Hypersequent) *If G is a hypersequent and q is a propositional variable then $[q]G$ is a focussed hypersequent with focus q . The interpretation of $[q]G$ is the same as for G ie $\models_X [q]G$ iff $\models_X G$.*

5.1 A Terminating Hypersequent Calculus for **A**

We present the following terminating hypersequent calculus for **A**.

Definition 57 (\mathbf{GA}_t) \mathbf{GA}_t consists of the axioms, logical rules and structural rules (*EW*) and (*M*) of **GA** with focussed hypersequents and the same focus for premises and conclusion, and also:

$$\begin{array}{l}
 \text{(shift)} \quad \frac{[q]G}{[p]G} \quad \text{where: } \begin{array}{l} q \text{ occurs in } G \\ p \text{ doesn't occur in } G \end{array} \\
 \\
 \text{(S')} \quad \frac{[p]G|m\Gamma_1, n\Gamma_2 \vdash m\Delta_1, n\Delta_2|S}{[p]G|\Gamma_1, np \vdash \Delta_1|\Gamma_2 \vdash \Delta_2, mp} \quad \text{where: } \begin{array}{l} \Gamma_1, \Gamma_2, \Delta_1, \Delta_2 \text{ atomic} \\ n > 0, m > 0 \\ p \notin \Gamma_1 \cup \Delta_1 \cup \Gamma_2 \cup \Delta_2 \\ S \text{ is } \Gamma_1, np \vdash \Delta_1 \text{ or } \Gamma_2 \vdash \Delta_2, mp \end{array}
 \end{array}$$

Consider the example below, noting that some of the more obvious steps in the proof have been left out to aid readability.

Example 58 *A proof of $((q + q + q) \wedge (p + p + p)) \rightarrow (p + q + q)$ in \mathbf{GA}_t :*

$$\frac{\frac{\frac{\frac{[q]p \vdash p \quad [q]p \vdash p}{[q]p, p \vdash p, p}}{[q]p, p \vdash p, p|q \vdash p}}{[q]q \vdash p|p, p \vdash q, q} \quad [1]}{[q]q, q, q \vdash p, q, q|p, p, p \vdash p, q}}{[q]q + q + q \vdash p, q, q|p + p + p \vdash p, q, q}}{[q](q + q + q) \wedge (p + p + p) \vdash p, q, q}}{[q](q + q + q) \wedge (p + p + p) \vdash p + q + q}}{[q] \vdash ((q + q + q) \wedge (p + p + p)) \rightarrow (p + q + q)}$$

[1] For the application of (S') at this point in the proof we have $n = 1$ occurrences of the focus q on the left hand side of the first component and $m = 2$ occurrences of q on the right hand side of the second component. Hence for the next line up in the proof we have a component with 2 copies of the first component put together with 1 copy of the second component.

Let us see why this calculus terminates. Clearly the logical rules serve to reduce any hypersequent to a hypersequent containing only atoms. The proof then *focuses* on a particular propositional variable, p say, and uses (S') , (M) , (ID) and (EW) to remove all occurrences of p . Each component has a difference between the number of ps on each side; (S') reduces the sum of differences for the whole hypersequent. When this sum is 0 only (M) , (ID) and (EW) can be used to remove further occurrences of p . If a propositional variable has been removed from the hypersequent completely the *(shift)* rule allows a change of focus to a different variable.

We now put this more formally.

Definition 59 (count) $count(\Gamma, p) = |p : p \in \Gamma|$

Definition 60 ($d([p]G)$) For a hypersequent $G = \Gamma_1 \vdash \Delta_1 | \dots | \Gamma_n \vdash \Delta_n$, $d([p]G) = \sum_{i=1}^n |count(\Gamma_i, p) - count(\Delta_i, p)|$.

Proposition 61 $d([p]G | m\Gamma_1, n\Gamma_2 \vdash m\Delta_1, n\Delta_2 | S) < d([p]G | \Gamma_1, np \vdash \Delta_1 | \Gamma_2 \vdash \Delta_2, mp)$ when $\Gamma_1, \Gamma_2, \Delta_1, \Delta_2$ are atomic, $n > 0$, $m > 0$, $p \notin \Gamma_1 \cup \Delta_1 \cup \Gamma_2 \cup \Delta_2$ and S is $\Gamma_1, np \vdash \Delta_1$ or $\Gamma_2 \vdash \Delta_2, mp$.

Proof. Suppose that $S = \Gamma_1, np \vdash \Delta_1$ then $d([p]G | m\Gamma_1, n\Gamma_2 \vdash m\Delta_1, n\Delta_2 | S) = d([p]G) + n < d([p]G) + n + m = d([p]G | \Gamma_1, np \vdash \Delta_1 | \Gamma_2 \vdash \Delta_2, mp)$. The case where $S = \Gamma_2 \vdash \Delta_2, mp$ is symmetrical. \square

Theorem 62 (Termination) \mathbf{GA}_t terminates for $[p]G$.

Proof. We proceed by induction on (c, n, d, s) ordered lexicographically where c is the multiset complexity of G not including atoms, n is the number of different propositional variables in $[p]G$ (including p), d is $d([p]G)$ and s is the number of symbols in G . We show that all the rules (read backwards) strictly *decrease* (c, n, d, s) . The logical rules all strictly reduce c so we turn our attention to the structural rules. *(shift)* does not increase c and decreases n . (S') does not increase c or n and by Proposition 61 strictly reduces d . (EW) and (M) do not increase c , n or d and strictly reduce s . \square

We now check that \mathbf{GA}_t is sound and complete.

Theorem 63 (Soundness) If $[p]G$ succeeds in \mathbf{GA}_t then $\models_{\mathbf{A}} G$.

Proof. We observe simply that (S') is a derived rule of \mathbf{GA} . \square

Although (S') is not invertible we have that if its conclusion is valid then its premise is valid for *one of the choices for S* .

Proposition 64 *If $\models_A G|\Gamma_1 \vdash \Delta_1|\Gamma_2 \vdash \Delta_2$ then either $\models_A G|\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2|\Gamma_1 \vdash \Delta_1$ or $\models_A G|\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2|\Gamma_2 \vdash \Delta_2$.*

Proof. Let $G = \Gamma'_1 \vdash \Delta'_1 | \dots | \Gamma'_n \vdash \Delta'_n$. By Proposition 37 we have that $\models_A G|\Gamma_1 \vdash \Delta_1|\Gamma_2 \vdash \Delta_2$ iff there exist $\lambda_1, \lambda_2, \mu_1, \dots, \mu_n$ where $\lambda_i > 0$ for $1 \leq i \leq 2$ or $\mu_i > 0$ for $1 \leq i \leq n$, such that $\sum_{i=1}^n \mu_i \Gamma'_i + \lambda_1 \Gamma_1 + \lambda_2 \Gamma_2 = \sum_{i=1}^n \mu_i \Delta'_i + \lambda_1 \Delta_1 + \lambda_2 \Delta_2$. Suppose that $\lambda_1 \leq \lambda_2$ then we have $\sum_{i=1}^n \mu_i \Gamma'_i + \lambda_1(\Gamma_1 + \Gamma_2) + (\lambda_2 - \lambda_1)\Gamma_2 = \sum_{i=1}^n \mu_i \Delta'_i + \lambda_1(\Delta_1 + \Delta_2) + (\lambda_2 - \lambda_1)\Delta_2$ which gives, again by Proposition 37, that $\models_A G|\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2|\Gamma_2 \vdash \Delta_2$. The case where $\lambda_1 \geq \lambda_2$ is symmetrical. \square

Now consider a valid hypersequent G where for a given propositional variable p there are never more occurrences of p on the right hand side of a component of G than the left. If we remove a component where p occurs *more* on the left than the right then we will still obtain a valid hypersequent G' .

Proposition 65 *Given an atomic hypersequent $G = \Gamma_1 \vdash \Delta_1 | \dots | \Gamma_n \vdash \Delta_n$ where $\models_A G$ and $\text{count}(\Gamma_1, p) > \text{count}(\Delta_1, p)$ and $\text{count}(\Gamma_i, p) \geq \text{count}(\Delta_i, p)$ for $i = 2 \dots n$ then $\models_A \Gamma_2 \vdash \Delta_2 | \dots | \Gamma_n \vdash \Delta_n$.*

Proof. By Proposition 37 $\models_A G$ iff there exist $\lambda_1, \dots, \lambda_n \in \mathbb{Z}^+$ such that $\lambda_i > 0$ for some i , $1 \leq i \leq n$ and $\cup_{i=1}^n \lambda_i \Gamma_i = \cup_{i=1}^n \lambda_i \Delta_i$. But if $\lambda_1 > 0$ then $\text{count}(\cup_{i=1}^n \lambda_i \Gamma_i, p) > \text{count}(\cup_{i=1}^n \lambda_i \Delta_i, p)$ a contradiction. Hence $\lambda_1 = 0$ and we have $\models_A \Gamma_2 \vdash \Delta_2 | \dots | \Gamma_n \vdash \Delta_n$. \square

Our completeness proof proceeds along the same lines as that of Theorem 39.

Theorem 66 (Completeness) *If $\models_A G$ then $[p]G$ succeeds in \mathbf{GA}_t .*

Proof. Suppose $\models_A G$ then by applying the (invertible) logical rules to $[p]G$ we get valid atomic hypersequents. It remains to show then that all valid atomic hypersequent $[p]G'$ are provable. We proceed by induction on (n, d) where n is the number of propositional variables in G' and d is $d([p]G')$. If $n = 0$ then we just apply (EW) and (\wedge) and we are done. For $n > 0$ if $d = 0$ then we apply (M) and (ID) until there are no occurrences of p left in the hypersequent and use (*shift*) to decrease n . If $d > 0$ then for some component $\Gamma_1 \vdash \Delta_1$ either $\text{count}(\Gamma_1, p) > \text{count}(\Delta_1, p)$ or $\text{count}(\Delta_1, p) > \text{count}(\Gamma_1, p)$. Suppose the former; if there is no component $\Gamma_2 \vdash \Delta_2$ in $[p]G'$ where $\text{count}(\Delta_2, p) > \text{count}(\Gamma_2, p)$ then by Proposition 65 we can apply (EW) to $[p]G'$ to remove $\Gamma_1 \vdash \Delta_1$ giving $[p]G''$ where $\models_A [p]G''$ and $d([p]G'') < d([p]G')$. Now suppose there is such a component $\Gamma_2 \vdash \Delta_2$ in $[p]G'$ where $\text{count}(\Delta_2, p) > \text{count}(\Gamma_2, p)$. We apply (M) and (ID) to $[p]G'$ giving $[p]G'''$ containing components $\Gamma'_1, np \vdash \Delta'_1$ and $\Gamma'_2 \vdash \Delta'_2, mp$ where $n > 0$, $m > 0$ and $p \notin \Gamma'_1, \Gamma'_2, \Delta'_1, \Delta'_2$. So by Proposition 64 we can apply (S') to obtain a hypersequent $[p]G''''$ where $\models_A [p]G''''$ and by Proposition 61 $d([p]G''') < d([p]G'') \leq d([p]G')$. \square

5.2 A Terminating Hypersequent Calculus for \mathbf{L}

A terminating hypersequent calculus for \mathbf{L} is developed in a similar way to \mathbf{GA}_t .

Definition 67 (\mathbf{GL}_t) \mathbf{GL}_t consists of the axioms, logical rules and structural rules (IW), (EW) and (M) of \mathbf{GL} with focussed hypersequents and the same focus for premises and conclusion, and also the rules ($shift$) and (S') of \mathbf{GA}_t .

Theorem 68 (Termination of \mathbf{GL}_t) \mathbf{GL}_t is terminating.

Proof. Similar to the proof of Theorem 62. Just observe that (IW) does not increase the multiset complexity of G not including atoms, the number of different atomic variables or $d([p]G)$ and strictly reduces the number of symbols of G . \square

Theorem 69 (Soundness of \mathbf{GL}_t) If $[p]G$ succeeds in \mathbf{GL}_t then $\models_L^* G$.

Proof. (S') is a derived rule of \mathbf{GL} so we are done. \square Completeness for \mathbf{GL}_t is proved in the same way as for \mathbf{GA}_t with the following proposition replacing Proposition 65. Note that for convenience we call formulae of the form q^* where q is atomic, *starred atoms*.

Proposition 70 Given a hypersequent $G = \Gamma_1 \vdash \Delta_1 | \dots | \Gamma_n \vdash \Delta_n$ containing only starred atoms where $\models_A G$ and $\text{count}(\Gamma_1, p^*) > \text{count}(\Delta_1, p^*)$ and $\text{count}(\Gamma_i, p^*) \geq \text{count}(\Delta_i, p^*)$ for $i = 2 \dots n$ then $\models_A \Gamma_1 - \{p^*\} \vdash \Delta_1 | \Gamma_2 \vdash \Delta_2 | \dots | \Gamma_n \vdash \Delta_n$.

Proof. G is valid iff the atomic hypersequents obtained by applying the \wedge and \vee rules of \mathbf{GA} to the starred atoms of G are valid. But these atomic hypersequents meet the conditions of Proposition 65 so we can remove the components containing a p resulting directly from the component $\Gamma_1 \vdash \Delta_1$. The resulting hypersequents are valid and therefore so is the hypersequent $\models_A \Gamma_1 - \{p^*\} \vdash \Delta_1 | \Gamma_2 \vdash \Delta_2 | \dots | \Gamma_n \vdash \Delta_n$.

Theorem 71 (Completeness of \mathbf{GL}_t) If $\models_L^* G$ then $[p]G$ succeeds in \mathbf{GL}_t .

Proof. Let \mathbf{GA}_t^* be the calculus \mathbf{GL}_t with formulae A replaced everywhere with A^* and the requirements for atoms in the structural rules changed to requirements for starred atoms. We have if $[p]G^*$ succeeds in \mathbf{GA}_t^* then $[p]G$ succeeds in \mathbf{GL}_t . Now suppose $\models_L^* G$, by Proposition 43, we have $\models_A G^*$. $[p]G^*$ is shown to succeed in \mathbf{GA}_t^* imitating the proof of Theorem 66. We just note that Propositions 64 and 61 hold when applied to starred atoms and that Proposition 65 can be replaced by Proposition 70. \square

6 Labelled Single Sequent Calculi

One drawback of our hypersequent calculi from a *computational* perspective is that components multiply exponentially with respect to occurrences of \vee and \wedge . Here we give a method for tackling this problem using *labels*. We represent a disjunction of several unlabelled components (ie a hypersequent) as a single *labelled* sequent and give rules that operate on all components *simultaneously*. The labels themselves are built up from a unit label 1 and atomic

labels x_1, x_2, \dots and each labelled formula in a sequent consists of a label and a formula. Unlabelled components may be obtained from labelled sequents via labelling functions that map each label into the set $\{0, 1\}$, removing formulae labelled with a 0 from the sequent and leaving those labelled with a 1. For example, the sequent $x : p, 1 : q \vdash 1 : p, x : q$ is mapped by a labelling function f to $q \vdash p$ if $f(x) = 0$ and to $p, q \vdash p, q$ if $f(x) = 1$, the corresponding hypersequent being $q \vdash p | p, q \vdash p, q$.

The following definitions make these notions precise:

Definition 72 ((Atomic) Labels, Labelled Formulae, Labelled Sequents)

The set of labels Lab is generated from the set of atomic labels $\{x_i\}_{i \in \mathbb{N}}$ as follows: (1) $1 \in Lab$, (2) $x_i \in Lab$ for all $i \in \mathbb{N}$, (3) if $x \in Lab$ and $y \in Lab$ then $xy \in Lab$. A labelled formula is of the form $x : A$ where $x \in Lab$ and A is a formula. A labelled sequent is a sequent $\Gamma \vdash \Delta$ where Γ and Δ contain only labelled formulae.

Definition 73 (Labelling Function) $f : Lab \rightarrow \{0, 1\}$ is a labelling function iff: (1) $f(1) = 1$, (2) $f(x_i) \in \{0, 1\}$ for all $i \in \mathbb{N}$, (3) $f(xy) = f(x).f(y)$. f is extended to multisets of labelled formulae and labelled sequents by the conditions: (4) $f(\Gamma) = \{A \mid x : A \in \Gamma \text{ and } f(x) = 1\}$, (5) $f(\Gamma \vdash \Delta) = f(\Gamma) \vdash f(\Delta)$.

Definition 74 (Γ^l, Γ^{ul}) Given a multiset of unlabelled formulae Γ , $\Gamma^l = \{1 : A \mid A \in \Gamma\}$. Given a multiset of labelled formulae Γ , $\Gamma^{ul} = \{A \mid x : A \in \Gamma\}$.

6.1 A Labelled Single Sequent Calculus for **A**

We interpret a labelled sequent S for **A** as the disjunction of all the unlabelled sequents obtained by applying labelling functions to S .

Definition 75 (Interpretation of Labelled Sequents for **A)** Let the interpretation of a labelled sequent S be $\phi^S = \bigvee \{\phi^{f(S)} \mid f \text{ a labelling function}\}$ so that $\models_A S$ iff $\models_A \phi^S$.

Observe that validity for labelled sequents coincides with the usual notion of validity \models_A for sequents where all formulae are labelled with a 1.

Proposition 76 For multisets Γ, Δ we have $\models_A \Gamma^l \vdash \Delta^l$ iff $\models_A \Gamma \vdash \Delta$.

Proof. Immediate from Definition 75.

We now present our labelled calculus for **A**. Note that recalling Proposition 22, we choose \Rightarrow as a primitive connective rather than \wedge and \vee . Our reasons are firstly that the rules for \Rightarrow are more uniform than those for \wedge and \vee (and looking ahead to Section 7 the rules for \wedge and \vee for the unlabelled calculus do not even obey the subformula property) and secondly that we want to exploit similarities with **L**.

Which gives:

$$\cup_{i=1}^2 f_i(\{xy : r, zw : r, 1 : p, 1 : q\}) = \cup_{j=1}^2 f_j(1 : r, x : p, xy : p, z : q, zw : q) = \{r, r, p, p, q, q\}$$

The (*success*) rule replaces the structural rules of **GA**. Interpreting each labelling function applied to the labelled sequent as a component it can be seen that (*success*) checks whether putting together any number of components (ie applying (*EC*) and (*S*)) gives a component of the form $\Gamma \vdash \Gamma$ (which succeeds in **GA** using (*M*) and (*ID*)). Note also that checking whether (*success*) can be applied is equivalent to checking the inconsistency of the set of inequations $\{f(\Gamma) > f(\Delta) \mid f \text{ a labelling function}\}$ over \mathbb{Q} ie to solving a *linear programming problem*.

We now consider the soundness and completeness of **GA₁**.

Theorem 79 (Soundness of GA₁) *If S succeeds in GA₁ then $\models_A S$.*

Proof We reason by induction on the length of a proof in **GA₁** and show that the logical rules and (*success*) are sound in \mathbb{Q} .

- Logical rules. Consider (\Rightarrow, l) ; if $\models_A \Gamma, xy : B \vdash \Delta, xy : A$ then given a valuation v we have that $v(f(\Gamma)) + v(f(xy : B)) \leq v(f(\Delta)) + v(f(xy : A))$ for some labelling function f . For $f(xy) = 0$ we have $v(f(\Gamma)) \leq v(f(\Delta))$ so we take a labelling function f' defined as $f'(z) = f(z)$ for $z \neq x$ and $f'(x) = 0$ and get $v(f'(\Gamma)) + v(f'(x : A \Rightarrow B)) \leq v(f'(\Delta))$. For $f(x) = f(y) = 1$ we have $v(f(\Gamma)) + v(B) \leq v(f(\Delta)) + v(A)$ so taking $f' = f$ we get $v(f'(\Gamma)) + v(f'(x : A \Rightarrow B)) \leq v(f'(\Delta))$. Proofs for the other logical rules are very similar.
- (*success*). By (*success*) there exist labelling functions f_1, \dots, f_n and $\lambda_1, \dots, \lambda_n$ where $\lambda_i > 0$ for some i $1 \leq i \leq n$ and $\cup_{i=1}^n \lambda_i \cdot f_i(\Gamma) = \cup_{i=1}^n \lambda_i \cdot f_i(\Delta)$ so by Proposition 37 we have $\models_A f_1(\Gamma) \vdash f_1(\Delta) \mid \dots \mid f_n(\Gamma) \vdash f_n(\Delta)$ as required. \square

Proposition 80 *The logical rules of GA₁ are invertible.*

Proof. We show the invertibility of (\Rightarrow, l) as an example. If $\models_A \Gamma, x : A \Rightarrow B \vdash \Delta$ then given a valuation v we have that $v(f(\Gamma)) + v(f(x : A \Rightarrow B)) \leq v(f(\Delta))$ for some labelling function f . For $f(x) = 0$ we take $f' = f$ and $f'(y) = 0$ and get $v(f'(\Gamma)) + v(f'(xy : B)) \leq v(f'(\Delta)) + v(f'(xy : A))$. For $f(x) = 1$, if $v(A) \leq v(B)$ then we have $v(f'(\Gamma)) \leq v(f'(\Delta))$ and we take $f' = f$ and $f'(y) = 0$; if $v(A) > v(B)$ then we have $v(f'(\Gamma)) + v(B) \leq v(f'(\Delta)) + v(B)$ and we take $f' = f$ and $f'(y) = 1$. \square

Theorem 81 (Completeness of GA₁) *If $\models_A S$ then S succeeds in GA₁.*

Proof As in the proof of Theorem 39 we apply the invertible (by Proposition 80) logical rules to S obtaining a set of valid labelled atomic sequents. We now show that all valid labelled atomic sequents $\Gamma \vdash \Delta$ are provable. We have that

there are labelling functions f_1, \dots, f_n such that $\models_A f_1(\Gamma) \vdash f_1(\Delta) \mid \dots \mid f_n(\Gamma) \vdash f_n(\Delta)$; now by Proposition 37 there exist $\lambda_1, \dots, \lambda_n$ where $\lambda_i > 0$ for some i $1 \leq i \leq n$ and $\cup_{i=1}^n \lambda_i \cdot f_i(\Gamma) = \cup_{i=1}^n \lambda_i \cdot f_i(\Delta)$ so we can apply (*success*). \square

An apparent problem for the complexity of \mathbf{GA}_1 is that checking the (*success*) rule is equivalent to checking the inconsistency of a set of equations of size exponential in the number of labels. As we show however, this set of equations can be transformed in linear time into a set of equations of linear size such that the new set is consistent iff the old set is consistent.

First however we introduce some new terminology.

Definition 82 (Labelled Inequation) *A labelled inequation is an inequation $\Gamma \triangleright \Delta$ where Γ and Δ contain only labelled formulae, $\triangleright \in \{>, \geq\}$. For a labelling function f , $f(\Gamma \triangleright \Delta) = f(\Gamma) \triangleright f(\Delta)$, $\triangleright \in \{>, \geq\}$.*

Definition 83 (Consistent Set of Labelled Inequations) *A set of labelled inequations U is consistent iff the set $\{f(\Gamma) \triangleright f(\Delta) \mid \Gamma \triangleright \Delta \in U, f \text{ a labelling function}\}$ is consistent over \mathbb{Q} .*

Definition 84 (Label-regular) *A labelled inequation is label-regular iff the set of atomic labels occurring in the inequation together with 1 form a tree with root 1 and each label occurring in the inequation is a path from 1 to a node.*

Definition 85 (Maximal Label) *Given a label-regular set of atomic labels, a maximal label is a child node of 1 in the tree.*

The idea is that proofs in \mathbf{GA}_1 generate labelled sequents of a particular form. Each new atomic label y is introduced in a proof as part of a complex label xy and will always occur alongside x in subsequent labelled sequents in the proof.

Proposition 86 *Given a branch of a proof in \mathbf{GA}_1 $[\Gamma^l \vdash \Delta^l, \dots, \Sigma \vdash \Pi]$ the labelled inequation $\Sigma > \Pi$ is label-regular.*

Proof. By induction on the length of a proof in \mathbf{GA}_1 . New labels are only introduced via (\Rightarrow, l) and it is easy to see that the new atomic label y can be added to the tree of atomic labels as a new node below those atomic labels occurring in x and that the label xy forms a path from the root to this node. \square

We now prove our main lemma.

Lemma 87 *Given a set U of m label-regular inequations with n atomic labels occurring in U where no atomic label occurs in more than one inequation, a set V of $2n + m$ unlabelled inequations can be found in $O(n)$ time such that V is consistent iff U is consistent.*

Proof. We proceed by induction on n . For $n = 0$ we take $V = \{\Gamma^{ul} \triangleright \Delta^{ul} \mid \Gamma \triangleright \Delta \in U\}$ and we are done. For $n > 0$ there must be an inequation S in U containing a maximal atomic label x , where $S = \Gamma, \Sigma \triangleright \Delta, \Pi$, $\triangleright \in \{>, \geq\}$, Σ and Π contain only

formulae in which x occurs and Γ and Δ contain only formulae in which x does not occur. Now we define new inequations $S_1 = \Gamma, 1 : \lambda_x \triangleright \Delta$, $S_2 = \Sigma \geq \Pi, 1 : \lambda_x$ and $S_3 = 0 \geq 1 : \lambda_x$ where λ_x is a new propositional variable and U is consistent iff $U' = U - \{S\} \cup \{S_1, S_2, S_3\}$ with $n - 1$ atomic labels is consistent. Now S_1 , S_2 and S_3 are all label-regular and since x is a maximal label there are no atomic labels occurring in more than one labelled inequation in U' . So by the induction hypothesis U' is consistent iff V is consistent where V contains $2(n - 1) + m + 2 = 2n + m$ unlabelled inequations. \square

Theorem 88 (Complexity of \mathbf{GA}_1) $\mathbf{GA}_1 \in co\text{-}NP$

Proof. Let n be the size of the input formula A . To refute A we choose a branch B of the proof tree non-deterministically. B is of $O(n)$ length (since each logical rule strictly reduces the number of connectives in the sequent) and ends with a labelled sequent $\Gamma \vdash \Delta$ with $O(n)$ atomic labels (since (\Rightarrow, l) introduces exactly one new atomic label). So applying Lemma 87 to the labelled and, by Proposition 86 label-regular, inequation $\Gamma > \Delta$ we obtain a set of unlabelled inequations U of $O(n)$ size in $O(n)$ time such that checking the (*success*) rule for $\Gamma \vdash \Delta$ is equivalent to checking the consistency of U over \mathbb{Q} ie a linear programming problem. Since linear programming can be solved in polynomial time we are done. \square

6.2 A Labelled Single Sequent Calculus for \mathbf{L}

Labelled sequents for \mathbf{L} are interpreted using the characteristic model $[-1, 0]_{\mathbf{L}}$.

Definition 89 (Interpretation of Labelled Sequents for \mathbf{L}) $\models_{\mathbf{L}}^* \Gamma \vdash \Delta$ iff for all $v : For \rightarrow [-1, 0]_{\mathbf{L}}$ there exists a labelling function f such that $\sum_{A \in \Gamma} v(f(A)) \leq \sum_{B \in \Delta} v(f(B))$.

We also extend the translation $*$ to labelled sequents.

Definition 90 (Translation $*$ for Labelled Sequents) Given a labelled multiset Γ , $\Gamma^* = \{x : A^* \mid x : A \in \Gamma\}$. Given a labelled sequent $S = \Gamma \vdash \Delta$, $S^* = \Gamma^* \vdash \Delta^*$.

Theorem 91 $\models_{\mathbf{L}}^* S$ iff $\models_A S^*$.

Proof. The proof of Theorem 24 is easily extended from formulae to labelled sequents. \square

To give a suitable (*success*) rule for \mathbf{L} we must first define a relation \subseteq^* that takes account of internal weakening and \perp .

Definition 92 (\subseteq^*) $\Delta \subseteq^* \Gamma$ iff (1) $\Delta \subseteq \Gamma$, or (2) $\Delta' \subseteq^* \Gamma'$ where $\Gamma = \Gamma' \cup \{\perp\}$ and $\Delta = \Delta' \cup \{q\}$ for some atom q .

We now present the following labelled calculus for \mathbf{L} .

Definition 93 (\mathbf{GL}_1) \mathbf{GL}_1 has the following rules:

$$(\Rightarrow, l) \frac{\Gamma, xy : B \vdash \Delta, xy : A}{\Gamma, x : A \Rightarrow B \vdash \Delta} \quad (\Rightarrow, r) \frac{\Gamma, x : A \vdash \Delta, x : B \quad \Gamma \vdash \Delta}{\Gamma \vdash \Delta, x : A \Rightarrow B}$$

y a new atomic label

$$(success) \quad \Gamma \vdash \Delta \quad \text{where } \Gamma \text{ and } \Delta \text{ are atomic and there exist labelling functions } f_1, \dots, f_n \text{ such that } \cup_{i=1}^n f_i(\Gamma) \subseteq^* \cup_{i=1}^n f_i(\Delta)$$

As in section 4, the soundness and completeness of \mathbf{GL}_1 are proved proof-theoretically using the translation from \mathbf{L} to \mathbf{A} .

Proposition 94 S succeeds in \mathbf{GL}_1 iff S^* succeeds in \mathbf{GA}_1 .

Proof. Similar to (and in fact easier than) the proof of Proposition 53. \square

Theorem 95 (Soundness and Completeness of \mathbf{GL}_1) S succeeds in \mathbf{GL}_1 iff $\models_{\mathbf{L}}^* S$.

Proof. By Proposition 94, Theorems 81 and 79, and Theorem 91 respectively we have that S succeeds in \mathbf{GL}_1 iff S^* succeeds in \mathbf{GA}_1 iff $\models_{\mathbf{A}} S^*$ iff $\models_{\mathbf{L}}^* S$. \square

Theorem 96 (Complexity of \mathbf{GL}_1) \mathbf{GL}_1 is co-NP-complete.

Proof. Similar to the proof of Theorem 88. \square

7 Unlabelled Single Sequent Calculi

In this section we obtain *unlabelled* single sequent calculi for \mathbf{A} and \mathbf{L} . Instead of maintaining several components in a hypersequent or introducing labels we add rules permitting contraction from a sequent $\Gamma, \Gamma \vdash \Delta, \Delta$ to $\Gamma \vdash \Delta$, derivable in \mathbf{GA} by applying (EC) and (S) . We also alter the (\Rightarrow, l) rules using the identity $A + (A \Rightarrow B) = B + (B \Rightarrow A)$, thereby obtaining sound and complete single sequent calculi for \mathbf{A} and \mathbf{L} .

7.1 An Unlabelled Single Sequent Calculus for \mathbf{A}

We present the following unlabelled single sequent calculus for \mathbf{A} .

Definition 97 (\mathbf{GA}_s) \mathbf{GA}_s has the following rules.

Axioms

$$(ID) \quad A \vdash A \quad (\mathbf{A}) \vdash$$

Structural Rules

$$(W) \quad \frac{\Gamma \vdash \Delta}{\Gamma, A \Rightarrow B \vdash \Delta} \quad (M) \quad \frac{\Gamma_1 \vdash \Delta_1 \quad \Gamma_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}$$

$$(C) \quad \frac{\overbrace{\Gamma, \dots, \Gamma}^n \vdash \overbrace{\Delta, \dots, \Delta}^n}{\Gamma \vdash \Delta} \quad n > 0$$

Logical Rules

$$\begin{array}{ll}
(t, l) & \frac{\Gamma \vdash \Delta}{\Gamma, t \vdash \Delta} \\
(\neg, l) & \frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta} \\
(\rightarrow, l) & \frac{\Gamma, B \vdash A, \Delta}{\Gamma, A \rightarrow B \vdash \Delta} \\
(+, l) & \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A + B \vdash \Delta} \\
(\Rightarrow, l) & \frac{\Gamma, B, B \Rightarrow A \vdash \Delta, A}{\Gamma, A \Rightarrow B \vdash \Delta} \\
(t, r) & \frac{\Gamma \vdash \Delta}{\Gamma \vdash t, \Delta} \\
(\neg, r) & \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta} \\
(\rightarrow, r) & \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \rightarrow B, \Delta} \\
(+, r) & \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A + B, \Delta} \\
(\Rightarrow, r) & \frac{\Gamma \vdash \Delta \quad \Gamma, A \vdash \Delta, B}{\Gamma \vdash \Delta, A \Rightarrow B}
\end{array}$$

Derived rules for \wedge and \vee are:

$$\begin{array}{ll}
(\wedge, l) & \frac{\Gamma, A, A \Rightarrow B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \\
(\vee, l) & \frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta} \\
(\wedge, r) & \frac{\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B} \\
(\vee, r) & \frac{\Gamma, B \Rightarrow A \vdash \Delta, A}{\Gamma \vdash \Delta, A \vee B}
\end{array}$$

To understand the differences between this calculus and **GA** compare the following proof of prelinearity with Example 31.

Example 98 A proof of $(A \rightarrow B) \vee (B \rightarrow A)$ in **GA_s**:

$$\begin{array}{c}
\frac{\frac{B \vdash B \quad B \vdash B}{B, B \vdash B, B} \quad \frac{A \vdash A \quad A \vdash A}{A, A \vdash A, A}}{B, B, A, A \vdash B, B, A, A} \\
\frac{B, B, A, A \vdash B, B, A, A}{B, A, A \vdash B, B, B \rightarrow A, A} \\
\frac{B, A, A \vdash B, B, B \rightarrow A, A}{A \rightarrow B, A, A \vdash B, B, B \rightarrow A} \\
\frac{A \rightarrow B, A, A \vdash B, B, B \rightarrow A}{(A \rightarrow B) \Rightarrow (B \rightarrow A), A \rightarrow B, A, A \vdash B, B, B \rightarrow A} \\
\frac{(A \rightarrow B) \Rightarrow (B \rightarrow A), A \rightarrow B, A, A \vdash B, B, B \rightarrow A}{(B \rightarrow A) \Rightarrow (A \rightarrow B), A, A \vdash B, B} \\
\frac{(B \rightarrow A) \Rightarrow (A \rightarrow B), (B \rightarrow A) \Rightarrow (A \rightarrow B), A, A \vdash B, B}{(B \rightarrow A) \Rightarrow (A \rightarrow B), A \vdash B} \\
\frac{(B \rightarrow A) \Rightarrow (A \rightarrow B), A \vdash B}{(B \rightarrow A) \Rightarrow (A \rightarrow B) \vdash A \rightarrow B} \\
\frac{(B \rightarrow A) \Rightarrow (A \rightarrow B) \vdash A \rightarrow B}{\vdash (A \rightarrow B) \vee (B \rightarrow A)}
\end{array}$$

Soundness is proved in the usual way.

Theorem 99 (Soundness of **GA_s)** If S succeeds in **GA_s** then $\models_A S$.

Proof. We follow the standard inductive proof and just check soundness in \mathbb{Q} for the rules different to those of **GA**.

- (W). If $v(\Gamma) \leq v(\Delta)$ then since $v(A \Rightarrow B) \leq 0$ we have $v(\Gamma) + v(A \Rightarrow B) \leq v(\Delta)$ for all valuations v .
- (C). If $\underbrace{v(\Gamma) + \dots + v(\Gamma)}_n \leq \underbrace{v(\Delta) + \dots + v(\Delta)}_n$ then dividing by n gives $v(\Gamma) \leq v(\Delta)$ for all valuations v .
- (\Rightarrow, l). For any valuation v , if $v(\Gamma) + v(B) + v(B \Rightarrow A) \leq v(\Delta) + v(A)$ then since $v(B) + v(B \Rightarrow A) - v(A) = v(A \Rightarrow B)$ we have $v(\Gamma) + v(A \Rightarrow B) \leq v(\Delta)$. \square

We now turn our attention to proving the completeness of \mathbf{GA}_s . Our strategy is to show that any sequent provable in the *labelled* calculus \mathbf{GA}_1 is provable in \mathbf{GA}_s . To this end we introduce an intermediate calculus \mathbf{GA}_i which performs proofs in \mathbf{GA}_1 but which also maintains a record of the formulae added by the (\Rightarrow, l) rule for \mathbf{GA}_s in a *store*; this allows us to prove inductively that any of the sequents represented by a labelled sequent $\Gamma \parallel \Pi \vdash \Delta$ (where Π is the store) can be reached in \mathbf{GA}_s from the unlabelled sequent $\Gamma^{ul}, \Pi \vdash \Delta^{ul}$.

Definition 100 (\Rightarrow -formula, $\text{atoms}(\Gamma)$, $\text{nonatoms}(\Gamma)$) *A \Rightarrow -formula is a formula of the form $A \Rightarrow B$, $\text{atoms}(\Gamma) = \{q \in \Gamma \mid q \text{ atomic}\}$, $\text{nonatoms}(\Gamma) = \{A \in \Gamma \mid A \text{ not atomic}\}$.*

Definition 101 (\mathbf{GA}_i) *The rules for \mathbf{GA}_i are exactly the same as for \mathbf{GA}_1 except that each premise and conclusion $\Gamma \vdash \Delta$ is replaced by $\Gamma \parallel \Pi \vdash \Delta$ and (\Rightarrow, l) becomes:*

$$\frac{\Gamma, xy : B \parallel \Pi, B \Rightarrow A \vdash \Delta, xy : A}{\Gamma, x : A \Rightarrow B \parallel \Pi \vdash \Delta} \quad \text{where:} \quad \begin{array}{l} (1) \ y \text{ is a new atomic label.} \\ (2) \ \Delta \text{ contains only atoms.} \\ (3) \ \Gamma \text{ contains only atoms and } \Rightarrow \text{-formulae.} \end{array}$$

We emphasize that formulae in the store are not used in \mathbf{GA}_i .

Proposition 102 $\Gamma \vdash \Delta$ *succeeds in \mathbf{GA}_1 iff $\Gamma \parallel \Pi \vdash \Delta$ succeeds in \mathbf{GA}_i .*

Proof. We simply observe that the unlabelled formulae in the store are not processed by the rules of the calculus and that the restrictions for (\Rightarrow, l) can easily be shown not to affect completeness. \square

Definition 103 (**Branch**) *A branch $[S_1, \dots, S_n]$ in a calculus \mathbf{G} is a sequence of sequents (or hypersequents) where for $i = 1 \dots n-1$, S_i is the conclusion and S_{i+1} a premise of a rule of \mathbf{G} .*

Proposition 104 *Given a branch $[\Gamma, \Delta \vdash \Gamma, \Delta \dots \Gamma, \Sigma^+ \vdash \Sigma^-, \Delta]$ in \mathbf{GA}_s containing only applications of the logical rules excepting (\Rightarrow, l), $\Gamma, \Sigma^+ \vdash \Sigma^-, \Delta$ succeeds in \mathbf{GA}_s .*

Proof. By induction on the multiset complexity of $\Gamma \cup \Delta$. \square

We now prove the main technical proposition for our completeness proof. The

idea is that at any point in a proof in \mathbf{GA}_i where only sequents and \Rightarrow -formulae occur in the sequent, and given any labelling function, we are able to divide the current sequent $S = \Gamma \parallel \Pi \vdash \Delta$ into three parts; the first is the result of applying the labelling function to S , the second is a sequent that succeeds (removing labels) in \mathbf{GA}_s and the third consists of \Rightarrow -formulae.

Proposition 105 *Given a branch of a proof in \mathbf{GA}_i $[\Gamma_1^l \parallel \emptyset \vdash \Delta_1^l \dots \Gamma \parallel \Pi \vdash \Delta]$ where Δ is atomic and Γ contains only atoms and \Rightarrow -formulae, then for every labelling function f there exist multisets $\Gamma_a, \Gamma_e, \Gamma_r, \Delta_a$ and Δ_e such that:*

- (1) $\Gamma^{ul} = \Gamma_a \cup \Gamma_e \cup \Gamma_r$ and $\Delta^{ul} = \Delta_a \cup \Delta_e$
- (2) $\Gamma_a = \text{atoms}(f(\Gamma))$ and $\Delta_a = f(\Delta)$
- (3) $\Gamma_e, \Pi \vdash \Delta_e$ succeeds in \mathbf{GA}_s
- (4) $\Gamma_e \subseteq \{A \mid x : A \in \Gamma, f(x) = 0\}$
- (5) Γ_r contains only \Rightarrow -formulae

Proof. By induction on the number of atomic labels, n , occurring in $\Gamma \parallel \Pi \vdash \Delta$. If $n = 0$ then we take $\Gamma_a = \text{atoms}(\Gamma^{ul})$, $\Gamma_e = \emptyset$, $\Gamma_r = \text{nonatoms}(\Gamma^{ul})$, $\Delta_a = \Delta^{ul}$ and $\Delta_e = \emptyset$ and the conditions hold. For $n > 0$ we consider the last introduction of a label via (\Rightarrow, l) :

From $Q = \Gamma, x : A \Rightarrow B \parallel \Pi \vdash \Delta$ to:

$$Q' = \Gamma, xy : B \parallel \Pi, B \Rightarrow A \vdash \Delta, xy : A$$

followed by applications of the logical rules excepting (\Rightarrow, l) giving:

$$Q'' = \Gamma, \Sigma^+ \parallel \Pi, B \Rightarrow A \vdash \Delta, \Sigma^-$$

where Σ^- contains only atoms, Σ^+ contains only atoms and \Rightarrow -formulae, and all formulae in Σ^- and Σ^+ have label xy .

Now for a given labelling function f we apply the induction hypothesis to Q , obtaining multisets $\Gamma_a, \Gamma_e, \Gamma_r, \Delta_a$ and Δ_e as specified above. We seek $\Gamma'_a, \Gamma'_e, \Gamma'_r, \Delta'_a$ and Δ'_e suitable for Q'' . There are two cases to consider:

- $f(x) = f(y) = 1$. We take $\Gamma'_a = \Gamma_a \cup \text{atoms}(\Sigma^+)^{ul}$, $\Gamma'_e = \Gamma_e$, $\Gamma'_r = \Gamma_r - \{A \Rightarrow B\} \cup \text{nonatoms}(\Sigma^+)^{ul}$, $\Delta'_a = \Delta_a \cup (\Sigma^-)^{ul}$ and $\Delta'_e = \Delta_e$. We have: (1) $(\Gamma \cup \Sigma^+)^{ul} = \Gamma'_a \cup \Gamma'_e \cup \Gamma'_r$ and $(\Delta \cup \Sigma^-)^{ul} = \Delta'_a \cup \Delta'_e$, (2) $\Gamma'_a = \text{atoms}(f(\Gamma \cup \Sigma^+))$ and $\Delta'_a = f(\Delta \cup \Sigma^-)$ since $f(xy) = 1$, (3) $\Gamma'_e, \Pi \vdash \Delta'_e$ succeeds in \mathbf{GA}_s and hence so does $\Gamma'_e, \Pi, B \Rightarrow A \vdash \Delta'_e$ by (W), (4) $\Gamma'_e \subseteq \{A \mid x : A \in \Gamma \cup \Sigma^+, f(x) = 0\} = \{A \mid x : A \in \Gamma, f(x) = 0\}$ and (5) Γ'_r contains only \Rightarrow -formulae.
- $f(x) = 0$ or $f(y) = 0$. We take $\Gamma'_a = \Gamma_a$, $\Delta'_a = \Delta_a$ and $\Delta'_e = \Delta_e \cup (\Sigma^-)^{ul}$ and then for $A \Rightarrow B \in \Gamma_e$, we take $\Gamma'_e = \Gamma_e - \{A \Rightarrow B\} \cup (\Sigma^+)^{ul}$ and $\Gamma'_r = \Gamma_r$, for $A \Rightarrow B \in \Gamma_r$, we take $\Gamma'_e = \Gamma_e \cup (\Sigma^+)^{ul}$ and $\Gamma'_r = \Gamma_r - \{A \Rightarrow B\}$. We have: (1) $(\Gamma \cup \Sigma^+)^{ul} = \Gamma'_a \cup \Gamma'_e \cup \Gamma'_r$ and $(\Delta \cup \Sigma^-)^{ul} = \Delta'_a \cup \Delta'_e$ by definition; (2) $\Gamma'_a = \text{atoms}(f(\Gamma \cup \Sigma^+))$ and $\Delta'_a = f(\Delta \cup \Sigma^-)$ since $f(xy) = 0$; (3) we have that $\Gamma_e, \Pi \vdash \Delta_e$ succeeds in \mathbf{GA}_s so (for $A \Rightarrow B \in \Gamma_e$) by Proposition

104 $\Gamma_e - \{A \Rightarrow B\}, (\Sigma^+)^{ul}, \Pi, B \Rightarrow A \vdash \Delta_e, (\Sigma^-)^{ul}$ succeeds in \mathbf{GA}_s , and (for $A \Rightarrow B \notin \Gamma_r$), since $B \Rightarrow A, (\Sigma^+)^{ul} \vdash (\Sigma^-)^{ul}$ succeeds in \mathbf{GA}_s using Proposition 104 so also by (M) does $\Gamma_e, B \Rightarrow A, (\Sigma^+)^{ul}, \Pi \vdash \Delta_e, (\Sigma^-)^{ul}$; (4) if $u : C \in \Gamma_e \cup (\Sigma^-)^{ul}$ then $f(u) = 0$ so $\Gamma'_e \subseteq \{C|u : C \in \Gamma \cup \Sigma^+, f(u) = 0\}$; (5) Γ'_r contains only \Rightarrow -formulae. \square

We now use this proposition to prove the completeness of \mathbf{GA}_s .

Theorem 106 (Completeness of \mathbf{GA}_s) *If $\models_A S$ then S succeeds in \mathbf{GA}_s .*

Proof. If $\models_A \Gamma_1 \vdash \Delta_1$ then by Theorem 81 we have that $\Gamma_1^l \vdash \Delta_1^l$ succeeds in \mathbf{GA}_1 and hence by Proposition 102 $\Gamma_1^l \|\emptyset \vdash \Delta_1^l$ also succeeds in \mathbf{GA}_1 . Given a proof of $\Gamma_1^l \|\emptyset \vdash \Delta_1^l$ in \mathbf{GA}_1 we apply the corresponding rules of \mathbf{GA}_s until the point on each branch when (*success*) is applied to a sequent $\Gamma \|\Pi \vdash \Delta$ ie where there are labelling functions f_1, \dots, f_n such that $\cup_{i=1}^n f_i(\Gamma) = \cup_{j=1}^n f_j(\Delta)$. Now by Proposition 105, for $i = 1 \dots n$ there exist multisets $\Gamma_a^i, \Gamma_e^i, \Gamma_r^i, \Delta_a^i, \Delta_e^i$ such that:

- (1) $\Gamma^{ul} = \Gamma_a^i \cup \Gamma_e^i \cup \Gamma_r^i$ and $\Delta^{ul} = \Delta_a^i \cup \Delta_e^i$
- (2) $\Gamma_a^i = \text{atoms}(f_i(\Gamma)) = f_i(\Gamma)$ and $\Delta_a^i = f_i(\Delta)$
- (3) $\Gamma_e^i, \Pi \vdash \Delta_e^i$ succeeds in \mathbf{GA}_s
- (4) $\Gamma_e^i \subseteq \{A \mid x : A \in \Gamma, f_i(x) = 0\}$
- (5) Γ_r^i contains only \Rightarrow -formulae

We show that the corresponding sequent $\Gamma^{ul}, \Pi \vdash \Delta^{ul}$ succeeds in \mathbf{GA}_s . First we use (C) to step to:

$$\underbrace{\Gamma^{ul}, \dots, \Gamma^{ul}}_n, \underbrace{\Pi, \dots, \Pi}_n \vdash \underbrace{\Delta^{ul}, \dots, \Delta^{ul}}_n$$

We then apply (W) repeatedly to formulae in Γ_r^i for $i = 1 \dots n$ obtaining:

$$\Gamma_a^1 \cup \Gamma_e^1, \dots, \Gamma_a^n \cup \Gamma_e^n, \underbrace{\Pi, \dots, \Pi}_n \vdash \Delta_a^1 \cup \Delta_e^1, \dots, \Delta_a^n \cup \Delta_e^n$$

Since $\cup_{i=1}^n f_i(\Delta) = \cup_{j=1}^n f_j(\Gamma)$ we have that $\Gamma_a^1, \dots, \Gamma_a^n \vdash \Delta_a^1, \dots, \Delta_a^n$ succeeds in \mathbf{GA}_s using (M) and (ID). Also $\Gamma_e^i, \Pi \vdash \Delta_e^i$ succeeds in \mathbf{GA}_s for $i = 1 \dots n$. So applying (M) repeatedly, we get that the whole sequent succeeds as required. \square

7.2 An Unlabelled Single Sequent Calculus for \mathbf{L}

We present the following unlabelled single sequent calculus for \mathbf{L} .

Definition 107 (\mathbf{GL}_s) *\mathbf{GL}_s has the following rules.*

Axioms

$$(ID) \quad A \vdash A \qquad (\Delta) \quad \vdash \qquad (\perp) \quad \perp \vdash A$$

Structural Rules

$$\begin{array}{l}
(W) \quad \frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} \qquad (M) \quad \frac{\Gamma_1 \vdash \Delta_1 \quad \Gamma_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2} \\
(C) \quad \frac{\overbrace{\Gamma, \dots, \Gamma}^n \vdash \overbrace{\Delta, \dots, \Delta}^n}{\Gamma \vdash \Delta} \quad n > 0
\end{array}$$

Logical Rules

$$(\Rightarrow, l) \quad \frac{\Gamma, B, B \Rightarrow A \vdash \Delta, A}{\Gamma, A \Rightarrow B \vdash \Delta} \quad (\Rightarrow, r) \quad \frac{\Gamma \vdash \Delta \quad \Gamma, A \vdash \Delta, B}{\Gamma \vdash \Delta, A \Rightarrow B}$$

We give the following proof of the characteristic axiom of \mathbf{L} for comparison with the \mathbf{GL} proof in Example 49.

Example 108 *A proof of $((A \Rightarrow B) \Rightarrow B) \Rightarrow ((B \Rightarrow A) \Rightarrow A)$ in \mathbf{GL}_1 :*

$$\begin{array}{c}
\frac{B \vdash B \quad A \Rightarrow B \vdash A \Rightarrow B}{B, A \Rightarrow B \vdash B, A \Rightarrow B} \\
\frac{B \Rightarrow (A \Rightarrow B), B, A \Rightarrow B \vdash B, A \Rightarrow B}{A \vdash A \quad (A \Rightarrow B) \Rightarrow B, A \Rightarrow B \vdash B} \\
\frac{\vdash \quad (A \Rightarrow B) \Rightarrow B, A \Rightarrow B, A \vdash A, B}{(A \Rightarrow B) \Rightarrow B \vdash (A \Rightarrow B) \Rightarrow B, B \Rightarrow A \vdash A} \\
\frac{\vdash \quad (A \Rightarrow B) \Rightarrow B \vdash (B \Rightarrow A) \Rightarrow A}{\vdash ((A \Rightarrow B) \Rightarrow B) \Rightarrow ((B \Rightarrow A) \Rightarrow A)}
\end{array}$$

As usual the soundness and completeness of \mathbf{GL}_s is proved proof-theoretically using the translation from \mathbf{L} to \mathbf{A} .

Proposition 109 *S succeeds in \mathbf{GL}_s iff S^* succeeds in \mathbf{GA}_s .*

Proof. For the left-to-right direction it is straightforward to show that the translated rules of \mathbf{GL}_s are either derivable or admissible in \mathbf{GA}_s . For the right-to-left direction we just sketch a proof. First we replace the axioms and the rules (W) and (M) of \mathbf{GL}_s with (AX) $\Gamma, t \dots, t \vdash \Delta, t \dots, t$ where $\Delta \subseteq^* \Gamma$. It is easy to show that the two versions prove the same theorems (replacing t with $q \Rightarrow q$ where appropriate). Now since the rules and axioms of \mathbf{GA}_s applying to \Rightarrow and propositional variables are a subset of those for \mathbf{GL}_s we have that if S^* succeeds in \mathbf{GA}_s then S^* succeeds in \mathbf{GL}_s . Hence we can then prove inductively that if S^* succeeds in \mathbf{GL}_s then S succeeds in \mathbf{GL}_s and we are done. \square

Theorem 110 (Soundness and Completeness of \mathbf{GL}_s) *S succeeds in \mathbf{GL}_s iff $\models_{\mathbf{L}}^* S$.*

Proof. By Proposition 109, Theorems 106 and 99, and Theorem 43 respectively we have that S succeeds in \mathbf{GL}_s iff S^* succeeds in \mathbf{GA}_s iff $\models_{\mathbf{A}} S^*$ iff $\models_{\mathbf{L}}^* S$. \square

8 Discussion

In this paper we have presented the first Gentzen-style analytic and internal sequent and hypersequent calculi for abelian logic \mathbf{A} and Lukasiewicz infinite-valued logic \mathbf{L} . We have also developed terminating versions of the hypersequent calculi and co-NP labelled sequent calculi matching the complexity bounds for both logics. The key step for achieving these results for \mathbf{L} has been a translation of \mathbf{L} into \mathbf{A} .

For abelian logic \mathbf{A} our work, providing the first comprehensive proof systems for this logic, extends that of Paoli [37] who gave a calculus for just the intensional part of \mathbf{A} . For Lukasiewicz logic \mathbf{L} , on the other hand, several proof systems have been presented in the literature. All of these however either fail to be analytic, that is they have cut as a uneliminable rule eg [39, 15], or they rely on external non-logical calculations such as solving mathematical programming problems [23, 35, 36]. One exception is the proof calculus of Aguzzoli and Ciabattoni [1] which exploits the fact that any formula valid in \mathbf{L} is also valid in \mathbf{L}_n where n is a function of the number of occurrences of variables in the formula. This calculus uses multiple sequents that seem to perform a similar role proof-theoretically to hypersequents but with a very different semantic interpretation. Although this approach provides a valuable perspective on the relationship between \mathbf{L} and the finite-valued logics we take the view that \mathbf{L} both can and should be viewed as *independent* of the finite-valued logics. This is supported by the fact that the calculi given in this paper, in particular \mathbf{GL} and \mathbf{GL}_s , are more direct and natural than those for \mathbf{L}_n . Moreover proof systems for the finite-valued logics may be obtained from our systems for \mathbf{L} by adding appropriate extra axioms or rules.

A more implementation-oriented approach to Lukasiewicz logic is introduced by Hähnle in [23] where a *labelled tableaux* reduction of \mathbf{L} to *mixed integer programming* is presented. Although similar in output (ie mathematical programming problems) to our calculus \mathbf{GL}_t and of the same complexity (co-NP) the method is significantly different to ours in that constraints (ie equations) are generated dynamically as a proof progresses rather than maintaining a single labelled equation for each branch. For this reason our calculus seems to be more *logical* and easier to understand at various stages of a proof. A further tableaux calculus has also been presented by Olivetti in [36] based on the Kripke semantics of \mathbf{L} that again performs a reduction to mathematical programming problems and is co-NP.

Finally we note that our approach of embedding fuzzy logics in comparative logics could prove successful in deriving sequent and hypersequent calculi for the other fuzzy logics. In particular we would like to find proof systems for Hájek's basic logic \mathbf{BL} and Product logic \mathbf{P} .

References

- [1] S. Aguzzoli and A. Ciabattani. Finiteness in infinite-valued logic. *Journal of Logic, Language and Information*, 9(1):5–29, 2000.
- [2] S. Aguzzoli and B. Gerla. Finite-valued reductions of infinite-valued logics. *Archive for Mathematical Logic*, 41(4):361–399, 2002.
- [3] A. R. Anderson and N. D. Belnap. *Entailment*, volume 1. Princeton University Press, Princeton, NJ, 1975.
- [4] A. Avellone, M. Ferrari, and P. Miglioli. Duplication-free tableau calculi and related cut-free sequent calculi for the interpolable propositional intermediate logics. *Logic Journal of the IGPL*, 7(4):447–480, 1999.
- [5] A. Avron. A constructive analysis of RM. *Journal of Symbolic Logic*, 52(4):939–951, 1987.
- [6] A. Avron. Hypersequents, logical consequence and intermediate logics for concurrency. *Annals of Mathematics and Artificial Intelligence*, 4(3–4):225–248, 1991.
- [7] A. Avron. The method of hypersequents in the proof theory of propositional non-classical logics. In W. Hodges, M. Hyland, C. Steinhorn, and J. Truss, editors, *Logic: from foundations to applications. Proc. Logic Colloquium, Keele, UK, 1993*, pages 1–32. Oxford University Press, New York, 1996.
- [8] A. Avron and B. Konikowska. Decomposition Proof Systems for Gödel-Dummett Logics. *Studia Logica*, 69(2):197–219, 2001.
- [9] M. Baaz, A. Ciabattani, C. G. Fermüller, and H. Veith. Proof theory of fuzzy logics: Urquhart’s C and related logics. In L. Brim, J. Gruska, and J. Zlatuska, editors, *Proc. 23rd International Symposium Mathematical Foundations of Computer Science, Brno, Czech Republic*, volume 1450 of *LNCS*, pages 203–212, 1998.
- [10] M. Baaz and C. G. Fermüller. Analytic calculi for projective logics. In *Proc. TABLEAUX ’99*, volume 1617, pages 36–50, 1999.
- [11] E. Casari. Comparative logics. In *Synthese 73*, pages 421–449. D. Reidel, 1987.
- [12] E. Casari. Comparative logics and abelian l-groups. In Ferro et al., editor, *Logic Colloquium ’88*, pages 161–190. Elsevier, 1989.
- [13] C. C. Chang. Algebraic analysis of many-valued logics. *Transactions of the American Mathematical Society*, 88:467–490, 1958.
- [14] A. Ciabattani. On Urquhart’s C logic. In *Proc. 30th International Symposium on Multiple-Valued Logics, Portland/OR, USA*, pages 113–118. IEEE CS Press, Los Alamitos, May 2000.

- [15] A. Ciabattoni and D. Luchi. Two connections between linear logic and Lukasiewicz logics. *LNCS*, 1289:128–139, 1997.
- [16] R. Cignoli, F. Esteva, L. Godo, and A. Torrens. Basic fuzzy logic is the logic of continuous t-norms and their residua. *Soft Computing*, 4(2):106–112, 2000.
- [17] R. L. O. Cignoli, I. M. Loffredo D’Ottaviano, and D. Mundici. *Algebraic Foundations of Many-Valued Reasoning*, volume 7 of *Trends in Logic*. Kluwer, Dordrecht, November 1999.
- [18] J. Michael Dunn and Robert K. Meyer. Algebraic completeness results for Dummett’s LC and its extensions. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 17:225–230, 1971.
- [19] R. Dyckhoff. A deterministic terminating sequent calculus for Gödel-Dummett logic. *Logic Journal of the IGPL*, 7(3):319–326, 1999.
- [20] F. Esteva and Lluís Godo. Monoidal t-norm based logic: towards a logic for left-continuous t-norms. *Fuzzy Sets and Systems*, 124:271–288, 2001.
- [21] A. Galli, R. A. Lewin, and M. Sagastume. The logic of equilibrium and abelian lattice ordered groups. To appear in *Transactions of the AMS*, 2002.
- [22] S. Gottwald. *A Treatise on Many-Valued Logics*, volume 9 of *Studies in Logic and Computation*. Research Studies Press, Baldock, 2000.
- [23] R. Hähnle. *Automated Deduction in Multiple-Valued Logics*. Oxford University Press, 1993.
- [24] P. Hájek. *Metamathematics of Fuzzy Logic*. Kluwer Academic Publishers, Dordrecht, 1998.
- [25] U. Höhle. Commutative, residuated ℓ -monoids. In U. Höhle and E. P. Klement, editors, *Non-Classical Logics and their Applications to Fuzzy Subsets*, pages 53–106. Kluwer, Dordrecht, 1995.
- [26] J. Lukasiewicz. *Jan Lukasiewicz, Selected Writings*. North-Holland, 1970. Edited by L. Borowski.
- [27] J. Lukasiewicz and A. Tarski. Untersuchungen über den Aussagenkalkül. *Comptes Rendus des Séances de la Société des Sciences et des Lettres de Varsovie, Classe III*, 23, 1930. Reprinted and translated in [26].
- [28] G. Malinowski. *Many-Valued Logics*, volume 25 of *Oxford Logic Guides*. Oxford University Press, 1993.
- [29] G. Metcalfe, N. Olivetti, and D. Gabbay. Analytic sequent calculi for abelian and Lukasiewicz logics. In *Proceedings of TABLEAUX 2002*, volume 2381 of *LNCS*, pages 191–205. Springer, 2002.

- [30] R. K. Meyer. Intuitionism, entailment, negation. In H. Leblanc, editor, *Truth, Syntax and Modality*, pages 168–198. North Holland, 1973.
- [31] R. K. Meyer and J. K. Slaney. Abelian logic from A to Z. In Graham Priest, Richard Sylvan, and Jean Norman, editors, *Paraconsistent Logic: Essays on the Inconsistent*, pages 245–288. Philosophia Verlag, 1989.
- [32] R. K. Meyer and J. K. Slaney. A, still adorable. In M. Coniglio W. Carnielli and I. M. L. D’Ottaviano, editors, *Paraconsistency, the logical way to the inconsistent*, pages 241–260. Marcel Dekker, 2002.
- [33] F. Montagna, G. M. Pinna, and E. B. P. Tiezzi. A tableau calculus for Hájek’s logic BL. *Submitted*, 2002.
- [34] D. Mundici. Satisfiability in many-valued sentential logic is NP-complete. *Theoretical Computer Science*, 52(1-2):145–153, 1987.
- [35] D. Mundici and N. Olivetti. Resolution and model building in the infinite-valued calculus of Łukasiewicz. *Theoretical Computer Science*, 200(1–2):335–366, 28 June 1998.
- [36] N. Olivetti. Tableaux for Łukasiewicz infinite-valued logic. To appear in *Studia Logica*.
- [37] F. Paoli. *Substructural Logics: A Primer*. Kluwer Academic Publishers, Dordrecht, 2002.
- [38] G. Pottinger. Uniform, cut-free formulations of T, S4 and S5 (abstract). *Journal of Symbolic Logic*, 48(3):900, September 1983.
- [39] A. Prijatelj. Bounded contraction and Gentzen-style formulation of Łukasiewicz logics. *Studia Logica*, 57(2–3):437–456, 1996.
- [40] A. Rose and J. B. Rosser. Fragments of many-valued statement calculi. *Transactions of the American Mathematical Society*, 87:1–53, 1958.
- [41] D. Scott. Completeness and Axiomatizability in Many-Valued Logic. In L. Henkin et al., editors, *Proceedings of Symposia in Pure Mathematics Volume XXV: Proceedings of the Tarski Symposium*, pages 411–435, 1974.
- [42] O. Sonobe. A Gentzen-type formulation of some intermediate propositional logics. *Journal of Tsuda College*, 7:7–14, 1975.
- [43] A. Urquhart. Many-valued logics. In D. M. Gabbay and F. Günthner, editors, *Handbook of Philosophical Logic*, volume 3, pages 71–116. Reidel, Dordrecht, 1986.
- [44] V. Weispfenning. The complexity of the word problem for Abelian l -groups. *Theoretical Computer Science*, 48(1):127–132, 1986.