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Some remarks concerning weakly disconjugate linear Hamiltonian systems ☆

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ABSTRACT

We study properties of weakly disconjugate linear Hamiltonian systems. We characterize this concept in terms of a nonoscillation condition. We then show how to approximate a weakly disconjugate system by one with an exponential dichotomy.

1. Introduction

In this paper, we study the concept of weak disconjugacy for linear, time-varying Hamiltonian differential systems. This notion was introduced in [6], and it was shown there that it is a natural generalization of the classical idea of disconjugacy. One advantage of weak disconjugacy as opposed to disconjugacy is that it can be studied under a very weak version of the condition of *identical normality*, which is often imposed when studying the classical disconjugacy concept [3,7,20].

Our goals are as follows. First, we show that, under the just-mentioned weak version of identical normality, the notion of weak disconjugacy can be characterized by a nonoscillation condition. Our starting point is the discussion in [6]; we extend the results given there. As in [6], we use an argument function of Yakubovich ([25,26]; see also Lidskii [17]).

Second, we give conditions sufficient that a weakly disconjugate system admits a principal solution (see also [6]). We then determine conditions under which the principal solution depends continuously on a parameter. We extend a result of [14] in this regard; as in [14], our parameter is of Atkinson type [2]. It should be noted that, even in the context of Hamiltonian linear systems which are disconjugate in the classical sense, the principal solution need not depend continuously on the coefficient matrix. This is true even if the coefficient matrix varies continuously in the topology of uniform convergence on all $\mathbb{R}$ [8].

As an essential part of our discussion of the continuous variation of the principal solution, we obtain a condition under which a weakly disconjugate linear Hamiltonian system can be approximated by a system which has an *exponential dichotomy* [4,22].

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A third goal is to state our hypotheses and results for a single linear nonautonomous Hamiltonian system, and not for an ergodic family of such systems. However, we will avail ourselves of results and methods which are valid in the ergodic framework [14,6].

This paper is organized as follows. In Section 2 we give basic definitions, then discuss the close relationship between weak disconjugacy and nonoscillation. In Section 3, we recall how to construct the principal solution of a weakly disconjugate system. We review the developments in [6], which are based on arguments of Coppel [3]. We also show that a large class of weakly disconjugate linear Hamiltonian systems admits approximation by systems which exhibit an exponential dichotomy. Finally, we discuss the continuity of the principal solution under variation of an Atkinson-type spectral parameter.

We close the Introduction by fixing some notation. Let $\mathbb{M}_n$ denote the set of $n \times n$ real matrices, and let $\mathbb{M}_n(\mathbb{C})$ be the set of $n \times n$ matrices with complex entries. If $A \in \mathbb{M}_n(\mathbb{C})$, let A^t denote its transpose, and let $A^* = \overline{A^t}$ denote its adjoint. Of course $A^* = A^t$ if $A \in \mathbb{M}_n$. Let $\langle\,,\rangle$ denote the Euclidean inner product on $\mathbb{R}^n$, and let $\|\cdot\|$ be the corresponding norm. We also use $\|\cdot\|$ to indicate the usual operator norm on $\mathbb{M}_n(\mathbb{C})$.

2. Weak disconjugacy and oscillation

Fix an integer $n \geqslant 1$. We write a vector $\mathbf{z} \in \mathbb{R}^{2n}$ in the form $\mathbf{z} = \binom{\mathbf{x}}{\mathbf{y}}$ where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Let J be the usual $2n \times 2n$ skew-symmetric matrix: thus $J = \left(\begin{smallmatrix} 0 & -I_n \\ I_n & 0 \end{smallmatrix}\right)$ where I_n is the $n \times n$ identity matrix. One has $J^t = J^{-1} = -J$.

Let $H : \mathbb{R} \to \mathbb{M}_{2n}$ be a continuous, bounded, $2n \times 2n$ matrix valued function with real entries, whose values are symmetric: $H(t)^t = H(t)$ for all $t \in \mathbb{R}$. Then $H(t)$ can be written in the block form

$$H(t) = \begin{pmatrix} H_{11}(t) & H_{12}(t) \\ H_{21}(t) & H_{22}(t) \end{pmatrix}$$

where $H_{11}(t)^t = H_{11}(t)$, $H_{22}(t)^t = H_{22}(t)$ and $H_{12}(t)^t = H_{21}(t)$ for all $t \in \mathbb{R}$. In what follows we will always assume that the following condition holds:

Hypothesis 2.1. $H_{22}(t) \geqslant 0$ for all $t \in \mathbb{R}$.

We introduce the nonautonomous Hamiltonian linear differential system

$$J\mathbf{z}' = H(t)\mathbf{z}, \quad \mathbf{z} \in \mathbb{R}^{2n}. \tag{1}$$

Let $Z(t)$ be a fundamental matrix solution of (1) satisfying $Z(0) = I_n$. Then it is well known that $Z(t)$ is an element of the symplectic group $Sp(n, \mathbb{R})$ for each $t \in \mathbb{R}$. More generally, if $Z(t)$ is a $2n \times 2n$ matrix solution of (1) such that $Z(0)$ lies in $Sp(n, \mathbb{R})$, then $Z(t)$ is symplectic for all $t \in \mathbb{R}$. We recall that $Sp(n, \mathbb{R}) = \{Z \in \mathbb{M}_{2n} \mid Z^t JZ = J\}$. Thus a $2n \times 2n$ real matrix $Z = \left(\begin{smallmatrix} U_1 & U_2 \\ V_1 & V_2 \end{smallmatrix}\right)$ lies in $Sp(n, \mathbb{R})$ if and only if the $n \times n$ blocks U_1, U_2, V_1, V_2 satisfy

$$U_1^t V_1 = V_1^t U_1, \qquad U_2^t V_2 = V_2^t U_2, \qquad U_1^t V_2 - V_1^t U_2 = I_n, \qquad V_2^t U_1 - U_2^t V_1 = I_n. \tag{2}$$

Definition 2.2. The system (1) is said to be *weakly disconjugate* if there exists a $t_* > 0$ such that, if $0 \neq \mathbf{y}_0 \in \mathbb{R}^n$, $\mathbf{z}_0 = \binom{0}{\mathbf{y}_0}$, and $Z(t)\mathbf{z}_0 = \binom{\mathbf{x}(t)}{\mathbf{y}(t)}$, then $\mathbf{x}(t) \neq 0$ for $t \geqslant t_*$.

We will relate the weak disconjugacy property to a classical nonoscillation condition on Eq. (1). For this we will use one of the well-known argument functions of Yakubovich [26]. Such a function can be viewed either as a multi-valued map from $Sp(n, \mathbb{R})$ to $\mathbb{R}$, or as a real-valued functional defined on continuous curves in $Sp(n, \mathbb{R})$. We will adopt the latter point of view.

Let $I \subset \mathbb{R}$ be an interval, and let $c : I \to Sp(n, \mathbb{R})$ be a continuous map. Let us write

$$c(t) = \begin{pmatrix} U_1(t) & U_2(t) \\ V_1(t) & V_2(t) \end{pmatrix}$$

where $U_j, V_j : I \to \mathbb{M}_n$, $j = 1, 2$, are continuous and satisfy the relations in (2). Set

$$W(t) = \big(U_1(t) - iU_2(t)\big)^{-1}\big(U_1(t) + iU_2(t)\big).$$

Following Lidskii [17], one can show that $W(\cdot)$ takes values in the complex unitary group

$$U(n) = \big\{W \in \mathbb{M}_n(\mathbb{C}) \bigm| W^*W = I_n\big\}.$$

Next, let $I = [t_1, t_2] \subset \mathbb{R}$, and let $c : I \to Sp(n, \mathbb{R})$ be a continuous map. Define the argument

$$\operatorname{Arg}(c) = \arg\det\bigl(U_1(t) + iU_2(t)\bigr)\big|_{t_1}^{t_2}.$$

This argument functional coincides with the functional Arg_3 of Yakubovich [26].

One can describe $\operatorname{Arg}(c)$ in another way. According to [15], it is possible to choose continuous functions $\rho_1(t), \dots, \rho_n(t)$ on I, with values in the unit circle $\mathbb{K} \subset \mathbb{C}$, such that the set of eigenvalues of $W(t)$ coincides with the unordered n-tuple $\{\rho_1(t), \dots, \rho_n(t)\}$, $t \in I$. Let $\varphi_k : I \to \mathbb{R}$ be a continuous function such that $\rho_k(t) = e^{i\varphi_k(t)}$, $1 \leqslant k \leqslant n$, $t \in I$. Then

$$\operatorname{Arg}(c) = \frac{1}{2}\sum_{k=1}^{n}\bigl[\varphi_k(t_2) - \varphi_k(t_1)\bigr].$$

Now we specialize the functional Arg to the fundamental matrix solution $Z(t)$ of (1). Let us now write

$$Z(t) = \begin{pmatrix} U_1(t) & U_2(t) \\ V_1(t) & V_2(t) \end{pmatrix}, \quad t \in \mathbb{R}.$$

For each $t \geqslant 0$, set $\operatorname{Arg}_Z(t) = \arg\det(U_1(t) + iU_2(t))|_0^t$. Thus, Arg_Z equals the functional Arg evaluated at the curve $c : [0,t] \to Sp(n,\mathbb{R})$, $s \mapsto c(s) = \left(\begin{smallmatrix} U_1(s) & U_2(s) \\ V_1(s) & V_2(s) \end{smallmatrix}\right)$.

Definition 2.3. The system (1) is said to be *nonoscillatory* if $|\operatorname{Arg}_Z(t)|$ is bounded on $0 \leqslant t < \infty$.

The following result will be useful in the sequel.

Proposition 2.4. *Assume that Hypothesis* 2.1 *is valid. Write $Z(t) = \left(\begin{smallmatrix} U_1(t) & U_2(t) \\ V_1(t) & V_2(t) \end{smallmatrix}\right)$ and $W(t) = (U_i(t) - iU_2(t))^{-1}(U_1(t) + iU_2(t))$. Let $\rho_1(t), \dots, \rho_n(t)$ be continuous determinations of the eigenvalues of $W(t)$, and let $\varphi_1(t), \dots, \varphi_n(t)$ be angles, i.e., continuous functions such that $\rho_k(t) = e^{i\varphi_k(t)}$, $1 \leqslant k \leqslant n$. Then each φ_k is a nondecreasing function of t.*

Proof. This proposition is stated by Yakubovich [26], as he notes the proof is essentially due to Lidskii [17]. For the reader's convenience we outline the argument.

First of all, we have already noted that there exist continuous determinations $\rho_1(t), \dots, \rho_n(t)$ of the eigenvalues of $W(t)$. In fact, according to [15, Theorem II.5.6] they may be chosen to be differentiable functions of t.

Next write $\phi(t) = U_1(t) + iU_2(t)$, so that $W(t) = \overline{\phi(t)^{-1}}\phi(t)$. Lidskii [17] shows that

$$\frac{dW}{dt} = iR(t)W$$

where $R(t) = 2\overline{\phi(t)^{-1}}H_{22}(t)(\overline{\phi(t)^{-1}})^*$. Clearly, $R(t) \geqslant 0$ and $R(t)^* = R(t)$ for each $t \in \mathbb{R}$.

We can write $\rho_k(t) = e^{i\varphi_k(t)}$ where $\varphi_k : \mathbb{R} \to \mathbb{R}$ is a continuous (in fact differentiable) function, $1 \leqslant k \leqslant n$. We claim that $\varphi_k(\cdot)$ is a nondecreasing function of t. To see this, we use an approximation method. Let $I \subset \mathbb{R}$ be a bounded open interval, and let $\varepsilon > 0$. There is a real-analytic function $R_\varepsilon : I \to \mathbb{M}_n(\mathbb{C})$, whose values are self-adjoint matrices, such that $\sup_{t\in I}\|R_\varepsilon(t) - R(t)\| \leqslant \varepsilon/2$. Consider the differential system

$$\frac{dW}{dt} = i\bigl(R_\varepsilon(t) + \varepsilon I_n\bigr)W. \tag{3}$$

Now $R_\varepsilon(t) + \varepsilon I_n$ is real-analytic and strictly positive definite on I. Let $t_0 \in I$, and let $W^\varepsilon(t)$ be the solution of (3) such that $W^\varepsilon(t_0) = W(t_0)$. Thus, $W^\varepsilon(\cdot)$ also takes values in the complex unitary group $U(n)$.

Using [15, Theorem II.1.10] together with the constructions in ([15, Section II.4.2]; also Daleckii–Krein [5, Chapter 4]) one can show that there are analytic families of eigenvalues $\rho_1^\varepsilon(t), \dots, \rho_n^\varepsilon(t)$ together with $\mathbf{v}_1^\varepsilon(t), \dots, \mathbf{v}_n^\varepsilon(t)$ analytic families of eigenvectors of $W^\varepsilon(t)$, $t \in I$. Write $\rho_k^\varepsilon(t) = e^{i\varphi_k^\varepsilon(t)}$ for analytic functions $\varphi_k^\varepsilon : I \to \mathbb{R}$. If ε is small enough, we can reorder the $\rho_k^\varepsilon(\cdot)$ and adjust the initial values $\varphi_k^\varepsilon(t_0)$ in such a way that $|\varphi_k^\varepsilon(t_0) - \varphi_k(t_0)| < 1$, $1 \leqslant k \leqslant n$. Having done these things, we now argue as on p. 266 of [26] to show that $\varphi_k^\varepsilon(t)$ is strictly increasing on I.

Choose a sequence of positive numbers $\varepsilon_j \to 0$. Write $\rho_k^j(t) = \rho_k^{\varepsilon_j}(t)$ for each $1 \leqslant k \leqslant n$, $t \in I$. The unordered set $\{\rho_1^j(t), \dots, \rho_n^j(t)\}$ converges to the unordered set $\{\rho_1(t), \dots, \rho_n(t)\}$ uniformly on I when $j \to \infty$. It follows that the continuous branches $\varphi_1(t), \dots, \varphi_n(t)$ are nondecreasing functions of t. $\square$

Recall now that Hypothesis 2.1 is in force. We will relate the concepts of nonoscillation and weak disconjugacy. It will turn out that weak disconjugacy implies nonoscillation but that the converse implication does not hold in general. The converse is true when an additional condition is valid (Hypothesis 2.7 below). This hypothesis can be viewed as a weak version of the classical condition of *identical normality*.

Let us now prove that, if the system (1) is weakly disconjugate, then it is nonoscillatory. For this, it is convenient to introduce the manifold of Lagrange planes in $\mathbb{R}^{2n}$. Let $l \subset \mathbb{R}^{2n}$ be a vector subspace of dimension n such that $\langle \mathbf{z}_1, J\mathbf{z}_2\rangle = 0$

for all $\mathbf{z}_1$, $\mathbf{z}_2 \in l$. Then l is called a *Lagrange plane* in $\mathbb{R}^{2n}$. Let $\mathcal{L}$ be the Grassmann-type manifold of all Lagrange planes in $\mathbb{R}^{2n}$. Then $\mathcal{L}$ is a real-analytic manifold of dimension $n(n+1)/2$ (see [19]).

Let $\{\mathbf{e}_1, \dots, \mathbf{e}_{2n}\}$ be the canonical basis in $\mathbb{R}^{2n}$ and let l_0 be the *vertical Lagrange plane*, i.e., $l_0 = \mathrm{Span}\{\mathbf{e}_{n+1}, \dots, \mathbf{e}_{2n}\} \in \mathcal{L}$. Let $\mathcal{C} \subset \mathcal{L}$ be the set of elements $l \in \mathcal{L}$ which intersect l_0 nontrivially. It turns out that $\mathcal{C}$ is a $\mathbb{Z}_2$-cycle in $\mathcal{L}$, of codimension 1. It also turns out that $\mathcal{L}$ defines a generator of the first cohomology group $\mathrm{H}^1(\mathcal{L}, \mathbb{Z}) \cong \mathbb{Z}$ in a well-known way [1].

Now let $l \in \mathcal{L}$, and let $\mathbf{z}_1, \dots, \mathbf{z}_n$ be n linearly independent vectors which span l. We associate with l the $2n \times n$ matrix $\binom{U}{V}$ whose columns are $\mathbf{z}_1, \dots, \mathbf{z}_n$; here U and V are of course $n \times n$ real matrices. It turns out that two $2n \times n$ matrices $\binom{U}{V}$ and $\binom{U_1}{V_1}$ are associated with the same Lagrange plane l if and only if there exists a real orthogonal $n \times n$ matrix O such that $U_1 = UO$ and $V_1 = VO$. In what follows, we will not always distinguish carefully a Lagrange plane l from an $2n \times n$ matrix $\binom{U}{V}$ which is associated to it.

Proposition 2.5. *(See [17].) Let $Z(t) = \begin{pmatrix} U_1(t) & U_2(t) \\ V_1(t) & V_2(t) \end{pmatrix}$ be the fundamental matrix solution of* (1) *and let $W(t) = (U_1(t) - iU_2(t))^{-1}(U_1(t) + iU_2(t))$, as before. Then $\det U_2(t) = 0$ if and only if 1 is an eigenvalue of $W(t)$. In fact, if $\mathbf{z} \in \mathbb{C}^n$, then $U_2(t)\mathbf{z} = 0$ if and only if $W(t)\mathbf{z} = \mathbf{z}$.*

Proposition 2.6. *If the system* (1) *is weakly disconjugate then it is nonoscillatory.*

Proof. Let $t_* > 0$ be the positive time in the definition of weak disconjugacy. If $t > t_*$ and $1 \leqslant k \leqslant n$, then the angle $\varphi_k(t)$ must take values in some interval $(2\pi m_k, 2\pi(m_k + 1))$ where $m_k \in \mathbb{Z}$. This follows from Proposition 2.5. In fact, if the stated condition does not hold, then 1 is an eigenvalue of $W(t)$ for some $t > t_*$. But in this case, there exists a nonzero vector $\mathbf{y}_0 \in \mathbb{R}^n$ such that $U_2(t)\mathbf{y}_0 = 0$, and then $\mathbf{z}(t) = Z(t)\binom{0}{\mathbf{y}_0}$ lies in l_0. This contradicts the assumption that (1) is weakly disconjugate. We conclude that $\mathrm{Arg}_Z(t) = \frac{1}{2}\sum_{k=1}^n [\varphi_k(t) - \varphi_k(0)]$ is bounded and the system (1) is nonoscillatory. $\square$

We turn to the converse statement. It is clear that nonoscillation is not in itself sufficient to guarantee that system (1) is weakly disconjugate (just set $H(t) \equiv 0$). So we introduce an appropriate

Hypothesis 2.7. Let $0 \neq \mathbf{y}_0 \in \mathbb{R}^n$, $\mathbf{z}_0 = \binom{0}{\mathbf{y}_0} \in \mathbb{R}^{2n}$, and $\binom{\mathbf{x}(t)}{\mathbf{y}(t)} = Z(t)\mathbf{z}_0$, $t \in \mathbb{R}$. There is a sequence $t_n \uparrow \infty$ (which may depend on $\mathbf{y}_0$) such that $\mathbf{x}(t_n) \neq 0$.

Proposition 2.8. *Suppose that Hypotheses* 2.1 *and* 2.7 *are valid. If system* (1) *is nonoscillatory, then it is weakly disconjugate.*

Proof. We introduce continuous determinations $\rho_1(t), \dots, \rho_n(t)$ of the eigenvalues of $W(t)$, and continuous functions $\varphi_1, \dots \varphi_n : \mathbb{R} \to \mathbb{R}$ such that $\rho_k(t) = e^{i\varphi_k(t)}$, $1 \leqslant k \leqslant n$. By Proposition 2.4, the angles $\varphi_1, \dots, \varphi_n$ are nondecreasing functions of t. By assumption, the quantity $\mathrm{Arg}_Z(t) = \frac{1}{2}\sum_{k=1}^n [\varphi_k(t) - \varphi_k(0)]$ is bounded as $t \to \infty$. Therefore, the limits $\lim_{t\to\infty} \varphi_k(t) = \varphi(\infty)$ exist and are finite, $1 \leqslant k \leqslant n$. It follows from Proposition 2.5 that there exists $t_0 > 0$ such that, if $t \geqslant t_0$, then the dimension $\dim[Z(t)l_0 \cap l_0]$ equals a constant p which does not depend on t. If $p = 0$, then the system (1) is weakly disconjugate. So we assume for contradiction that $p \geqslant 1$.

We will show that there is a fixed vector subspace $m_0 \subset l_0$, of dimension p, such that $Z(t)l_0 \cap l_0 = Z(t)m_0$ for all $t \geqslant t_0$. Let us assume for the moment that this has been done. Then, if $0 \neq \mathbf{z}_0 = \binom{0}{\mathbf{y}_0} \in m_0$, the vector $\mathbf{z}_0$ does not satisfy the condition imposed in Hypothesis 2.7. Hence the proof that system (1) is weakly disconjugate will be complete when we have proved the existence of a subspace $m_0 \subset l_0$ which satisfies the above conditions.

We turn to the proof of the existence of m_0. As before, we write the fundamental matrix solution of (1) in the form $Z(t) = \begin{pmatrix} U_1(t) & U_2(t) \\ V_1(t) & V_2(t) \end{pmatrix}$. Let us identify $\mathbb{R}^n$ with l_0 via the map $\mathbf{y} \to \binom{0}{\mathbf{y}}$, then define $m(t) \subset l_0$ to be the subspace of eigenvectors of $U_2(t)$ which correspond to the eigenvalue zero, $t \geqslant t_0$. We now apply Proposition 2.5: $m(t)$ can be identified with the space of real eigenvectors of $W(t) = (U_1(t) - iU_2(t))^{-1}(U_1(t) + iU_2(t))$ which correspond to the eigenvalue 1; moreover the 1-eigenspace of $W(t)$ in $\mathbb{C}^n$ equals the complexification of $m(t)$. (The last statement also follows from the symmetry of $W(t)$ [17].) We have in particular that $\dim m(t) = p$ for all $t \geqslant t_0$.

Using results of [15], we conclude that the orthogonal projection $P_t : \mathbb{R}^n \to m(t)$ is C^1 as a function of $t \in [t_0, \infty)$. We can and will view $m(t)$ as an element of the Grassmannian manifold $Gr_{p,n}$ of p-dimensional vector subspaces of $\mathbb{R}^n$. Then $t \mapsto m(t) : [t_0, \infty) \to Gr_{p,n}$ is a C^1-map. Let us consider the following

Condition C. If $I \subset [t_0, \infty)$ is a nondegenerate interval, and if $f : I \to \mathbb{R}^n$ is a C^1 map such that $f(t) \neq 0$ and $f(t) \in m(t)$ for all $t \in I$, then $f'(t) \in m(t)$ for all $t \in I$.

First we show that, if Condition C holds, then $m(t)$ is constant on $[t_0,\infty)$. Let $t_* > t_0$ and let $I \subset [t_0,\infty)$ be an open interval containing t_* with the following property: there exist C^1 functions $b_1,\dots,b_k : I \to \mathbb{R}^n$ such that $\{b_1(t),\dots,b_k(t)\}$ is a basis of $m(t)$ for each $t \in I$. By Condition C, $b_1'(t) \in m(t),\dots,b_k'(t) \in m(t)$ for each $t \in I$. Thus

$$b_i'(t) = \sum_{j=1}^{k} c_{ij}(t) b_j(t), \quad t \in I,\ 1 \leqslant i \leqslant k,$$

where $c_{ij} : I \to \mathbb{R}$ is continuous, $1 \leqslant i, j \leqslant k$. Let $C(t) = (c_{ij}(t))_{i,j=1}^{k}$ and consider the differential system

$$\mathbf{x}' = C(t)\mathbf{x}, \quad \mathbf{x} \in \mathbb{R}^k.$$

Let $\Phi(t) = (\phi_{ij}(t))_{i,j=1}^{k}$ be the fundamental matrix solution of this system with $\Phi(t_*) = I_k$. By uniqueness of solutions we deduce that

$$b_i(t) = \sum_{j=1}^{k} \Phi_{ij}(t) b_j(t_*), \quad t \in I,$$

and $m(t) = \mathrm{Span}\{b_1(t),\dots,b_k(t)\} = \mathrm{Span}\{b_1(t_*),\dots,b_k(t_*)\}$ for all $t \in I$. But then it follows immediately from the definition of $m(t)$ that $m(t_0) = m(t) = Z(t)l_0 \cap l_0$ for all $t \geqslant t_0$. So the subspace $m_0 = m(t_0) \subset l_0$ satisfies the condition we require.

We proceed to show that Condition C is indeed valid. Let $f : I \in \mathbb{R}^n$ be a C^1-map such that $f(t) \neq 0$ and $f(t) \in m(t)$ for all $t \in I$. There is no loss of generality in assuming that $\|f(t)\| = 1$, $t \in I$. We have $U_2(t)f(t) = 0$ for $t \in I$, so

$$U_2 f' = -U_2' f = -H_{22} g \quad \text{where } g = V_2 f$$

for all $t \in I$.

There is a fixed set $\{j_1,\dots,j_p\}$ of indices in $\{1,\dots,n\}$ such that $\rho_{j_k}(t) = 1$ for $t \geqslant t_1$, $1 \leqslant k \leqslant p$. For each $j \in \{j_1,\dots,j_p\}$ we can view f as a C^1 family of eigenvectors of the family W, which corresponds to the eigenvalue $\rho_j = \rho_j(t) = 1$. Then $W(t)f(t) = f(t) = e^{i\varphi_j(t)} f(t)$ where we can set $\varphi_j(t) = 0$, $t \in I$. Hence

$$0 = \dot{\varphi}_j(t) = \langle R(t)f(t), f(t)\rangle$$

where as before $R = 2\overline{\phi^{-1}} H_{22} (\overline{\phi^{-1}})^*$ and $\phi = U_1 + iU_2$; see the proof of Proposition 2.4.

We claim that $(\overline{\phi^{-1}})^* f = g$. This uses calculations of Lidskii [17] which we repeat. First of all, Z is symplectic, so $U_1^t V_2 - V_1^t U_2 = I_n$ and $U_2^t V_2 = V_2^t U_2$, $t \in I$. Hence $U_1^t V_2 f = f + V_1^t U_2 f$ and $U_2^t V_2 f = V_2^t U_2 f = 0$, so

$$f = \big(U_1^t + iU_2^t\big) V_2 f = (U_1 - iU_2)^* g, \quad t \in I.$$

This proves the claim. We conclude that $g(t) = V_2(t)f(t)$ lies in the kernel of $H_{22}(t)$ for each $t \in I$. This means that $f'(t)$ lies in the kernel of $U_2(t)$ for each $t \in I$; that is, $f'(t) \in m(t)$, $t \in I$. This completes the proof. □

We close this section with an example which illustrates the significance of Hypothesis 2.7. In fact one might conjecture that, if $H_{22}(t) \geqslant 0$ for all $t \geqslant 0$, and if Hypothesis 2.7 holds, then there is a sequence $t_k \to \infty$ such that $Z(t_k)l_0 \cap l_0 = \{0\}$. This conjecture is indeed true if $n = 1$, but need not be true if $n = 2$, as we now illustrate.

Example 2.9. Consider the 4-dimensional differential system

$$Jz' = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} z = H(t)z \tag{4}$$

where $\mathbf{z} = \binom{\mathbf{x}}{\mathbf{y}}$, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, and B, C are real diagonal 2×2 matrices. In the previous notation we have $H_{22} \equiv 0$.

We introduce piecewise-continuous, 2π-periodic matrix functions $B(\cdot)$, $C(\cdot)$ as follows:

$$B(t) = \begin{cases} \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, & 0 \leqslant t < \pi, \\ \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, & \pi \leqslant t < 2\pi, \end{cases} \qquad C(t) = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, & 0 \leqslant t < \pi, \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, & \pi \leqslant t < 2\pi. \end{cases}$$

Let $\mathbf{z}_3(t)$ be the solution of (4) such that $\mathbf{z}_3(0) = \mathbf{e}_3 = (0,0,1,0)^t$. Then $\mathbf{z}_3(t)$ rotates through an angle of π radians in the $\mathbf{e}_1$–$\mathbf{e}_3$ plane as t varies between 0 and π. On the other hand, the solution $\mathbf{z}_4(t)$ of (4) such that $\mathbf{z}_4(0) = \mathbf{e}_4 = (0,0,0,1)^t$ rotates through an angle of π radians in the $\mathbf{e}_2$–$\mathbf{e}_4$ plane as t varies between π and 2π. One can check that Hypothesis 2.7 is valid for system (4). However $Z(t)l_0 \cap l_0 \neq \{0\}$ for all $t \geqslant 0$. In geometric terms, the curve $t \mapsto Z(t)l_0 : \mathbb{R} \to \mathcal{L}$ remains in the Maslov cycle $\mathcal{C}$ for all $t \geqslant 0$, in fact for all $t \in \mathbb{R}$.

It is clear that one can modify $H(\cdot)$ in such a way that H is continuous and π-periodic, Hypotheses 2.1 and 2.7 hold, and $Z(t)l_0 \in \mathcal{C}$ for all $t \in \mathbb{R}$. One can make further modification so as to ensure that the above conditions hold, and also $H_{22}(t) > 0$ for some (but not all) $t \in [0,2\pi]$.

Note that the system (4) is not weakly disconjugate, so by Proposition 2.8, $\mathrm{Arg}_Z(t)$ must be unbounded.

3. The principal solution

It is well known that a disconjugate linear Hamiltonian system admits a principal solution, in other words a *minimal* isotropic solution. On the other hand, simple examples show that a weakly disconjugate system (1) need not admit a principal solution.

In this section, we show that, roughly speaking, if the hypotheses imposed in Section 2 are required to hold uniformly with respect to translations of the argument t of the coefficient matrix $H(t)$, then (1) admits a principal solution. In fact, each system in the topological hull of (1) is weakly disconjugate and admits a principal solution.

We then introduce a perturbation of (1) which depends on an Atkinson-type spectral parameter λ. We show that, under the uniformity conditions mentioned above, the perturbed system is weakly disconjugate and admits an exponential dichotomy for negative values of λ. In particular, it admits a principal solution for $\lambda < 0$. We show that, as $\lambda \to 0^-$, the principal solution of the perturbed system converges to that of the original system (1).

We begin the discussion by returning to the linear Hamiltonian system (1)

$$J\mathbf{z}' = H(t)\mathbf{z}, \quad \mathbf{z} \in \mathbb{R}^{2n},$$

where $H : \mathbb{R} \to \mathbb{M}_{2n}$ is bounded, continuous, and assumes symmetric values. We write $H = \left(\begin{smallmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{smallmatrix}\right)$ where $H_{11}^t = H_{11}$, $H_{22}^t = H_{22}$ and $H_{12}^t = H_{21}$. Let $Z(t)$ be the fundamental matrix solution of (1).

To avoid interruption of the later discussion, we formulate two basic results concerning the system (1).

Lemma 3.1. *Assume that the system* (1) *satisfies Hypotheses* 2.1 *and* 2.7. *This means that* $H_{22}(t) \geqslant 0$ *for all* $t \geqslant 0$, *and that each solution* $\mathbf{z}(t) = \left(\begin{smallmatrix} \mathbf{x}(t) \\ \mathbf{y}(t) \end{smallmatrix}\right) = Z(t)\mathbf{z}_0$ *of* (1) *such that* $0 \neq \mathbf{z}_0 = \left(\begin{smallmatrix} 0 \\ \mathbf{y}_0 \end{smallmatrix}\right)$ *admits a sequence* $t_k \uparrow \infty$ *such that* $\mathbf{x}(t_k) \neq 0$.

(a) *Let* $\Psi(t)$ *be the fundamental matrix solution of* $\mathbf{x}' = H_{21}(t)\mathbf{x}$. *There exist numbers* $T > 0$, $\delta > 0$ *such that, if* $\mathbf{y}_0 \in \mathbb{R}^n$, *then*

$$\int_0^T \left\| H_{22}(t)\Psi^t(t)^{-1}\mathbf{y}_0 \right\|^2 dt \geqslant \delta \|\mathbf{y}_0\|^2. \tag{C}$$

(b) *There exist numbers* $T > 0$, $\delta > 0$ *with the following property. If* $\mathbf{z}_0 \in \mathbb{R}^{2n}$ *and if* $\left(\begin{smallmatrix} \mathbf{x}(t) \\ \mathbf{y}(t) \end{smallmatrix}\right) = Z(t)\mathbf{z}_0$, *then*

$$\int_0^T \left\|\mathbf{x}(t)\right\|^2 dt \geqslant \delta \|\mathbf{z}_0\|^2. \tag{A}$$

Proof. (a) Suppose for contradiction that there exist sequences $t_k \uparrow \infty$ and $\{\mathbf{y}_k\} \subset \mathbb{R}^n$ such that $\|\mathbf{y}_k\| = 1$ and

$$\int_0^{t_k} \left\| H_{22}(t)\Psi^t(t)^{-1}\mathbf{y}_k \right\|^2 dt \leqslant \frac{1}{k}.$$

Passing to a subsequence if necessary, we can assume that $\mathbf{y}_k \to \mathbf{y}_0 \in \mathbb{R}^n$, where of course $\|\mathbf{y}_0\| = 1$. Then $\int_0^\infty \|H_{22}(t) \times \Psi^t(t)^{-1}\mathbf{y}_0\|^2 dt = 0$, and $H_{22}(t)\Psi^t(t)^{-1}\mathbf{y}_0 = 0$ for each $t \geqslant 0$. Set $\mathbf{z}_0 = \left(\begin{smallmatrix} 0 \\ \mathbf{y}_0 \end{smallmatrix}\right)$, then set $\mathbf{z}(t) = \left(\begin{smallmatrix} \mathbf{x}(t) \\ \mathbf{y}(t) \end{smallmatrix}\right) = Z(t)\mathbf{z}_0$. One has $\mathbf{x}' = H_{21}\mathbf{x} + H_{22}\mathbf{y}$, $\mathbf{y}' = -H_{12}\mathbf{y} - H_{11}\mathbf{x}$, and hence $\mathbf{z}(t) = \left(\begin{smallmatrix} 0 \\ \Psi^t(t)^{-1}\mathbf{y}_0 \end{smallmatrix}\right)$. This clearly contradicts Hypothesis 2.7.

(b) If the thesis is false, then there exist sequences $t_k \uparrow \infty$ and $\{\mathbf{z}_k\} \subset \mathbb{R}^{2n}$ with the following property: if $\left(\begin{smallmatrix} \mathbf{x}_k(t) \\ \mathbf{y}_k(t) \end{smallmatrix}\right) = Z(t)\mathbf{z}_k$, then $\int_0^{t_k} \|\mathbf{x}_k(t)\|^2 dt \leqslant (1/k)\|\mathbf{z}_k\|$. Assume without loss of generality that $\|\mathbf{z}_k\| = 1$ for $k \geqslant 1$, and that $\mathbf{z}_k \to \mathbf{z}_0 \in \mathbb{R}^{2n}$. Let $\left(\begin{smallmatrix} \mathbf{x}_0(t) \\ \mathbf{y}_0(t) \end{smallmatrix}\right) = Z(t)\mathbf{z}_0$. Then

$$\int_0^\infty \left\|\mathbf{x}_0(t)\right\|^2 dt = 0 \quad \Rightarrow \quad \mathbf{x}_0(t) = 0 \quad (t \geqslant 0).$$

This contradicts Hypothesis 2.7 and finishes the proof. □

We remark that Lemma 3.1(a) states that the control system $\mathbf{x}' = H_{21}\mathbf{x} + H_{22}\mathbf{u}$ is null controllable, while Lemma 3.1(b) states that the ($2n$-dimensional) control system $\mathbf{z}' = -H^t J\mathbf{z} + \left(\begin{smallmatrix} I_n & 0 \\ 0 & 0 \end{smallmatrix}\right)\mathbf{u}$ is null controllable. See, e.g., [16]. Condition (A) is to be viewed as an Atkinson condition, of which more later.

Next we define the concept of principal solution of the system (1).

Definition 3.2. Let $\binom{U(t)}{V(t)}$ be a $2n \times n$ matrix solution (1) which defines a Lagrange plane $l(t)$ for each $t \in \mathbb{R}$, that is, $\binom{U(t)}{V(t)}$ is isotropic (also called *conjoined solution*). We say that $\binom{U(t)}{V(t)}$ is a *principal solution* of (1) if the following conditions are satisfied. First, $\det U(t) \neq 0$ for all $t \in \mathbb{R}$. Second, the matrix $J(t) = \int_0^t U(s)^{-1} H_{22}(t) U^t(s)^{-1}\, ds$ is strictly positive definite for all sufficiently large $t > 0$. Third, $\lim_{t\to\infty} J(t)^{-1} = 0$.

Let us note that the first condition in Definition 3.2 states that $l(t)$ lies off the Maslov cycle $\mathcal{C}$ for all $t \in \mathbb{R}$.

Our first goal is to introduce sufficient conditions under which system (1) admits a principal solution. To this end, we introduce certain hypotheses which regard the family of Hamiltonian systems obtained by translating the argument of $H(\cdot)$. For each $s \in \mathbb{R}$ consider

$$J\mathbf{z}' = H(s+t)\mathbf{z}, \quad \mathbf{z} \in \mathbb{R}^{2n}. \tag{1_s}$$

Clearly $Z_s(t) = Z(t+s)Z(s)^{-1}$ is the fundamental matrix solution of (1_s). For each $s \in \mathbb{R}$, $Z_s(t)$ gives rise to an argument function $\mathrm{Arg}_s(t) = \mathrm{Arg}_{Z_s}(t)$; see Section 2.

We introduce the following hypotheses, which will hold throughout Section 3. Notice that the second, third, and fourth hypotheses are uniform versions of those imposed in Section 2, while the first merely repeats Hypothesis 2.1.

Hypothesis 3.3.

(0) $H_{22}(t) \geqslant 0$ for all $t \in \mathbb{R}$.
(i) $H(t)$ is a uniformly continuous (and uniformly bounded) function of $t \in \mathbb{R}$.
(ii) For each $0 \neq \mathbf{y}_0 \in \mathbb{R}^n$ there exist numbers $T_0 > 0$, $\delta_0 > 0$ (which may depend on $\mathbf{y}_0$) with the following property: if $s \in \mathbb{R}$ and $\binom{\mathbf{x}(t)}{\mathbf{y}(t)} = Z_s(t)\binom{0}{\mathbf{y}_0}$, then there is a time $t_s \in [0, T_0]$ such that $\|x(t_s)\| \geqslant \delta_0$.
(iii) Eq. (1) admits a $2n \times n$ isotropic matrix solution $\binom{U(t)}{V(t)}$ such that $\det U(t) \neq 0$ for all $t \in \mathbb{R}$ and $V(t)U(t)^{-1}$ is bounded on $\mathbb{R}$.

Let us explain the significance of Hypothesis 3.3(ii). It follows from Hypothesis 2.7 that there exists $T_0 > 0$ such that, if $0 \neq \mathbf{y}_0 \in \mathbb{R}^n$ and $\binom{\mathbf{x}(t)}{\mathbf{y}(t)} = Z(t)\binom{0}{\mathbf{y}_0}$, then $\mathbf{x}(t_0) \neq 0$ for some $t_0 \in [0, T_0]$. So it is clear that Hypothesis 3.3(ii) *uniformizes* Hypothesis 2.7 with respect to $s \in \mathbb{R}$.

As for Hypothesis 3.3(iii), it implies that, for each $s \in \mathbb{R}$, Eq. (1_s) admits the $2n \times n$ matrix solution $\binom{U_s(t)}{V_s(t)} = \binom{U(t+s)}{V(t+s)}$ where $\det U_s(t) \neq 0$ for all $t \in \mathbb{R}$. Using Theorem 4 of [3], one see that $\mathrm{Arg}_s(t)$ is bounded for each $s \in \mathbb{R}$. In fact a bit more work shows that there exists a fixed $M > 0$ such that $0 \leqslant \mathrm{Arg}_s(t) \leqslant M$ for all $s \in \mathbb{R}$, $t \geqslant 0$. In this sense, Hypothesis 3.3(iii) uniformizes the nonoscillation condition imposed in the statement of Proposition 2.8.

Now we introduce the topological hull $\Omega = \Omega_H$ of the bounded uniformly continuous function H. We review the construction of Ω. Let $BC = C(\mathbb{R}, \mathbb{M}_{2n})$ be the space of bounded continuous functions form $\mathbb{R}$ to $\mathbb{M}_{2n}$. Give BC the compact-open topology. Introduce the Bebutov (translation) flow on BC: if $c \in BC$, set $\tau_t(c)(\cdot) = c(\cdot + t)$, $t \in \mathbb{R}$. Define $\Omega = \Omega_H = \mathrm{cls}\{\tau_t(H) \mid t \in \mathbb{R}\}$. Then Ω is compact because H is uniformly continuous. Clearly Ω is invariant with respect to the flow $\{\tau_t \mid t \in \mathbb{R}\}$.

One can extend H to Ω in the following way. Set $\widetilde{H}(\omega) = \omega(0)$, $\omega \in \Omega$. Then $\widetilde{H} : \Omega \to \mathbb{M}_{2n}$ is continuous. Let us write $H(\cdot) = \omega_0 \in \Omega$. We observe that $\widetilde{H}(\tau_t(\omega_0)) = H(t)$, $t \in \mathbb{R}$. Let us abuse notation, and write H instead of $\widetilde{H}$. Then the system (1) is an element of the family of nonautonomous linear Hamiltonian systems

$$J\mathbf{z}' = H\big(\tau_t(\omega)\big)\mathbf{z}, \quad \mathbf{z} \in \mathbb{R}^{2n},\ \omega \in \Omega. \tag{5_ω}$$

In fact, we recover Eq. (1) by setting $\omega = \omega_0$, i.e., $(1) = (5_{\omega_0})$. We write $H(\omega) = (H_{ij}(\omega))_{1\leqslant i,j\leqslant 2}$, and observe that $H_{22}(\omega) \geqslant 0$ for all $\omega \in \Omega$. Let $Z_\omega(t)$ be the fundamental matrix solution of (5_ω), $\omega \in \Omega$.

The next result show that Hypotheses 3.3 also hold for the family of systems (5_ω).

Proposition 3.4. *Assume that Hypotheses* 3.3 *hold. Then*

(i) *Hypothesis* 2.7 *holds for each Eq.* (5_ω)*, i.e., if* $0 \neq \mathbf{y}_0 \in \mathbb{R}^n$ *and if* $\binom{\mathbf{x}(t)}{\mathbf{y}(t)} = Z_\omega(t)\binom{0}{\mathbf{y}_0}$ *then* $\mathbf{x}(t_n) \neq 0$ *for a sequence* $t_n \uparrow \infty$.
(ii) *Eq.* (5_ω) *admits a* $2n \times n$ *isotropic matrix solution* $\binom{U_\omega(t)}{V_\omega(t)}$ *such that* $\det U_\omega(t) \neq 0$ *for all* $t \in \mathbb{R}$.

Proof. (i) Assume on the contrary that there is a $\widetilde{\omega} \in \Omega$ and a solution $\binom{\mathbf{x}(t)}{\mathbf{y}(t)} = Z_{\widetilde{\omega}}(t)\binom{0}{\mathbf{y}_0}$ such that $\mathbf{x}(t) = 0$ for each $t \geqslant 0$. Let T_0 and δ_0 be the constants of Hypothesis 3.3(ii) for $\mathbf{y}_0$. Since $\widetilde{\omega} = \lim \tau_{t_k}(\omega_0)$ for some sequence $\{t_k\}_{k\in\mathbb{N}}$, we deduce that $\mathbf{x}(t) = \lim_{k\to\infty} \mathbf{x}_k(t)$ on $[0, T_0]$, where $\binom{\mathbf{x}_k(t)}{\mathbf{y}_k(t)} = Z_{t_k}(t)\binom{0}{\mathbf{y}_0}$. However, from Hypothesis 3.3(ii), for each k there is an $s_k \in [0, T_0]$ such that $\|\mathbf{x}_k(s_k)\| \geqslant \delta_0$, which contradict that $\mathbf{x} \equiv 0$ in $[0, T_0]$.

(ii) Fix $\omega \in \Omega$. Then $\omega = \lim \tau_{t_k}(\omega_0)$ for some sequence $\{t_k\}_{k\in\mathbb{N}}$. By Hypothesis 3.3(iii), there is a $2n \times n$ isotropic matrix solution $\binom{U(t)}{V(t)}$ such that $\det U(t) \neq 0$ for all $t \in \mathbb{R}$ and $M(t) = V(t)U(t)^{-1}$ is bounded on $\mathbb{R}$. Now $M(t)$ is a symmetric $n \times n$

matrix valued solution of the Riccati equation

$$M' + H_{11}(t) + H_{12}(t)M + MH_{21}(t) + MH_{22}(t)M = 0.$$

Since M is bounded the sequence of functions $\{M(\cdot + t_k)\}$ is bounded and equicontinuous on each compact subinterval of $\mathbb{R}$. Using the theorem of Arzelà–Ascoli we can find a bounded solution of the Riccati equation

$$M' + H_{11}\big(\tau_t(\omega)\big) + H_{12}\big(\tau_t(\omega)\big)M + MH_{21}\big(\tau_t(\omega)\big) + MH_{22}\big(\tau_t(\omega)\big)M = 0.$$

Using Lemma 7 in Chapter 2 of [3], we conclude that Eq. (5_ω) admits a $2n \times n$ isotropic matrix solution $\binom{U_\omega(t)}{V_\omega(t)}$ such that $\det U_\omega(t) \neq 0$ for all $t \in \mathbb{R}$ and $V_\omega(t)U_\omega(t)^{-1}$ is bounded on $\mathbb{R}$. □

Lemma 3.5. *Assume that Hypotheses* 3.3 *are valid. There exist positive numbers T and δ with the following property. Let $\Psi_\omega(t)$ be the fundamental matrix solution of*

$$\mathbf{x}' = H_{21}\big(\tau_t(\omega)\big)\mathbf{x}, \quad \mathbf{x} \in \mathbb{R}^n.$$

Let $\mathbf{y}_0 \in \mathbb{R}^n$. Then

$$\int_0^T \big\| H_{22}\big(\tau_t(\omega)\big)\Psi^t_\omega(t)^{-1}\mathbf{y}_0\big\|^2\, dt \geqslant \delta \|\mathbf{y}_0\|^2. \tag{C_ω}$$

Proof. One uses well-known arguments to pass from the statement of Lemma 3.1(a) to that of Lemma 3.5. For the reader's convenience we sketch these arguments.

First note that the thesis of Lemma 3.1(a) is valid for each system (5_ω) because from Proposition 3.4 it satisfies Hypothesis 2.7, and therefore there exist positive numbers T_ω, δ_ω such that the controllability condition (C_ω) holds with T_ω in place of T and δ_ω in place of δ for each $\omega \in \Omega$.

Recall that a subset $M \subset \Omega$ is called minimal if it is invariant and, for each $\omega \in M$, the orbit $\{\tau_t(\omega) \mid t \in \mathbb{R}\}$ is dense in M. Using [10], one can show that the condition (C_ω) holds uniformly on M in the sense that constants T_ω, δ_ω can be chosen to be independent of $\omega \in M$.

Now one applies Lemma 2.5 of [12] to show that there exist positive constants T and δ, which do not depend on $\omega \in \Omega$, such that the controllability condition (C_ω) is valid with these values of T, δ, for all $\omega \in \Omega$. This completes the proof. □

The next result shows that, under Hypotheses 3.3, the systems of the family (5_ω) are uniformly weakly disconjugate in the sense that the constant of the Definition 2.2 of weak disconjugacy does not depend on $\omega \in \Omega$.

Proposition 3.6. *Assume that Hypotheses* 3.3 *are valid and let T be the positive constant of Lemma* 3.5. *If $0 \neq \mathbf{y}_0 \in \mathbb{R}^n$, and $\binom{\mathbf{x}(t)}{\mathbf{y}(t)} = Z_\omega(t)\binom{0}{\mathbf{y}_0}$, then $\mathbf{x}(t) \neq 0$ for all $t \geqslant T$, and thus, the family of systems (5_ω) is uniformly weakly disconjugate.*

Proof. First, although it is not necessary for the proof, we show how to deduce that each system is weakly disconjugate with the nonoscillation techniques from the previous section. Fix $\omega \in \Omega$. Let $\mathrm{Arg}_\omega(t) = \mathrm{Arg}_{Z_\omega}(t)$ be the argument function of Eq. (5_ω). Using Hypothesis 3.3(iii) and the succeeding discussion, one checks that $0 \leqslant \mathrm{Arg}_\omega(t) \leqslant M$ for all $t \geqslant 0$, and system (5_ω) is nonoscillatory. Moreover, from Proposition 3.4, Hypothesis 2.7 holds for (5_ω), and by Proposition 2.8 we conclude that the system (5_ω) is weakly disconjugate.

Next we show that the constant of Definition 2.2 of weak disconjugacy coincides with the constant T in Lemma 3.5, and hence it does not depend on $\omega \in \Omega$. Let $\binom{\mathbf{x}(t)}{\mathbf{y}(t)} = Z_\omega(t)\binom{0}{\mathbf{y}_0}$ be a nontrivial solution of (5_ω). From Proposition 3.4 each Eq. (5_ω) admits a $2n \times n$ isotropic matrix solution $\binom{U_\omega(t)}{V_\omega(t)}$ such that $\det U_\omega(t) \neq 0$ for all $t \in \mathbb{R}$. We assume that $U_\omega(0) = I_n$. Let us set

$$J(t,\omega) = \int_0^t U_\omega(s)^{-1} H_{22}\big(\tau_s(\omega)\big) U^t_\omega(s)^{-1}\, ds. \tag{6}$$

We use the controllability condition (C_ω) and the proof of [3, Proposition 2, p. 38] to see that $J(t,\omega)$ is strictly positive definite for each $t \geqslant T$. Moreover, it is easy to check that

$$\mathbf{x}(t) = U_\omega(t) J(t,\omega)\mathbf{y}_0,$$
$$\mathbf{y}(t) = \big[V_\omega(t) J(t,\omega) + U^t_\omega(t)^{-1}\big]\mathbf{y}_0,$$

and we conclude that $\mathbf{x}(t) \neq 0$ for each $t \geqslant T$, as stated. □

We can now prove the existence and uniqueness of the principal solution. We explain the meaning of uniqueness in this context. Let $\binom{U(t)}{V(t)}$ resp. $\binom{\widetilde{U}(t)}{\widetilde{V}(t)}$ be two principal solutions of (5_ω); then the Lagrange planes $l(t)$ resp. $\widetilde{l}(t)$ which they determine are equal for all $t \in \mathbb{R}$. Note that, when this uniqueness condition holds, the symmetric matrix $n \times n$ matrix $N = V(0)U^{-1}(0)$ is independent of the chosen principal solution. So we can regard N as a parametrization of the set of principal solutions of (5_ω). In fact, we can also regard N as a parametrization of the uniquely defined Lagrange plane $l(0)$ which corresponds to the set of principal solutions of (5_ω). Notice that N depends on ω. In the sequel we will denote $N(\omega)$ when needed.

Theorem 3.7. *Suppose that system* (1), *which coincides with system* (5_{ω_0}), *satisfies Hypotheses* 3.3. *Then for each* $\omega \in \Omega$, *the system* (5_ω) *admits a unique principal solution.*

Proof. Some of the arguments used in this proof follow well-known lines, so we only sketch them.

From Proposition 3.4 we know that Eq. (5_ω) admits a $2n \times n$ isotropic matrix solution $\binom{U_\omega(t)}{V_\omega(t)}$ such that $\det U_\omega(t) \neq 0$ for all $t \in \mathbb{R}$. As in Proposition 3.6, if T denotes the positive number of Lemma 3.5, the matrix $J(t,\omega)$ defined on (6) is strictly positive definite for each $t \geqslant T$, and $\{J(t,\omega)^{-1}\}_{t \geqslant T}$ is nonincreasing in t. Thus, there exists the limit $L_0 = \lim_{t\to\infty} J(t,\omega)^{-1}$.

We claim that $I_n - J(t,\omega)L_0$ is invertible for each $t \in \mathbb{R}$. If $t \geqslant T$

$$I_n - J(t,\omega)L_0 = J(t,\omega)\big[J(t,\omega)^{-1} - L_0\big]$$

which is a product of invertible matrices because from $J(t,\omega) < J(t+T,\omega)$ we deduce that $J(t,\omega)^{-1} - L_0 > J(t+T,\omega)^{-1} - L_0 \geqslant 0$.

Let $0 \leqslant t \leqslant T$. We have $I_n - J(t,\omega)L_0 = \lim_{s\to\infty}(I_n - J(t,\omega)J(s,\omega)^{-1})$. The matrices $I_n - J(t,\omega)J(s,\omega)^{-1}$ and $I_n - J(s,\omega)^{-1/2}J(t,\omega)J(s,\omega)^{-1/2}$ have the same eigenvalues and

$$I_n - J(s,\omega)^{-1/2}J(t,\omega)J(s,\omega)^{-1/2} > I_n - J(s,\omega)^{-1/2}J(t+T,\omega)J(s,\omega)^{-1/2}.$$

From this we can compare the eigenvalues of $I_n - J(t,\omega)L_0$ with those of $I_n - J(t+T,\omega)L_0$ which are strictly positive and conclude that $I_n - J(t,\omega)L_0$ is also invertible. The case $t \leqslant 0$ is completely analogous.

Next write

$$\widetilde{U}_\omega(t) = U_\omega(t)\big[I_n - J(t,\omega)L_0\big],$$
$$\widetilde{V}_\omega(t) = V_\omega(t)\big[I_n - J(t,\omega)L_0\big] - U^t_\omega(t)^{-1}L_0$$

for $t \in \mathbb{R}$. Then $\big[\begin{smallmatrix}\widetilde{U}_\omega(t)\\ \widetilde{V}_\omega(t)\end{smallmatrix}\big]$ is a $2n \times n$ matrix valued solution of (5_ω), which determines a Lagrange plane $\widetilde{l}_\omega(t)$ and satisfies $\det \widetilde{U}_\omega(t) \neq 0$ for each $t \in \mathbb{R}$. Therefore

$$\widetilde{J}(t,\omega) = \int_0^t \widetilde{U}_\omega(s)^{-1} H_{22}\big(\tau_s(\omega)\big)\widetilde{U}^t_\omega(s)^{-1}\,ds$$

is strictly positive definite for each $t \geqslant T$, and as in [3] it can be shown that $\widetilde{J}(t,\omega)^{-1} = J(t,\omega)^{-1} - L_0$ and $\lim_{t\to\infty} \widetilde{J}(t,\omega)^{-1} = 0$.

Let $\big[\begin{smallmatrix}\widehat{U}(t,\omega)\\ \widehat{V}(t,\omega)\end{smallmatrix}\big]$ be a second $2n \times n$ matrix valued solution of (5_ω), which parametrizes a Lagrange plane $\widehat{l}_\omega(t)$ such that $\det \widehat{U}_\omega(t) \neq 0$ for each $t \in \mathbb{R}$ and

$$\lim_{t\to\infty}\left(\int_0^t \widehat{U}_\omega(s)^{-1} H_{22}\big(\tau_s(\omega)\big)\widehat{U}^t_\omega(s)^{-1}\,ds\right)^{-1} = 0.$$

Then, arguing as in p. 42 of [3] one shows that $\big[\begin{smallmatrix}\widetilde{U}_\omega(t)\\ \widetilde{V}_\omega(t)\end{smallmatrix}\big] = \big[\begin{smallmatrix}\widehat{U}_\omega(t)M\\ \widehat{V}_\omega(t)M\end{smallmatrix}\big]$ for each $t \in \mathbb{R}$ and some invertible matrix M. Hence, the Lagrange planes $\widetilde{l}_\omega(t)$ and $\widehat{l}_\omega(t)$ coincide for all $t \in \mathbb{R}$ and a unique principal solution is obtained. $\square$

Moreover, as in the disconjugate case, see p. 44 of [3] and p. 1060 of [13], the principal solutions can be constructed in the following way:

Proposition 3.8. *Assume that Hypotheses* 3.3 *hold. The principal solution of* (5_ω) *can be constructed as*

$$\begin{pmatrix} U(t)\\ V(t)\end{pmatrix} = \lim_{r\to\infty}\begin{pmatrix} U_r(t)\\ V_r(t)\end{pmatrix},$$

where $\binom{U_r(t)}{V_r(t)}$ *is a* $2n \times n$ *nontrivial matrix solution of* (5_ω) *with* $U_r(r) = 0$ *irrespective of a given fixed value of* $V_r(r)$.

As a consequence of Theorem 3.7 and comparison theorems for Riccati equations, we obtain an extension to linear Hamiltonian systems of Sturm's comparison theorem, similar to the one obtained in [3] for disconjugate systems, as well as a comparison result for the corresponding Lagrange planes obtained from the principal solutions.

Proposition 3.9. *Consider two families of linear Hamiltonian systems satisfying Hypotheses* 3.3(0)–(ii)*,*

$$J\mathbf{z}' = H^1\bigl(\tau_t(\omega)\bigr)\mathbf{z}, \quad \omega \in \Omega, \tag{7_ω}$$

$$J\mathbf{z}' = H^2\bigl(\tau_t(\omega)\bigr)\mathbf{z}, \quad \omega \in \Omega, \tag{8_ω}$$

with $H^1(\omega) \leqslant H^2(\omega)$. Then if (8_ω) satisfies Hypothesis 3.3(iii) *so do the systems (7_ω). In particular, they are weakly disconjugate and denoting by $N_1(\omega)$ and $N_2(\omega)$ the symmetric $n \times n$ matrix-valued functions obtained from the respective principal solutions we have*

$$N_1(\omega) \leqslant N_2(\omega).$$

Proof. From Theorem 3.7, we can consider the $n \times n$ symmetric matrices $N_2(\omega)$ defined in terms of the principal solution $\binom{U_2(t)}{V_2(t)}$ for (8_ω) by $N_2(\omega) = V_2(0)U_2^{-1}(0)$. Let M satisfy $N_2(\omega) \leqslant M$ and let $M_1(t)$ be the solution of the Riccati equation

$$M' + H_{11}^1 + H_{12}^1 M + M H_{21}^1 + M H_{22}^1 M = 0$$

with initial condition $M_1(0) = M$. From Proposition 9, p. 52 of [3], $M_1(t)$ is defined for each $t \in \mathbb{R}$ and $N_2(\tau_t(\omega)) \leqslant M_1(t)$ for $t \geqslant 0$. This implies that system (7_ω) admits a solution $\binom{U_1(t)}{V_1(t)}$ which determines a Lagrange plane and satisfies $\det U_1(t) \neq 0$ for all $t \in \mathbb{R}$. Hence, from Theorem 4 of [26] we deduce that systems (7_ω) are nonoscillatory and from Proposition 2.8 we conclude that they are weakly disconjugate. The rest of the proof is completely similar to the one of the Corollary to Theorem 8, p. 54 of [3]. □

Now we recall the definition of exponential dichotomy, first for the single equation $(1) = (5_{\omega_0})$, see [4], and then for the family (5_ω), see [21].

One says that Eq. (1) has an *exponential dichotomy* (ED for short) if there exist positive numbers K and δ together with a projection $P : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that

$$\begin{aligned} &\bigl\| Z(t) P Z(s)^{-1} \bigr\| \leqslant K e^{-\gamma(t-s)}, \quad t \geqslant s, \\ &\bigl\| Z(t)(I_n - P) Z(s)^{-1} \bigr\| \leqslant K e^{\gamma(t-s)}, \quad t \leqslant s. \end{aligned}$$

One says that the family (5_ω) admits an ED (over Ω) if there are positive numbers K and δ, together with a continuous family of projections $\omega \to P_\omega = P_w^2 : \Omega \to \mathbb{M}_{2n}$ such that

$$\begin{aligned} &\bigl\| Z_\omega(t) P_\omega Z_\omega(s)^{-1} \bigr\| \leqslant K e^{-\gamma(t-s)}, \quad t \geqslant s, \\ &\bigl\| Z_\omega(t)(I_n - P_\omega) Z_\omega(s)^{-1} \bigr\| \leqslant K e^{\gamma(t-s)}, \quad t \leqslant s. \end{aligned}$$

Note that the orbit $\{\tau_t(\omega_0) \mid t \in \mathbb{R}\}$ is dense in Ω. This implies that the system (1) admits an ED if and only if the family of systems (5_ω) admits an ED over Ω.

It is time to introduce the following Atkinson-type spectral problem:

$$J\mathbf{z}' = \left[H(t) + \lambda \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix} \right] \mathbf{z}, \quad \mathbf{z} \in \mathbb{R}^{2n}, \tag{1_λ}$$

where 0 denotes the n-dimensional zero matrix. The parameter λ will take on real values. We will show that, if $\lambda < 0$, then the system (1_λ) is weakly disconjugate and admits an ED. Actually, it will be no harder to work with the family

$$J\mathbf{z}' = \left[H\bigl(\tau_t(\omega)\bigr) + \lambda \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix} \right] \mathbf{z}, \quad \mathbf{z} \in \mathbb{R}^{2n},\ \omega \in \Omega. \tag{$5_{\omega,\lambda}$}$$

We will show that the family $(5_{\omega,\lambda})$ is weakly disconjugate, and it admits an ED over Ω when $\lambda < 0$. The first step in proving these assertions is to show that the Atkinson condition (A) holds for each $\omega \in \Omega$. In fact,

Proposition 3.10. *There are positive constants T and δ, which do not depend on $\omega \in \Omega$, such that for each $\omega \in \Omega$*

$$\int_0^T \left\| \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix} Z_\omega(t) \mathbf{z}_0 \right\|^2 dt \geqslant \delta \|\mathbf{z}_0\|^2. \tag{A_ω}$$

Proof. Let us first note that, as shown in the proof of Proposition 3.4, Hypothesis 2.7 holds for each system (5_ω). Therefore we can apply the argument used in proving Lemma (3.1)(b) to each system (5_ω). We conclude that, for each fixed $\omega \in \Omega$, there are positive constants T_ω and δ_ω such that

$$\int_0^{T_\omega} \left\| \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix} Z_\omega(t)\mathbf{z}_0 \right\|^2 dt \geqslant \delta_\omega \|\mathbf{z}_0\|^2$$

for all $\mathbf{z}_0 \in \mathbb{R}^{2n}$.

Now we interpret condition (A_ω) as the controllability condition for the control system

$$\mathbf{z}' = -H^t\big(\tau_t(\omega)\big) J\mathbf{z} + \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}\mathbf{u}, \quad \omega \in \Omega.$$

Arguing as in the proof of Lemma 3.5, we can determine positive constants T and δ, which do not depend on $\omega \in \Omega$, such that the Atkinson condition (A_ω) holds for all $\omega \in \Omega$. This completes the proof. □

Theorem 3.11. *Suppose that system* (1)*, which coincides with system* (5_{ω_0})*, satisfies Hypotheses* 3.3 *and let* $\lambda < 0$*. Then for each* $\omega \in \Omega$ *the linear Hamiltonian system* $(5_{\omega,\lambda})$ *is weakly disconjugate. Moreover, the family of systems* $(5_{\omega,\lambda})$*,* $\omega \in \Omega$*, admits an exponential dichotomy over* Ω*.*

Proof. From Proposition 3.9 we deduce that system $(5_{\omega,\lambda})$ is weakly disconjugate whenever $\omega \in \Omega$ and $\lambda < 0$. Moreover, it turns out that the controllability condition (C_ω) holds also for $\lambda \in \mathbb{R}$, with the same values of T and δ as in Lemma 3.5. Therefore, from Proposition 3.6 the family $(5_{\omega,\lambda})$ is uniformly weakly disconjugate with constant T.

Let $\nu(dt)$ be the spectral matrix of the Atkinson problem

$$\begin{cases} J\mathbf{z}' - H(\tau_t(\omega))\mathbf{z} = \lambda\left(\begin{smallmatrix} I_n & 0 \\ 0 & 0 \end{smallmatrix}\right)\mathbf{z}, \\ \mathbf{z}(-\infty) = \mathbf{z}(\infty) = 0. \end{cases}$$

This object is defined in Chapter 9 of [2]. In the present circumstances, it can be defined to be the weak-* limit $\lim_{r\to\infty} \nu_r(dt)$, where $\nu_r(dt)$ is the spectral matrix of the boundary value problem

$$\begin{cases} J\mathbf{z}' - H(\tau_t(\omega))\mathbf{z} = \lambda\left(\begin{smallmatrix} I_n & 0 \\ 0 & 0 \end{smallmatrix}\right)\mathbf{z}, \quad \mathbf{z} = \left(\begin{smallmatrix} \mathbf{x} \\ \mathbf{y} \end{smallmatrix}\right) \in \mathbb{R}^{2n}, \\ \mathbf{x}(-r) = \mathbf{x}(r) = 0. \end{cases} \tag{9_ω}$$

It is shown in [11] that, if $\lambda \in \mathbb{R}$ and if $\mathbf{z}(t)$ is a nonzero bounded solution of the system $(5_{\omega,\lambda})$, then the spectral matrix $\nu(dt)$ is not constant in any neighborhood of λ. That is, if $\varepsilon > 0$, then $\int_{\lambda-\varepsilon}^{\lambda+\varepsilon} \nu(dt) \neq 0$.

On the other hand, suppose that $\lambda < 0$. Since, as explained above, the family of systems $(5_{\omega,\lambda})$ is uniformly weakly disconjugate with constant T, the following statement is true. Let $r > T/2$; then the boundary value problem (9_ω) admits no nontrivial solution. This means that, if $I \subset (-\infty, 0)$ is an open interval, and if $r > T/2$, then the spectral matrix $\nu_r(dt)$ vanishes on I.

We have shown that, if $\omega \in \Omega$ and $\lambda < 0$, then the system $(5_{\omega,\lambda})$ admits no nontrivial bounded solution. We can now show that, if $\lambda < 0$, then the family $(5_{\omega,\lambda})$ admits an ED over Ω.

To do this, fix $\lambda < 0$. Let M be a minimal subset of Ω. If $\omega \in \Omega$, then the system $(5_{\omega,\lambda})$ admits no nontrivial bounded solutions. According to a theorem of Sacker and Sell ([21]; also Selgrade [23]), the family $(5_{\omega,\lambda})$ admits an ED over M. Let $\omega \in M$, and let P_ω be the corresponding projection. It was shown in [9] that $\dim \operatorname{Im} P_\omega = n$, which of course does not depend on the choice of the minimal set M. Using another result of Sacker and Sell [22], we conclude that the family $(5_{\omega,\lambda})$ admits an ED over Ω. This completes the proof of Theorem 3.11. □

We complete the discussion with a proposition concerning the left continuity of the principal solution of Eq. $(5_{\omega,\lambda})$ as $\lambda \to 0^+$. Before doing so, we make the following remarks.

First, let $\lambda < 0$. Then each system $(5_{\omega,\lambda})$ is weakly disconjugate and admits a principal solution. Also, the family $(5_{\omega,\lambda})$ admits an ED over Ω. Let $\left(\begin{smallmatrix} U(t) \\ V(t) \end{smallmatrix}\right)$ be the principal solution of $(5_{\omega,\lambda})$ for some $\omega \in \Omega$. Then one can show that the Lagrange plane $l(t)$ determined by $\left(\begin{smallmatrix} U(t) \\ V(t) \end{smallmatrix}\right)$ is the image of the dichotomy projection $P_{\tau_t(\omega)}$.

Second, if $\lambda < 0$, then the Lagrange plane $l(\omega) = \operatorname{Im} P_\omega$ varies continuously with $\omega \in \Omega$. This sort of continuity need not hold if $\lambda = 0$. More precisely, let $\left(\begin{smallmatrix} U(t) \\ V(t) \end{smallmatrix}\right)$ be the principal solution of Eq. $(5_\omega) = (5_{\omega,0})$. Let $l(\omega)$ be the Lagrange plane determined by $\left(\begin{smallmatrix} U(0) \\ V(0) \end{smallmatrix}\right)$. Then the map $\omega \to l(\omega)$ need not be continuous. Examples for which $\omega \to l(\omega)$ is discontinuous can be constructed using a method of Millionščikov ([18], also Vinograd [24]). See [8] for the details.

Having made these remarks, we return to our one-sided continuity result. Fix $\omega \in \Omega$ and $\lambda \geqslant 0$. Let $\left(\begin{smallmatrix} U(t) \\ V(t) \end{smallmatrix}\right)$ be the principal solution of Eq. $(5_{\omega,\lambda})$ and let $l(\omega, \lambda)$ be the Langrange plane determined by $\left(\begin{smallmatrix} U(0) \\ V(0) \end{smallmatrix}\right)$. Then $l(\omega, \lambda)$ is uniquely determined by the $n \times n$ symmetric matrix $N(\omega, \lambda) = V(0)U^{-1}(0)$.

Proposition 3.12. *With the notation as above: for each $\omega \in \Omega$, one has*

$$\lim_{\lambda \to 0^-} l(\omega, \lambda) = l(\omega, 0)$$

where the convergence is in the Grassmann sense.

Proof. The same proof of point (ii) in [14, Theorem 4.4] applies here. We include it for completeness. It is enough to check that

$$\lim_{\lambda \to 0^-} N(\omega, \lambda) = N(\omega, 0).$$

From Proposition 3.9 we deduce that for $\lambda \leqslant \lambda'$, $N(\omega, \lambda) \leqslant N(\omega, \lambda')$. Therefore, there exist the limits

$$\lim_{\lambda \to 0^-} N(\omega, \lambda) = N_0(\omega)$$

and they are finite. In order to show the coincidence of $N_0(\omega)$ with $N(\omega, 0)$, as in Proposition 3.8 it can be shown that

$$N(\omega, \lambda) = \lim_{r \to \infty} M_r(\omega, \lambda),$$

where $M_r(\omega, \lambda) = V_r(0, \omega, \lambda) U_r^{-1}(0, \omega, \lambda) \leqslant N(\omega, \lambda)$ and $\binom{U_r(t,\omega,\lambda)}{V_r(t,\omega,\lambda)}$ is the solution of $(5_{\omega,\lambda})$ with initial conditions $U_r(r, \omega, \lambda) = 0$, $V_r(r, \omega, \lambda) = I_n$. Hence

$$N(\omega, 0) = \lim_{r \to \infty} M_r(\omega, 0),$$
$$N(\omega, 0) \leqslant N_0(\omega),$$
$$N(\omega, 0) \geqslant M_r(\omega, 0) \geqslant M_r(\omega, \lambda), \quad \lambda \leqslant 0, \ r > 0.$$

Thus,

$$\begin{aligned} 0 &\leqslant N_0(\omega) - N(\omega, 0) \leqslant N_0(\omega) - M_r(\omega, 0) \\ &\leqslant N_0(\omega) - N(\omega, \lambda) + N(\omega, \lambda) - M_r(\omega, \lambda), \end{aligned}$$

and, since $N(\omega, \lambda) \uparrow N_0(\omega)$ as $\lambda \to 0^-$ and $M_r(\omega, \lambda) \uparrow N(\omega, \lambda)$ as $r \to \infty$, we conclude that $N(\omega, 0) = N_0(\omega)$. □

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