Research Article

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Series solution to fractional contact problem using Caputo's derivative

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Abstract: In this article, contact problem with fractional derivatives is studied. We use fractional derivative in the sense of Caputo. We deploy penalty function method to degenerate the obstacle problem into a system of fractional boundary value problems (FBVPs). The series solution of this system of FBVPs is acquired by using the variational iteration method (VIM). The performance as well as precision of the applied method is gauged by means of significant numerical tests. We further study the convergence and residual errors of the solutions by giving variation to the fractional parameter, and graphically present the solutions and residual errors accordingly. The outcomes thus obtained witness the high effectiveness of VIM for solving FBVPs.

Keywords: fractional contact problem, obstacle, variational iteration method, Caputo's fractional derivative

1 Introduction

Recently, numerous issues in physics, potential theory, fluid mechanics, and economics have been transformed in variational inequality form [1]. Variational inequality has close linkage with diverse fields such as optimal control problem governed by PDE, bi-level programming, and free BVPs [2]. As a result of these diverse applications, implementing numerical techniques for variational inequalities has received much attention by numerous engineers and mathematicians.

Obstacle problem has its own importance in core domain of variational inequalities. The fundamental concern in discussing obstacle-type problems is to identify the equilibrium point of elastic layer resting over a hypothetical obstacle. Some of the issues in applied areas can be demonstrated as obstacle problems, and some notable referrals are refs [3–5]. The existence, uniqueness, and regularity of the obstacle problems can found in refs [6,1]. Due to highly nonlinear nature of obstacle problem the task of finding the exact solution is difficult. Many researchers solved the obstacle problems numerically by different methods including boundary element method [7], projection method [8,9], and VIM [10–12].

In 1695, Leibniz made known the first ever notation for qth order derivative of the function, i.e., Leibniz queried to Mathematician D. Hospital that what it be in the event that we take the order as a fraction. Later on non-integer/fractional order derivatives gained immense value to portray numerous problems faced in rheology, physics, control systems, damping laws, fluid mechanics, biomathematics, computational chemistry, control theory, engineering science, and finance. With the passage of time these applications motivated many mathematicians and physicists to create various definitions of this concept of fractional differential operators to model complicated phenomena (see, e.g., [13–21]). In the book of Oldham and Spanier [22], we can find the initial efforts put by many mathematicians and researchers in the field of fractional calculus.

Mostly, analytical and numerical schemes have been developed to solve fractional order differential equations as it is nevertheless not easy to compute their exact solutions. Due to their invaluable involvement in almost every field, scientists developed many numerical as well as analytical algorithms with much stability, see

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refs [18,23–34,35] for solution of these types of equations. Among them, Martin [29] has discussed the stability of algorithm that is a combination of VIM and Laplace transformation. With the help of weak formulation, Heidarkhani et al. [36] succeeded to solve system of fractional order differential equations. Nowadays, many researchers [37,38] have shown their interest to tackle contact problems of fractional nature involving an obstacle because of their immense importance in mathematical physics and engineering. Recently, in ref. [39], an advanced Caputo fractional derivative approach is used to study the generalized model for quantitative analysis of sediments loss. In ref. [40], Caputo derivatives are deployed to study the magnetohydrodynamic (MHD) flow over a shrinking sheet and heat transfer with viscous dissipation. Kavitha et al. [41] studied the existence of mild solutions for the Hilfer fractional evolution system with infinite delay via measures of noncompactness. Some of the most relevant recent developments are presented in refs [42–51]. However, to the best of our knowledge, the analytical solution of system of fractional boundary value problems (FBVPs) using VIM has not been discussed in the literature so far. Inspired by this, we attempt to present the analytical solution of system FBVPs in this proposed study.

We organized this study as follows: Section 2 provides some definitions and preliminaries. A short introduction of the VIM introduced by Inokuti et al. [52] is also presented here for readers. To estimate the viability of applied VIM we did some numerical tests for various cases in Section 3. Results and discussions are given in Section 4.

2 Preliminaries

Here at first we give the definition of fractional derivative which will be used later. There are many definitions of fractional derivative which can be found in refs [13,16,53], but in our work we will use the definition Caputo of fractional order $\alpha > 0$, see ref. [54].

2.1 Definition

Let $n \geq \alpha$ be the least integer, Caputo's fractional derivative of order $\alpha > 0$ is defined as

$$\frac{\mathrm{d}^{\alpha}f(\xi)}{\mathrm{d}\xi^{\alpha}} = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_0^{\xi}(\xi-s)^{(n-\alpha-1)} f^{(n)}(s)\mathrm{d}s, & n-1<\alpha<n \\ \dfrac{\mathrm{d}^n f(\xi)}{\mathrm{d}\xi^n}, & \alpha = n, \end{cases} \tag{1}$$

where α is a real number and Γ denotes the gamma function.

2.2 Problem

BVP incorporated with an obstacle is the most classical type of free BVPs. Consider a membrane is attached between two fixed points. Effect of the gravitational force is negligibly small. In a still position, i.e., when the membrane is at rest, this problem resembles with the problem of a string in 1D. When we push up this membrane with the help of some non-flat object, called obstacle, we can witness that membrane touches the obstacle at some points, while at other points, obstacle stays below the membrane. The collection of points at which membrane and obstacle do not touch each other is called free boundary. Now, we will present the mathematical formulation of the obstacle-type problem. Assume ψ as an obstacle function and the membrane u is fixed on the boundary of domain D. Moreover, assume that the membrane is forced to stay above the obstacle. The set

$$\Lambda = \{\xi \in D : u(\xi) = \psi(\xi)\}$$

is called the coincidence set. If we set $\Omega = D\backslash\Lambda$, then the set

$$\Gamma = \partial\Lambda \cap D = \partial\Omega \cap D$$

is the corresponding free boundary which is a priory unknown. Figure 1 illustrates the obstacle problem. In the equilibrium situation, the function u is harmonic outside the contact set, i.e., $\frac{\mathrm{d}^{\alpha}u}{\mathrm{d}\xi^{\alpha}} \leq 0$ in Λ^c, otherwise $u(\xi) = \psi(\xi)$.

We consider a system of boundary value problems as:

$$-\frac{\mathrm{d}^{\alpha}u}{\mathrm{d}\xi^{\alpha}} = \begin{cases} f(u,\xi) + u(x)g(u,\xi) + r, & a \leq \xi \leq c \\ f(u,\xi), & c \leq \xi \leq d \\ f(u,x) + u(\xi)g(u,\xi) + r, & d \leq \xi \leq b, \quad 1 < \alpha \leq 2, \end{cases} \tag{2}$$

with boundary conditions

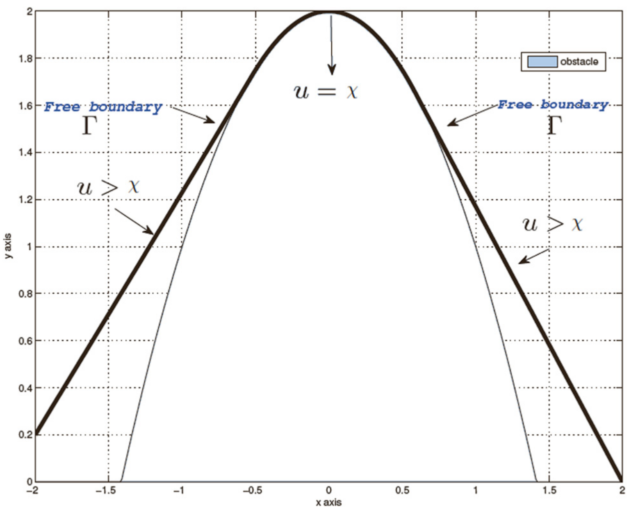

Figure 1: The obstacle $\psi(\xi)$ touches the membrane u and we are looking for the free boundary Γ and the solution u.

$$u(\xi)|_{\xi=a} = 0, \quad u(\xi)|_{\xi=b} = 0 \tag{3}$$

having continuity conditions of $u(\xi)$, $\frac{du}{d\xi}$ at a and b. Here, parameters r, a, b, and c are real constants and $f(u,\xi)$, $g(u,\xi)$ are continuous functions on $[a,b]$. Systems of type (2) arise in the mathematical modeling of contact, obstacle, unilateral, moving, and free boundary value problems. These problems have important applications in pure and applied sciences; see ref. [55] and references therein. For simplicity, we consider a second-order obstacle boundary value problems of the type:

$$\begin{cases} -\dfrac{d^\alpha u}{d\xi^\alpha} \geq f(\xi), & \xi \in [c,d] \\ u(\xi) \geq \kappa(\xi), & \xi \in [c,d] \\ \left[\dfrac{d^\alpha u}{d\xi^\alpha} - f(\xi)\right][u(\xi) - \kappa(\xi)] = 0, & \xi \in [c,d], \\ & \alpha = 2 \ \& \ c,d \quad \in \mathbb{R} \end{cases} \tag{4}$$

subject to the BCs

$$u(\xi)|_{\xi=c} = 0, \quad u(\xi)|_{\xi=d} = 0, \tag{5}$$

where $f(\xi)$ is a continuous functions, $\kappa(\xi)$ is an obstacle function and at the end of the domain, and $\mathbb{R}$ denotes a set of real numbers.

Equation (4) describes geometrically an elastic string pulled at the ends and having constraint to lie over an elastic obstacle $\kappa(\xi)$ in the equilibrium position.

If $u(\xi) = \kappa(\xi)$, then problems (4) reduce to finding such that

$$-\frac{d^\alpha u}{d\xi^\alpha} = f(\xi), \quad \alpha = 2, \tag{6}$$

with boundary conditions

$$u(\xi)|_{\xi=c} = 0, \quad u(\xi)|_{\xi=d} = 0. \tag{7}$$

A large class of problems arising in harmonic motion, oscillatory vertical motion, solid-state physics, nuclear charge in heavy atoms, and other analogous systems can be formulated as problem (6), see ref. [56] and references therein.

We study the problem (4) along with (5) in the framework of variational inequalities. For this purpose, we define the set K as

$$K = \{u(\xi) : u(\xi) \in H_0^1 : u(\xi) \geq \kappa(\xi) \quad \text{on } \xi \in [a,b]\}.$$

One can associate an energy functional $I[u]$ with the obstacle problem (4) using the technique of Tonti [57] as:

$$\begin{aligned} I[u] &= \int_a^b \left\{-\frac{d^2 u}{d\xi^2} - 2f(\xi)\right\} u(\xi) d\xi, \quad \forall u \in K \\ &= \int_a^b \frac{du}{d\xi}\frac{du}{d\xi} d\xi - 2\int_a^b f(\xi)u(\xi)d\xi \\ &= \langle Tu, u\rangle - 2\langle f, u\rangle, \end{aligned} \tag{8}$$

where

$$\langle Tu, v\rangle = \int_a^b \frac{du}{dx}\frac{dv}{dx} d, \quad \forall u, v \in K, \tag{9}$$

and

$$\langle f, u\rangle = \int_a^b f(\xi)u(\xi)d\xi, \quad \forall u, v \in K.$$

One can show that T defined by (9) is a linear, symmetric and positive operator. It is well known [58,59] that a minimum of functional $I[u]$ defined by (8), on the closed and convex set K in $H_0^1[a,b]$ can be characterized by a variational inequality of the type

$$\langle Tu, v-u\rangle \geq \langle f, v-u\rangle, \quad \forall v \in K. \tag{10}$$

Thus, we conclude that the obstacle boundary value problem (4) is equivalent to solving the variational inequality problem (10). This equivalence has been used to study the existence of a unique solution of (4), see ref. [58]. Utilizing the method of penalty function [60], the problem (4) can be rewritten as follows:

$$-\frac{d^\alpha u}{d\xi^\alpha} - \mu\{u(\xi) - \kappa(\xi)\}(u(\xi) - \kappa(\xi)) = f(\xi). \tag{11}$$

In (11), the penalty function is denoted by $\mu\{.\}$. Let us suppose the penalty function as follows:

$$\mu(\eta) = \begin{cases} 2, & \eta \geq 0, \\ 0, & \eta < 0. \end{cases} \tag{12}$$

In this article, the obstacle function $\kappa(\xi)$ is given as follows:

$$\kappa(\xi) = \begin{cases} -\dfrac{1}{2}, & a \leq \xi \leq c, \\ \dfrac{1}{2}, & c \leq \xi \leq d, \\ -\dfrac{1}{2}, & d \leq \xi \leq b. \end{cases} \tag{13}$$

Using (12) and (13) in (11), the following system of boundary value problems is obtained

$$-\frac{d^\alpha u}{d\xi^\alpha} = \begin{cases} f(\xi) + 2u + 1, & a \leq \xi \leq c \\ f(\xi), & c \leq \xi \leq d \\ f(\xi) + 2u + 1, & d \leq \xi \leq d, \quad \alpha = 2, \end{cases} \tag{14}$$

with the boundary conditions as given in (3) and continuity conditions of $u(\xi)$, $\frac{du}{d\xi}$ at c and d, which is of the type (2).

2.3 Variational iteration method (VIM)

The main task of the method is to find $u(\xi)$ by considering the problem such that

$$L(u) + N(u) = g(\xi), \tag{15}$$

where L and N are linear and nonlinear differential operators, respectively, and $g(\xi)$ is a nonhomogeneous term. For a given u_0, an approximate solution u_{p+1} of problem (8) can be obtained as follows:

$$u_{p+1}(\xi) = u_p(\xi) + \int_0^\xi \lambda(s, \xi)[Lu_p(s) + Nu_p(s) - g(s)]ds, \quad p = 0, 1, 2, \ldots, \tag{16}$$

where λ is known as the Lagrange multiplier. This λ function can be found by taking variation δ on both sides of equation (9) with respect to the variable u_p.

$$\delta u_{p+1}(\xi) = \delta u_p(\xi) + \delta \int_0^\xi \lambda(s, \xi)[Lu_p(s) + N\widetilde{u_p(s)} - g(s)]ds,$$

where $\widetilde{u_p(s)}$ is a restricted term which means $\delta\widetilde{u_p(s)} = 0$. An unknown function $\lambda(s, \xi)$ is found by using the optimality conditions, see ref. [55]. We may obtain the exact solution $u(\xi)$, when

$$u(\xi) = \lim_{p\to\infty} u_p(\xi). \tag{17}$$

This method of finding an approximate solution is named as VIM. In this method, proper selection of initial approximation leads to the fast converging solution, see refs [61,62] and references therein for better clarifications.

3 Implementation of VIM

In this section, we give an example of systems of FBVPs of type (2) to show the implementation and efficiency of VIM.

3.1 Example

By taking $f(u, \xi) = 0$, $g(u, \xi) = 2$, $r = 1$, $a = \frac{1}{4}$, $b = \frac{6}{4}$, $c = \frac{3}{4}$, and $d = 1$, problem (2) becomes

$$-\frac{d^\alpha u}{d\xi^\alpha} = \begin{cases} 2u + 1, & \dfrac{1}{4} \leq \xi \leq \dfrac{3}{4} \\ 0, & \dfrac{3}{4} \leq \xi \leq 1 \\ 2u + 1, & 1 \leq \xi \leq \dfrac{3}{2}, \quad 1 < \alpha \leq 2, \end{cases} \tag{18}$$

with boundary conditions

$$u(\xi)|_{\xi=\frac{1}{4}} = 0, \quad u(\xi)|_{\xi=\frac{3}{2}} = 0 \tag{19}$$

having continuity conditions of $u(\xi)$, $\frac{du}{d\xi}$ at $\frac{3}{4}$ and 1.

Using VIM, one can construct a correct functional of equation (18) as follows:

$$u_{p+1}(\xi) = \begin{cases} u_p(\xi) + \int_0^\xi \lambda(s, \xi)[u_p^{(\alpha)}(s) + 2\widetilde{u}_p(s) + 1]ds, & \dfrac{1}{4} \leq \xi \leq \dfrac{3}{4} \\ u_p(\xi) + \int_0^\xi \lambda(s, \xi)[u_p^{(\alpha)}(s)]ds, & \dfrac{3}{4} \leq \xi \leq 1 \\ u_p(\xi) + \int_0^\xi \lambda(s, \xi)[u_p^{(\alpha)}(s) + 2\widetilde{u}_p(s) + 1]ds, & 1 \leq \xi \leq \dfrac{3}{2}, \quad p = 0, 1, 2, \ldots. \end{cases} \tag{20}$$

We find $\lambda(s,\xi)$ for $\alpha = 2$. Using the optimality conditions [61,62], one has

$$\lambda(s,\xi) = s - \xi. \tag{21}$$

This parameter $\lambda(s,\xi) = s - \xi$, works for all values of α in the domain $1 < \alpha \le 2$. To show its validity, residual errors are plotted for each solution. We consider the initial approximations as:

$$u_0(\xi) = \begin{cases} a_0\xi + a_1, & \frac{1}{4} \le \xi \le \frac{3}{4} \\ a_2\xi + a_3, & \frac{3}{4} \le \xi \le 1 \\ a_4\xi + a_5, & 1 \le \xi \le \frac{3}{2}. \end{cases} \tag{22}$$

Case 1. In this case, we take $\alpha = 1.90$ in equation (18). Using (21) and (22) in (20), we find the approximate solutions as:

$$u_1(\xi) = \begin{cases} 0.93750\ a_1 - 0.010417\ a_0 - 0.031250 + (1.0625\ a_0 + 0.5\ a_1 + 0.25)\xi + (-1.0\ a_1 - 0.5)\xi^2 & \frac{1}{4} \le \xi \le \frac{3}{4} \\ \quad - 0.33333\ a_0\xi^3, & \\ a_3 + a_2\xi, & \frac{3}{4} \le \xi \le 1 \\ -0.5 - 0.66667\ a_4 + (2.0\ a_4 + 2.0\ a_5 + 1.0)\xi + (-1.0\ a_5 - 0.5)\xi^2 - 0.33333\ a_4\xi^3, & 1 \le \xi \le \frac{3}{2}, \end{cases}$$

$$u_2(\xi) = \begin{cases} 0.93012\ a_1 - 0.012318\ a_0 - 0.034941 + (1.0735\ a_0 + 0.57366\ a_1 + 0.28683)\xi & \frac{1}{4} \le \xi \le \frac{3}{4} \\ + (-1.9375\ a_1 - 0.96875 + 0.010417\ a_0)\xi^2 + (0.91004\ a_1 + 0.45502)\xi^{\frac{21}{10}} & \\ + (-0.6875\ a_0 - 0.16667\ a_1 - 0.083333)\xi^3 + 0.29356\ a_0\xi^{\frac{31}{10}} & \\ + (0.16667\ a_1 + 0.083333)\xi^4 + 0.033333\ a_0\xi^5, & \\ a_3 + a_2\xi, & \frac{3}{4} \le \xi \le 1 \\ -0.41614 - 0.58354\ a_4 + 0.16771\ a_5 + (1.59\ a_4 + 1.4222\ a_5 + 0.71112)\xi & 1 \le \xi \le \frac{3}{2}. \\ + (-1.0\ a_5 - 0.5 + 0.66665\ a_4)\xi^2 + (0.91004\ a_5 + 0.45502)\xi^{\frac{21}{10}} & \\ + (-1.0\ a_4 - 0.66667\ a_5 - 0.33333)\xi^3 + 0.29356\ a_4\xi^{\frac{31}{10}} & \\ + (0.16667\ a_5 + 0.083333)\xi^4 + 0.033333\ a_4\xi^5, & \end{cases}$$

$\vdots$

Using the boundary conditions (19) and continuity conditions on $u_{20}(\xi)$, one has a system of linear equations. Solving that system using Maple software, one has:

$$\begin{aligned} a_0 &= 0.620186560993483232717, \\ a_1 &= -0.155046640248370808 2, \\ a_2 &= -0.011189585534569415, \\ a_3 &= 0.164486654638228502167, \\ a_4 &= -0.0111895855345694144 55, \\ a_5 &= 0.164486654638228502 17. \end{aligned}$$

The graph obtained by 20th iteration $u_{20}(\xi)$ of problem (18) is shown in Figure 2.

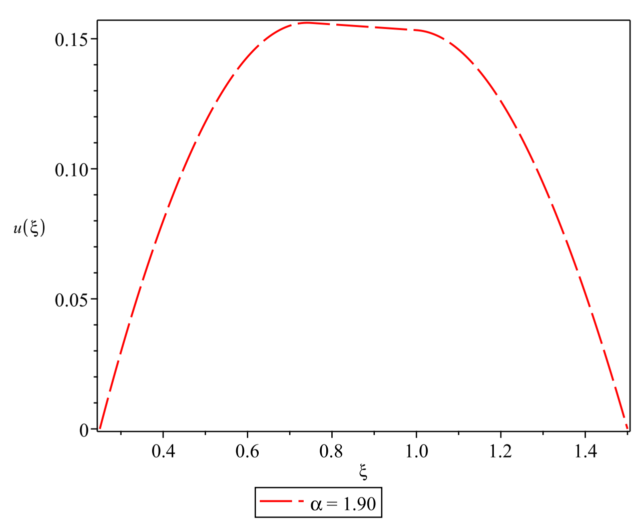

Figure 2: Graph of the approximate solution $u_{20}(\xi)$ for $\alpha = 1.90$.

The residual error $r_{20}(\xi)$ of problem (18), for $\alpha = 1.9$ is given as follows:

$$r_{20}(\xi) = \begin{cases} \dfrac{d^{1.90}u_{20}(\xi)}{d\xi^{1.90}} + 2u_{20} + 1, & \dfrac{1}{4} \le \xi \le \dfrac{3}{4} \\ \dfrac{d^{1.90}u_{20(\xi)}}{d\xi^{1.90}}, & \dfrac{3}{4} \le \xi \le 1 \\ \dfrac{d^{1.90}u_{20}(\xi)}{d\xi^{1.90}} + 2u_{20} + 1, & 1 \le \xi \le \dfrac{3}{2}. \end{cases} \tag{23}$$

and is plotted in Figure 3.

From this figure, it is clear that residual error is very small close to zero. Its maximum error is 2.5×10^{-13} at $\xi = 1$.

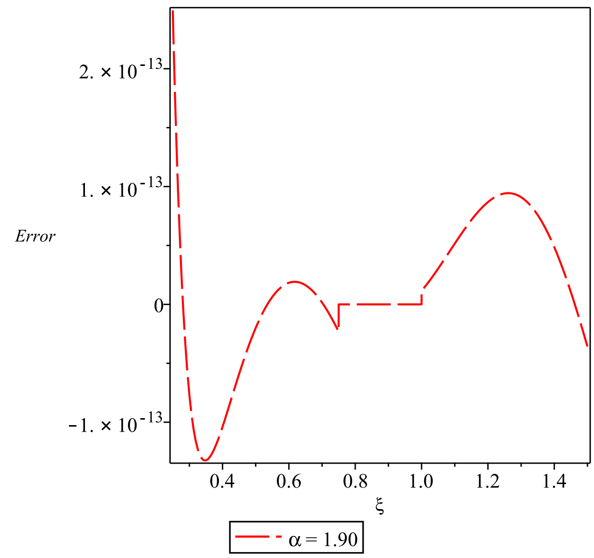

Figure 3: Graph of the residual error $r_{20}(\xi)$ for $\alpha = 1.90$.

Case 2. In this case, we take $\alpha = 1.8$, in equation (18), we find the approximate solutions as:

$$u_1(\xi) = \begin{cases} 0.93750\ a_1 - 0.010417\ a_0 - 0.031250 + (1.0625\ a_0 + 0.5\ a_1 + 0.25)\xi & \dfrac{1}{4} \le \xi \le \dfrac{3}{4} \\ + (-1.0\ a_1 - 0.5)\xi^2 - 0.33333\ a_0\xi^3, & \\ a_2\xi + a_3, & \dfrac{3}{4} \le \xi \le 1 \\ -0.5 - 0.66667\ a_4 + (2.0\ a_4 + 2.0\ a_5 + 1.0)\xi + (-1.0\ a_5 - 0.5)\xi^2 - 0.33333\ a_4\xi^3, & 1 \le \xi \le \dfrac{3}{2}, \end{cases}$$

$$u_2(\xi) = \begin{cases} 0.92255\ a_1 - 0.013986\ a_0 - 0.038724 + (1.084\ a_0 + 0.64566\ a_1 + 0.32282)\xi & \dfrac{1}{4} \le \xi \le \dfrac{3}{4} \\ + (-1.9375\ a_1 - 0.96875 + 0.010417\ a_0)\xi^2 + (0.82511\ a_1 + 0.41258)\xi^{\frac{11}{5}} & \\ + (-0.6875\ a_0 - 0.16667\ a_1 - 0.083333)\xi^3 + 0.25784\ a_0\xi^{\frac{16}{5}} & \\ + (0.16667\ a_1 + 0.083333)\xi^4 + 0.033333\ a_0\xi^5, & \\ a_2\xi + a_3, & \dfrac{3}{4} \le \xi \le 1 \\ -0.42158 - 0.63277\ a_4 + 0.15680\ a_5 + (1.6749\ a_4 + 1.5181\ a_5 + 0.759)\xi & 1 \le \xi \le \dfrac{3}{2}. \\ + (-1.0\ a_5 - 0.5 + 0.66665\ a_4)\xi^2 + (0.82511\ a_5 + 0.41258)\xi^{\frac{11}{5}} & \\ + (-1.0\ a_4 - 0.66667\ a_5 - 0.33333)\xi^3 + 0.25784\ a_4\xi^{\frac{16}{5}} & \\ + (0.16667\ a_5 + 0.083333)\xi^4 + 0.033333\ a_4\xi^5, & \end{cases}$$

⋮

Using the given boundary conditions (19) and continuity conditions on $u_{20}(\xi)$, one has a system of linear equations. Solving that system, one has:

$$\begin{aligned} a_0 &= -0.770518737432793778 84, \\ a_1 &= 0.19262968435819845182, \\ a_2 &= -0.17195715916729766102, \\ a_3 &= -0.10390004452240088474, \\ a_4 &= 0.069133691137254366 32, \\ a_5 &= -0.31600378256024247227. \end{aligned}$$

The graph obtained by VIM of the problem (19) for $\alpha = 1.8$ is shown in Figure 4.

The residual error is

$$r_{20}(\xi) = \begin{cases} \dfrac{d^{1.8}u_{20}}{d\xi^{1.8}} + 2u_{20} + 1, & \dfrac{1}{4} \le \xi \le \dfrac{3}{4} \\ \dfrac{d^{1.8}u_{20}}{d\xi^{1.8}}, & \dfrac{3}{4} \le \xi \le 1 \\ \dfrac{d^{1.8}u_{20}}{d\xi^{1.8}} + 2u_{20} + 1, & 1 \le \xi \le \dfrac{3}{2}. \end{cases} \tag{24}$$

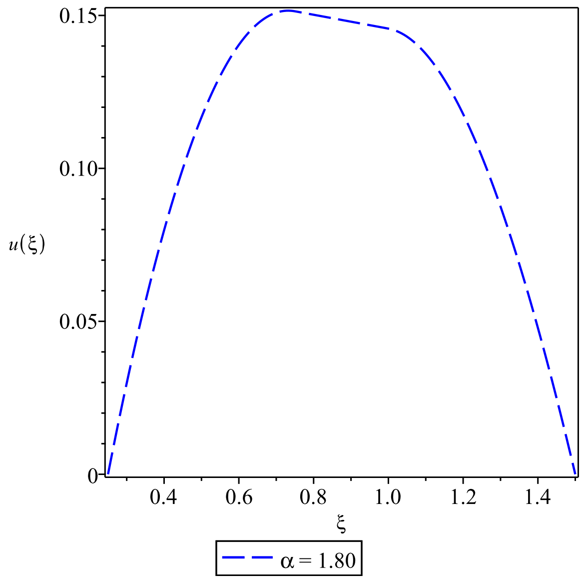

Figure 4: Graph of the solution $u_{20}(\xi)$, when $\alpha = 1.8$.

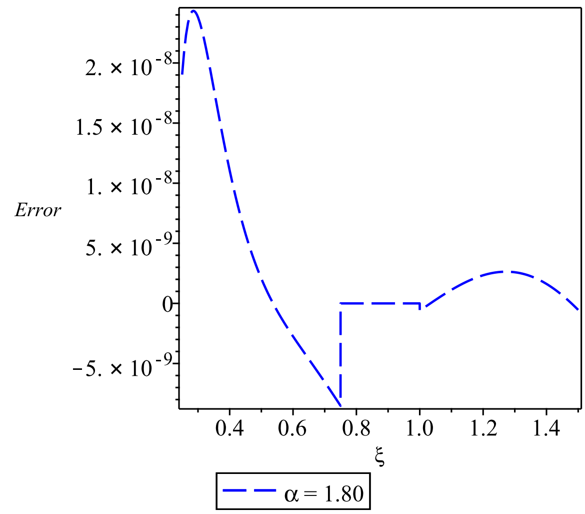

Figure 5: Graph of the residual error $r_{20}(\xi)$ for $\alpha = 1.8$.

The graph of residual error (24) is plotted in Figure 5. From this figure, it is clear that residual error is very small close to zero. Its maximum error is 3×10^{-8}.

Case 3. In this case, we take $\alpha = 1.7$ in equation (18). We find the approximate solutions as:

$$u_1(\xi) = \begin{cases} 0.93750\ a_1 - 0.010417\ a_0 - 0.031250 + (1.0625\ a_0 + 0.5\ a_1 + 0.25)\xi & \frac{1}{4} \le \xi \le \frac{3}{4} \\ + (-1.0\ a_1 - 0.50000)\xi^2 - 0.33333\ a_0\xi^3, & \\ a_2\xi + a_3, & \frac{3}{4} \le \xi \le 1 \\ -0.5 - 0.66667\ a_4 + (2.0\ a_4 + 2.0\ a_5 + 1.0)\xi + (-1.0\ a_5 - 0.5)\xi^2 & 1 \le \xi \le \frac{3}{2}. \\ -0.33333\ a_4\xi^3, & \end{cases}$$

$$u_2(\xi) = \begin{cases} 0.91560\ a_1 - 0.015349\ a_0 - 0.042199 + (1.0923\ a_0 + 0.70684\ a_1 + 0.35343)\xi & \frac{1}{4} \le \xi \le \frac{3}{4} \\ + (-1.9375\ a_1 - 0.96875 + 0.010417\ a_0)\xi^2 + (0.74532\ a_1 + 0.37264)\xi^{\frac{23}{10}} & \\ + (-0.6875\ a_0 - 0.16667\ a_1 - 0.083333)\xi^3 + 0.22585\ a_0\xi^{\frac{33}{10}} & \\ + (0.16667\ a_1 + 0.083333)\xi^4 + 0.033333\ a_0\xi^5, & \\ a_2\xi + a_3, & \frac{3}{4} \le \xi \le 1 \\ -0.43223 - 0.68057\ a_4 + 0.13558\ a_5 + (1.7547\ a_4 + 1.6191\ a_5 + 0.80959)\xi & 1 \le \xi \le \frac{3}{2}. \\ + (-1.0\ a_5 - 0.5 + 0.66665\ a_4)\xi^2 + (0.74532\ a_5 + 0.37264)\xi^{\frac{23}{10}} & \\ + (-1.0\ a_4 - 0.66667\ a_5 - 0.33333)\xi^3 + 0.22585\ a_4\xi^{\frac{33}{10}} & \\ + (0.16667\ a_5 + 0.083333)\xi^4 + 0.033333\ a_4\xi^5, & \end{cases}$$

⋮

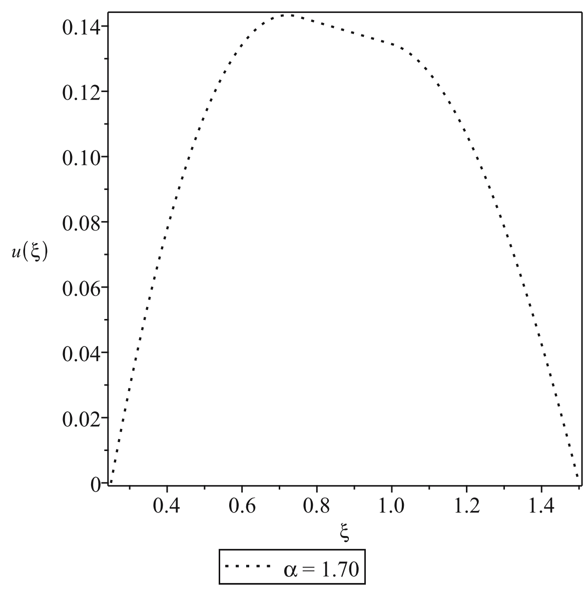

Figure 6: Graph of the approximate solution $u_{20}(\xi)$ for $\alpha = 1.7$.

Using the given boundary conditions (19) and continuity conditions on $u_{20}(\xi)$, one has a system of linear equations. Solving that system, one has:

$$\begin{aligned} a_0 &= -0.734299770795797, \\ a_1 &= 0.183574942698949, \\ a_2 &= -0.187080946262860, \\ a_3 &= -0.0859249043624337, \\ a_4 &= 0.0709589933588242, \\ a_5 &= -0.312633230648770. \end{aligned}$$

The graphs obtained by VIM is shown in Figure 6.

The residual error is

$$r_{20}(\xi) = \begin{cases} \dfrac{d^{1.6}u_{20}}{d\xi^{1.7}} + 2u_{20} + 1, & \dfrac{1}{4} \le \xi \le \dfrac{3}{4} \\ \dfrac{d^{1.7}u_{20}}{d\xi^{1.7}}, & \dfrac{3}{4} \le \xi \le 1 \\ \dfrac{d^{1.7}u_{20}}{d\xi^{1.7}} + 2u_{20} + 1, & 1 \le \xi \le \dfrac{3}{2}. \end{cases} \tag{25}$$

The graph of residual error (25) is plotted in Figure 7.

From this figure, it is clear that residual error is very small close to zero. Its maximum error is $1.282195070 \times 10^{-6}$ at $\xi = 1$.

4 Results and discussions

Physically, the solutions represent an elastic string lying over an elastic obstacle in an equilibrium state having fixed boundaries with some external forces. If we decrease the value of fractional parameter, the bending of the string decreases as shown in Figure 8.

Figure 9 gives the graphical representation of residual errors of the solution with different fractional parameters. This reveals that the suggested algorithm gives satisfactory results for the FBVPs whose orders of derivative are close to second order. After checking the viability, we can easily deduce that the technique of VIM is very meticulous, clear-cut, and meek for solving obstacle FBVPs. It is further observed that suggested algorithm is fit

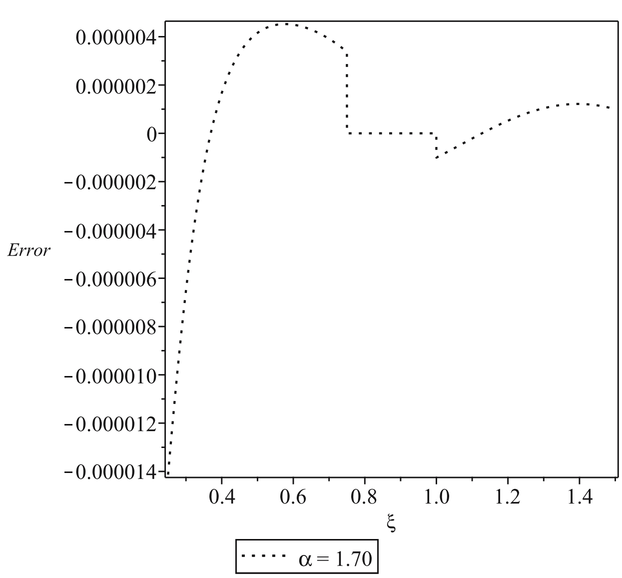

Figure 7: Graph of the residual error $r_{20}(\xi)$ for $\alpha = 1.7$.

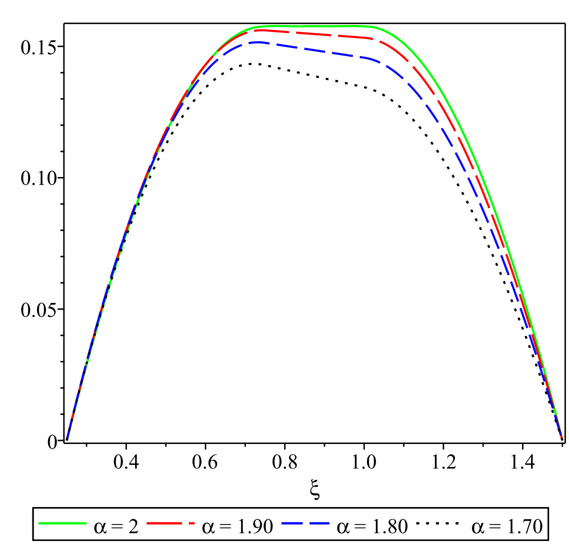

Figure 8: Comparison of solutions.

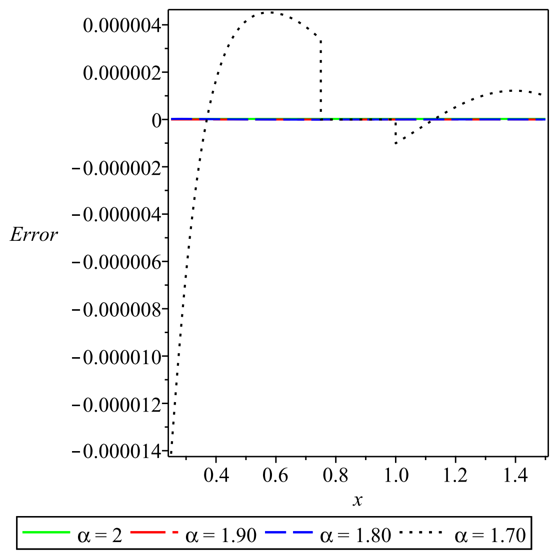

Figure 9: Comparison of residual errors.

for FBVPs with orders of derivative close to 2. We further bring about that VIM algorithm is greatly operative, accurate, and meek for the purpose of solving obstacle FBVPs.

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