Algorithmic causets

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Abstract. In the context of quantum gravity theories, several researchers have proposed causal sets as appropriate discrete models of spacetime. We investigate families of causal sets obtained from two simple models of computation – 2D Turing machines and network mobile automata – that operate on 'high-dimensional' supports, namely 2D arrays of cells and planar graphs, respectively. We study a number of quantitative and qualitative emergent properties of these causal sets, including dimension, curvature and localized structures, or 'particles'. We show how the possibility to detect and separate particles from background space depends on the choice between a global or local view at the causal set. Finally, we spot very rare cases of pseudo-randomness, or deterministic chaos; these exhibit a spontaneous phenomenon of 'causal compartmentation' that appears as a prerequisite for the occurrence of *anything* of physical interest in the evolution of spacetime.

1. Introduction

The purpose of this paper is to progress investigation in an area of fundamental physics that we locate at the intersection between the *Causal Set Programme* and the *Computational Universe Conjecture*. Causality among events is widely recognized as a most fundamental aspect of spacetime, and the concept of *causal set* has been formulated in the homonymous research programme [1, 2, 3] in "an attempt to combine the twin ideas of discreteness and order to produce a structure on which a theory of quantum gravity can be based" [4]. The fundamental idea at the root of the Causal Set Programme is summarized by the conceptual equation:

"spacetime geometry = order + number".

A *causal set* (or 'causet') is a finitary partially ordered set, that is, a set of events provided with a binary relation '≺' which is reflexive, antisymmetric and transitive, and such that the number of elements between any two events is finite. A causet is meant to represent an instance of discrete spacetime in which causality is reflected by the '≺' relation ('order'), and the volume of a given region is measured by the count of nodes in it ('number'). Causets are important because the order and number information that they encode is sufficient for completely determining the metric tensors of General Relativity (see e.g. [2]).

The idea of a discrete universe is also adopted in the Computational Universe Conjecture, which, in its most extreme form, suggests that all the complexity we observe in the physical universe should emerge from the iteration of a few simple transition rules, and could be fully reproduced by running a short computer program. In this case, the conceptual equation is:

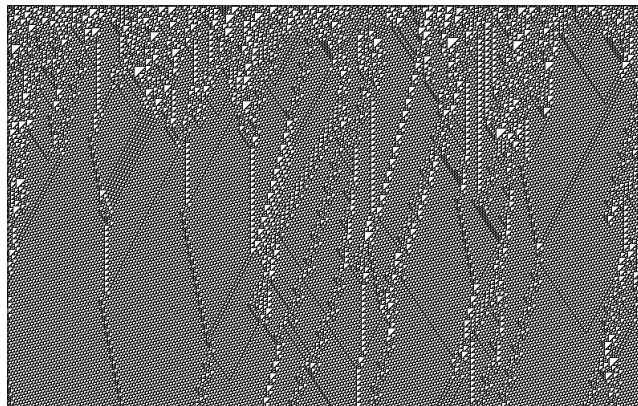"complexity in physics = emergence in computation".

Figure 1. 500-step computation of Wolfram's Elementary Cellular Automaton 110. Interacting 'particles' always emerge, from any random initial configuration.

The fact that complex patterns resembling those found in nature can originate from simple computations is widely recognized today, and has been investigated and popularized in particular by S. Wolfram, with his extensive analysis of the 'spontaneous' behavior of cellular automata and other simple models [5]. An example of a structure emerging from the computation of an elementary cellular automaton is illustrated in Figure 1. In the depicted bit array, the first row is random and each bit in the subsequent rows is computed by a boolean function of three variables:

$$b_{i+1,j} = (\neg b_{i,j-1} \wedge b_{i,j} \wedge b_{i,j+1}) \oplus b_{i,j} \oplus b_{i,j+1},$$

where $\oplus$ is exclusive OR and column indices rotate, the last on the right being followed by the first on the left. Cellular automata diagrams are indeed commonly interpreted as space-time structures, with space expanding horizontally and time flowing downward. However, unlike in causets, neither events nor causal links are explicitly represented in this type of diagram.

We believe that causets represent an excellent opportunity for progressing research on the computational universe conjecture, due to the fact that they come with a clear, explicit physical interpretation, and due to their relevance in current quantum gravity research. Thus, our main objectives in this paper are: (i) to explore, by computer simulations, the causets that can be obtained from the 'spontaneous' computations of some simple *deterministic* algorithms, (ii) to analyze some of their quantitative and qualitative properties, with the idea to compare them with those considered in the Causal Set Programme. Our hope is to promote some cross-fertilization between the two approaches in terms of techniques and specific research items.

2. Causets from simple models of computation

The idea of describing computations as networks of causally related events has been first introduced, to our knowledge, by Levin: "A causal net can be interpreted as the time-space history of all elementary operations accomplished in the computing process, with their mutual dependencies indicated" [6]. The purpose of these nets, however, was to characterize computable functions; it is only by the more recent work of Wolfram that these directed graphs are viewed as possible instances of physical spacetime.

In [7] (see also [8]) we have described a general criterion for deriving causets from virtually any model of computation. In summary, the idea is as follows. A computation is defined as a sequence of composite states, and a computation step as a pair of adjacent states. Each computation step becomes an event in the causet. Then, a causal link is created from event e_i to event e_j, with $i < j$, whenever e_j is the first event occurring after e_i to read a global state

component that was affected by e_i: that state component has acted as a *causality mediator* between the two events. [1] Finally, letting R denote the relation defined by the above links, the full causality relation is obtained by taking the *transitive closure* of R. Note that the above relation is antisymmetric by construction. It may be sometimes useful to also consider the *transitive reduction* of R, which is the minimal relation whose transitive closure is the same as the transitive closure of R. When R is antisymmetric and finite, the transitive reduction is unique.

In [7] we have explored causet families associated with a number of models, including various classes of one-dimensional Turing machines, mobile automata on tape, string rewrite systems and tag systems. All those models operate on a *one-dimensional* support, such as a tape or a string of symbols. In this paper we extend our exploration by considering two models of computation that operate on *higher-dimensional* supports, namely 2-D arrays of cells and planar trivalent graphs. As with our previous investigation, our attitude is one of avoiding predetermined expectations, while being open to anything interesting that might *emerge*. Two obvious items of interest, however, are the emergent dimensionality (or 'dimension') of the causet, and its curvature. Thus, before illustrating our algorithmic causet families, we briefly discuss how we have addressed these quantitative features.

2.1. Causet dimension and curvature by shortest-path node-shell growth
A simple way to assign a dimension to an *undirected* graph consists in considering the size $S(d)$ of the node-shells at progressive distance d from a generic node, where distance is measured by the edge count of the *shortest-path* between nodes. If $S(d) = O(d^n)$, we assign dimension $n + 1$ to the graph. In the case of undirected graphs this concept is defined, for example, in [9].

We apply the same idea to causets, that are *directed*, acyclic graphs. One can define a quantity:

$$D_x(d) := log(S_x(d))/log(d) + 1$$

relative to some node x of the causet, where $S_x(d)$ is the size of the node-shell at distance d from x. Then the causet has *node-shell dimension* $D_x := lim_{d->\infty} D_x(d)$, *starting from node x*, when such limit exists, and it has *node-shell dimension D* whenever $D_x = D$ for all x.

What if node-shell growth is not polynomial? In our experience, if we exclude cases in which large fluctuations prevent the identification of a definite growth rate, the only alternative is exponential growth. We interpret it as a sign of *negative curvature*, while suspending judgement about dimension. This choice is justified by an analogy with the continuous case: a circumference of radius r in a hyperbolic surface with negative Gaussian curvature K has length $c(r) = 2\pi r sinh(r/R)$, where $R = 1/\sqrt{-K}$, and in the expansion of the hyperbolic sine function $sinh(x) = 1/2 * (e^x - e^{-x})$ the predominant term is indeed a positive exponential.

2.2. Causets from two-dimensional Turing machines
A *two-dimensional Turing machine* is a Turing machine in which the control head operates on a 2-D array of cells, containing symbols from a given alphabet. For each value p of the current control head state, and each symbol a read in the current cell c, the state transition table describing the control head behavior provides a triple (p', a', d), indicating the new control head state p', the new symbol a' to be written in c, and the move of the control head, expressed as a vertical or horizontal, unit-length displacement d from c. By convention, we start computations with the array filled uniformly, say with 0's. No termination conditions are imposed.

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[1] The total order of computation steps does not represent physical time; the latter, as well as space, can only emerge (if ever) from the growing structure of the causet, which does not keep explicit track (e.g. by node labels) of that total order.
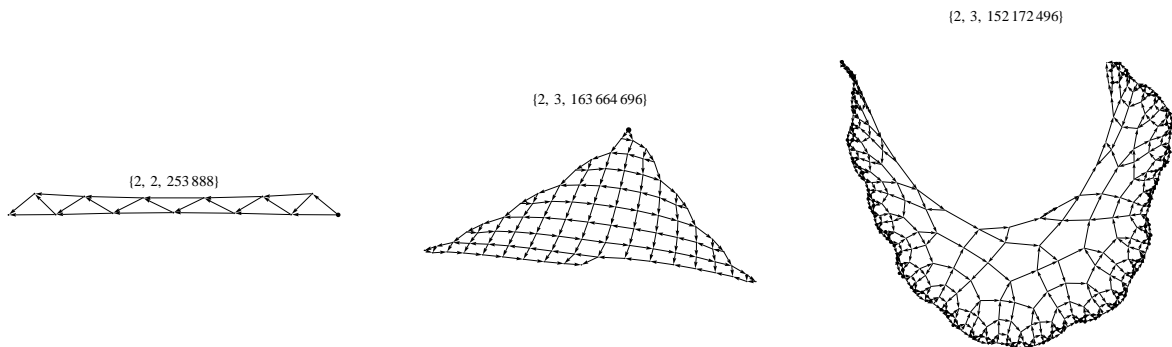
Figure 2. Three causets from simple 2-D Turing machines in class ($s = 2, k = 2$) and class ($s = 2, k = 3$). 1-D (left), 2-D flat (center), negatively curved (right).

The instantiation of the general causet construction method for this model is simple. Causal links are mediated by two global state components: tape cells and control state. Thus:

- Every step of the machine corresponds to an event of the causet.
- A directed edge connects event e_i to event e_j iff e_j reads a cell that was written by e_i, and by no other event in between.
- A directed edge connects each event e_i to its successor e_{i+1}, since the latter reads the control head state that is set by the former.

A consequence of the last bullet is that these causets will always be *totally ordered*. [2]

Using shortest-path node-shell growth analysis, we have examined families of causets from classes of 2-D Turing machines characterized by various settings of parameters s (number of control states) and k (tape alphabet size). The number of different 2-D Turing machines for given s and k is $(4sk)^{sk}$. The simplest class we consider is ($s = 2, k = 2$), consisting of 65,536 elements. We find that all of the causets in this class are 1-D; an example is shown in Figure 2 (left). By considering class ($s = 2, k = 3$), we still find that the vast majority of causets are 1-D, with exceptionally rare cases of two different types: regular, flat 2-D grids and negatively curved causets (see Figure 2). [3]

In search for more interesting behaviors, we have considered *turmites* [10]. They are a subclass of 2-D Turing machines for which the motion of the control head is described in terms of left and right turns relative to the current head orientation. The most famous turmite is Langton's ant [11]. In [10], 44 turmites with particularly interesting behavior have been identified; we have derived causets for all of them, thus discovering interesting quantitative and qualitative features. Two examples are shown in Figure 3.

The highest causet dimensions we have found in this family fall around 3.5, which is higher than any value found in [7]. For establishing these values we have computed the sizes of the node-shells at progressive distance d from the root, and then found a best fit for a prefix of this numeric sequence, for excluding boundary effects. The fitting is based on the non-integer monomial function $a * d^b$, where parameters a and b are positive real numbers: the estimated dimension is then $b + 1$. We also detect a case of exponential node-shell growth; this is not shown, for space reasons.

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[2] We discuss this potential limitation in the next subsection, in connection with particles.

[3] These results closely resemble those found in [7] for 1-D Turing machines.
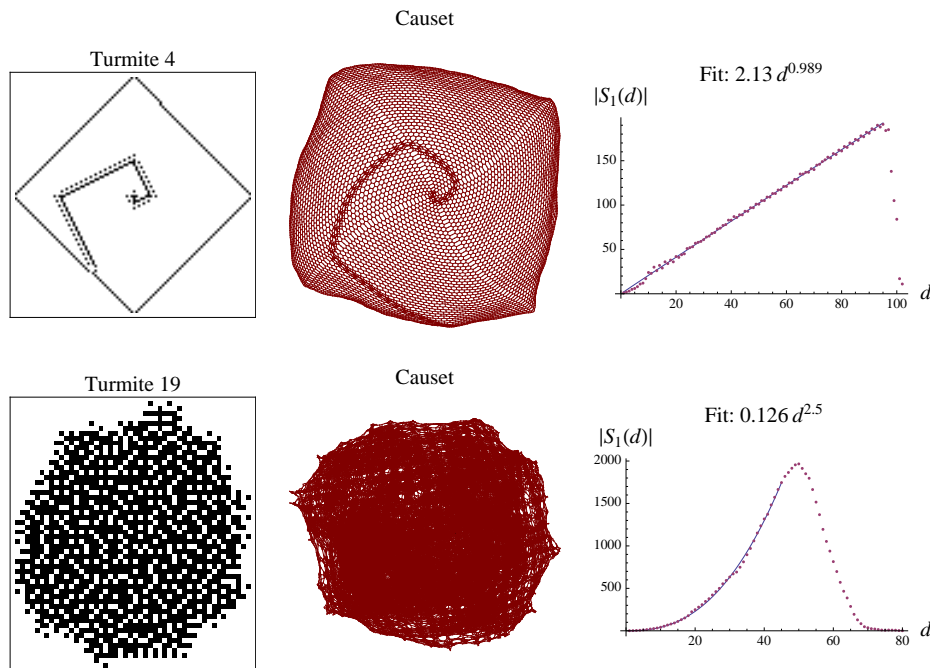
Figure 3. Two turmites: final configuration (left), causet (center), and best fit approximation of node-shell sizes at progressive distance d from the causet root, using non-integer monomial $a * d^b$. Turmite code numbers as in [10]. Computation lengths: 10000 and 50000 steps.

2.3. Particle detection: global vs. local view

Let us turn to qualitative aspects of turmite causets. One the most attractive features of cellular automata is the emergence of localized structures, or 'particles' (see Figure 1). In our previous experiments [7] our search for particles was successful only in part: causets exhibited at most one particle each, in the form of a spacetime trajectory at the border between two periodic 2-D regions. We have now the more elaborate case of turmite 4 in Figure 3. The causet for this machine consists of four triangular sectors formed by regular hexagonal grids, separated by four 'radii' formed by octagons. On top of this structure, a localized perturbation creates a spiral. By looking at both the spiral and the four radii as particles, the causet appears as a sort particle collision diagram. Note that the graph is not planar; the non-planarity is introduced precisely by the spiraling particle. We can however determine the *combinatorial curvature* for each node in the planar region. Recall that, for a planar graph, the combinatorial curvature of node x is defined as

$$cc(x) := 1 - degree(x)/2 + \sum_{f \sim x} (1/size(f))$$

where summation is over all faces f incident with x. In our graph, the four grid sectors are flat, their nodes having null combinatorial curvature, but the nodes of the octagons have three different non-null curvature values, namely -1/24, -1/12, +1/4, suggesting a possible relation between the motion of the spiraling particle and the curvature of its 'background'.

So far we have used the term 'particle' in a rather abstract way, for referring to emergent localized structures that we can visually spot in causet diagrams. In doing so we have sloppily ignored edge orientation, and, more importantly, we have considered causets exactly as obtained by the described construction criterion, while ignoring the fact that the transitive reduction of these graphs collapses into a trivial chain of spacetime events. We are not addressing here the

problem of retrieving a notion of (dynamic) space from a generic causet; but it is clear that a necessary condition for appearance of space is the presence of pairs of events in a *spacelike* relation, or anti-chains, and these cannot be found in a totally ordered set.

Should we conclude that Turing machine computations are unable to produce causets of any interest for physics? Let c be the causet directly obtained from a Turing machine computation, and let $TC(c)$ and $TR(c)$ denote, respectively, the transitive closure and transitive reduction of c; clearly $TR(c) \subseteq c \subseteq TC(c)$. Then, the question is whether the 'redundant' edges in $c - TR(c)$ can be assigned any physical role. We believe they can: their meaning becomes apparent when switching from a global to a local view at the causet structure. When we compute the transitive reduction of the a totally ordered causet with redundant edges, these are washed away. But when the reduction is applied locally, to small regions of the causet, then (some of) these edges might survive, and reveal some spatial structure.

In the subsequent figures we illustrate this idea by focusing on small circular regions of causets from turmite computations. A disk with center node c and radius r shall include all nodes found at a distance at most r from c, regardless of edge orientation. When these disks are centered at the causet root, the total order of the included spacetime events is (almost) completely represented inside the disk, and the reduction process basically destroys all structure. But when the disk is moved off the root, information on causal relations is only partial; this affects the reduction process and may expose local space and/or particles.

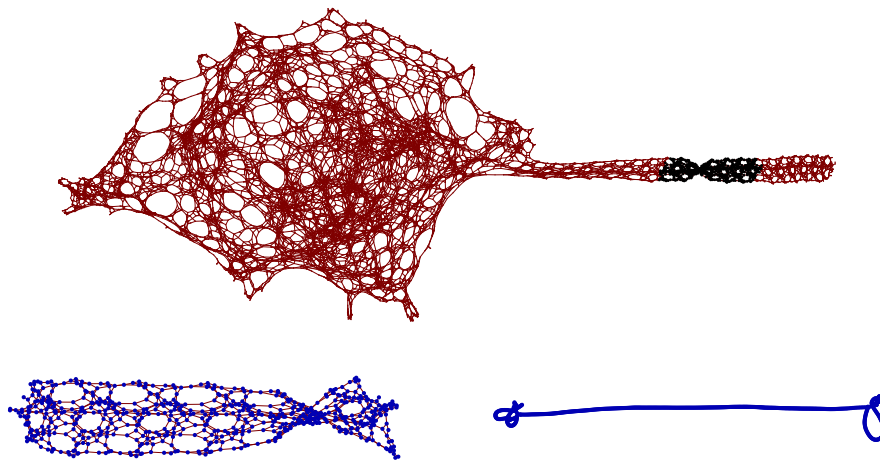In Figure 4 we show the causet for Langton's ant. The ant (the TM control head) settles to

Figure 4. Causet from 12000-step computation of turmite 1 - 'Langton's ant' (upper), disk from the causet arm (left), and transitive reduction of the disk (right).

a periodic motion – a 'highway' – after about 11000 steps. This is reflected in the arm growing from the main body of the causet. The picture shows the transitive reduction of a disk centered at some event in the arm: this is essentially a plain chain. We conclude that, at the considered scale, the causet arm does not extend spatially.

Figure 5 provides an example in which space and particle are preserved by the local transitive reduction, unless the disk is centered to the causet root. Note that the particle pattern is simplified by the reduction.

Figure 6 provides a surprising example. The grid-like spatial structure is preserved by the transitive reduction, unless a particle is present: in this case the grid basically disappears,
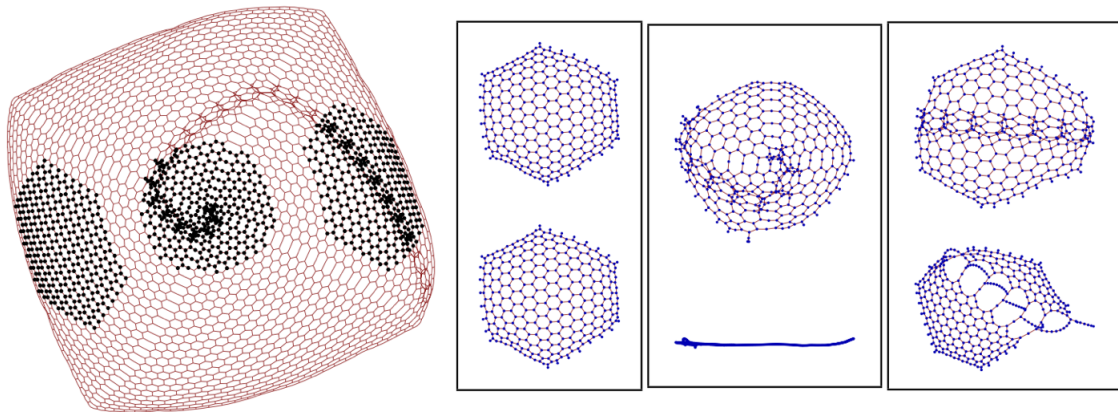
Figure 5. Causet from 4000-step computation of turmite 4, with three highlighted disks (left); the same disks and their transitive reductions (right).
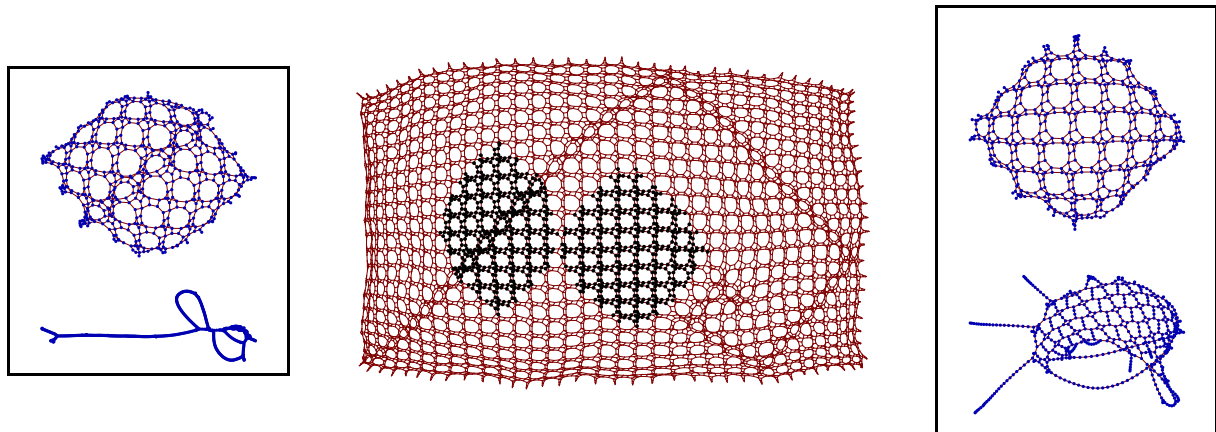

Figure 6. Causet from 10000-step computation of turmite 18, with two highlighted disks (center); the same disks and their transitive reductions (sides).

except for a portion corresponding to the 'shadow' of the original particle trajectory (this is barely visible in the lower-left diagram).

Figure 7 is the only case in which we initialize the cell array with a non-uniform 5×5 region. This computation, when observed on the cell array, exhibits the emergence of a 'two-way' highway (lower-left diagram) that is more interesting than the 'one-way' highway of Langton's ant: the ant now keeps oscillating between the two opposite endpoints of the linear structure, while progressively extending it in both directions, thus implementing a sort of wave behavior.

When taking local views at this causet, we find that basically all the structure is washed away by the transitive reduction, as with Langton's ant; the only space-like cells left (see lower-right diagram) are organized in two sequences that correspond to the ant entering and leaving the central, random-like region of the cell array.

We believe that the detection of particles in causets from computations is an interesting, original result of this paper; localized structures may appear also in turmite diagrams, but the two phenomena are not always coupled. The causet, not the 2D-grid of a Turing machine, or any other computational support, is the right place where to look for particles, or their worldlines.
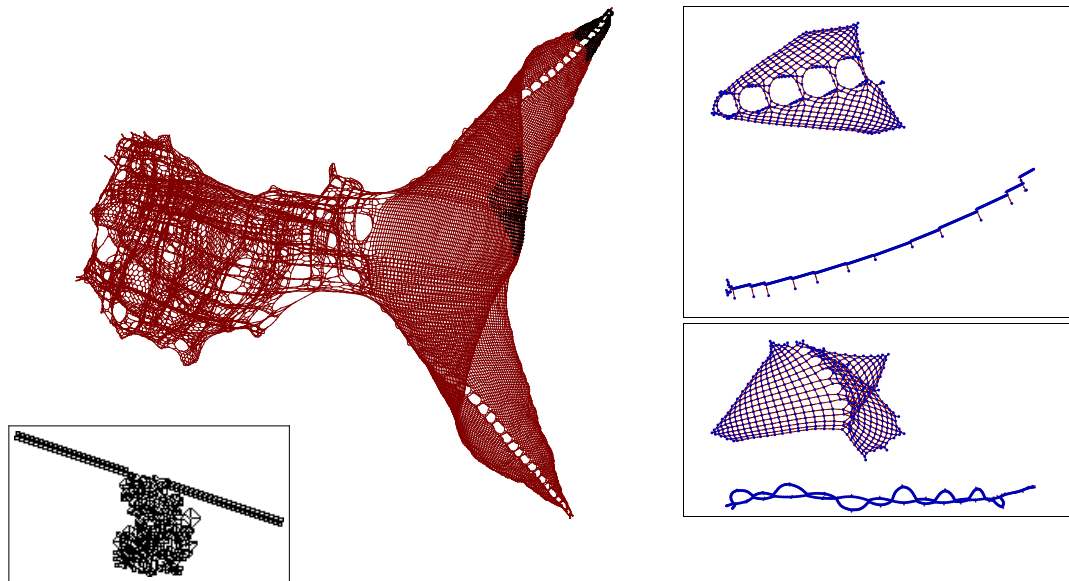
Figure 7. Causet from 26000-step computation of turmite 21, with two highlighted disks (center), and the final configuration of the cell array (lower left); the two disks and their transitive reductions (right).

2.4. Causets from network mobile automata on trivalent planar graphs

Network mobile automata have been introduced and shortly discussed in [5], and further studied in [12, 13]. A *trinet* is a planar, undirected, trivalent graph. A *trinet mobile automaton* is similar to a 2-D Turing machine, except that the stateless control head moves on a trinet rather than on an array of cells. At each computation step the control head performs a local graph rewriting and moves to some nearby location. Both the rewrite rule and the next location are chosen by some deterministic criterion. In our investigations we have always used 2-D Pachner moves, sometimes called *Expand*, *Contract* and *Exchange*, that find application also in Loop Quantum Gravity (see, e.g., [14]). Each of these rules affects three or four faces of the planar graph.

The instantiation of the general causet construction method is simple also in this case.

- Every step of the machine corresponds to an event of the causet.
- A directed edge connects event e_i to event e_j iff the rewriting performed by e_j affects a planar face of the trinet that has been previously affected by e_i (without further modifications in between).

In [12, 13] we have explored two variants of the model, in both of which rule *Contract* is not used. A first variant, qualified as *three-connectivity preserving*, is defined as follows:

 (i) Start with a trinet consisting of two nodes connected by three parallel arcs. (This is the smallest possible three-connected graph. A connected graph is *n*-connected when n is the smallest number of edges one has to remove in order to disconnect it.)

 (ii) Choose rule *Exchange* whenever it does not violate three-connectivity, otherwise choose *Expand*.

(iii) Move the control head to a new nearby location which depends on the applied rule and on a fixed parameter (see [13] for details).
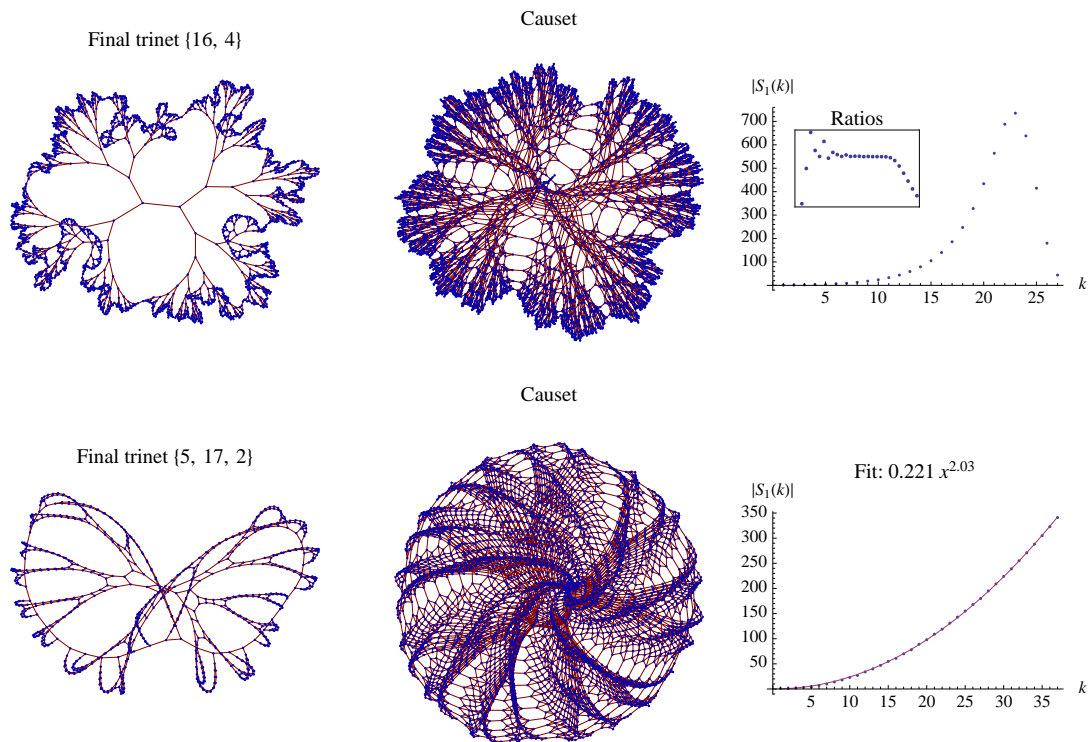
Figure 8. Two network mobile automata: final configuration (left), causet (center), and analysis of node-shell sizes at progressive distance d from the causet root. Numeric codes as in [12, 13].

In the second variant, qualified as *threshold-based*, point (ii) is modified as follows: choose *Exchange* whenever it does not create trinet faces with less than k sides, otherwise choose *Expand*. The two-parameter policy for control head moves is the same in the two variants, and always involves *short* steps, in analogy with Turing machines. As a consequence, the faces affected by a step always partially overlap with those affected by the next step, so that events e_i and e_{i+1} are always causally related, and the causet is, again, totally ordered. However, due to the construction rule, each node has 3 or 4 incoming, and 3 or 4 outgoing arcs, except for the first node and those at the growth boundary, so that causet structures can still be far from trivial, and they would remain so even after *local* transitive reduction.

Even with computations from this model, the large majority of causets are 1-D graphs of no interest. However, several more interesting cases are found; two of them are illustrated in Figure 8. In the first case the causet is negatively curved: the exponential node-shell growth is revealed by the flat region of the small inset diagram, showing the ratios of consecutive shell size values. In the second case the causet exhibits an approximate node-shell dimension 3 from the root, as revealed by the exponent of the non-integer monomial fit.

The causets considered above manifest a considerable degree of regularity, or predictability; this becomes even more apparent when tracing the motion of the control head (not shown here). With the objective of modeling spacetime in mind, we are certainly interested in more chaotic behaviors. In [12, 13] only three computations, out of a few thousand, could be singled out for the surprising pseudo-random character of control head motion and resulting trinet graphs. Their causets turn out to be rather similar, at least at small scales, and quite different from those in Figure 8. Their dimensionality analysis based on node-shell growth is made problematic
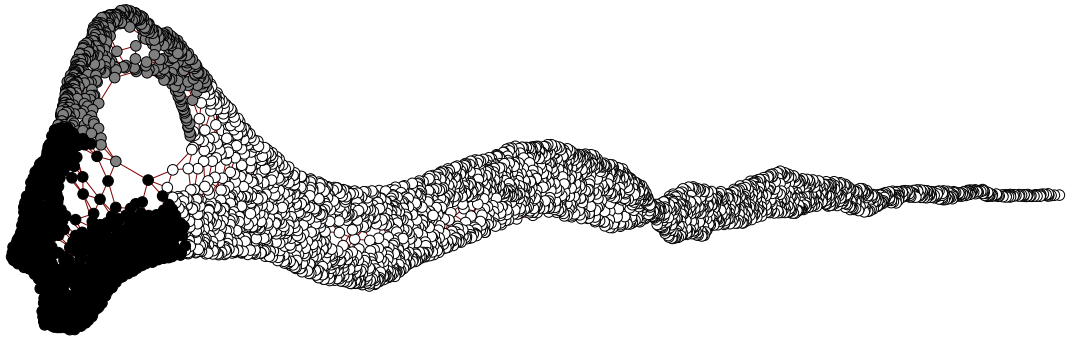
Figure 9. Pseudo-random causet from a computation of a trinet mobile automaton (code 4, 17, 8 in [12]). Nodes are white up to event 4020, then gray up to event 4649, then black up to event 6000.

by the wide oscillations of shell sizes. However, one of these pseudo-random computations lends itself to the illustration of a remarkable, emergent *qualitative* property, as discussed below.

2.5. Emergent compartmentation

What type of feature might we expect to see emerge in our computational spacetime models, and in particular in those with pseudorandom character, on larger and larger scales, beside particles and curvature? We suggest that one of the key features would be the self-organization into components that achieve some form of *independence* from one another, as a necessary basis for building further complexity and obtaining a multiplicity of phenomena (in principle, up to chemistry and biology). In Figure 9 we show a remarkable, pseudo-random causet. The directed graph is partitioned into three sub-segments, colored in white, gray, and black, in a progression that reflects the sequence of computation steps. All grey events have occurred between white and black events, and yet the white and black portions are in direct causal contact. Thus, the 'hole' appearing in the graph is not a pure accident of the specific layout algorithm. This phenomenon reflects the confinement of the 'ant' inside some region of the growing trinet for a relatively large number of steps.

As one would expect, in experiments we have conducted with 'genuinely random' causets, no spontaneous formation of similar macroscopic compartments has been observed.

3. Conclusions

In this paper we have provided some experimental evidence that causets created by simple deterministic algorithms exhibit a variety of interesting emergent properties of physical significance, that can not be obtained by probabilistic approaches.

Several items require further investigation. The node-shell growth analysis based on graph-theoretic, shortest-path distance is weak in two aspects: it does not separate dimension and curvature, and it does not match spacetime distance, which corresponds to the longest-path between points. We have started using longest-path distance in our most recent experiments; but we still need estimators that decouple dimension and curvature. We have identified localized structures in some algorithmic causets, and have called them 'particles', following terminology commonly used for cellular automata; but we still have to discover whether and how these structures exhibit wave/particle duality. We plan to experiment with ant-based models of

computation that operate directly on the causet structure, rather than on an underlying memory support. We expect this approach to possibly account for important phenomena from quantum mechanics, e. g. Bell inequalities, that are out of the reach of our current models (this idea has been suggested by Alex Lamb.)

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