Advances in Differential Equations Volume 10, Number 10 (2005), 1131–1164

# $L^p$–REGULARITY FOR ELLIPTIC OPERATORS WITH UNBOUNDED COEFFICIENTS

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(Submitted by: G. Da Prato)

**Abstract.** Under suitable conditions on the functions $a \in C^1(\mathbb{R}^N, \mathbb{R}^{N^2})$, $F \in C^1(\mathbb{R}^N, \mathbb{R}^N)$, and $V : \mathbb{R}^N \to [0, \infty)$, we show that the operator $Au = \nabla(a\nabla u) + F \cdot \nabla u - Vu$ with domain $W^{2,p}_V(\mathbb{R}^N) = \{u \in W^{2,p}(\mathbb{R}^N) : Vu \in L^p(\mathbb{R}^N)\}$ generates a positive analytic semigroup on $L^p(\mathbb{R}^N)$, $1 < p < \infty$. Analogous results are also established in the spaces $L^1(\mathbb{R}^N)$ and $C_0(\mathbb{R}^N)$. As an application we show that the generalized Ornstein–Uhlenbeck operator $A_{\Phi,G}u = \Delta u - \nabla\Phi \cdot \nabla u + G \cdot \nabla u$ with domain $W^{2,p}(\mathbb{R}^N, \mu)$ generates an analytic semigroup on the weighted space $L^p(\mathbb{R}^N, \mu)$, where $1 < p < \infty$ and $\mu(dx) = e^{-\Phi(x)}dx$.

## 1. Introduction

In recent years there has been an increasing interest in differential operators with unbounded coefficients on $\mathbb{R}^N$ arising in the analytic treatment of stochastic differential equations; see [6], [7], [10], [11], [12], [13], [14], [21], [23], [24], [25], [27], and the references therein. An important example of such operators is the *generalized Ornstein–Uhlenbeck operator*

$$A_{\Phi,G}u = \Delta u - \nabla\Phi \cdot \nabla u + G \cdot \nabla u.$$

This operator has good properties in the space $L^p(\mathbb{R}^N, \mu)$ with weighted measure $\mu(dx) = e^{-\Phi(x)}dx$. We show in Theorem 7.4 that $A_{\Phi,G}$ with the natural domain $W^{2,p}(\mathbb{R}^N, \mu)$ generates an analytic $C_0$–semigroup on $L^p(\mathbb{R}^N, \mu)$ if

Accepted for publication: May 2005.
AMS Subject Classifications: 35J70, 35K65, 47D06.

$1 < p < \infty$, where in the simpler case $G = 0$ we only require that $e^{-\Phi}$ is integrable and that $|D^2\Phi|$ is small compared to $|\nabla\Phi|^2$ (Corollary 7.8). Our theorem extends and simplifies recent results by Da Prato and Vespri [13] (for $G = 0$) and the authors [27] (for quadratic $\Phi$ and linear $G$); see also [7], [11], [12], [21], and in particular [10] (where a description of the domain is given if $G = 0$ and $\Phi$ is convex).

Via the transformation $v = e^{-\Phi/p}u$, the operator $A_{\Phi,G}$ on $L^p(\mathbb{R}^N, \mu)$ is similar to an operator of the form

$$Av = \Delta v + F \cdot \nabla v - Vv, \quad \text{or more generally,} \quad Av = \nabla(a\nabla v) + F \cdot \nabla v - Vv$$

in the unweighted space $L^p(\mathbb{R}^N)$, cf. [13]. In fact, most of our paper deals with the operator $A$ aiming at a precise description of the domain on which $A$ generates an analytic semigroup. We make the following assumptions on the coefficients of $A$, where $p \in [1, \infty)$ is given.

**(H1)** $a_{hk} \in C_b^1(\mathbb{R}^N)$ are real-valued functions with $a_{hk} = a_{kh}$ and

$$\sum_{h,k=1}^{N} a_{hk}(x)\xi_h\xi_k \geq \nu|\xi|^2$$

for all $x$, $\xi \in \mathbb{R}^N$ and some constant $\nu > 0$.

**(H2)** $U \in C^1(\mathbb{R}^N)$ is a function such that $U \geq c_0 > 0$ and $|\nabla U| \leq \gamma U^{\frac{3}{2}} + C_\gamma$ for some constants $c_0, \gamma > 0$ and $C_\gamma \geq 0$.

**(H3)** $V : \mathbb{R}^N \to \mathbb{R}$ is measurable and $U \leq V \leq c_1 U$ for some constant $c_1 \geq 1$.

**(H4)** The function $F \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ satisfies $|F| \leq \kappa U^{\frac{1}{2}}$ for some constant $\kappa > 0$.

**(H5)** There is a constant $\theta < p$ such that $\theta U + \operatorname{div} F \geq 0$.

We will also require below that $\gamma > 0$ be sufficiently small. The auxiliary potential $U$ is mostly used to simplify the resulting hypotheses on the coefficients of $A_{\Phi,G}$. In addition, it allows us to avoid assumptions on the oscillation of $V$ itself.

There are several approaches to construct semigroups whose generator extends the closure of $A$ defined on test functions; see also the references in [6], [11], [13], [23], [25], [33]. Assuming essentially (H1) and (H5), one can imitate the arguments in [25, Section 4] yielding a contractive $C_0$–semigroup on $L^p(\mathbb{R}^N)$ for $p \geq 1$, cf. Remark 2.7. If one adds hypothesis (H4), then the quadratic form corresponding to $A$ on the natural form domain is closed, positive, and sectorial, see Section 6. Due to [33] the resulting analytic $C_0$–semigroup on $L^2(\mathbb{R}^N)$ induces contractive $C_0$–semigroups on $L^p(\mathbb{R}^N)$ for $p \geq 1$, which are analytic for $p > 1$; see also [15]. It is further possible to solve first a related stochastic equation assuming only a dissipativity condition and use this solution to define a semigroup acting on, say, bounded Borel functions. This approach is exposed in, e.g., [6] for operators such as the generalized Ornstein–Uhlenbeck operator $A_{\Phi,G}$. One can then extend the semigroup to weighted $L^p$–spaces under suitable hypotheses as is done in [11]. In sup–norm context the semigroup can also be constructed using apriori estimates of Schauder type and an approximation procedure, [23]. The assumptions in [6], [11], [23] differ from ours.

However, we point out that the approaches mentioned in the above paragraph do not (directly) allow one to compute the domain of $A$ (or $A_{\Phi,G}$) on $L^p$. (In [11] the case $p = 2$ is treated by additional arguments.)

In our main results Theorems 3.4, 4.4, and 5.2 we establish that $A$ with the natural domain $D(\Delta) \cap D(V)$ generates an analytic and contractive $C_0$–semigroup on $L^p(\mathbb{R}^N)$, $1 \le p < \infty$, and $C_0(\mathbb{R}^N)$ (where $a_{hk} = \delta_{hk}$ if $p = 1, \infty$). The precise description of the domain corresponds to good apriori estimates for the elliptic problem $\lambda u - Au = f$ stated in Corollaries 4.5 and 4.6. This semigroup then solves the parabolic partial differential equation corresponding to the elliptic operator $A$. Here we improve results due to Okazawa [28], who studied Schrödinger operators where $F = 0$ in $L^p(\mathbb{R}^N)$ with $1 < p < \infty$, and results due to Cannarsa and Vespri [4], [5], who replaced our condition (H5) by an additional bound on the constant $\kappa$ in (H4). We note that the results in [13] on $A_{\Phi,G}$ rely on [5] and that the additional restriction on $\kappa$ in [5] leads to difficulties in [13] (cf. Example 2.4 and Theorem 2.7 in [13]).

Cannarsa and Vespri first show generation in $L^2(\mathbb{R}^N)$, then in Morrey–Campanato spaces, then in $C_0(\mathbb{R}^N)$ and $L^1(\mathbb{R}^N)$, and finally in $L^p(\mathbb{R}^N)$. Our approach is much more direct. In Propositions 3.1 and 3.2 we prove the crucial apriori estimates in $L^p$ employing only integration by parts and related elementary techniques (besides standard regularity properties of the diffusion part, cf. (2.6)). We combine these apriori estimates with known results and methods in semigroup theory to obtain the semigroups on $L^p(\mathbb{R}^N)$, $1 \le p < \infty$, which are analytic for $p > 1$. The Stewart–Masuda technique then gives the generation result in $C_0(\mathbb{R}^N)$. A duality argument yields analyticity for $p = 1$ (if also $\operatorname{div} F \le c_2 U$).

As can be seen from the proofs of Propositions 2.3 and 3.2, hypothesis (H2) is the essential ingredient to determine the domain. In fact, in Example 3.7 we present a Schrödinger operator $A = \Delta - V$ on $L^2(\mathbb{R}^3)$ such that (H2) holds with a too large constant $\gamma$ and $D(A) \not\subseteq D(V)$. Condition (H4) combined with (H2) and (H3) allows us to control the drift term by the diffusion term and the potential, employing Proposition 2.3. Assumptions (H4) and (H5) further lead to the sectoriality of $A$, while (H5) already guarantees that $A$ is dissipative; see Proposition 3.2 and Lemma 2.6.

In the next section we give some preliminary results. The following three sections deal with generation in $L^p$, $C_0$, and $L^1$, respectively. In Section 6 we show additional qualitative properties of the semigroups, such as positivity, compactness, and maximal regularity of type $L^q$ for the inhomogeneous parabolic problem. Section 7 is devoted to the Ornstein–Uhlenbeck operator $A_{\Phi,G}$.

**Notation.** $C_0^\infty(\mathbb{R}^N)$ means the space of $C^\infty$-functions with compact support and $C_b(\mathbb{R}^N)$ $(C_0(\mathbb{R}^N))$ the space of bounded continuous functions (vanishing at infinity). For $1 \le p \le \infty$ and $k \in \mathbb{N}$, $W^{k,p}(\mathbb{R}^N)$ denotes the usual Sobolev space. $B(x,r)$ designates the open ball in $\mathbb{R}^N$ with center $x$ and radius $r$. The norm of $L^p(\mathbb{R}^N)$ is denoted by $\|\cdot\|_p$ and that of $L^p(B(x,r))$ by $\|\cdot\|_{p,r}$, thus without indicating the center of the ball. We set $A_0u = \sum_{h,k=1}^N D_h(a_{hk}D_ku)$ and $a(\xi,\eta) = \sum_{h,k=1}^N a_{hk}(\cdot)\xi_h\eta_k$ for $\xi$, $\eta \in \mathbb{R}^N$.

## 2. Preliminary results

First of all, let us observe that if a function $U$ satisfies assumption (H2), then $U+\lambda$ satisfies (H2) with $C_\gamma = 0$ for $\lambda$ large enough. In this case, (H2) is equivalent to the inequality $|\nabla U^{-\frac{1}{2}}| \le \gamma/2$. From the mean value theorem it follows that

$$(1-\frac{\delta\gamma}{2})U(x)^{\frac{1}{2}} \le U(x_0)^{\frac{1}{2}} \le (1+\frac{\delta\gamma}{2})U(x)^{\frac{1}{2}} \tag{2.1}$$

if $|x-x_0| \le r$ and $r = \delta U(x_0)^{-\frac{1}{2}}$. Some of the following results are valid assuming only (2.1) but we prefer to keep (H2) in order to simplify the exposition.

If $N = 1$ (or $N > 1$ and $U$ is rotational symmetric), (H2) holds whenever $U$ does not oscillate too fast. For example, it is satisfied by all polynomials and functions like $e^P$, with $P$ a polynomial. The picture is more complicated in several variables, if $U$ is not rotational symmetric. For example, the function $1+x^2y^2$ fails (H2). However, (H2) is valid if $U$ is a polynomial whose homogeneous part of maximal degree is positive definite or if, more generally, $U$ is hypoelliptic, see [19, Chapter 11].

The interpolation result stated in Proposition 2.3 below will be crucial throughout the paper. To establish this result, we need the following well-known fact; its proof is given for the sake of completeness.

**Lemma 2.1.** *For every $u \in C_0^\infty(\mathbb{R}^N)$ and $1 \le p \le \infty$, we have*

$$\|\nabla u\|_p \le C\,\|\Delta u\|_p^{\frac{1}{2}}\,\|u\|_p^{\frac{1}{2}} \tag{2.2}$$

*with a constant $C > 0$ depending only on $N$.*

**Proof.** For $\lambda > 0$ and $u \in C_0^\infty(\mathbb{R}^N)$ we set $f = \lambda u - \Delta u$. Then

$$u(x) = \int_0^\infty e^{-\lambda t}\big(G(t)f\big)(x)\,dt,$$

where

$$\big(G(t)f\big)(x) = (4\pi t)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4t}} f(y)\,dy$$

for $t > 0$ and $x \in \mathbb{R}^N$. Differentiating under the integral sign and using Young's inequality, we obtain

$$\|\nabla G(t)f\|_p \le \frac{c}{\sqrt{t}}\,\|f\|_p\,,$$

with $c = c(N)$, whence

$$\|\nabla u\|_p \le \frac{c'}{\sqrt{\lambda}}\|f\|_p \le \frac{c'}{\sqrt{\lambda}}\Big(\lambda\|u\|_p + \|\Delta u\|_p\Big)$$

for each $\lambda > 0$. The assertion follows if we take $\lambda = \|\Delta u\|_p\,\|u\|_p^{-1}$. $\square$

We next show a variant of Besicovitch's covering theorem, needed in the proof of Proposition 2.3.

**Lemma 2.2.** *There exists a natural number $K(N)$ with the following property. Let $\{B(x, r_x) : x \in \mathbb{R}^N\}$ be a collection of balls such that $0 < r_x \le C$ and $|r_x - r_y| \le \frac{1}{2}|x-y|$ for every $x, y \in \mathbb{R}^N$. Then there exists a countable subcovering $\{B(x_n, \frac{r_{x_n}}{2})\}$ of $\mathbb{R}^N$ such that at most $K(N)$ among the balls $B(x_n, r_{x_n})$ overlap.*

**Proof.** Due to the Besicovitch covering theorem, [36, Theorem 1.3.5], there exists a countable subcovering $\{B(x_n, \frac{r_n}{2})\}$ of $\mathbb{R}^N$ such that at most $\xi(N)$ among the balls $B(x_n, \frac{r_n}{2})$ overlap. Here we write $r_n$ for $r_{x_n}$ and remark that the number $\xi(N)$ depends only on the dimension $N$. We now have to estimate the number of overlapping doubled balls $B(x_n, r_n)$, where we follow the proof of [36, Theorem 1.3.4]. Fix $x = x_k$ for some $k$ and assume that $B(x, r_x)$ and $B(x_j, r_j)$ overlap for $j \in J$. Then $|x - x_j| \le r_x + r_j$, hence $|r_x - r_j| \le \frac{1}{2}(r_x + r_j)$ by the assumption. This yields $r_x \le 3r_j$ and $r_j \le 3r_x$.

Thus the balls $B(x_j, \frac{r_j}{2})$, $j \in J$, are contained in $B(x, 5r_x)$. Since at most $\xi(N)$ of the balls $B(x_j, \frac{r_j}{2})$ overlap, we obtain

$$6^{-N}|J|r_x^N \le \sum_{j\in J} 2^{-N} r_j^N \le \xi(N)5^N r_x^N$$

and hence $|J| \le 30^N \xi(N)$. This yields $K(N) \le 1 + 30^N \xi(N)$. □

**Proposition 2.3.** *Let $U$ be a function satisfying (H2). Then there are two constants $\alpha, \varepsilon_0 > 0$ (depending only on $\gamma$, $C_\gamma$, $c_0$) such that for all $0 < \varepsilon \le \varepsilon_0$, $1 \le p \le \infty$, and $v \in C_0^\infty(\mathbb{R}^N)$, we have*

$$\|U^{\frac{1}{2}} \nabla v\|_p \le \varepsilon \|\Delta v\|_p + \frac{\alpha}{\varepsilon}\|Uv\|_p. \tag{2.3}$$

**Proof.** We replace $U$ by $U + \lambda$ for $\lambda \ge 0$ such that (H2) holds for $U + \lambda$ with $C_\gamma = 0$. Since $U \ge c_0 > 0$, estimate (2.3) for $U + \lambda$ implies (2.3) for $U$ (with a different $\alpha$). So we may assume that $C_\gamma = 0$ in (H2). Fix $x_0 \in \mathbb{R}^N$ and choose $\eta \in C_0^\infty(\mathbb{R}^N)$ such that $0 \le \eta \le 1$, $\eta = 1$ in $B(x_0, r/2)$, supp $\eta \subset B(x_0, r)$, $|\nabla \eta(x)| \le c/r$, and $|D^2\eta(x)| \le c/r^2$ for a constant $c$ independent of $x_0$ and $r$. Using (2.2) for $\eta v$ and Young's inequality, we estimate

$$\begin{aligned} \|U(x_0)^{\frac{1}{2}}\nabla v\|_{p,\frac{r}{2}} &\le \|U(x_0)^{\frac{1}{2}}\nabla(\eta v)\|_p \le C\|\Delta(\eta v)\|_p^{\frac{1}{2}}\, \|U(x_0)\eta v\|_p^{\frac{1}{2}} \\ &\le \varepsilon\Big(\|\Delta v\|_{p,r} + \frac{1}{r}\|\nabla v\|_{p,r} + \frac{1}{r^2}\|v\|_{p,r}\Big) + \frac{C_1}{\varepsilon}\|U(x_0)v\|_{p,r} \end{aligned} \tag{2.4}$$

for a constant $C_1$ and each $\varepsilon > 0$. We fix $r = \delta U(x_0)^{-\frac{1}{2}}$ and $\delta = 1/\gamma$. Then (2.1) shows that $\frac{1}{2}U(x)^{\frac{1}{2}} \le U(x_0)^{\frac{1}{2}} \le \frac{3}{2}U(x)^{\frac{1}{2}}$ if $|x - x_0| \le r$. Thus, (2.4) yields

$$\frac{1}{2}\|U^{\frac{1}{2}}\nabla v\|_{p,\frac{r}{2}} \le \varepsilon\|\Delta v\|_{p,r} + \varepsilon C_2\|U^{\frac{1}{2}}\nabla v\|_{p,r} + \frac{C_2}{\varepsilon}\|Uv\|_{p,r} \tag{2.5}$$

for a constant $C_2 > 0$. Let us first deal with the simpler situation where $p = \infty$. In this case, varying $x_0$ in $\mathbb{R}^N$, equation (2.5) implies

$$\frac{1}{2}\|U^{\frac{1}{2}}\nabla v\|_\infty \le \varepsilon\|\Delta v\|_\infty + C_2\varepsilon\|U^{\frac{1}{2}}\nabla v\|_\infty + \frac{C_2}{\varepsilon}\|Uv\|_\infty\,.$$

Thus the asserted estimate holds with $\varepsilon_0 = (4C_2)^{-1}$. Suppose now that $1 \le p < \infty$. Applying Lemma 2.2, we find a countable covering $\{B(x_n, \frac{r_n}{2})\}$ of $\mathbb{R}^N$ with $r_n = \delta U(x_n)^{-\frac{1}{2}}$ such that at most $K(N)$ among the balls $B(x_n, r_n)$ overlap. Taking the $p$-th power of (2.5) relative to each ball $B(x_n, \frac{r_n}{2})$ and summing up, we arrive at

$$\frac{1}{2}\|U^{\frac{1}{2}}\nabla v\|_p \le 3^{\frac{1}{p'}}K(N)^{\frac{1}{p}}\Big(\varepsilon\|\Delta v\|_p + C_2\varepsilon\|U^{\frac{1}{2}}\nabla v\|_p + \frac{C_2}{\varepsilon}\|Uv\|_p\Big),$$

where $\frac{1}{p'}+\frac{1}{p}=1$. The asserted estimate follows with $\varepsilon_0 = (12C_2K(N))^{-1}$. $\square$

**Remark 2.4.** Let $1 < p < \infty$. Thanks to the elliptic estimate

$$\|v\|_{W^{2,p}(\mathbb{R}^N)} \le C_p\left(\|A_0 v\|_p + \|v\|_p\right) \tag{2.6}$$

for $v \in L^p(\mathbb{R}^N)$, see e.g. [18, Chapter 9], for uniformly elliptic operators (and $U \ge c_0 > 0$) one can replace $\Delta v$ with $A_0 v$ in Proposition 2.3. This will be used systematically in the next section. If $1 < p < \infty$, there is also a simpler proof of Proposition 2.3 with $D^2 v$ instead of $\Delta v$ involving Taylor's formula.

We now introduce the spaces

$$\begin{aligned}
\mathcal{D}_1 &= \{v \in L^1(\mathbb{R}^N) : \Delta v \in L^1(\mathbb{R}^N),\ Uv \in L^1(\mathbb{R}^N)\}.\\
\mathcal{D}_p &= W^{2,p}_U(\mathbb{R}^N) = \{v \in W^{2,p}(\mathbb{R}^N) : Uv \in L^p(\mathbb{R}^N)\}, \qquad 1 < p < \infty,\\
\mathcal{D}_\infty &= \{v \in C_0(\mathbb{R}^N) : v \in W^{2,q}_{\mathrm{loc}}(\mathbb{R}^N) \text{ for all}\\
&\qquad\qquad q < \infty,\ \Delta v \in C_0(\mathbb{R}^N),\ Uv \in C_0(\mathbb{R}^N)\}.
\end{aligned}$$

Here $\Delta$ is understood in the sense of distributions. These spaces are Banach spaces endowed with the norms

$$\begin{aligned}
\|v\|_{\mathcal{D}_p} &= \|v\|_{W^{2,p}(\mathbb{R}^N)} + \|Uv\|_{L^p(\mathbb{R}^N)}, \quad 1 < p < \infty,\\
\|v\|_{\mathcal{D}_p} &= \|v\|_{L^p(\mathbb{R}^N)} + \|\Delta v\|_{L^p(\mathbb{R}^N)} + \|Uv\|_{L^p(\mathbb{R}^N)}, \quad p = 1, \infty.
\end{aligned}$$

$C_0^\infty(\mathbb{R}^N)$ is a dense subspace of $\mathcal{D}_p$ as we show in the next lemma. Therefore, (2.2) and (2.3) also hold for $v \in \mathcal{D}_p$. In particular, $U^{\frac{1}{2}}|\nabla v| \in L^p(\mathbb{R}^N)$ for $v \in \mathcal{D}_p$, $1 \le p < \infty$, and $U^{\frac{1}{2}}|\nabla v| \in C_0(\mathbb{R}^N)$ for $v \in \mathcal{D}_\infty$.

**Lemma 2.5.** *The space $C_0^\infty(\mathbb{R}^N)$ is dense in $\mathcal{D}_p$ for $1 \le p \le \infty$.*

**Proof.** Let $\eta$ be a cutoff function such that $0 \le \eta \le 1$, $\eta = 1$ in $B(0,1)$, supp $\eta \subset B(0,2)$. Define $\eta_n(x) = \eta(x/n)$. For $f \in \mathcal{D}_p$ it is easy to see that $\eta_n f \to f$ as $n \to \infty$ in $\mathcal{D}_p$. This shows that the set of all functions in $\mathcal{D}_p$ having compact support is dense. On the other hand, if $f \in \mathcal{D}_p$ has compact support, a standard convolution argument shows the existence of a sequence of smooth functions with compact support converging to $f$ in $\mathcal{D}_p$. $\square$

In the following lemma we establish the dissipativity of $A$ under rather weak assumptions.

**Lemma 2.6.** *Assume that assumption (H1) is satisfied.*

*a) If $F \in C^1(\mathbb{R}^N, \mathbb{R}^N)$, $V \in L^p_{loc}(\mathbb{R}^N)$, and $V + \frac{1}{p}\mathrm{div}\, F \geq 0$ for some $1 \leq p < \infty$, then $(A, C_0^\infty(\mathbb{R}^N))$ is dissipative in $L^p(\mathbb{R}^N)$.*

*b) If $F \in C(\mathbb{R}^N, \mathbb{R}^N)$, $V \in C(\mathbb{R}^N, \mathbb{R})$, and $V \geq 0$, then $(A, C_0^\infty(\mathbb{R}^N))$ is dissipative in $C_0(\mathbb{R}^N)$.*

**Proof.** (a) Let $1 \leq p < \infty$ and $u \in C_0^\infty(\mathbb{R}^N)$. It is known that $A_0$ is dissipative in $L^p(\mathbb{R}^N)$; that is,

$$Re\Big(\int_{\mathbb{R}^N} (A_0 u)\overline{u}|u|^{p-2}\, dx\Big) \leq 0.$$

Moreover, from $(\nabla u)\overline{u}|u|^{p-2} = \frac{1}{p}(\nabla |u|^p) + iIm(\overline{u}\nabla u)|u|^{p-2}$ it follows that

$$Re\Big(\int_{\mathbb{R}^N} F \cdot (\nabla u)\overline{u}|u|^{p-2}\, dx\Big) = -\frac{1}{p}\int_{\mathbb{R}^N} |u|^p \mathrm{div}\, F\, dx.$$

As a result,

$$Re\Big(\int_{\mathbb{R}^N} (Au)\overline{u}|u|^{p-2}\, dx\Big) \leq -\int_{\mathbb{R}^N} (V + \frac{1}{p}\mathrm{div}\, F)|u|^p\, dx \leq 0.$$

(b) The dissipativity of $A$ in $C_0(\mathbb{R}^N)$ is a standard consequence of the maximum principle. □

**Remark 2.7.** Suppose that the assumptions of Lemma 2.6 hold for some $p \in [1, \infty)$ and that $V$ is locally Hölder continuous. Then there exists a contractive $C_0$–semigroup on $L^p(\mathbb{R}^N)$ whose generator extends (the closure of) $A$ defined on test functions. This semigroup can be constructed as the limit of the contraction semigroups on $L^p(B(0,n))$ which are generated by the operator on $L^p(B(0,n))$ induced by $A$ with Dirichlet boundary conditions; compare [25, Section 4].

## 3. GENERATION OF ANALYTIC SEMIGROUPS ON $L^p(\mathbb{R}^N)$, $1 < p < \infty$.

In this section we establish that, under the assumptions (H1)–(H5), the operator $A$ with domain $\mathcal{D}_p = W^{2,p}_U(\mathbb{R}^N)$ generates an analytic semigroup on $L^p(\mathbb{R}^N)$ if $1 < p < \infty$. We start by showing that $A$ is *regularly dissipative*; that is, for some $\phi \in (0, \pi/2)$ the operator $e^{\pm i\phi}A$ is dissipative. This property is clearly equivalent to the estimate (3.4) below (with $\delta = \cot\phi$).

**Proposition 3.1.** *Let $1 < p < \infty$ and assume that (H1), (H4), and (H5) are satisfied with $U$ replaced by $V \in L^p_{loc}(\mathbb{R}^N)$. Then the operator $A$ defined on $C_0^\infty(\mathbb{R}^N)$ is regularly dissipative in $L^p(\mathbb{R}^N)$ with angle $\phi_p > 0$ only depending on $p$ and the constants in (H1), (H4), and (H5).*

**Proof.** We first assume that $p \geq 2$. Let $u \in C_0^\infty(\mathbb{R}^N)$ and set $u^* := \overline{u}|u|^{p-2}$. Integrating by parts, one computes

$$- Re\Big(\int_{\mathbb{R}^N}(Au)u^*\,dx\Big) = (p-1)\int_{\mathbb{R}^N}|u|^{p-4}a(Re(\overline{u}\nabla u), Re(\overline{u}\nabla u))\,dx \quad (3.1)$$
$$+ \int_{\mathbb{R}^N}|u|^{p-4}a(Im(\overline{u}\nabla u), Im(\overline{u}\nabla u))\,dx + \int_{\mathbb{R}^N}\Big(V + \frac{1}{p}\mathrm{div}\,F\Big)|u|^p\,dx.$$

If we put

$$B^2 := \int_{\mathbb{R}^N}|u|^{p-4}a(Re(\overline{u}\nabla u), Re(\overline{u}\nabla u))\,dx,$$
$$C^2 := \int_{\mathbb{R}^N}|u|^{p-4}a(Im(\overline{u}\nabla u), Im(\overline{u}\nabla u))\,dx, \qquad D^2 := \int_{\mathbb{R}^N}V|u|^p\,dx,$$

then we deduce from (H5)

$$-Re\Big(\int_{\mathbb{R}^N}(Au)u^*\,dx\Big) \geq (p-1)B^2 + C^2 + D^2(1-\frac{\theta}{p}) \geq 0.$$

On the other hand, proceeding in a similar way for the imaginary part, we have

$$\Big|Im\int_{\mathbb{R}^N}(Au)u^*\,dx\Big| \leq |p-2|\int_{\mathbb{R}^N}|u|^{p-4}a(Re(\overline{u}\nabla u), Im(\overline{u}\nabla u))\,dx$$
$$+ \int_{\mathbb{R}^N}|F||u|^{p-2}|Im(\overline{u}\nabla u)|\,dx. \quad (3.2)$$

Conditions (H1) and (H4) imply

$$\int_{\mathbb{R}^N}|F||u|^{p-2}|Im(\overline{u}\nabla u)|\,dx \leq \kappa\int_{\mathbb{R}^N}V^{\frac{1}{2}}|Im(\overline{u}\nabla u)||u|^{\frac{p}{2}}|u|^{\frac{p-4}{2}}\,dx$$
$$\leq \kappa\Big(\int_{\mathbb{R}^N}V|u|^p\,dx\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{R}^N}|u|^{p-4}|Im(\overline{u}\nabla u)|^2\Big)^{\frac{1}{2}} \quad (3.3)$$
$$\leq \frac{\kappa}{\sqrt{\nu}}\Big(\int_{\mathbb{R}^N}V|u|^p\,dx\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{R}^N}|u|^{p-4}a(Im(\overline{u}\nabla u), Im(\overline{u}\nabla u))\,dx\Big)^{\frac{1}{2}}.$$

Therefore,

$$\Big|Im\Big(\int_{\mathbb{R}^N}(Au)u^*\,dx\Big)\Big| \leq |p-2|BC + \frac{\kappa}{\sqrt{\nu}}CD.$$

Taking $\delta_p = \delta$ such that $\delta^2 = \frac{|p-2|^2}{4(p-1)} + \frac{\kappa^2}{4\nu(1-\theta/p)}$, we see that

$$\Big|Im\Big(\int_{\mathbb{R}^N}(Au)u^*\,dx\Big)\Big| \leq \delta\Big[-Re\Big(\int_{\mathbb{R}^N}(Au)u^*\,dx\Big)\Big]. \quad (3.4)$$

This shows the assertion for $p \geq 2$. If $p \in (1,2)$, we replace $|u|$ by $u_\varepsilon = \sqrt{|u|^2+\varepsilon}$ for $\varepsilon > 0$ in the calculations involving $A_0$. Passing to the limit

$\varepsilon \to 0$, one then establishes (3.1) and (3.2) (in particular, all integrands are integrable). Thus estimate (3.3) is also valid and one can deduce (3.4) as above. □

Next we prove the closedness of the operator $(A, W^{2,p}_U(\mathbb{R}^N))$ for $1 < p < \infty$.

**Proposition 3.2.** *Let $1 < p < \infty$. Assume that the assumptions (H1), (H3), (H4), and (H5) are satisfied. If (H2) holds with $\gamma$ satisfying*

$$\frac{\theta}{p} + (p-1)\gamma\Big[\frac{\kappa}{p} + \frac{M\gamma}{4}\Big] < 1, \tag{3.5}$$

*where $M^2 := \sup_{x\in\mathbb{R}^N} \max_{|\xi|=1} a(\xi,\xi)$, then $(A, W^{2,p}_U(\mathbb{R}^N))$ is closed. Moreover, there exist constants $C, C' \geq 0$ depending only on $M$, $p$, on the constants in (H1)–(H5), and on $C_p$ from (2.6) such that*

$$\|u\|_{W^{2,p}_U(\mathbb{R}^N)} \leq C\,\|u - Au\|_p \leq C'\,\|u\|_{W^{2,p}_U(\mathbb{R}^N)} \qquad \textit{for} \;\; u \in W^{2,p}_U(\mathbb{R}^N).$$

**Proof.** We assume preliminarily that (H2) is satisfied with $C_\gamma = 0$. Proposition 2.3, (H3), and (H4) imply that $A : W^{2,p}_U(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$ is continuous. Since $C_0^\infty(\mathbb{R}^N)$ is dense in $W^{2,p}_U(\mathbb{R}^N)$ and $A$ is dissipative, it remains to show the estimates

$$\|Uu\|_p,\ \|D^2u\|_p \leq C\|Au\|_p \quad \text{ for } u \in C_0^\infty(\mathbb{R}^N).$$

We consider first the case $p \geq 2$. For a fixed real $u \in C_0^\infty(\mathbb{R}^N)$ we set

$$f := -Au = -A_0u - F\cdot\nabla u + Vu. \tag{3.6}$$

Integrating by parts, one deduces

$$\begin{aligned}
&\int_{\mathbb{R}^N} U^{p-1} F\cdot(\nabla u)u|u|^{p-2}\,dx = \frac{1}{p}\int_{\mathbb{R}^N} U^{p-1}F\cdot\nabla(|u|^p)\,dx\\
&= -\frac{1}{p}\int_{\mathbb{R}^N} (\operatorname{div} F)U^{p-1}|u|^p\,dx - \big(1-\frac{1}{p}\big)\int_{\mathbb{R}^N} U^{p-2}|u|^pF\cdot\nabla U\,dx,\\
&\int_{\mathbb{R}^N} (A_0u)U^{p-1}u|u|^{p-2}\,dx = -\sum_{h,k=1}^N \int_{\mathbb{R}^N} a_{hk}(x)D_huD_k\big(U^{p-1}u|u|^{p-2}\big)\,dx\\
&= -(p-1)\int_{\mathbb{R}^N}\sum_{h,k=1}^N a_{hk}(x)U^{p-1}|u|^{p-2}D_huD_ku\,dx\\
&-(p-1)\int_{\mathbb{R}^N}\sum_{h,k=1}^N a_{hk}(x)U^{p-2}u|u|^{p-2}D_huD_kU\,dx.
\end{aligned} \tag{3.7}$$

If we multiply (3.6) by $U^{p-1}u|u|^{p-2}$ and integrate, we thus obtain

$$\int_{\mathbb{R}^N} VU^{p-1}|u|^p\,dx = -(p-1)\int_{\mathbb{R}^N}\sum_{h,k=1}^N a_{hk}(x)U^{p-1}|u|^{p-2}D_huD_ku\,dx$$
$$-(p-1)\int_{\mathbb{R}^N}\sum_{h,k=1}^N a_{hk}(x)u|u|^{p-2}U^{p-2}D_huD_kU\,dx$$
$$-\frac{1}{p}\int_{\mathbb{R}^N}(\operatorname{div}F)U^{p-1}|u|^p\,dx - \big(1-\frac{1}{p}\big)\int_{\mathbb{R}^N}U^{p-2}|u|^pF\cdot\nabla U\,dx$$
$$+\int_{\mathbb{R}^N} fU^{p-1}u|u|^{p-2}\,dx.$$

Conditions (H3) and (H5) yield

$$V+\frac{1}{p}\operatorname{div}F \geq \big(1-\frac{\theta}{p}\big)U,$$

so that

$$\int_{\mathbb{R}^N}\big(1-\frac{\theta}{p}\big)U^p|u|^p\,dx + (p-1)\int_{\mathbb{R}^N}|u|^{p-2}U^{p-1}a(\nabla u,\nabla u)\,dx$$
$$\leq (p-1)\int_{\mathbb{R}^N}U^{p-2}|u|^{p-1}a(\nabla u,\nabla U)\,dx$$
$$+\big(1-\frac{1}{p}\big)\int_{\mathbb{R}^N}U^{p-2}|u|^p\,|F|\,|\nabla U|\,dx + \int_{\mathbb{R}^N} fU^{p-1}|u|^{p-1}\,dx.$$

Using (H2) and (H4), we estimate

$$\int_{\mathbb{R}^N}U^{p-2}|u|^p|F|\,|\nabla U|\,dx \quad\leq\quad \kappa\gamma\int_{\mathbb{R}^N}U^p|u|^p\,dx.$$

Moreover, (H2) implies that

$$\int_{\mathbb{R}^N}U^{p-2}|u|^{p-1}|a(\nabla u,\nabla U)|dx \leq \int_{\mathbb{R}^N}U^{p-2}|u|^{p-1}a(\nabla u,\nabla u)^{\frac{1}{2}}a(\nabla U,\nabla U)^{\frac{1}{2}}dx$$
$$\leq \gamma\sqrt{M}\int_{\mathbb{R}^N}U^{p-\frac{1}{2}}|u|^{p-1}a(\nabla u,\nabla u)^{\frac{1}{2}}\,dx$$
$$=\gamma\sqrt{M}\int_{\mathbb{R}^N}U^{\frac{p-1}{2}}|u|^{\frac{p-2}{2}}a(\nabla u,\nabla u)^{\frac{1}{2}}|u|^{\frac{p}{2}}U^{\frac{p}{2}}\,dx$$
$$\leq \gamma\sqrt{M}\Big(\int_{\mathbb{R}^N}|u|^{p-2}U^{p-1}a(\nabla u,\nabla u)\,dx\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{R}^N}|u|^pU^p\,dx\Big)^{\frac{1}{2}}. \tag{3.8}$$

From Hölder's and Young's inequalities it follows that for each $\varepsilon > 0$ there is $c_\varepsilon > 0$ such that

$$\Big[1 - \frac{\theta}{p} - \kappa\gamma(1 - \frac{1}{p})\Big] \int_{\mathbb{R}^N} U^p |u|^p \, dx + (p-1) \int_{\mathbb{R}^N} |u|^{p-2} U^{p-1} a(\nabla u, \nabla u)\, dx$$
$$\le (p-1)\gamma\sqrt{M}\Big( \int_{\mathbb{R}^N} |u|^{p-2} U^{p-1} a(\nabla u, \nabla u)\, dx \Big)^{\frac{1}{2}} \Big( \int_{\mathbb{R}^N} U^p |u|^p \, dx \Big)^{\frac{1}{2}}$$
$$+ \varepsilon \int_{\mathbb{R}^N} U^p |u|^p \, dx + c_\varepsilon \int_{\mathbb{R}^N} |f|^p \, dx.$$

Setting

$$B^2 := \int_{\mathbb{R}^N} U^p |u|^p \, dx \quad \text{and} \quad D^2 := \int_{\mathbb{R}^N} |u|^{p-2} U^{p-1} a(\nabla u, \nabla u)\, dx,$$

we arrive at

$$\Big[1 - \frac{\theta}{p} - \kappa\gamma(1 - \frac{1}{p}) - \varepsilon\Big] B^2 + (p-1) D^2 \le (p-1)\gamma\sqrt{M} BD + c_\varepsilon \|f\|_p^p.$$

Thanks to (3.5) we may choose a small $\varepsilon$ to deduce

$$\|Uu\|_p \le C\|f\|_p.$$

Then (H4), (H2), Proposition 2.3, (2.6), and (H3) imply that

$$\begin{aligned} \|F \cdot \nabla u\|_p &\le \kappa(\varepsilon\|\Delta u\|_p + \frac{\alpha}{\varepsilon}\|Uu\|_p) \le \kappa C_p \varepsilon \|A_0 u\|_p + c_\varepsilon \|Uu\|_p \\ &\le \kappa C_p \varepsilon \left(\|f\|_p + \|F \cdot \nabla u\|_p\right) + c'_\varepsilon \|Uu\|_p \end{aligned}$$

for $\varepsilon \le \varepsilon_0$. Thus, for sufficiently small $\varepsilon$, we have

$$\|F \cdot \nabla u\|_p \le C(\|Uu\|_p + \|f\|_p) \le \tilde{C}\|f\|_p.$$

Finally,

$$\begin{aligned} \|D^2 u\|_p &\le C_p\big(\|A_0 u\|_p + \|u\|_p\big) \\ &\le C_p\big(\|f\|_p + \|F \cdot \nabla u\|_p + \|Vu\|_p + \|u\|_p\big) \le C'\|f\|_p. \end{aligned}$$

If $p < 2$, then one can verify as in the proof of Proposition 3.1 that (3.7) holds and, in particular, that $D^2$ is a finite number. Thus estimate (3.8) is also valid and one can conclude the proof as above.

In order to remove the assumption $C_\gamma = 0$ we fix a large $\lambda$ such that $U + \lambda$ satisfies (H2) with $C_\gamma = 0$ and apply the previous estimates to the operator $A - \lambda$. Then

$$\begin{aligned} \|u\|_{W_U^{2,p}(\mathbb{R}^N)} &\le C\, \|(\lambda + 1)u - Au\|_p \\ &\le C\left(\|u - Au\|_p + \lambda\|u\|_p\right) \le C(1 + \lambda)\, \|u - Au\|_p, \end{aligned}$$

by the dissipativity of $A$. $\square$

**Remark 3.3.** If $F \equiv 0$ and $a_{hk} = \delta_{hk}$, then condition (3.5) becomes $\gamma^2 < \frac{4}{p-1}$, which was already used by N. Okazawa in [28, Theorem 2.5].

We now come to the main result of this section.

**Theorem 3.4.** *Assume that the assumptions (H1), (H3), (H4), and (H5) are satisfied. If (H2) holds with $\gamma$ satisfying*

$$\frac{\theta}{p} + (p-1)\gamma\Big[\frac{\kappa}{p} + \frac{M\gamma}{4}\Big] < 1,$$

*where $M^2 := \sup_{x\in\mathbb{R}^N} \max_{|\xi|=1} a(\xi,\xi)$, then $(A, W^{2,p}_U(\mathbb{R}^N))$ generates an analytic $C_0$–semigroup $T_p(\cdot)$ in $L^p(\mathbb{R}^N)$, $1 < p < \infty$, such that $\|T_p(z)\| \le 1$ for $|\arg z| \le \phi_p$ and some $\phi_p > 0$.*

**Proof.** It suffices to show that 1 belongs to the resolvent set $\rho(A)$ of $A$. In fact, Propositions 3.1, 3.2, and the Lumer–Phillips theorem then imply that the operators $e^{\pm i\phi_p}A$ generate contractive $C_0$–semigroups for some $\phi_p > 0$. This yields the assertion by [17, Theorem II.4.9].

To verify $1 \in \rho(A)$, we employ the continuity method, cf. [18, Theorem 5.2]. For $t \in [0,1]$ and $u \in W^{2,p}_U(\mathbb{R}^N)$ we set $L_t u := A_0 u + tF\cdot\nabla u - Vu$. Note that these operators satisfy the assertions of Propositions 3.1 and 3.2 with uniform constants. In particular

$$\|u\|_{W^{2,p}_U(\mathbb{R}^N)} \le C\|u - L_t u\|_p$$

for every $u \in W^{2,p}_U(\mathbb{R}^N)$, with $C$ independent of $t \in [0,1]$. Then $1 \in \rho(L_0)$ if and only if $1 \in \rho(L_1) = \rho(A)$. To establish that $1 \in \rho(L_0)$, we use as in [28] the approximating potentials $U_\varepsilon := \frac{U}{1+\varepsilon U}$ and $V_\varepsilon := \frac{V}{1+\varepsilon V}$, where $\varepsilon > 0$. Then we have

$$\frac{c_0}{1+\varepsilon c_0} \le U_\varepsilon \le V_\varepsilon \le c_1 U_\varepsilon \le \frac{c_1}{\varepsilon} \quad \text{and} \quad |\nabla U_\varepsilon| \le \gamma U_\varepsilon^{\frac{3}{2}} + C_\gamma. \tag{3.9}$$

Let $f \in L^p(\mathbb{R}^N)$. Observe that $A_0 - V_\varepsilon$ with domain $D(A_0 - V_\varepsilon) = W^{2,p}(\mathbb{R}^N)$ generates a positive contraction $C_0$-semigroup on $L^p(\mathbb{R}^N)$. Thus the equation

$$u - A_0 u + V_\varepsilon u = f$$

has a unique solution $u_\varepsilon \in W^{2,p}(\mathbb{R}^N)$ satisfying

$$\|u_\varepsilon\|_p \le \|f\|_p \qquad \text{and} \qquad \|U_\varepsilon u_\varepsilon\|_p \le C\|f\|_p,$$

where the last inequality follows from the Proposition 3.2. We remark that the constant $C$ does not depend on $\varepsilon$ due to (3.9). From (2.6) we thus deduce

$$\|u_\varepsilon\|_{W^{2,p}(\mathbb{R}^N)} \le C_p(\|A_0 u_\varepsilon\|_p + \|u_\varepsilon\|_p) \le C\|f\|_p.$$

Therefore, there exists a sequence $(u_{\varepsilon_n})$ converging weakly to a function $u$ in $W^{2,p}(\mathbb{R}^N)$ as $\varepsilon_n \to 0$. The Rellich–Kondrachov theorem implies that a subsequence of $(u_{\varepsilon_n})$ tends strongly to $u$ in $W^{1,p}_{loc}(\mathbb{R}^N)$ and therefore we may assume that $u_{\varepsilon_n}(x) \to u(x)$ almost everywhere in $\mathbb{R}^N$. This shows that $\|Uu\|_p \le C\|f\|_p$, and hence $u \in W^{2,p}_U(\mathbb{R}^N)$. Passing to the limit in $L^p_{\text{loc}}(\mathbb{R}^N)$ in the equality $u_{\varepsilon_n} - A_0 u_{\varepsilon_n} + V_{\varepsilon_n} u_{\varepsilon_n} = f$, we derive

$$u - A_0 u + Vu = f.$$

This concludes the proof. □

**Remark 3.5.** Let $c'$ be a positive constant. By considering $V + c'$ instead of $V$, Theorem 3.4 shows that $(A, \mathcal{D}_p)$ generates an analytic semigroup on $L^p(\mathbb{R}^N)$, $1 < p < \infty$, if we replace (H3) by $U \le V + c' \le c_1' U$.

**Remark 3.6.** As already pointed out in the introduction, condition (H4) is crucial to treat the drift term $F \cdot \nabla$ as a perturbation of the Laplacian and of the multiplication operator $u \mapsto Vu$. This is done through Proposition 2.3. We remark, however, that the operator $F \cdot \nabla$ is not a small perturbation of the Schrödinger operator $\Delta - V$. If this were true, the generated semigroup $T(\cdot)$ would be analytic of angle $\pi/2$ and the same would hold for the Ornstein-Uhlenbeck semigroups in $L^p(\mathbb{R}^N, \mu)$, treated in Section 7 (see the proof of Theorem 7.4). However, this is false even for $p = 2$, see [24].

It is clear that (H4) is, in general, necessary for the domain characterization given in Theorem 3.4. In the following example we show that the closedness of $(A, W^{2,2}_U)$ may fail even if $F = 0$ and $U$ fulfills (H2) with a (too) large constant $\gamma$.

**Example 3.7.** We consider the Schrödinger operator $B = \Delta - V$ with $V(x) = \frac{3}{4}|x|^{-2}$ in $L^2(\mathbb{R}^3)$. Since $0 \le V \in L^1_{\text{loc}}(\mathbb{R}^3)$, $B$ can be defined by the quadratic from

$$b(u) = \int_{\mathbb{R}^3} \left(|\nabla u|^2 + V|u|^2\right) dx$$

for $u \in D(b) = \{u \in W^{1,2}(\mathbb{R}^3) : V|u|^2 \in L^1(\mathbb{R}^3)\}$. Observe that the function $w(x) = |x|^{1/2}$ satisfies $\Delta w - Vw = 0$ on $\mathbb{R}^3 \setminus \{0\}$. Let $u = \eta w$, where $\eta \in C_0^\infty(\mathbb{R}^3)$ is equal to 1 near the origin. It is easy to see that $u \in D(b) \cap W^{2,1}(\mathbb{R}^3)$ and that $\Delta u - Vu = f \in C_0^\infty(\mathbb{R}^3)$. It follows that $u \in D(B)$ and $Bu = f$. Observe, however, that neither $\Delta u$ nor $Vu$ belong to $L^2(\mathbb{R}^3)$.

Since $|\nabla V(x)| = \gamma V(x)^{3/2}$ with $\gamma = 4/\sqrt{3}$, it is not difficult to construct bounded positive potentials $V_n \in C^1(\mathbb{R}^N)$ such that $0 \le V_n \le V_{n+1}$, $V_n(x) \to V(x)$ as $n \to \infty$, and $|\nabla V_n(x)| \le \gamma V_n(x)^{3/2}$ (it is sufficient to regularize $r^{-2}$ for $r \le 1/n$). Since $V_n$ converges monotonically to $V$, we have $(1 - \Delta +$

$V_n)^{-1}g \to (1-\Delta+V)^{-1}g$ for every $g \in L^2(\mathbb{R}^3)$, see [32, Theorem S. 14]. Taking $g = u - f$ and possibly considering a subsequence $(n_k)$, we obtain from Fatou's lemma

$$\infty = \|Vu\|_2 \le \liminf_{k\to\infty} \|V_{n_k}(1-\Delta+V_{n_k})^{-1}(u-f)\|_2$$

and therefore $2c_k := \|V_{n_k}(1-\Delta+V_{n_k})^{-1}\| \to \infty$ as $k \to \infty$. Since $C_0^\infty(\mathbb{R}^3)$ is a core for $\Delta - V_{n_k}$, we find $u_k \in C_0^\infty(B(0,R_k))$, $R_k \ge 1$, such that

$$\|V_{n_k}u_k\|_2 \ge c_k\|u_k + \Delta u_k - V_{n_k}u_k\|_2. \tag{3.10}$$

Take now points $x_k \in \mathbb{R}^3$ and define $W_k(x) = V_{n_k}(x-x_k)$, $v_k(x) = u_k(x-x_k)$, $\eta_k(x) = \eta((x-x_k)R_k^{-1})$, where $\eta$ is a cutoff function such that $\eta(x) = 1$ for $|x| \le 1$ and $\eta(x) = 0$ for $|x| \ge 2$. One can choose the points $x_k$ with large distances $|x_k - x_h|$ in such a way that the balls $B(x_k, 2R_k)$ do not overlap. Hence, the positive potential $W = \sum_k \eta_k W_k \in C^1(\mathbb{R}^N)$ satisfies $|\nabla W| \le \gamma W^{3/2} + c'$. On the other hand, (3.10) yields

$$\|Wv_k\|_2 \ge c_k\|v_k + \Delta v_k - Wv_k\|_2$$

for every $k$. The operator $W(1-\Delta+W)^{-1}$ is thus unbounded in $L^2(\mathbb{R}^3)$, and $W^{2,2}(\mathbb{R}^3) \cap D(W)$ is not the domain of $\Delta - W$.

We remark that the constant $c = 3/4$ in the definition of $V$ (hence $\gamma = 4/\sqrt{3}$) is the best possible constant for such a counterexample. In fact $\Delta - c|x|^{-2}$ is closed on $W^{2,2}(\mathbb{R}^3) \cap D(|x|^{-2})$ for every $c > 3/4$, see [29, Theorem 3.6] where a very detailed analysis of the potential $|x|^{-2}$ is carried out in $L^p(\mathbb{R}^N)$. However, the hypotheses of Theorem 3.4 hold for $\gamma < 2$ for Schrödinger operators in $L^2(\mathbb{R}^N)$.

## 4. Generation of an analytic semigroup on $C_0(\mathbb{R}^N)$

In this section we use the Stewart–Masuda localization technique to prove that the operator $A$ with the domain

$$\mathcal{D}_\infty = \{v \in C_0(\mathbb{R}^N) : v \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^N) \text{ for all } p < \infty,\ \Delta v \in C_0(\mathbb{R}^N),\ Uv \in C_0(\mathbb{R}^N)\}$$

generates an analytic semigroup in $C_0(\mathbb{R}^N)$. For simplicity, we assume that the principal part of $A$ is the Laplacian and that (H2) holds for all $\gamma > 0$. We first show in two steps that $A$ generates a contraction semigroup in $C_0(\mathbb{R}^N)$.

**Proposition 4.1.** *Assume that (H2) holds for all $\gamma > 0$ and that (H4) is satisfied, where $U = V$. Then $A = \Delta + F\cdot\nabla - V$ with $D(A) = \mathcal{D}_\infty$ is closed in $C_0(\mathbb{R}^N)$. Moreover,*

$$\|u\|_\infty + \|\Delta u\|_\infty + \|Vu\|_\infty \le C\,\|u - Au\|_\infty \le C'\left(\|u\|_\infty + \|\Delta u\|_\infty + \|Vu\|_\infty\right)$$

*for every $u \in \mathcal{D}_\infty$, where $C$ and $C'$ only depend on $\gamma, C_\gamma, c_0, \kappa$.*

**Proof.** Proposition 2.3 shows the second estimate in the assertion. Considering $V+\lambda$ instead of $V$ for a sufficiently large $\lambda = \lambda(\gamma) \geq 0$, and using the dissipativity of $A$, we may assume that $|\nabla V| \leq \gamma V^{\frac{3}{2}}$, as in the proof of Proposition 3.2. The parameter $\gamma$ will be fixed below.

Let $u \in C_0^\infty(\mathbb{R}^N)$, $f = Au$, and fix $x_0 \in \mathbb{R}^N$. We take a smooth cutoff function $\eta$ such that $0 \leq \eta \leq 1$, $\eta = 1$ in $B(x_0, r/2)$, supp $\eta \subset B(x_0, r)$, $|\nabla \eta(x)| \leq c/r$, and $|D^2\eta(x)| \leq c/r^2$ for some constant $c > 0$. Observe that

$$\Delta(\eta u) + F\cdot\nabla(\eta u) - V(x_0)\eta u = \eta f + u\Delta\eta + 2\nabla u\cdot\nabla\eta + uF\cdot\nabla\eta + (V - V(x_0))\eta u.$$

Since $V(x_0) > 0$, the dissipativity of $\Delta + F\cdot\nabla$ on $C_0^\infty(\mathbb{R}^N)$ yields

$$\begin{aligned}\|V(x_0)\eta u\|_{\infty,r} \leq \|f\|_{\infty,r} + \frac{c}{r^2}\|u\|_{\infty,r} + \frac{c}{r}\|\nabla u\|_{\infty,r} + \frac{c\kappa}{r}\|V^{\frac{1}{2}}u\|_{\infty,r}\\ + \|(V - V(x_0))u\|_{\infty,r}\,,\end{aligned}$$

where we have also used (H4). Let $\gamma \leq 1$. We choose now $r = (3\gamma)^{-1}V(x_0)^{-\frac{1}{2}}$ so that (2.1) yields $\frac{5}{6}V(x)^{\frac{1}{2}} \leq V(x_0)^{\frac{1}{2}} \leq \frac{7}{6}V(x)^{\frac{1}{2}}$ for $|x - x_0| \leq r$. Hence, $|V(x) - V(x_0)| \leq \frac{13}{36}V(x)$ for $|x - x_0| \leq r$ and

$$\frac{25}{36}\|Vu\|_{\infty,\frac{r}{2}} \leq \|f\|_\infty + \gamma C_1(1+\kappa)\|Vu\|_{\infty,r} + \gamma C_1\|V^{\frac{1}{2}}\nabla u\|_{\infty,r} + \frac{13}{36}\|Vu\|_{\infty,r}\,,$$

where $C_1 := 13c$. Letting $x_0$ vary in $\mathbb{R}^N$ and then taking $\gamma \leq \min\{1, (6(1+\kappa)C_1)^{-1}\}$, we obtain

$$\|Vu\|_\infty \leq 6\,\|f\|_\infty + 6\gamma C_1\|V^{\frac{1}{2}}\nabla u\|_\infty.$$

Now the equation $Au = f$ and (H4) imply

$$\|\Delta u\|_\infty \leq C_2\,(\|f\|_\infty + \|V^{\frac{1}{2}}\nabla u\|_\infty)$$

for $C_2 := \max\{7, \kappa + 6C_1\}$. At this point we use Proposition 2.3 for $0 < \varepsilon \leq \varepsilon_0$ and estimate

$$\|V^{\frac{1}{2}}\nabla u\|_\infty \leq \varepsilon\|\Delta u\|_\infty + \frac{\alpha}{\varepsilon}\|Vu\|_\infty \leq c_\varepsilon\|f\|_\infty + \left(\varepsilon C_2 + \frac{6\alpha C_1\gamma}{\varepsilon}\right)\|V^{\frac{1}{2}}\nabla u\|_\infty\,.$$

Setting $\varepsilon := \gamma^{\frac{1}{2}}$ and choosing $\gamma$ small enough, we arrive at $\|V^{\frac{1}{2}}\nabla u\|_\infty \leq C\|f\|_\infty$ and, by the previous inequalities, $\|\Delta u\|_\infty, \|Vu\|_\infty \leq C\|f\|_\infty$, with $C$ independent of $f$. These facts show that

$$\|u\|_{\mathcal{D}_\infty} \leq C\|Au\|_\infty \leq 2C\|u - Au\|_\infty,$$

for $u \in C_0^\infty(\mathbb{R}^N)$, using the dissipativity of $A$. Since $C_0^\infty(\mathbb{R}^N)$ is dense in $\mathcal{D}_\infty$, the proof is complete. $\square$

**Proposition 4.2.** *Assume that (H3) and (H4) hold, that $V$ is continuous, and that (H2) is satisfied for all $\gamma > 0$. Then $A = \Delta + F \cdot \nabla - V$ with $D(A) = \mathcal{D}_\infty$ generates a contraction semigroup $T_\infty(\cdot)$ on $C_0(\mathbb{R}^N)$.*

**Proof.** First we assume that $U = V$ and we show that the operator $I - (\Delta - V) : \mathcal{D}_\infty \to C_0(\mathbb{R}^N)$ has dense range. Therefore $\Delta - V$ generates a contraction semigroup by Lemma 2.6, Proposition 4.1, and the Lumer–Phillips theorem. The result for $F \neq 0$ is then deduced applying the continuity method to the operators $\Delta + tF \cdot \nabla - V$, cf. [18, Theorem 5.2], using the fact that the apriori estimate from Proposition 4.1 is independent of $t \in [0, 1]$.

Let $p > N$ and $\gamma$ be so small that the assumptions of Theorem 3.4 and Proposition 4.1 are satisfied. We may assume that $C_\gamma = 0$ replacing $V$ by $V + \lambda$. We define $V_\varepsilon = \frac{V}{1+\varepsilon V} \in C_b(\mathbb{R}^N)$ for $\varepsilon \in (0, 1]$ and fix $f \in C_0^\infty(\mathbb{R}^N)$. Note that $\Delta - V_\varepsilon$ with $D(\Delta - V_\varepsilon) = D(\Delta)$ generates a contraction semigroup on $L^p(\mathbb{R}^N)$ and $C_0(\mathbb{R}^N)$. Thus there exists $u_\varepsilon \in W^{2,p}(\mathbb{R}^N) \hookrightarrow C_0(\mathbb{R}^N)$ such that

$$u_\varepsilon - \Delta u_\varepsilon + V_\varepsilon u_\varepsilon = f \tag{4.1}$$

and $\|u_\varepsilon\|_r \leq \|f\|_r$ for $r = p, \infty$. Moreover, $\Delta u_\varepsilon \in C_0(\mathbb{R}^N)$. Since $V_\varepsilon$ fulfills (H2) and (H4) with uniform constants, Propositions 3.2 and 4.1 yield

$$\|\Delta u_\varepsilon\|_r\,,\ \|V_\varepsilon u_\varepsilon\|_r \leq C_1\,\|f\|_r$$

for $r = p, \infty$. (Here and below the constants do not depend on $\varepsilon$.) Lemma 2.1 then gives $\|\nabla u_\varepsilon\|_\infty \leq C_2 \|f\|_\infty$. For a suitable sequence $(\varepsilon_n)$, $u_{\varepsilon_n}$ thus converges uniformly on compact sets to a continuous function $u$. Combined with Lemma 2.1, this fact yields that $u \in C^1(\mathbb{R}^N)$. Due to (4.1), also $\Delta u_{\varepsilon_n}$ converges uniformly on compact sets. Local elliptic regularity (see e.g. Theorem 8.8 and Lemma 9.16 of [18]) now implies that $u \in W^{2,q}_{\mathrm{loc}}(\mathbb{R}^N)$ for every $q < \infty$. Moreover,

$$u - \Delta u + Vu = f$$

and $\|Vu\|_r, \|\Delta u\|_r \leq C_1 \|f\|_r$ for $r = p, \infty$. Therefore $u \in W^{2,p}(\mathbb{R}^N)$, and hence $u, \nabla u \in C_0(\mathbb{R}^N)$ by Sobolev's embedding theorem. We next show that $Vu$ belongs to $C_0(\mathbb{R}^N)$. Take $\eta \in C^\infty(\mathbb{R}^N)$ such that $0 \leq \eta \leq 1$, $\eta = 1$ in $B(x_0, R)$, $\eta = 0$ outside $B(x_0, 2R)$, $|\nabla \eta| \leq c/R$, and $|D^2 \eta| \leq c/R^2$. Then

$$\eta u - \Delta(\eta u) + V\eta u = \eta f - 2\nabla\eta \cdot \nabla u - u\Delta\eta$$

and Proposition 4.1 (applied on $\eta u \in \mathcal{D}_\infty$) shows that

$$|V(x_0)u(x_0)| \leq \|V\eta u\|_\infty \leq C_3\Big[\|\eta f\|_\infty + \frac{1}{R}\|\nabla u\|_\infty + \frac{1}{R^2}\|u\|_\infty\Big].$$

Since $u, f \in C_0(\mathbb{R}^N)$, the above inequality implies that $Vu \in C_0(\mathbb{R}^N)$. Thus $\Delta u = u + Vu - f \in C_0(\mathbb{R}^N)$ and $u \in \mathcal{D}_\infty$. This proves that $I - A$ has dense range and concludes the proof in the case where $U = V$.

Finally, we deal with the general case $U \le V \le c_1 U$. Let $V_t = U + t(V - U)$, $A_t = \Delta + F \cdot \nabla - V_t$, $D(A_t) = \mathcal{D}_\infty$, $0 \le t \le 1$, and observe that $A_t = A_0 + (V_t - U)$. Since $\|U(1 - A_0)^{-1}\| \le C$ by Proposition 4.1, we have $\|(V_t - U)(1 - A_0)^{-1}\| \le t(c_1 - 1)C$. Thus standard perturbation theory for contraction semigroups shows that $A_t$ generates a contraction semigroup $C_0(\mathbb{R}^N)$ if $t(c_1 - 1)C < 1$.

Due to $V_t \ge U$, the maximum principle [22, Proposition 3.1.10] yields $0 \le (1 - A_t)^{-1} \le (1 - A_0)^{-1}$ whenever $(1 - A_t)^{-1}$ exists. Then

$$0 \le (V_s - V_t)(1 - A_t)^{-1} \le (c_1 - 1)(s - t)U(1 - A_0)^{-1}$$

if $s \ge t$, and hence $\|(V_s - V_t)(1 - A_t)^{-1}\| \le (s - t)(c_1 - 1)C$. The same argument as before applies and proves that $A_t$ generates a contraction semigroup in $C_0(\mathbb{R}^N)$ for $t(c_1 - 1)C < 2$. Iterating this procedure a finite number of times, the proof is complete. $\square$

In the following theorem we show that the semigroup is analytic in $C_0(\mathbb{R}^N)$ if one more condition holds. In the proof we need the next lemma, which is of interest in itself.

**Lemma 4.3.** *Let $1 < p < \infty$, $1 < q \le \infty$, $A_p = (A, \mathcal{D}_p)$, $A_q = (A, \mathcal{D}_q)$ and assume that the hypotheses of Theorem 3.4 hold for $A_p$ if $1 < p < \infty$ and that those of Theorem 3.4, respectively Proposition 4.2, hold for $A_q$ if $q < \infty$, respectively $q = \infty$. If $q < \infty$ and $f \in L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ or if $q = \infty$ and $f \in L^p(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)$, then $(\lambda - A_p)^{-1} f = (\lambda - A_q)^{-1} f$ for* $\operatorname{Re} \lambda > 0$, *and hence $T_p(t) = T_q(t)$.*

**Proof.** First let $q < \infty$. Then the proof of Theorem 3.4 applies and shows that the operator $\lambda - A$, $\operatorname{Re} \lambda > 0$, with domain $W_U^{2,p}(\mathbb{R}^N) \cap W_U^{2,q}(\mathbb{R}^N)$ is invertible in $L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$. Hence the equation $\lambda u - Au = f$ with $f \in L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ has a unique solution $u \in W_U^{2,p}(\mathbb{R}^N) \cap W_U^{2,q}(\mathbb{R}^N)$ and the assertions easily follow. If $q = \infty$, the proof is similar using Theorem 3.4 and Proposition 4.2. $\square$

**Theorem 4.4.** *Assume that (H3) and (H4) hold, that $V$ is continuous, and that (H2) is satisfied for all $\gamma > 0$. If $\theta U + \operatorname{div} F \ge 0$ for some $\theta \ge 0$ then $A = \Delta + F \cdot \nabla - V$ with $D(A) = \mathcal{D}_\infty$ generates a bounded analytic $C_0$–semigroup $T_\infty(\cdot)$ on $C_0(\mathbb{R}^N)$.*

**Proof.** Again we assume that $C_\gamma = 0$, replacing $A$ by $A - w$ for some $w = w(\gamma) \ge 0$, where we fix $\gamma > 0$ such that the hypotheses of Theorem 3.4

and Proposition 4.2 hold. We fix $p \in (N, \infty)$ with $p > \theta$ such that Theorem 3.4 is valid. Thus $(A, W_U^{2,p}(\mathbb{R}^N))$ generates an analytic semigroup in $L^p(\mathbb{R}^N)$. Let $f \in C(\mathbb{R}^N)$ have compact support and $\operatorname{Re}\lambda > 0$. Due to Proposition 4.2 there exists $u \in \mathcal{D}_\infty$ such that $\lambda u - Au = f$. The above lemma shows that $u \in W_U^{2,p}(\mathbb{R}^N)$ and hence $|\lambda|\,\|u\|_p + \|Au\|_p \le C\|f\|_p$ by Theorem 3.4. Proposition 3.2 further yields $\|D^2u\|_p + \|Uu\|_p \le C\|f\|_p$. Using these estimates as well as Hölder's and the Gagliardo–Nirenberg inequalities, we conclude that $|\lambda|^{\frac12}\|U^{\frac12}u\|_p \le C\|f\|_p$ and $|\lambda|^{\frac12}\|\nabla u\|_p \le C\|f\|_p$. Taking into account Proposition 2.3, we have shown that

$$\|D^2u\|_p + \|Uu\|_p + \|U^{\frac12}\nabla u\|_p + |\lambda|^{\frac12}\|\nabla u\|_p + |\lambda|^{\frac12}\|U^{\frac12}u\|_p + |\lambda|\,\|u\|_p \le C\|f\|_p. \tag{4.2}$$

(Here and below $C$ is a generic constant independent of $\lambda$ and $f$.) Fix $x_0 \in \mathbb{R}^N$ and a smooth function $\eta$ such that $0 \le \eta \le 1$, $\eta = 1$ in $B(x_0, r/2)$, $\operatorname{supp}\eta \subset B(x_0, r)$, $|\nabla\eta(x)| \le c/r$, and $|D^2\eta(x)| \le c/r^2$. Observe that $\lambda(\eta u) - A(\eta u) = \eta f - u\Delta\eta - 2\nabla u\cdot\nabla\eta - uF\cdot\nabla\eta$. Applying (4.2) to $\eta u$ and employing (H4), we obtain

$$\begin{aligned}
&\|D^2u\|_{p,\frac r2} + \|Uu\|_{p,\frac r2} + \|U^{\frac12}\nabla u\|_{p,\frac r2} + |\lambda|^{\frac12}\|\nabla u\|_{p,\frac r2} + |\lambda|^{\frac12}\|U^{\frac12}u\|_{p,\frac r2} + |\lambda|\|u\|_{p,\frac r2}\\
&\le C\Big(\|f\|_{p,r} + \frac{1}{r^2}\|u\|_{p,r} + \frac1r\|\nabla u\|_{p,r} + \frac1r\|U^{\frac12}u\|_{p,r}\Big)\\
&\le Cr^{N/p}\Big(\|f\|_\infty + \frac{1}{r^2}\|u\|_{\infty,r} + \frac1r\|\nabla u\|_{\infty,r} + \frac1r\|U^{\frac12}u\|_{\infty,r}\Big).
\end{aligned}$$

From the compactness of the embedding $W^{1,p}(B_1) \hookrightarrow C(\bar B_1)$ and a rescaling argument it follows that for every $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that

$$\begin{aligned}
\|v\|_{\infty,\frac r2} &\le C_1 r^{-N/p}\Big(\varepsilon r\|\nabla v\|_{p,\frac r2} + C_\varepsilon\|v\|_{p,\frac r2}\Big)\\
\|\nabla v\|_{\infty,\frac r2} &\le C_1 r^{-N/p}\Big(\varepsilon r\|D^2 v\|_{p,\frac r2} + C_\varepsilon\|\nabla v\|_{p,\frac r2}\Big)
\end{aligned}$$

for $v$ belonging to the respective spaces, where $C_1$ does not depend on $\varepsilon, r$. Hypothesis (H2) further yields

$$\|\nabla(U^{\frac12}u)\|_{p,\frac r2} \le \|U^{\frac12}\nabla u\|_{p,\frac r2} + C_2\|Uu\|_{p,\frac r2}$$

so that

$$\|U^{\frac12}u\|_{\infty,\frac r2} \le C_1 r^{-N/p}\Big(\varepsilon r\|U^{\frac12}\nabla u\|_{p,\frac r2} + C_2\varepsilon r\|Uu\|_{p,\frac r2} + C_\varepsilon\|U^{\frac12}u\|_{p,\frac r2}\Big).$$

Using the above inequalities, one deduces

$$|\lambda|^{\frac12}\|\nabla u\|_{\infty,\frac r2} + |\lambda|^{\frac12}\|U^{\frac12}u\|_{\infty,\frac r2} + |\lambda|\,\|u\|_{\infty,\frac r2}$$

$$\leq (C_3\varepsilon r|\lambda|^{\frac{1}{2}} + C'_\varepsilon)\Big(\|f\|_\infty + \frac{1}{r^2}\|u\|_\infty + \frac{1}{r}\|\nabla u\|_\infty + \frac{1}{r}\|U^{\frac{1}{2}}u\|_\infty\Big).$$

We set $r = M|\lambda|^{-\frac{1}{2}}$ and let $x_0$ vary in $\mathbb{R}^N$. Then we have

$$\begin{aligned}&|\lambda|^{\frac{1}{2}}\|\nabla u\|_\infty + |\lambda|^{\frac{1}{2}}\|U^{\frac{1}{2}}u\|_\infty + |\lambda|\,\|u\|_\infty\\ &\leq (C_3\varepsilon M + C'_\varepsilon)\Big(\|f\|_\infty + \frac{|\lambda|}{M^2}\|u\|_\infty + \frac{|\lambda|^{\frac{1}{2}}}{M}\|\nabla u\|_\infty + \frac{|\lambda|^{\frac{1}{2}}}{M}\|U^{\frac{1}{2}}u\|_\infty\Big).\end{aligned}$$

Taking first a small $\varepsilon$ and then a large $M$, we arrive at

$$|\lambda|^{\frac{1}{2}}\|\nabla u\|_\infty + |\lambda|^{\frac{1}{2}}\|U^{\frac{1}{2}}u\|_\infty + |\lambda|\,\|u\|_\infty \leq C'\|f\|_\infty. \tag{4.3}$$

In view of the rescaling made at the beginning, this establishes the theorem for $A-w$ by density, that is, with $f = \lambda u + \omega u - Au$ in (4.3). Since $V \geq c_0 > 0$, the operator $A + c_0/2$ satisfies the assumptions of Proposition 4.2 and thus generates a contraction semigroup on $C_0(\mathbb{R}^N)$. It is then easy to verify (4.3) also for $A$, that is, with $f = \lambda u - Au$. $\square$

We state two useful consequence of the above proof refining our previous apriori estimates.

**Corollary 4.5.** *Under the hypotheses of Theorem* 4.4, *if* $\operatorname{Re}\lambda > 0$ *and* $\lambda u - Au = f$, *then*

$$\|\Delta u\|_\infty + \|U^{\frac{1}{2}}\nabla u\|_\infty + \|Uu\|_\infty + |\lambda|^{\frac{1}{2}}\|\nabla u\|_\infty + |\lambda|^{\frac{1}{2}}\|U^{\frac{1}{2}}u\|_\infty + |\lambda|\|u\|_\infty \leq C\|f\|_\infty.$$

**Proof.** Since $V \geq c_0 > 0$, the operator $A$ is invertible on $C_0(\mathbb{R}^N)$ and the closedness of $(A, \mathcal{D}_\infty)$ yields

$$\|\Delta u\|_\infty + \|Uu\|_\infty \leq C\|Au\|_\infty \leq C\big(\|\lambda u - Au\|_\infty + |\lambda|\,\|u\|_\infty\big).$$

It now suffices to combine (4.3) for $A$ with Proposition 2.3 to conclude the proof. $\square$

**Corollary 4.6.** *Under the hypotheses of Theorem* 3.4, *if* $\operatorname{Re}\lambda > 0$ *and* $\lambda u - Au = f$, *then*

$$\|\Delta u\|_p + \|U^{\frac{1}{2}}\nabla u\|_p + \|Uu\|_p + |\lambda|^{\frac{1}{2}}\|\nabla u\|_p + |\lambda|^{\frac{1}{2}}\|U^{\frac{1}{2}}u\|_p + |\lambda|\,\|u\|_p \leq C\|f\|_p.$$

**Proof.** In (4.2) we have proved the statement for $A - w$, continuous $f$ with compact support and $p > N$. The proof given in Theorem 4.4 extends to all $p > 1$ and all $f \in L^p(\mathbb{R}^N)$. Moreover, as above the initial rescaling does not matter since $A + \frac{(p-\theta)c_0}{2p}$ is dissipative. $\square$

## 5. Generation of an analytic semigroup on $L^1(\mathbb{R}^N)$

Also in this section we assume that $A_0 = \Delta$. We observe that Proposition 2.3 shows that $A = \Delta + F \cdot \nabla - V$ with domain

$$\mathcal{D}_1 = \{v \in L^1(\mathbb{R}^N) : \Delta v \in L^1(\mathbb{R}^N),\, Uv \in L^1(\mathbb{R}^N)\}$$

is well defined in $L^1(\mathbb{R}^N)$. Recall that $C_0^\infty(\mathbb{R}^N)$ is dense in $\mathcal{D}_1$ by Lemma 2.5. It is known that $\mathcal{D}_1$ embeds continuously into $W^{1,p}(\mathbb{R}^N)$ for every $p < \frac{N}{N-1}$, see e.g. [34, Theorem 5.8].

**Proposition 5.1.** *Assume that (H2)–(H5) hold for some $\gamma > 0$ and $\theta < 1$. Then $(A, \mathcal{D}_1)$ generates a contraction semigroup $T_1(t)$ in $L^1(\mathbb{R}^N)$. If also the assumptions of Theorem* 3.4 *for some $p \in (1, \infty)$, respectively of Proposition* 4.2 *if $p = \infty$, are satisfied, then $T_1(t)f = T_p(t)f$ for $f \in L^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$, respectively $f \in L^1(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)$.*

**Proof.** Let $u \in C_0^\infty(\mathbb{R}^N)$ be real and set $f = -\Delta u - F \cdot \nabla u + Vu$. If we multiply this equation by $\operatorname{sign} u$, integrate by parts, and use the dissipativity of the Laplacian on $L^1(\mathbb{R}^N)$, we estimate

$$\begin{aligned}\int_{\mathbb{R}^N} (V + \operatorname{div} F)|u|\, dx &\le -\int_{\mathbb{R}^N} (\Delta u) \operatorname{sign} u\, dx + \int_{\mathbb{R}^N} (Vu - F \cdot \nabla u) \operatorname{sign} u\, dx \\ &= \int_{\mathbb{R}^N} f \operatorname{sign} u\, dx \le \|f\|_1\end{aligned}$$

and therefore $\|(1-\theta)Uu\|_1 \le \|f\|_1$. Taking into account Proposition 2.3 and proceeding as in the proof of Proposition 3.2, we obtain the closedness of $(A, \mathcal{D}_1)$. Moreover, $(A, \mathcal{D}_1)$ is dissipative by Lemmas 2.5 and 2.6.

As before we may assume that $C_\gamma = 0$ to show the surjectivity of $I - A$. We first assume that $F = 0$. The general case is then deduced by means of the continuity method, cf. [18, Theorem 5.2]. We keep the notation introduced in the proof of Theorem 3.4. For $f \in L^1(\mathbb{R}^N)$ there is $u_\varepsilon \in L^1(\mathbb{R}^N)$ such that $u_\varepsilon - \Delta u_\varepsilon + V_\varepsilon u_\varepsilon = f$. Then

$$\|u_\varepsilon\|_1 \le \|f\|_1, \quad \|U_\varepsilon u_\varepsilon\|_1 \le C\|f\|_1, \tag{5.1}$$

with $C$ independent of $\varepsilon$ since $V_\varepsilon$ and $U_\varepsilon$ satisfy the assumptions with uniform constants. It follows that $\|\Delta u_\varepsilon\|_1 \le (2+C)\|f\|_1$. We fix $1 < p < \frac{N}{N-1}$. Then $\|u_\varepsilon\|_{W^{1,p}(\mathbb{R}^N)} \le C_1\|f\|_1$ and so there is a sequence $(\varepsilon_n)$ converging to 0 such that $u_n := u_{\varepsilon_n}$ tends weakly in $W^{1,p}(\mathbb{R}^N)$ to a function $u$. By the Rellich-Kondrachov theorem, a subsequence $(u_{n_k})$ converges to $u$ in $L^1_{loc}(\mathbb{R}^N)$ and after relabeling we have $u_n \to u$ almost everywhere. We infer from (5.1) that $\|u\|_1 \le \|f\|_1$ and $\|Uu\|_1 \le C\|f\|_1$. Moreover, $\Delta u_n \to \Delta u$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$

and therefore $u-\Delta u+Vu=f$ and $u\in\mathcal{D}_1$. The last assertion can be shown as in Lemma 4.3. $\square$

In order to establish the analyticity of $T(\cdot)$ on $L^1(\mathbb{R}^N)$ we employ a duality argument taken from [31, Theorem 7.3.10], see also [1]. Under the assumptions of the previous proposition, we define the formal adjoint of $A$ by

$$A'v = \Delta v - F\cdot\nabla v - (V+\mathrm{div}F)v.$$

This operator has the same structure as $A$ with the new drift $\tilde{F}=-F$ and the new potential $\tilde{V}=V+\mathrm{div}F$. Set $\tilde{U}=(1-\theta)U$. Hypotheses (H2)–(H5) yield $\tilde{V}\geq\tilde{U}$ and that $\tilde{F}$ and $\tilde{U}$ satisfy (H2) and (H4). In addition, we suppose that $\mathrm{div}F\leq c_2U$ for some constant $c_2\geq 0$ and $V$ is continuous. Then (H3) holds for $\tilde{U}$ and $\tilde{V}$ and

$$\frac{c_1+c_2}{1-\theta}\,\tilde{U}+\mathrm{div}\tilde{F}\geq\tilde{V}+\mathrm{div}\tilde{F}=V\geq 0.$$

We can therefore apply Theorem 4.4 and thus $A'$ induces a generator $A'_\infty$ of a bounded analytic semigroup on $C_0(\mathbb{R}^N)$. Observe further that

$$\int_{\mathbb{R}^N}(Au)v\,dx=\int_{\mathbb{R}^N}u(A'v)\,dx$$

for $u\in C_0^\infty(\mathbb{R}^N)$ and $v\in W^{2,1}_{loc}(\mathbb{R}^N)$. Due to [31, Lemma 7.3.8] we have

$$\|u\|_1=\sup\Big\{\int_{\mathbb{R}^N}u\varphi\,dx:\ \varphi\in C_0^\infty(\mathbb{R}^N),\ \|\varphi\|_\infty\leq 1\Big\} \tag{5.2}$$

for $u\in L^1(\mathbb{R}^N)$.

**Theorem 5.2.** *Assume that (H2)–(H5) hold for all $\gamma>0$ and some $\theta<1$, that $V$ is continuous, and that $\mathrm{div}F\leq c_2U$ for some constant $c_2\geq 0$. Then $(A,\mathcal{D}_1)$ generates a bounded analytic semigroup on $L^1(\mathbb{R}^N)$.*

**Proof.** Let $u\in C_0^\infty(\mathbb{R}^N)$ and $\mathrm{Re}\,\lambda>0$. As observed above, the formal adjoint $A'$ induces a generator $A'_\infty$ of a bounded analytic semigroup on $C_0(\mathbb{R}^N)$ by Theorem 4.4. In (5.2) we replace $\varphi$ by $\varphi=(\lambda-A')v$ with $v\in D(A'_\infty)$ and $\|\varphi\|_\infty\leq 1$. Since $\|v\|_\infty\leq M/|\lambda|$, we obtain

$$\begin{aligned}\|u\|_1&\leq\sup\Big\{\int_{\mathbb{R}^N}u(\lambda-A')v\,dx: v\in D(A'_\infty),\ \|v\|_\infty\leq\frac{M}{|\lambda|}\Big\}\\&=\sup\Big\{\int_{\mathbb{R}^N}v(\lambda-A)u\,dx: v\in D(A'_\infty),\ \|v\|_\infty\leq\frac{M}{|\lambda|}\Big\}\\&\leq\frac{M}{|\lambda|}\,\|(\lambda-A)u\|_1\,.\end{aligned}$$

This inequality implies the assertion because of Proposition 5.1 and an approximation argument. □

## 6. Consequences

In this section we establish several regularity properties of the semigroups obtained above. In order to treat the scale of $L^p$ spaces simultaneously, and for the sake of simplicity, we assume here that

$$Au = \Delta u + F \cdot \nabla u - Vu,$$

that (H2)–(H5) are satisfied for every $\gamma > 0$ and some $\theta < 1$, and that $V$ is continuous. When we need the analyticity in $L^1(\mathbb{R}^N)$, we also have to suppose that $\mathrm{div}F \le c_2 U$ for some $c_2 \ge 0$. Theorem 3.4 and Proposition 5.1 show that $A$ with domain $D(A_p) = \mathcal{D}_p$ generates a contraction semigroup $(T_p(t))_{t\ge 0}$ on $L^p(\mathbb{R}^N)$ for $1 \le p < \infty$ and on $C_0(\mathbb{R}^N)$ for $p = \infty$. Moreover, $T_p(t)f = T_q(t)f$ if $f \in L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$, by Lemma 4.3 and Proposition 5.1. Therefore we drop the index $p$ and write simply $T(t)$.

Observe that the analyticity estimate $\|AT(t)f\|_p \le Ct^{-1}\|f\|_p$ for $t > 0$ and the domain characterization of the previous sections yield $\|\Delta T(t)f\|_p \le Ct^{-1}\|f\|_p$. Hence the gradient estimate $\|\nabla T(t)f\|_p \le Ct^{-1/2}\|f\|_p$ holds, which is well known for uniformly elliptic operators, and is also satisfied by certain operators with unbounded coefficients, see e.g. [23].

We next associate a sesquilinear form $a$ in $L^2(\mathbb{R}^N)$ with the operator $(A, \mathcal{D}_2)$ setting

$$a(u, v) = \int_{\mathbb{R}^N} (\nabla u \cdot \nabla \overline{v} + V u \overline{v} - \overline{v} F \cdot \nabla u)\, dx \tag{6.1}$$

for $u, v \in D(a) = \{f \in W^{1,2}(\mathbb{R}^N) : U^{\frac{1}{2}} f \in L^2(\mathbb{R}^N)\}$. The form is well defined thanks to (H4). Moreover, as in Lemma 2.5 it can be verified that $C_0^\infty(\mathbb{R}^N)$ is dense in $D(a)$. Integrating by parts and using (H5), with $p = 2$, we deduce

$$\mathrm{Re}\, a(u, u) = \int_{\mathbb{R}^N} \left(|\nabla u|^2 + V|u|^2 + \tfrac{1}{2}\,\mathrm{div}F\, |u|^2\right) dx \ge \|\nabla u\|_2^2 + (1 - \tfrac{\theta}{2})\, \|U^{\frac{1}{2}} u\|_2^2 . \tag{6.2}$$

This shows that $a$ is a closed positive form. Since $a(u, v) = (-Au, v)$ for $u \in \mathcal{D}_2$ and $v \in D(a)$, $a$ is sectorial by Proposition 3.1 and the m–sectorial operator induced by $a$ coincides with $-A$, since we know that $\mathcal{D}_2 \subseteq D(a)$. If $\mathrm{divF} \le c_2 U$, the adjoint form

$$a'(u, v) = \overline{a(v, u)} = \int_{\mathbb{R}^N} (\nabla u \cdot \nabla \overline{v} + (V + \mathrm{divF}) u \overline{v} + \overline{v} F \cdot \nabla u)\, dx,$$

with $D(a') = D(a)$ has the same properties as $a$ and defines the adjoint operator $(A', \mathcal{D}_2)$, where $A'v = \Delta v - F \cdot \nabla v - (V + \operatorname{div} F)v$.

We point out that this approach yields the analyticity of our semigroup $T(\cdot)$ in $L^p(\mathbb{R}^N)$ for $1 < p < \infty$ (see [33, Theorem 1.1]) but not the description of the domain of the generator. We refer to [15], [33], and the references therein for recent developments of these methods using rather weak assumptions.

In the proof of Proposition 4.2 we have already proved the positivity of $T(\cdot)$, employing the maximum principle for functions in $\mathcal{D}_\infty$, [22, Proposition 3.1.10], to show the positivity of the resolvent on $C_0(\mathbb{R}^N)$. Here we give an alternative proof, based on form methods.

**Proposition 6.1.** *The semigroup $T(\cdot)$ is positive; i.e., $f \geq 0$ implies*

$$T(t)f \geq 0.$$

**Proof.** It suffices to treat the case $p = 2$, due to Lemma 4.3 and Proposition 5.1. Observe that the form $a$ defined in (6.1) is real and if $u \in D(a)$ is real valued, then $u^+, u^- \in D(a)$ and $a(u^+, u^-) = 0$. The positivity of $T(t)$ on $L^2(\mathbb{R}^N)$ thus follows from the non–symmetric first Beurling–Deny criterion, see [30, Theorem 2.4], since its generator $(A, \mathcal{D}_2)$ is the operator associated to $a$. □

The semigroup further improves the integrability of the initial datum.

**Proposition 6.2.** *For every $t > 0$ and $1 < p < \infty$, $T(t)$ maps $L^p(\mathbb{R}^N)$ into $C_0(\mathbb{R}^N)$ and thus also into $L^q(\mathbb{R}^N)$ for $q \in (p, \infty)$.*

**Proof.** It suffices to show that $T(t)$ maps $L^p(\mathbb{R}^N)$ into $C_0(\mathbb{R}^N)$. Let $p > 1$ and $f \in L^p(\mathbb{R}^N)$. Since $T(\cdot)$ is analytic, $T(t)f \in W^{2,p}_U(\mathbb{R}^N) \subset W^{2,p}(\mathbb{R}^N)$. If $p > N/2$ we are done due to Sobolev's embedding theorem. If $p \leq N/2$ we use Sobolev's embedding theorem (and the semigroup law) finitely many times to deduce that $T(t)f \in L^r(\mathbb{R}^N)$ for some $r > N/2$ and conclude as above. □

In order to deal with the case $p = 1$ and to give an estimate of the operator norm of $T(t)$ acting from $L^\infty(\mathbb{R}^N)$ into $L^1(\mathbb{R}^N)$, we use form methods as in the proof of Proposition 6.1. The operator norm of $T(t) : L^p(\mathbb{R}^N) \to L^q(\mathbb{R}^N)$ can then easily be estimated by interpolation.

**Proposition 6.3.** *There is a positive constant $M$ such that $\|T(t)f\|_2 \leq Mt^{-\frac{N}{4}}\|f\|_1$ for $t > 0$ and $f \in L^1(\mathbb{R}^N)$. If $\operatorname{div} F \leq c_2 U$, then $\|T(t)f\|_\infty \leq Mt^{-\frac{N}{2}}\|f\|_1$ for $t > 0$ and $f \in L^1(\mathbb{R}^N)$.*

**Proof.** Let us first show that

$$\|T(t)f\|_2 \leq Mt^{-N/4}\|f\|_1, \tag{6.3}$$

without assuming that $\mathrm{div}F \leq c_2U$. To this aim we employ the sesquilinear form $a$ introduced in (6.1) and observe that $\mathrm{Re}\, a(u,u) \geq \|\nabla u\|_2^2$ for every $u \in W_U^{2,2}(\mathbb{R}^N)$ due to (6.2). Therefore, Sobolev's inequality yields

$$\|u\|_2^{2+4/N} \leq Ca(u,u)\|u\|_1^{4/N}$$

for every $u \in W_U^{2,2}(\mathbb{R}^N)$. Now (6.3) follows as in [15, Theorem 2.4.6] since $T(\cdot)$ is analytic in $L^2(\mathbb{R}^N)$.

Next we assume that $\mathrm{div}F \leq c_2U$ and consider the adjoint $A'v = \Delta v - F\cdot\nabla v - (V + \mathrm{div}F)v$ with domain $\mathcal{D}_2$. It is associated with the adjoint form $a'$ and therefore it is the generator of the adjoint semigroup $(T(t)')_{t\geq 0}$. Therefore (6.3) holds with $T(t)'$ instead of $T(t)$ and yields, by duality,

$$\|T(t)f\|_\infty \leq Mt^{-N/4}\|f\|_2.$$

By the semigroup law, the proof is complete. □

It is well known that the estimate $\|T(t)f\|_\infty \leq Ct^{-N/2}\|f\|_1$ implies the existence of a bounded kernel $k_t(x,y)$ such that

$$\big(T(t)f\big)(x) = \int_{\mathbb{R}^N} k_t(x,y)f(y)\,dy$$

and, moreover, $\sup_{x,y\in\mathbb{R}^N} k_t(x,y) \leq Ct^{-N/2}$. This kernel is positive by Proposition 6.1.

In the following proposition we show the compactness of $T(\cdot)$ assuming that the potential $V$ tends to infinity as $|x| \to \infty$.

**Proposition 6.4.** *Assume that* $\lim_{|x|\to\infty} V(x) = \infty$. *Then* $\mathcal{D}_p$ *is compactly embedded in* $L^p(\mathbb{R}^N)$ *if* $1 \leq p < \infty$ *and in* $C_0(\mathbb{R}^N)$ *if* $p = \infty$. *The spectrum of* $(A, \mathcal{D}_p)$ *is independent of* $p \in [1,\infty]$. *Moreover,* $T(t)$, $t > 0$, *is compact in* $L^p(\mathbb{R}^N)$ *for* $1 < p < \infty$ *and in* $C_0(\mathbb{R}^N)$. $T(t)$, $t > 0$, *is compact in* $L^1(\mathbb{R}^N)$ *if* $\mathrm{div}F \leq c_2U$.

**Proof.** Let $\mathcal{F}$ be a bounded set in $\mathcal{D}_p$ and assume that $1 \leq p < \infty$. By the assumption, given $\varepsilon > 0$, we can find $R > 0$ such that

$$\int_{|x|>R} |f(x)|^p\,dx \leq \varepsilon$$

for every $f \in \mathcal{F}$. At this point the compactness of $\mathcal{F}$ in $L^p(\mathbb{R}^N)$ easily follows from the compactness of the embedding of $W^{2,p}(B_R)$ into $L^p(B_R)$. A similar argument works for $p = \infty$. The $p$-independence of the spectrum is now a consequence of e.g. [3, Proposition 2.6]. Since $T(\cdot)$ is analytic, its compactness is equivalent to the compactness of the embedding of $\mathcal{D}_p$ into $L^p(\mathbb{R}^N)$, respectively $C_0(\mathbb{R}^N)$. □

An analytic semigroup $S(\cdot)$ on a Banach space $X$ with generator $B$ has *maximal regularity of type* $L^q$ $(1 < q < \infty)$ if for each $f \in L^q([0,T],X)$ the function $t \mapsto u(t) = \int_0^t S(t-s)f(s)\,ds$ belongs to $W^{1,q}([0,T],X) \cap L^q([0,T],D(B))$. This means that the mild solution of the evolution equation

$$u'(t) = Bu(t) + f(t), \quad t > 0, \qquad u(0) = 0,$$

is in fact a strong solution and has the best regularity one can expect. It is known that this property does not depend on $1 < q < \infty$ and $T > 0$. In recent years this concept has thoroughly been studied and applied in various directions, see e.g. [2], [16], [35], and the references therein. For our purposes we only need the following facts. Let $X = L^p(\mathbb{R}^N)$ for some $1 < p < \infty$. Then the operator $B$ has maximal regularity of type $L^q$ if its imaginary powers satisfy $\|(-B)^{is}\| \le Me^{a|s|}$ for some $a \in [0,\pi/2)$ and all $s \in \mathbb{R}$ thanks to the Dore–Venni theorem, see e.g. [2, Theorem II.4.10.7]. If $p = 2$ and $B$ is maximal dissipative and invertible, then $\|(-B)^{is}\| \le Me^{\pi|s|/2}$ by a result due to Kato, [20, Theorem 5]. Hence, if $p = 2$ and $B$ is regularly dissipative, then $\|(-B)^{is}\| \le Me^{a|s|}$ for $a = \pi/2 - \phi$ and some $\phi \in (0,\pi/2]$. Moreover, if $B$ generates a positive contraction semigroup on $L^p(\mathbb{R}^N)$, then $\|(-B)^{is}\| \le M_\varepsilon \exp((\varepsilon + \pi/2)|s|)$ for each $\varepsilon > 0$ and $s \in \mathbb{R}$ because of the transference principle [9, Section 4], see [8, Theorem 5.8]. If we combine these facts with the Riesz-Thorin interpolation theorem, Proposition 3.1, Theorem 3.4, and Proposition 6.1, we obtain the following result. (See also [5, Theorem 7.1].)

**Proposition 6.5.** *$A$ has maximal regularity of type $L^q$ on $L^p(\mathbb{R}^N)$ for all $1 < p,q < \infty$.*

## 7. Generalized Ornstein–Uhlenbeck operators

In this section we investigate the operator

$$A_{\Phi,G}u = \Delta u - \nabla\Phi \cdot \nabla u + G \cdot \nabla u$$

on $L^p(\mathbb{R}^N,\mu)$, $1 < p < \infty$, where $\mu(dx) = e^{-\Phi}dx$. Under the following assumptions on $\Phi : \mathbb{R}^N \to [0,\infty)$ and $G : \mathbb{R}^N \to \mathbb{R}^N$ we show that $A_{\Phi,G}$ with domain $W^{2,p}(\mathbb{R}^N,\mu)$ generates a positive analytic semigroup $T(\cdot)$ on $L^p(\mathbb{R}^N,\mu)$ employing our previous results.

**(A1)** $\Phi \in C^2(\mathbb{R}^N,\mathbb{R})$, $G \in C^1(\mathbb{R}^N,\mathbb{R}^N)$, and $\int_{\mathbb{R}^N} e^{-\Phi(x)}dx < \infty$.

**(A2)** For each $\varepsilon > 0$ there is $c_\varepsilon > 0$ with
$$|\operatorname{div} G| + |D^2\Phi| \le \varepsilon|\nabla\Phi|^2 + c_\varepsilon.$$
**(A3)** There is a constant $w \in \mathbb{R}$ such that $G \cdot \nabla\Phi - \operatorname{div} G \le w$.
**(A3')** $\operatorname{div} G = G \cdot \nabla\Phi$.
**(A4)** There is a constant $d > 0$ such that $|G| \le d(|\nabla\Phi| + 1)$.

Below we will see that (A1) and (A3') say that the finite measure $\mu(dx)$ is, up to a multiplicative constant, the unique invariant measure of $T(\cdot)$. In fact, (A3') is only needed for this purpose. Moreover, (A2) with $\varepsilon = 1$ and (A3) imply that
$$\nabla\Phi \cdot (-\nabla\Phi + G) \le w + c_1 =: w';$$
i.e., $\Phi$ is a Liapunov function for the ODE corresponding to the shifted vector field $-\nabla\Phi + G - w'$. Lemma 7.3 also shows that (A3) implies the dissipativity of $A_{\Phi,G} - \frac{w}{p}$. We can omit (A3) if the constant $d$ in (A4) is small enough, see Remark 7.5. The growth assumptions (A2) and (A4) allow us to use Theorem 3.4 and to compute the domain. Notice that the above hypotheses simplify considerably if $G = 0$, i.e., for the (symmetric) operator $A_\Phi := A_{\Phi,0}$, see Corollary 7.8. It can be proved that the analyticity of the semigroup fails in some cases where (A4) is not satisfied, see [26].

We need some properties of the weighted Sobolev spaces
$$W^{k,p}(\mathbb{R}^N, \mu) = \Big\{u \in W^{k,p}_{\mathrm{loc}}(\mathbb{R}^N) : D^\alpha u \in L^p(\mathbb{R}^N, \mu) \text{ if } |\alpha| \le k\Big\}.$$
First of all, we observe that $C_0^\infty(\mathbb{R}^N)$ is dense in $W^{k,p}(\mathbb{R}^N, \mu)$. This can be seen as in Lemma 2.5. In the following lemma we do not need the full strength of hypothesis (H2) but only a weaker, one–sided estimate.

**Lemma 7.1.** *Let $1 < p < \infty$, assume that (A1) holds, and that*
$$\Delta\Phi + (p-2)(1 + |\nabla\Phi|^2)^{-1} \langle D^2\Phi\nabla\Phi, \nabla\Phi\rangle \le \varepsilon\, |\nabla\Phi|^2 + c_\varepsilon$$
*for some $\varepsilon < 1$. Then the map $u \mapsto |\nabla\Phi| u$ is bounded from $W^{1,p}(\mathbb{R}^N, \mu)$ to $L^p(\mathbb{R}^N, \mu)$ and the map $u \mapsto |\nabla\Phi||\nabla u|$ is bounded from $W^{2,p}(\mathbb{R}^N, \mu)$ to $L^p(\mathbb{R}^N, \mu)$. In particular, $A_{\Phi,G} : W^{2,p}(\mathbb{R}^N, \mu) \to L^p(\mathbb{R}^N, \mu)$ is bounded in view of (A4).*

**Proof.** Fix $p \in (1, \infty)$. Since $C_0^\infty(\mathbb{R}^N)$ is dense in $W^{2,p}(\mathbb{R}^N, \mu)$, it suffices to prove that
$$\|\nabla\Phi\, u\|_{L^p_\mu} \le c(\|\nabla u\|_{L^p_\mu} + \|u\|_{L^p_\mu})$$
for $u \in C_0^\infty(\mathbb{R}^N)$ and some constant $c > 0$. Observe that there are two constants $a$, $b > 0$ such that
$$|\nabla\Phi|^p \le a(1 + |\nabla\Phi|^2)^{\frac{p}{2}-1}|\nabla\Phi|^2 + b,$$

so that we only have to estimate $\int_{\mathbb{R}^N}(1+|\nabla\Phi|^2)^{\frac{p}{2}-1}|\nabla\Phi|^2|u|^p e^{-\Phi}dx$. An integration by parts yields

$$\begin{aligned}
&\int_{\mathbb{R}^N}(1+|\nabla\Phi|^2)^{\frac{p}{2}-1}|\nabla\Phi|^2|u|^p e^{-\Phi}dx \\
&= -\int_{\mathbb{R}^N}|u|^p(1+|\nabla\Phi|^2)^{\frac{p}{2}-1}\nabla\Phi\cdot\nabla(e^{-\Phi})\,dx \\
&= \int_{\mathbb{R}^N}|u|^p(1+|\nabla\Phi|^2)^{\frac{p}{2}-1}\Delta\Phi\,\mu(dx) \\
&\quad + (p-2)\int_{\mathbb{R}^N}|u|^p(1+|\nabla\Phi|^2)^{\frac{p}{2}-2}\langle D^2\Phi\nabla\Phi,\nabla\Phi\rangle\,\mu(dx) \\
&\quad + p\int_{\mathbb{R}^N}u|u|^{p-2}(1+|\nabla\Phi|^2)^{\frac{p}{2}-1}\nabla\Phi\cdot\nabla u\,\mu(dx) \\
&\le \varepsilon\int_{\mathbb{R}^N}(1+|\nabla\Phi|^2)^{\frac{p}{2}-1}|\nabla\Phi|^2|u|^p\,\mu(dx) + c_\varepsilon\int_{\mathbb{R}^N}(1+|\nabla\Phi|^2)^{\frac{p}{2}-1}|u|^p\,\mu(dx) \\
&\quad + p\int_{\mathbb{R}^N}(1+|\nabla\Phi|^2)^{\frac{p}{2}-1}|\nabla\Phi|\,|u|^{p-1}|\nabla u|\,\mu(dx).
\end{aligned}$$

Using the inequality $(1+t^2)^{\frac{p}{2}-1} \le \eta(1+t^2)^{\frac{p}{2}-1}t^2 + c$ with $\eta = \frac{1-\varepsilon}{2c_\varepsilon}$ and Hölder's inequality, we obtain

$$\begin{aligned}
&\frac{1-\varepsilon}{2}\int_{\mathbb{R}^N}(1+|\nabla\Phi|^2)^{\frac{p}{2}-1}|\nabla\Phi|^2|u|^p\mu(dx) \\
&\le p\Big(\int_{\mathbb{R}^N}(1+|\nabla\Phi|^2)^{p'(\frac{p}{2}-1)}|\nabla\Phi|^{p'}|u|^p\mu(dx)\Big)^{\frac{1}{p'}} \\
&\times\Big(\int_{\mathbb{R}^N}|\nabla u|^p\,\mu(dx)\Big)^{\frac{1}{p}} + C\int_{\mathbb{R}^N}|u|^p\,\mu(dx)
\end{aligned}$$

($p'$ is the conjugate of $p$). Now, since $(1+t^2)^{p'(\frac{p}{2}-1)}t^{p'} \le c_1(1+t^2)^{\frac{p}{2}-1}t^2 + c_2$ for certain constants $c_1, c_2 > 0$, the conclusion follows from Young's inequality. □

The following lemma can be proved supposing only the one–sided estimate

$$\Delta\Phi + 2(p-1)(1+|\nabla\Phi|^2)^{-1}\langle D^2\Phi\nabla\Phi,\nabla\Phi\rangle \le \varepsilon|\nabla\Phi|^2 + c_\varepsilon$$

for some $\varepsilon < 1$. This can be done imitating the proof of Lemma 7.1. Since we need hypothesis (A2) in the sequel, we prefer to assume it to shorten the proof.

**Lemma 7.2.** *Suppose that $1 < p < \infty$ and that hypotheses (A1) and (A2) hold. Then the map $u \mapsto |\nabla\Phi|^2 u$ is bounded from $W^{2,p}(\mathbb{R}^N,\mu)$ to $L^p(\mathbb{R}^N,\mu)$.*

**Proof.** We apply Lemma 7.1 to the (vector) function $u\nabla\Phi$ and we obtain

$$\begin{aligned}\|u|\nabla\Phi|^2\|_{L^p_\mu} \le C\|u\nabla\Phi\|_{W^{1,p}_\mu} &\le C\left(\|u\nabla\Phi\|_{L^p_\mu} + \|uD^2\Phi\|_{L^p_\mu} + \|\nabla u\cdot\nabla\Phi\|_{L^p_\mu}\right)\\ &\le C\left(\|u\|_{W^{2,p}_\mu} + \varepsilon\|u|\nabla\Phi|^2\|_{L^p_\mu} + c_\varepsilon\|u\|_{L^p_\mu}\right).\end{aligned}$$

In the second inequality we have used Lemma 7.1 twice and (A2). Taking a small $\varepsilon$, we establish the assertion. □

**Lemma 7.3.** *Assume that (A1) and (A3) hold. For $u \in C_0^\infty(\mathbb{R}^N)$ and $v \in W^{1,1}_{loc}(\mathbb{R}^N)$ we have*

$$\begin{aligned}\int_{\mathbb{R}^N}(A_{\Phi,G}u)v\,\mu(dx) = &-\int_{\mathbb{R}^N}\nabla u\cdot\nabla v\,\mu(dx) + \int_{\mathbb{R}^N}(G\cdot\nabla\Phi - \mathrm{div}G)\,uv\,\mu(dx)\\ &-\int_{\mathbb{R}^N}uG\cdot\nabla v\,\mu(dx).\end{aligned}\tag{7.1}$$

*In particular, $A_{\Phi,G} - \frac{w}{p}$ (defined on $C_0^\infty(\mathbb{R}^N)$) is dissipative in $L^p(\mathbb{R}^N,\mu)$ for $1 < p < \infty$. Moreover, $A_\Phi$ is symmetric in $L^2(\mathbb{R}^N,\mu)$.*

**Proof.** The first assertion is straightforward to check and implies the last one. Let $p \ge 2$. Take a real $u \in C_0^\infty(\mathbb{R}^N)$ and set $u^* = u\,|u|^{p-2}$. Then $\nabla u^* = (p-1)|u|^{p-2}\,\nabla u$ and therefore

$$\begin{aligned}\int_{\mathbb{R}^N}(A_{\Phi,G}u)u^*\,\mu(dx) &= -(p-1)\int_{\mathbb{R}^N}|u|^{p-2}\|\nabla u\|^2\,\mu(dx)\\ &+\int_{\mathbb{R}^N}(G\cdot\nabla\Phi - \mathrm{div}G)\,|u|^p\,\mu(dx) - (p-1)\int_{\mathbb{R}^N}u^*\,G\cdot\nabla u\,\mu(dx).\end{aligned}\tag{7.2}$$

This equality and (A3) imply that

$$p\int_{\mathbb{R}^N}(A_{\Phi,G}u)u^*\,\mu(dx) \le w\,\|u\|_p^p,$$

as claimed. If $p \in (1,2)$, we replace $|u|$ by $(\eta + |u|^2)^{1/2}$. By similar calculations and letting $\eta \to 0$, one also obtains (7.2), and thus the dissipativity of $A_{\Phi,G} - w/p$. □

We now come to the main result of this section. We say that a finite Borel measure $\nu$ on $\mathbb{R}^N$ is an *invariant measure* for a semigroup $T(\cdot)$ on $C_b(\mathbb{R}^N)$ if

$$\int_{R^N}T(t)f\,\nu(dx) = \int_{R^N}f\,\nu(dx) \qquad \text{for} \quad f \in C_b(\mathbb{R}^N),\ t \ge 0.\tag{7.3}$$

**Theorem 7.4.** *Assume that assumptions (A1), (A2), (A3), and (A4) are satisfied. Then the operator*

$$A_{\Phi,G} = \Delta - \nabla\Phi\cdot\nabla + G\cdot\nabla$$

*with domain $D(A_{\Phi,G}) = W^{2,p}(\mathbb{R}^N,\mu)$ generates a positive analytic $C_0$–semigroup $T(\cdot)$ on $L^p(\mathbb{R}^N,\mu)$, $1<p<\infty$, such that $\|T(t)\|_p \le e^{tw/p}$, where $\mu(dx) = e^{-\Phi(x)}dx$. Further, $\mu$ is an invariant measure of $T(\cdot)$ if and only if (A3′) holds in addition. Moreover, $T(t)$ is symmetric if $p=2$ and $G=0$. Finally, if $|\nabla\Phi(x)| \to \infty$ as $|x|\to\infty$, then $T(t)$, $t>0$, is compact in $L^p(\mathbb{R}^N,\mu)$ and the spectrum of $A_{\Phi,G}$ is independent of $p$ for $1<p<\infty$.*

**Proof.** Fix $p \in (1,\infty)$. We make a change of variable in order to work with an operator on $L^p(\mathbb{R}^N, dx)$ instead of $L^p(\mathbb{R}^N,\mu)$. Namely we define the isometry

$$J : L^p(\mathbb{R}^N,\mu) \to L^p(\mathbb{R}^N), \qquad Ju = e^{-\frac{\Phi}{p}}u.$$

A straightforward computation shows that

$$A_{\Phi,G}u = J^{-1}AJu \;\text{ for }\; u \in C_0^\infty(\mathbb{R}^N),$$

where

$$Av = \Delta v + \Big[\Big(\frac{2}{p}-1\Big)\nabla\Phi + G\Big]\cdot\nabla v - \frac{1}{p}\Big[\Big(1-\frac{1}{p}\Big)|\nabla\Phi|^2 - \Delta\Phi - G\cdot\nabla\Phi\Big]v.$$

We define $F = (\frac{2}{p}-1)\nabla\Phi + G$ and

$$V = \frac{1}{p}\Big[\Big(1-\frac{1}{p}\Big)|\nabla\Phi|^2 - \Delta\Phi - G\cdot\nabla\Phi\Big]. \tag{7.4}$$

We now want to show that $A$ satisfies assumptions (H2)-(H5) for a suitable function $U$. Taking $\varepsilon' = \frac{1}{2}(1-\frac{1}{p})$ in (A2) and using (A3), we see that

$$V \ge \frac{1}{2p}\Big(1-\frac{1}{p}\Big)|\nabla\Phi|^2 - \frac{1}{p}(c_{\varepsilon'} + w). \tag{7.5}$$

We set therefore $U = c_0 + \frac{1}{2p}(1-\frac{1}{p})|\nabla\Phi|^2$ for a number $c_0 > 0$ to be determined below. Assumption (A2) implies that (H2) holds for every $\gamma > 0$. Set $c' = c_0 + \frac{1}{p}(c_{\varepsilon'} + w)$. Due to (A2) and (A4), there is a positive number $c_1 \ge 0$ such that $0 < c_0 \le U \le V + c' \le c_1 U$; i.e., (H3) holds for $V + c'$. It is clear that (H4) follows from (A4). It remains to choose $c_0$ so that (H5) is satisfied. Using (A2) with $\varepsilon'' = \frac{p-1}{8p}$, we obtain

$$\begin{aligned} -\mathrm{div}\, F = -\mathrm{div}\, G - (\frac{2}{p}-1)\Delta\Phi &\le \varepsilon''\Big(1+\big|\frac{2}{p}-1\big|\Big)|\nabla\Phi|^2 + c_{\varepsilon''}\Big(1+\big|\frac{2}{p}-1\big|\Big) \\ &\le 2\varepsilon''|\nabla\Phi|^2 + 2c_{\varepsilon''} \le \frac{p}{2}U \end{aligned}$$

if $c_0 = \frac{4c_{\varepsilon''}}{p}$. Theorem 3.4 (and Remark 3.5) thus shows that $(A, W^{2,p}_U(\mathbb{R}^N))$ generates an analytic $C_0$–semigroup $S(\cdot)$ on $L^p(\mathbb{R}^N)$ for $1<p<\infty$, which is positive and compact (if $|\nabla\Phi(x)| \to \infty$ as $|x| \to \infty$) due to Propositions 6.1 and 6.4, respectively. As a result,

$$A_{\Phi,G} = J^{-1}AJ$$

with domain $D(A_{\Phi,G}) := \{u \in L^p(\mathbb{R}^N, \mu) : Ju \in W_U^{2,p}(\mathbb{R}^N)\}$ generates a positive analytic $C_0$–semigroup on $L^p(\mathbb{R}^N, \mu)$, $1 < p < \infty$, which is compact if $|\nabla\Phi(x)| \to \infty$ as $|x| \to \infty$. The independence of the spectrum of $1 < p < \infty$ is a consequence of the compactness of $T(\cdot)$ as in Proposition 6.4.

Suppose that $u \in D(A_{\Phi,G})$. Then $v := Ju \in W^{2,p}(\mathbb{R}^N)$ and $|\nabla\Phi|^2 v \in L^p(\mathbb{R}^N)$. Hence

$$e^{-\frac{\Phi}{p}} D_j u = \frac{1}{p} v D_j\Phi + D_j v \in L^p(\mathbb{R}^N),$$

since $|\nabla\Phi| \le 1 + |\nabla\Phi|^2$. Moreover, from (A2) and Proposition 2.3 we deduce

$$e^{-\frac{\Phi}{p}} D_{ij} u = \frac{1}{p} v D_{ij}\Phi + D_{ij} v + \frac{1}{p} D_j v D_i\Phi + \frac{1}{p} D_i v D_j\Phi + \frac{1}{p^2} v D_i\Phi D_j\Phi \in L^p(\mathbb{R}^N);$$

i.e., $u \in W^{2,p}(\mathbb{R}^N, \mu)$. Conversely, take $u \in W^{2,p}(\mathbb{R}^N, \mu)$ and set $v := Ju$. Then, by Lemma 7.1,

$$D_j v = e^{-\frac{\Phi}{p}} \Big( -\frac{1}{p} u D_j\Phi + D_j u \Big) \in L^p(\mathbb{R}^N).$$

Lemma 7.2 further implies that $|\nabla\Phi|^2 v \in L^p(\mathbb{R}^N)$. Using Lemma 7.1 and (A2), we obtain

$$D_{ij} v = e^{-\frac{\Phi}{p}} \big( D_{ij} u - \frac{1}{p} u D_{ij}\Phi + \frac{1}{p^2} u D_j\Phi D_i\Phi - \frac{1}{p} D_j u D_i\Phi - \frac{1}{p} D_i u D_j\Phi \big) \in L^p(\mathbb{R}^N).$$

Thus $u \in D(A_{\Phi,G})$ and we have established $D(A_{\Phi,G}) = W^{2,p}(\mathbb{R}^N, \mu)$.

In particular, $C_0^\infty(\mathbb{R}^N)$ is a core of $A_{\Phi,G}$. Therefore Lemma 7.3 yields the asserted estimate and, if $G = 0$, the symmetry of $T(t)$. Moreover, $\mu$ is an invariant measure of $T(\cdot)$ if and only if $\int_{\mathbb{R}^N} A_{\Phi,G} u \, d\mu = 0$ for each $u \in C_0^\infty(\mathbb{R}^N)$. This property is equivalent to (A3') by (7.1). $\square$

**Remark 7.5.** In the above proof we have used (A3) only to deduce (7.5) and to obtain the asserted estimate. Therefore $A_{\Phi,G}$ with domain $W^{2,p}(\mathbb{R}^N, \mu)$ also generates a positive analytic $C_0$–semigroup on $L^p(\mathbb{R}^N, \mu)$ if we assume that (A1) and (A2) hold and that $|G| \le d\,|\nabla\Phi| + \hat{c}$ for some constants $0 < d < 1 - \frac{1}{p}$ and $\hat{c} > 0$.

**Remark 7.6.** Observe that in the above theorem the domains of generators $A_{\Phi,G}$ become smaller if we increase $p \in (1, \infty)$. Thus the semigroups $T(\cdot)$ generated by $A_{\Phi,G}$ on $L^p(\mathbb{R}^N, \mu)$ and $L^q(\mathbb{R}^N, \mu)$ coincide on $L^p(\mathbb{R}^N, \mu)$ if

$p > q$. As in Remark 2.7 one further sees that these semigroups also extend the semigroup on $C_b(\mathbb{R}^N)$ constructed in [25, Section 4]. Corollary 4.7 in [25] thus shows that the semigroups $T(\cdot)$ in Theorem 7.4 are irreducible and have the strong Feller property, As a result, $\mu$ is the (up to a multiplicative constant) unique invariant measure of $T(\cdot)$ if (A3') holds due to [14, Theorem 4.2.1].

**Remark 7.7.** Observe that the change of variable used in the proof of Theorem 7.4 and the resulting operator $A$ and the auxiliary potential $U$ depend on $p$. Moreover, in the limiting cases $p = 1, \infty$ hypotheses (H2)–(H5) may fail. This is not only a technical problem, but some of the conclusions of Theorem 7.4 can in fact be wrong for $p = 1, \infty$. For example, $T(\cdot)$ is, in general, not analytic and even the inclusion $D(A_{\Phi,G}) \subset W^{1,1}(\mathbb{R}^N, \mu)$ may be false, see e.g. [24].

**Corollary 7.8.** *Let $\Phi : \mathbb{R}^N \to \mathbb{R}$ be a $C^2$ function satisfying $\int_{\mathbb{R}^N} e^{-\Phi(x)} dx < \infty$. Assume that for each $\varepsilon > 0$ there is $c_\varepsilon > 0$ such that $|D^2\Phi| \le \varepsilon |\nabla\Phi|^2 + c_\varepsilon$. Then the operator $A_\Phi u = \Delta u - \nabla\Phi \cdot \nabla u$ with domain $D(A_\Phi) = W^{2,p}(\mathbb{R}^N, \mu)$ generates an analytic semigroup in $L^p(\mathbb{R}^N, \mu)$ for $1 < p < \infty$.*

Corollary 7.8 improves Theorems 2.2 and 2.7 in [13]. These results require condition (H2) for all $\gamma > 0$ for the potential $V$ (with $G = 0$) defined in (7.4) and additional growth assumptions on $\nabla\Phi$ and $D^2\Phi$. See also [11] for further developments in the case $p = 2$. Observe that if $\Phi(x) = \varphi(|x|)$ for $|x| \ge r_0 > 0$ and $G = 0$, then (H2) says that $|\varphi''| \le \varepsilon |\varphi'|^2 + C_\varepsilon$ on $[r_0, \infty)$. This holds e.g. if $\varphi(r) = r^\alpha$, $\alpha > 0$.

As a particular case of the Theorem 7.4, we consider the Ornstein–Uhlenbeck operator

$$Lu(x) = Tr(QD^2u(x)) + \langle Bx, \nabla u(x) \rangle, \quad x \in \mathbb{R}^N,\ u \in C_0^\infty(\mathbb{R}^N),$$

where $Q$ is a real, symmetric, positive definite $N \times N$–matrix and $B$ is a real $N \times N$–matrix whose eigenvalues are contained in the open left half plane. The Ornstein–Uhlenbeck semigroup is given by

$$(T(t)\phi)(x) = (4\pi)^{-\frac{N}{2}} (\det Q_t)^{-\frac{1}{2}} \int_{\mathbb{R}^N} \phi(e^{tB}x - y) e^{-\frac{1}{4}\langle Q_t^{-1} y, y\rangle} dy,$$

$x \in \mathbb{R}^N$, $t > 0$, $\phi \in C_b(\mathbb{R}^N)$, where $Q_t := \int_0^t e^{sB} Q e^{sB^*} ds$, $t > 0$. We know from [27] that, by a change of variables, $L$ is similar to the operator

$$A_{\Phi,G} u = \Delta u - \nabla\Phi \cdot \nabla u + G \cdot \nabla u,$$

where $\Phi(x) := \sum_{j=1}^{N} \frac{x_j^2}{4\lambda_j}$ and $G(x) := B_1 x$ for $x \in \mathbb{R}^N$. Here $B_1$ is a real $N \times N$–matrix satisfying

$$B_1 D_\lambda = -D_\lambda B_1^*, \tag{7.6}$$

where $D_\lambda := \operatorname{diag}(\lambda_1, \ldots, \lambda_N)$ and $\lambda_1, \ldots, \lambda_N > 0$. Condition (7.6) implies that $\operatorname{div} G = G \cdot \nabla\Phi = 0$. It is easy to see that $\Phi$ and $G$ satisfy the assumptions (A1), (A2), and (A4). Thus we obtain the following result, which was proved in [27, Theorem 3.4] using completely different arguments. We also refer to [21], where the case $p = 2$ was treated, and to [7], where the symmetric case $B_1 = 0$ was studied with $\mathbb{R}^N$ replaced by an separable Hilbert space.

**Corollary 7.9.** *Suppose that $Q$ is a real, symmetric, positive definite $N \times N$-matrix and that $B$ is a real $N \times N$-matrix with eigenvalues in the open left half plane. Then the Ornstein-Uhlenbeck operator $L$ with domain $W^{2,p}(\mathbb{R}^N, \mu)$, where $\mu(dx) = e^{-\Phi(x)}dx$, generates the Ornstein-Uhlenbeck semigroup $T(\cdot)$ on $L^p(\mathbb{R}^N, \mu)$ for $1 < p < \infty$. Moreover, $\mu$ is the invariant measure of $T(\cdot)$.*

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