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# Uniformizing (proximal) $\Delta$-topologies

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**Abstract**

Beer and Tamaki investigated necessary and sufficient conditions for the uniformizability of (proximal) $\Delta$-topologies.

Their proofs involved construction of special Urysohn functions. In this paper we attack the same problem using as a useful tool a uniform topology with reference to a Hausdorff uniformity patterned after the one related to the Attouch–Wets topology. We also study $\Delta U$-topologies, proximal $\Delta U$-topologies which are natural generalizations of the $U$-topology discovered by Costantini and Vitolo.


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## 1. Introduction

Poppe [19,20] initiated the study of abstract Vietoris-type hyperspace topologies on $CL(X)$, the family of all nonempty closed subsets of a topological space $(X, \tau)$, corresponding to a family $\Delta \subseteq CL(X)$. He was motivated by an attempt to generalize the Fell topology, in which case $\Delta$ equals the family of all nonempty compact subsets (see [1] for a comprehensive account where further references will be found). Di Concilio,

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Naimpally and Sharma [10] introduced proximal hypertopologies on $CL(X)$. Then Beer and Tamaki [5,6] investigated the uniformizability of (proximal) $\Delta$-topologies. In this paper, we study the same problem using as a useful tool the Attouch–Wets or AW uniformities [1,3].

In [7] Costantini and Vitolo introduced a new hypertopology which they called the $U$-*topology* which is useful in the study of the infimum of the Hausdorff metric topologies on $CL(X)$ associated with a metrizable space $X$. This topology is finer than the Fell topology and for the upper part uses the subbase $\{U^+\colon U \in \tau\}$ where $U^c$ or $\mathrm{cl}\, U$ is compact (see below for precise definitions). We also study $\Delta U$-topologies, proximal $\Delta U$-topologies which are natural and interesting generalizations.

Let $(X, \tau)$ be a $T_1$ space, $\delta$ a compatible LO-proximity on $X$ and $\delta_0$ the finest compatible LO-proximity on $X$ defined by $A\delta_0 B$ iff $\mathrm{cl}\, A \cap \mathrm{cl}\, B \neq \emptyset$.

Note that $\delta_0$ is not necessarily EF and it is so if and only if $(X, \tau)$ is normal (Urysohn's theorem).

For each $U \in \tau$, we use the following notation:

$$U^+ = \{E \in CL(X)\colon E \subset U\},$$
$$U_\delta^{++} = \{E \in CL(X)\colon E \ll_\delta U\},$$

where $E \ll_\delta U$ means $E \not\delta U^c$ (we will omit reference to $\delta$ if this is clear from the context),

$$U^- = \{E \in CL(X)\colon E \cap U \neq \emptyset\}.$$

We refer to [1,13,18] for all undefined terms.

We assume that $\Delta$ is a subfamily of $CL(X)$ which is a *cover* of $X$ (i.e., $\Delta$ is *closed under finite unions, closed hereditary* and *contains the singletons*), unless otherwise explicitly stated.

We will do this to display trasparent statements and make theory much simpler, and also because the most important subfamilies $\Delta$ satisfy the above conditions as we see from the examples below:

(i) the family $K(X)$ of all nonempty compact subsets of $X$;
(ii) the family of all totally bounded subsets of $X$ (when $\tau$ is uniformizable);
(iii) the family of all $d$-bounded subsets of a metric space $(X, d)$;
(iv) the family of all finite subsets of $X$;
(v) the family of all pseudocompact subsets of $X$;
(vi) the family of all $\Gamma$-bounded subsets of $X$, where $\Gamma \subset C(X)$, i.e., $\{A \in CL(X)\colon$ for every $f \in \Gamma$, $f(A)$ is a bounded subset of $\mathbb{R}\}$;
(vii) the family of all countably compact subsets of $X$;
(viii) the family of all Lindelöf subsets of $X$;
(ix) the family of all topologically bounded subsets of $X$, i.e., $\{A \in CL(X)\colon$ every open cover of $X$ has a finite subfamily covering $A\}$ [15];
(x) the family of all subsets of $X$ of measure zero (if $X$ has a measure);
(xi) the family of all subsets of $X$ of finite measure (if $X$ has a measure);
(xii) the family of all subsets of $X$ of first category;
(xiii) the family of all nowhere dense subsets of $X$.

**Definition 1.1.** We recall and define various topologies on $CL(X)$:

(a) The *proximal $\Delta$-topology* $\sigma(\Delta,\delta)$ has a subbase consisting of the *upper* part $\{U_\delta^{++}\colon U^c \in \Delta\}$ and the *lower* part $\{U^-\colon U \in \tau\}$. In particular we have:
the *proximal topology* $\sigma(\delta) = \sigma(\Delta,\delta)$ (see [4] or [10]) when $\Delta = CL(X)$;
the *proximal Fell topology* $\sigma(F,\delta) = \sigma(\Delta,\delta)$ when $\Delta = K(X)$.

(b) *The $\Delta$-topology* $\tau(\Delta)$ has a subbase consisting of the *upper* part $\{U^+\colon U^c \in \Delta\}$ and the *lower* part $\{U^-\colon U \in \tau\}$. (We note that this too can be considered as a proximal $\Delta$-topology. In fact $\tau(\Delta) = \sigma(\Delta,\delta_0)$.)
In particular we obtain:
the *Vietoris topology* $\tau(V) = \tau(\Delta)$ (see [16]) when $\Delta = CL(X)$;
the *Fell topology* $\tau(F) = \tau(\Delta)$ (see [14]) when $\Delta = K(X)$ (note that $\tau(F) = \sigma(F,\delta)$ if either $\delta = \delta_0$ or $\delta$ is EF (cf. [8])).

(c) The *proximal $\Delta U$-topology* $\sigma(\Delta U,\delta)$ has a subbase consisting $\{U_\delta^{++}\colon U^c \in \Delta$ or $\operatorname{cl} U \in \Delta\}$ and $\{U^-\colon U \in \tau\}$.
If $\Delta = K(X)$, then $\sigma(\Delta U,\delta)$ is the *proximal U-topology* $\sigma(U,\delta)$.

(d) The *$\Delta U$-topology* $\tau(\Delta U)$ has a subbase consisting $\{U^+\colon U^c \in \Delta$ or $\operatorname{cl} U \in \Delta\}$ and $\{U^-\colon U \in \tau\}$.
If $\Delta = K(X)$, then $\tau(\Delta U)$ is the *U-topology* $\tau(U)$ (see [7]); furthermore $\tau(U) = \sigma(U,\delta)$ if either $\delta = \delta_0$ or $\delta$ is EF (cf. [8]).
Moreover, if $X$ is a uniformizable space, we have:

(e) The *Hausdorff uniformity* $\mathcal{U}_H$ on $CL(X)$ corresponding to a uniformity $\mathcal{U}$ on $X$ has a base $\{W_H\colon W \in \mathcal{U}\}$ where $W_H = \{(A_1,A_2) \in CL(X) \times CL(X)\colon A_1 \subset W(A_2)$ and $A_2 \subset W(A_1)\}$. (Some authors call this the Bourbaki uniformity.)

(f) The *$\Delta$-Attouch–Wets topology* $\tau(\Delta AW)$. For each $D \in \Delta$ and $W \in \mathcal{U}$ set $[D,W] = \{(A_1,A_2) \in CL(X) \times CL(X)\colon A_1 \cap D \subset W(A_2)$ and $A_2 \cap D \subset W(A_1)\}$.
The family $\{[D,W]\colon D \in \Delta$ and $W \in \mathcal{U}\}$ is a base for a filter $\mathcal{U}_\Delta$ on $CL(X)$ called the *$\Delta$-Attouch–Wets filter*. $\mathcal{U}_\Delta$ induces the topology $\tau(\mathcal{U}_\Delta)$ (cf. [2,3]).

The following result is well known [10]:

**Theorem 1.2.** *If $\delta$ is a compatible EF-proximity on a Tychonoff space $(X,\tau)$, then the corresponding proximal topology $\sigma(\delta)$ on $CL(X)$ is always Tychonoff. In fact, it is the topology induced on $CL(X)$ by the Hausdorff uniformity $\mathcal{U}_{wH}$ which is derived from the unique totally bounded uniformity $\mathcal{U}_w$ on $X$ compatible with $\delta$.*

**Definition 1.3.** Let $(X,\tau)$ be a $T_1$ space with a compatible LO-proximity $\delta$ and $\Delta \subseteq CL(X)$.

(a) $\Delta$ is *$\delta$-Urysohn* iff for each $D \in \Delta$ and $A \in CL(X)$ with $D \not\delta A$, there is an $S \in \Delta$ such that $D \ll_\delta S \ll_\delta A^c$ (see also [9]).

(b) $\Delta$ is *Urysohn* iff for each $D \in \Delta$ and $A \in CL(X)$ with $D \cap A = \emptyset$, there is an $S \in \Delta$ such that $D \subset \operatorname{int} S \subset S \subset A^c$ (or equivalently $\Delta$ is $\delta_0$-Urysohn).

(c) $\Delta$ is *local* iff for each $x \in X$ and $V \in \tau$ with $x \in V$ there is a $D \in \Delta$ such that $x \in \operatorname{int} D \subset D \subset V$.

**Remark 1.4.** Note that if $\Delta$ is ($\delta$-) Urysohn, then it is also local since $\Delta$ contains the singletons.

By imitating the construction of the coarsest EF-proximity $\delta_1$ in a locally compact space (where $A \not\delta_1 B$ iff $A \not\delta_0 B$ and either $\operatorname{cl} A$ or $\operatorname{cl} B \in K(X)$) we give the following definition:

**Definition 1.5.** Let $(X, \tau)$ be a $T_1$ space with a compatible LO-proximity $\delta$ and $\Delta \subseteq CL(X)$ be $\delta$-Urysohn. The relation $\delta'$ on the power set of $X$ defined by

$$A \not\delta' B \quad \text{iff} \quad \text{either } \operatorname{cl} A \in \Delta \text{ or } \operatorname{cl} B \in \Delta \text{ and } A \not\delta B \tag{$\star$}$$

is called the *$\Delta$-Wallman proximity associated to $\delta$*.

**Theorem 1.6.** *Let $(X, \tau)$ be a $T_1$ space with a compatible LO-proximity $\delta$. Let $\Delta \subseteq CL(X)$ be $\delta$-Urysohn and $\delta'$ the $\Delta$-Wallman proximity associated to $\delta$. Then*

(a) *$\delta'$ is a compatible EF-proximity on $X$ coarser than $\delta$;*
(b) *$\Delta$ is $\delta$-Urysohn iff it is $\delta'$-Urysohn.*

**Proof.** We prove (a). To show $\delta'$ is an EF-proximity only two axioms need verification viz:

(i) $A \not\delta' B$ and $A \not\delta' C$ implies $A \not\delta' (B \cup C)$ (*union axiom*) and
(ii) whenever $A \not\delta' B$, there exists an $E \subset X$ such that $A \not\delta' E$ and $E^c \not\delta' B$ (*EF axiom*).

To verify (i) suppose $A \not\delta' B$ and $A \not\delta' C$.

($i_1$) If $\operatorname{cl} A \in \Delta$, then $A \not\delta B$ and $A \not\delta C$ and so $A \not\delta (B \cup C)$. By ($\star$) $A \not\delta' (B \cup C)$.

($i_2$) If $\operatorname{cl} A \notin \Delta$, then $\operatorname{cl} B \in \Delta$, $\operatorname{cl} C \in \Delta$ and $A \not\delta B$ and $A \not\delta C$. Then $\operatorname{cl}(B \cup C) \in \Delta$ and $A \not\delta (B \cup C)$ and hence from ($\star$) $A \not\delta' (B \cup C)$.

To verify (ii) suppose $A \not\delta' B$. We may assume $\operatorname{cl} A \in \Delta$ and $A \not\delta B$, i.e., $A \ll_\delta B^c$. Since $\Delta$ is $\delta$-Urysohn, then there is an $E \in \Delta$ with $A \ll_\delta E \ll_\delta B^c$. By ($\star$) $A \not\delta' E^c$ and $E \not\delta' B$.

Observe that $\delta'$ is a compatible proximity since $\Delta$ contains the singletons and it is clearly coarser than $\delta$.

To show (b) note that from ($\star$), whenever $D \in \Delta$ and $A \in CL(X)$, $D \not\delta A$ if and only if $D \not\delta' A$. Hence $S \in \Delta$ with $D \ll_\delta S \ll_\delta A^c$ is equivalent to $S \in \Delta$ with $D \ll_{\delta'} S \ll_{\delta'} A^c$. □

**Remarks 1.7.** (a) In the case $\delta = \delta_0$, the local compactness of the space $X$ (which guarantees that $\delta_1$ is EF) is equivalent to $\Delta = K(X)$ be local. So, in the construction of $\delta'$ we have replaced $K(X)$ by $\Delta$ and local compactness by assuming $\Delta$ to be ($\delta$-) Urysohn and so local by Remark 1.4.

(b) Note that even if the starting proximity $\delta$ is just LO, the new proximity $\delta'$ is compatible and it is always EF as above theorem shows. As a byproduct of this result, we have that if the base space $X$ admits a proximity $\delta$ and a family $\Delta$ which is a cover of $X$ and $\delta$-Urysohn, then it is automatically completely regular. Thus, in this case we restrict our attention to Tychonoff spaces. We point out that Tychonoff spaces admit compatible LO-proximities which are not EF: a prototype is the proximity $\delta_0$ which is EF if and only if $X$ is normal. So, we have a procedure that allow us to construct an EF-proximity on a Tychonoff space $X$ by using as a seed a given LO-proximity.

Now we return to the hypertopologies $\sigma(\Delta,\delta)$, $\sigma(\Delta U,\delta)$, $\tau(\Delta)$ and $\tau(\Delta U)$. From Definition 1.1 it follows that $\sigma(\Delta,\delta)\leqslant\sigma(\Delta U,\delta)$ as well as $\tau(\Delta)\leqslant\tau(\Delta U)$.

We characterize coincidence when $\Delta$ is assumed just closed under finite unions.

**Theorem 1.8.** *Let $(X,\tau)$ be a Tychonoff space with a compatible LO-proximity $\delta$ and $\Delta\subseteq CL(X)$* closed under finite unions. *Then the following are equivalent*:

(a) *either $X$ has no open set $V$ with $\operatorname{cl} V\in\Delta$ or for each open set $V$ with $\operatorname{cl} V\in\Delta$ and each $A\in CL(X)$ with $A\ll_\delta V$ there exists an $S\in\Delta$ with $A\ll_\delta S^c\subset V$ and hence $X\in\Delta$*;
(b) $\sigma^+(\Delta U,\delta)\leqslant\sigma^+(\Delta,\delta)$ *on $CL(X)$*;
(c) $\sigma^+(\Delta U,\delta)=\sigma^+(\Delta,\delta)$ *on $CL(X)$*.

**Proof.** Only (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (a) need some comments, since (b) $\Leftrightarrow$ (c) it is obvious.

We start with (a) $\Rightarrow$ (b). Let $A\in CL(X)$ and $V_\delta^{++}$ a $\sigma^+(\Delta U,\delta)$-neighbourhood at $A$. Then either $V^c\in\Delta$ or $\operatorname{cl} V\in\Delta$ and $A\not\delta V^c$.

If $V^c\in\Delta$, then we are done (since $V_\delta^{++}$ it is also a $\sigma^+(\Delta,\delta)$-neighbourhood at $A$).

If $\operatorname{cl} V\in\Delta$ and $A\not\delta V^c$, then $A\ll_\delta V$ and by assumption there is an $S\in\Delta$ with $A\ll_\delta S^c\subset V$. Hence $(S^c)_\delta^{++}$ is a $\sigma^+(\Delta,\delta)$- neighbourhood at $A$ with $A\in(S^c)_\delta^{++}\subset V_\delta^{++}$.

(b) $\Rightarrow$ (a). Let $V$ be an open set with $\operatorname{cl} V\in\Delta$ and let $A\in CL(X)$ with $A\ll_\delta V$. Then $V_\delta^{++}$ is a $\sigma^+(\Delta U,\delta)$-neighbourhood at $A$. By assumption there exists a $\sigma^+(\Delta,\delta)$ neighbourhood $\mathcal{A}=(S^c)_\delta^{++}$ (for some $S\in\Delta$) at $A$ with $A\in(S^c)_\delta^{++}\subset V_\delta^{++}$. Clearly $D\not\delta S$ and it is easy to check that $V^c\subset S$.

Hence $X=\operatorname{cl} V\cup S\in\Delta$, since $\Delta$ is closed under finite unions. □

**Corollary 1.9.** *Let $(X,\tau)$ be a Tychonoff space and $\Delta\subseteq CL(X)$* closed under finite unions. *Then the following are equivalent*:

(a) *either $X$ has no open set $V$ with $\operatorname{cl} V\in\Delta$ or for each open set $V$ with $\operatorname{cl} V\in\Delta$ and each $A\in CL(X)$ with $A\subset V$ there exists an $S\in\Delta$ with $A\subset S^c\subset V$ and hence $X\in\Delta$*;
(b) $\tau^+(\Delta U,\delta)\leqslant\tau^+(\Delta,\delta)$ *on $CL(X)$*;
(c) $\tau^+(\Delta U,\delta)=\tau^+(\Delta,\delta)$ *on $CL(X)$*.

**Proof.** Use above theorem with $\delta=\delta_0$. □

**Remark 1.10.** Note that if in the above theorem or corollary $\Delta$ is also local, then $\sigma(\Delta U,\delta)=\sigma(\Delta,\delta)$ (respectively, $\tau(\Delta U)=\tau(\Delta)$) if and only if $X\in\Delta$ and for each $V\in\tau$ with $\operatorname{cl} V\in\Delta$ and each $A\in CL(X)$ with $A\ll_\delta V$ there exists an $S\in\Delta$ with $A\ll_\delta S^c\subset V$ (respectively, $X\in\Delta$ and for each $V\in\tau$ with $\operatorname{cl} V\in\Delta$ and each $A\in CL(X)$ with $A\subset V$ there exists an $S\in\Delta$ with $A\subset S^c\subset V$).

A prototype of corollaries that we can deduce from Theorem 1.8, Corollary 1.9 and Remark 1.10 is the following.

**Corollary 1.11.** *Let $(X,\tau)$ be a locally compact Hausdorff space (and $\delta$ a compatible LO-proximity on $X$), then the $U$-topology $\tau(U)$ (the proximal $U$-topology $\sigma(U,\delta)$) on $CL(X)$ equals the Fell topology $\tau(F)$ (the proximal Fell topology $\sigma(F,\delta)$) iff $X$ is compact.*

The interested reader can easily deduce corollaries corresponding to each example (i)–(xii) listed previously.

We point out that when $\Delta$ is local and a *cover* of $X$, then $\tau(\Delta U)=\tau(\Delta)$ (resp. $\sigma(\Delta U,\delta)=\sigma(\Delta,\delta)$) if and only if $\Delta = CL(X)$, i.e., coincidence occurs when the $\Delta U$-topology $\tau(\Delta U)$ (resp. the proximal $\Delta U$-topology $\sigma(\Delta U,\delta)$) is the Vietoris topology $\tau(V)$ (resp. the proximal topology $\sigma(\delta)$) on $CL(X)$.

## 2. Uniformizing (proximal) $\Delta$-topologies and (proximal) $\Delta U$-topologies

We recall that if $(X,\tau)$ is a Tychonoff space with a compatible EF-proximity $\delta$, then a uniformity $\mathcal{U}$ on $X$ is called *compatible w.r.t.* $\delta$ iff the proximity relation $\delta(\mathcal{U})$ defined by $A\delta(\mathcal{U})B$ iff $A\cap U[B]\neq\emptyset$ for each $U\in\mathcal{U}$ equals $\delta$ (see [18]). $\delta$ admits a unique compatible totally bounded uniformity $\mathcal{U}_w(\delta)$ [18] and we will omit reference to $\delta$ if this is clear from the context.

**Theorem 2.1.** *Let $(X,\tau)$ be a Tychonoff space with a compatible EF-proximity $\delta$, $\mathcal{U}_w$ the unique totally bounded uniformity which induces $\delta$ and $\Delta\subseteq CL(X)$ a cover of $X$. Then the following are equivalent*:

(a) *$\Delta$ is $\delta$-Urysohn*;
(b) (1) *the $\Delta$-Attouch–Wets filter $\mathcal{U}_{w\Delta}$ (cf.* (f) *in Definition* 1.1) *is a Hausdorff uniformity*;
 (2) *the proximal $\Delta$-topology $\sigma(\Delta,\delta)$ equals $\tau(\mathcal{U}_{w\Delta})$.*

**Proof.** (a) $\Rightarrow$ (b) We start showing (1). It suffices to show that the subbase filter $\Psi=\{[D,U]\colon D\in\Delta$ and $U\in\mathcal{U}\}$ of $\mathcal{U}_{w\Delta}$, where $[D,U]=\{(A_1,A_2)\in CL(X)\times CL(X)\colon A_1\cap D\subset U(A_2)$ and $A_2\cap D\subset U(A_1)\}$, is a subbase for a Hausdorff uniformity on $CL(X)$.

Without loss of generality we may assume that *all entourages $U\in\mathcal{U}_w$ are open and symmetric*.

We claim that whenever $[D,U]\in\Psi$, there is some $[S,V]\in\Psi$ such that $[S,V]\circ[S,V]\subset[D,U]$.

So, let $[D,U]\in\Psi$. Then $D\in\Delta$ and $U\in\mathcal{U}_w$. Without loss of generality, we may assume that $U(D)\neq X$. Set $A=[U(D)]^c$. Then $A\not\delta D$. By assumption there is an $S\in\Delta$ such that $D\ll_\delta S\subset A^c$. Let $V\in\mathcal{U}_w$ be such that $V\circ V\subset U$ and $V(D)\subset S$. Clearly, $[S,V]\in\Psi$. We claim that $[S,V]\circ[S,V]\subset[D,U]$. So, let $(E_1,E_2)$ and $(E_2,E_3)\in[S,V]$. We have to consider two cases:

(i) both $E_1\cap D=\emptyset$ and $E_3\cap D=\emptyset$;
(ii) either $E_1\cap D\neq\emptyset$ or $E_3\cap D\neq\emptyset$.

If (i) occurs, then clearly $(E_1, E_3) \in [D, U]$. So, suppose (ii) occurs and let $x \in E_1 \cap D$. Since $V(D) \subset S$ and $E_1 \cap S \subset V(E_2)$ there exists a $y \in E_2$ such that $y \in E_2 \cap S$ and $y \in V(x)$. Again, since $V(D) \subset S$ and $E_2 \cap S \subset V(E_3)$ there exist a $z \in E_3$ such that $z \in V(y)$. But $V \circ V \subset U$ and so $x \in U(E_3)$. Thus, $E_1 \cap D \subset U(E_3)$. Similarly, we have $E_3 \cap D \subset U(E_1)$. So, $\mathcal{U}_{w\Delta}$ is a uniformity.

Then, let $A_1, A_2 \in CL(X)$ with $A_1 \neq A_2$ and without loss of generality assume $a_1 \in A_1 \setminus A_2$. Let $U \in \mathcal{U}_w$ with $a_1 \notin U(A_2)$. By assumption $a_1 \in \Delta$. Clearly, $[a_1, U] \in \Psi$ and $(A_1, A_2) \notin [a_1, U]$ and so $\mathcal{U}_{w\Delta}$ is Hausdorff, too.

Now, we prove (2). So, let $A_\lambda$ be a net converging to $A$ w.r.t. the topology $\tau(\mathcal{U}_{w\Delta})$.

(i) If $A \in V^-$, where $V \in \tau$, then there exist $a \in A \cap V$ and a $W \in \mathcal{U}_w$ such that $W(a) \subset V$. Since $A \in [\{a\}, W](A) \subset V^-$, $A_\lambda \in [\{a\}, W](A) \subset V^-$, eventually.

(ii) If $A \in (D^c)^{++}_\delta$, where $D \in \Delta$, then $D \ll_\delta A^c$ and hence there is an $S \in \Delta$ such that $D \ll_\delta S \ll_\delta A^c$. Hence there is a $W \in \mathcal{U}_w$ such that $W(A) \cap S = \emptyset$. Eventually $A_\lambda \in [S, W](A)$, i.e., $A_\lambda \in (D^c)^{++}_\delta$. Thus $\sigma(\Delta, \delta) \leqslant \tau(\mathcal{U}_{w\Delta})$.

On the other hand, let $A_\lambda$ be a net converging to $A$ w.r.t. the topology $\sigma(\Delta, \delta)$, $D \in \Delta$ and $W \in \mathcal{U}_w$. Let $V \in \mathcal{U}_w$ such that $V^2 \subset W$. We have to consider two cases:

(i) $A \in (D^c)^{++}_\delta$. Then eventually $A_\lambda \in (D^c)^{++}_\delta$ and obviously, $\emptyset = A_\lambda \cap D \subset W(A)$ and $\emptyset = A \cap D \subset W(A_\lambda)$.
(ii) $A \notin (D^c)^{++}_\delta$. Then $V(A) \cap D \neq \emptyset$.

Since $V$ is totally bounded, there are $x_j \in A,\ 1 \leqslant j \leqslant n$, such that $A \subset \bigcup_{j=1}^n V(x_j) \subset V^2(A)$. Since $A \cap V(x_j) \neq \emptyset$ for each $j$, eventually $A_\lambda \cap V(x_j) \neq \emptyset$ and so $x_j \in V(A_\lambda)$. Hence,

$$A \cap D \subset \bigcup_{j=1}^{n} V(x_j) \subset V^2(A_\lambda) \subset W(A_\lambda), \quad \text{eventually.}$$

We note that $(D \cap V(A)^c) \in \Delta$ and $A \in (D^c \cup V(A))^{++}_\delta \in \sigma(\Delta, \delta)$. So, $A_\lambda \in (D^c \cup V(A))^{++}_\delta$, eventually.

Therefore $A_\lambda \cap D = [A_\lambda \cap D \cap V(A)] \subset W(A)$, eventually. Thus, $A_\lambda$ converges to $A$ in the topology $\tau(\mathcal{U}_{w\Delta})$.

Hence, $\tau(\mathcal{U}_{w\Delta}) \leqslant \sigma(\Delta, \delta)$. Combining the earlier part we get $\tau(\mathcal{U}_{w\Delta}) = \sigma(\Delta, \delta)$.

(b) $\Rightarrow$ (a). By assumption the $\Delta$-Attouch–Wets topology associated to $\mathcal{U}_w$ is Tychonoff and it coincides with the proximal $\Delta$-topology $\sigma(\Delta, \delta)$. So, $\sigma(\Delta, \delta)$ is regular and by using Theorem 4.4.5 in [1] the claim. □

**Theorem 2.2.** *Let $(X, \tau)$ be a Tychonoff space with a compatible LO-proximity $\delta$ and $\Delta \subseteq CL(X)$ a cover of $X$. If $\Delta$ is $\delta$-Urysohn, then the relation $\delta'$ on the power set of $X$ defined by*

($\star$) $A \not\delta' B$ *iff either* $\mathrm{cl}\, A \in \Delta$ *or* $\mathrm{cl}\, B \in \Delta$ *and* $A \not\delta B$

*is a compatible EF-proximity on $X$ coarser that $\delta$. Further, we have*:

(a) *The proximal $\Delta$-topologies $\sigma(\Delta,\delta)$ and $\sigma(\Delta,\delta')$ and the topology $\tau(\mathcal{U}_{w\Delta})$ induced by the $\Delta$-Attouch–Wets uniformity $\mathcal{U}_{w\Delta}$, where $\mathcal{U}_w$ is the unique totally bounded uniformity on $X$ compatible w.r.t. $\delta'$, all coincide. Thus $\sigma(\Delta,\delta)$ is Tychonoff.*
(b) *The proximal $\Delta U$-topology $\sigma(\Delta U,\delta)$ equals the proximal topology $\sigma(\delta')$. Thus $\sigma(\Delta U,\delta)$ is Tychonoff.*

*Conversely, if either $\sigma(\Delta,\delta)$ or $\sigma(\Delta U,\delta)$ is Tychonoff, then $\Delta$ is local and $\delta$-Urysohn.*

**Proof.** By Theorem 1.6 $\delta'$ defined as in ($\star$) is a compatible EF-proximity coarser than $\delta$ as well as $\Delta$ is $\delta'$-Urysohn. Let $\mathcal{U}_w(\delta')=\mathcal{U}_w$ the unique totally bounded uniformity which induces $\delta'$.

To show (a) note:

(1) By Theorem 2.1 the corresponding $\Delta$-Attouch–Wets topology $\tau(\mathcal{U}_{w\Delta})$ is Tychonoff and it equals the proximal $\Delta$-topology $\sigma(\Delta,\delta')$.
(2) From ($\star$) it follows that whenever $U\in\tau$ and $U^c\in\Delta$, for $E\subset X$, $E\not\delta U^c$ if and only if $E\not\delta' U^c$. So, $(U^c)^{++}_{\delta}=(U^c)^{++}_{\delta'}$ and thus $\sigma(\Delta,\delta)$ equals $\sigma(\Delta,\delta')$.

Combining (1) and (2) we get $\sigma(\Delta,\delta)=\sigma(\Delta,\delta')=\tau(\mathcal{U}_{w\Delta})$ and hence the claim.

To show (b) it suffices to consider the upper parts.

Let $A\in U^{++}_{\delta}\in\sigma(\Delta U,\delta)$. Then either $U^c\in\Delta$ and $A\not\delta U^c$ or $\operatorname{cl}U\in\Delta$ and $A\not\delta U^c$.

If $U^c\in\Delta$, then $U^{++}_{\delta}=U^{++}_{\delta'}\in\sigma(\delta')$.

If $\operatorname{cl}U\in\Delta$, then $A\in\Delta$ (since $\Delta$ is closed hereditary) and $A\not\delta U^c$. By ($\star$) $A\not\delta' U^c$. Since $\Delta$ is also $\delta'$-Urysohn there is an $S\in\Delta$ with $A\ll_{\delta'} S\ll_{\delta'} U$ By ($\star$) we have also $A\ll_{\delta} S\ll_{\delta} U$. Clearly, $A\in(S^c)^{++}_{\delta'}\in\sigma(\delta')$ and $(S^c)^{++}_{\delta'}\subset U^{++}_{\delta}$. Thus $\sigma(\Delta U,\delta)\leqslant\sigma(\delta')$.

On the other hand, let $A\in U^{++}_{\delta'}\in\sigma(\delta')$. Then either $U^c\in\Delta$ and $A\not\delta U^c$ or $U^c\notin\Delta$ but $A\in\Delta$ and $A\not\delta U^c$.

If $U^c\in\Delta$, then $U^{++}_{\delta'}=U^{++}_{\delta}$.

If $U^c\notin\Delta$ and $A\in\Delta$, then (since $\delta'$ satisfies the EF-axiom) there is an $S\in CL(X)$ such that $A\ll_{\delta'} S\ll_{\delta'} U$. By ($\star$) we have $S\in\Delta$ and $A\ll_{\delta} S\ll_{\delta} U$. Clearly, $A\in(\operatorname{int}S)^{++}_{\delta}\subset U^{++}_{\delta'}$ and $(\operatorname{int}S)^{++}_{\delta}\in\sigma(\Delta U,\delta)$, showing thereby $\sigma(\delta')\leqslant\sigma(\Delta U,\delta)$ and hence $\sigma(\delta')=\sigma(\Delta U,\delta)$.

Since $\sigma(\delta')$ is Tychonoff (cf. Theorem 1.2) $\sigma(\Delta U,\delta)$ is Tychonoff.

For the converse we just study the case $\sigma(\Delta U,\delta)$ is Tychonoff, since the case $\sigma(\Delta,\delta)$ has been considered in [9].

So, let $\sigma(\Delta U,\delta)$ be Tychonoff. We claim $\Delta$ is a $\delta$-Urysohn family. Let $A\in CL(X)$, $D\in\Delta$ and $A\not\delta D$. By assumption there exists a $\sigma(\Delta U,\delta)$-basic neighbourhood $\mathcal{V}=U^{++}_{\delta}\cap\bigcap_{i=1}^{n}V_i^{-}$ of $A$ such that $A\in\mathcal{V}\subset\operatorname{cl}_{\sigma(\Delta U,\delta)}(\mathcal{V})\subset(D^c)^{++}_{\delta}$. Then, there are two cases:

(i) $A\not\delta U^c$ with $U^c\in\Delta$.
(ii) $A\not\delta U^c$ with $\operatorname{cl}U\in\Delta$ and $U^c\notin\Delta$.

If (i) occurs, then take $S=U^c$ and using similar argument as in [9] (cf. (d) $\Rightarrow$ (a) in Theorem 4.9) we have $D\ll_{\delta} S\ll_{\delta} A^c$.

If (ii) occurs, then $A \in \Delta$, $A \not\delta U^c$ and $D \subset U^c$. By assumption there exists a $\sigma(\Delta U, \delta)$-neighbourhood $\mathcal{W} = W_\delta^{++} \cap \bigcap_{j=1}^m H_j^-$ at $U^c$ such that $U^c \in \mathcal{W} \subset \mathrm{cl}_{\sigma(\Delta U,\delta)}(\mathcal{W}) \subset (A^c)_\delta^{++}$. Hence $U^c \not\delta W$. We claim that $W^c \in \Delta$. Assume not, then $\mathrm{cl}\, W^c \in \Delta$ and hence $U^c \in \Delta$; a contradiction with $U^c \notin \Delta$. Hence, $U^c \not\delta W$ and $W^c \in \Delta$. So, putting $S = W^c$ we have $A \ll_\delta S \ll_\delta U$ (Theorem 4.9 in [9]). Since $U \subset D^c$ we have $A \ll_\delta S \ll_\delta D^c$. □

**Corollary 2.3.** *Let $(X, \tau)$ be a Tychonoff and $\Delta \subseteq CL(X)$ a cover of $X$ which is Urysohn. Then the relation $\delta'$ on the power set of $X$ defined by*

($\star\star$) $A \not\delta' B$ *iff either* $\mathrm{cl}\, A \in \Delta$ *or* $\mathrm{cl}\, B \in \Delta$ *and* $A \not\delta_0 B$

*is a compatible EF-proximity on $X$ with $\delta' \leqslant \delta_0$. Further we have*:

(a) $\tau(\Delta) = \sigma(\Delta, \delta')$ *and if $\mathcal{U}_w$ is the unique totally bounded uniformity on $X$ compatible with $\delta'$, then the $\Delta$-topology $\tau(\Delta)$ is the topology $\tau(\mathcal{U}_{w\Delta})$ induced by the $\Delta$-Attouch–Wets uniformity $\mathcal{U}_{w\Delta}$ and hence is Tychonoff.*
(b) $\tau(\Delta U)$ *equals* $\sigma(\delta')$. *Thus $\tau(\Delta U)$ is Tychonoff.*

*Conversely, if either $\tau(\Delta)$ or $\tau(\Delta U)$ are Tychonoff, then $\Delta$ is Urysohn.*

**Corollary 2.4.** *Let $(X, \tau)$ be a Hausdorff space. The following are equivalent*:

(a) *$X$ is locally compact*;
(b) *the U-topology $\tau(U)$ is uniformizable*;
(c) *$\tau(U)$ is the proximal topology $\sigma(\delta_1)$, where $\delta_1$ is the proximity induced by the one-point-compactification of $X$* (*see Remark* 1.4).

## 3. First and second countability of (proximal) $\Delta U$-topologies

We start with the following lemma and remark and point out that $\Delta$ is just a subfamily of $CL(X)$ containing the singletons.

**Lemma 3.1** (cf. Lemma 5.3 in [11]). *Let $(X, \tau)$ be a $T_1$ space with a compatible LO-proximity $\delta$ and $\Delta \subseteq CL(X)$. If $(CL(X), \tau(\Delta U))$* (*respectively $(CL(X), \sigma(\Delta U, \delta))$*) *is first countable, then every $A \in CL(X)$ is separable.*

**Remark 3.2.** If $(X, \tau)$ is a $T_1$ space with a compatible LO-proximity $\delta$ and $\Delta \subseteq CL(X)$, then $\tau(\Delta U)$ (respectively $\sigma(\Delta U, \delta)$) is admissible; i.e., the assignment $x \to \{x\}$ is a topological embedding of $X$ into $(CL(X), \tau(\Delta U))$ (respectively of $X$ into $(CL(X), \sigma(\Delta U, \delta))$).

Now, we assume that $\Delta$ is also a ring, i.e., it is closed under finite unions and finite intersection, unless otherwise explicitly stated.

**Definition 3.3.** Let $(X, \tau)$ be a $T_1$ space with a compatible LO-proximity $\delta$, $A$ a closed nonempty subset of $X$ and $\Delta \subseteq CL(X)$ a ring. Then:

(a) A family $\Delta'_A \subset \Delta$ is a (*proximal*) *local* $\Delta U$*-base* at $A$, $A \neq X$, if whenever $A \subset U$ ($A \ll_\delta U$) with $U^c$ or $\mathrm{cl}\, U \in \Delta$, there is a $V$ with $V^c$ or $\mathrm{cl}\, V \in \Delta'_A$ and $A \subset V \subset U$ ($A \ll_\delta V \subset U$).
(b) A family $\Delta' \subset \Delta$ is a (*proximal*) $\Delta U$*-base* if for each $A \subset U$ ($A \ll_\delta U$), $A \neq X$, with $U^c$ or $\mathrm{cl}\, U \in \Delta$ and $A \in CL(X)$, there is a $V$ with $V^c$ or $\mathrm{cl}\, V \in \Delta'$ and $A \subset V \subset U$ ($A \ll_\delta V \subset U$).

**Theorem 3.4.** *Let $(X, \tau)$ be a $T_1$ space with a compatible LO-proximity $\delta$ and $\Delta \subseteq CL(X)$ a ring. The following are equivalent*:

(a) $(CL(X), \sigma(\Delta U, \delta))$ *is first countable*;
(b) *$X$ is first countable, every closed set $A$ is separable and every $A \in CL(X)$, $A \neq X$, has a countable proximal local $\Delta U$-base $\Delta'_A$ at $A$.*

**Proof.** (a) $\Rightarrow$ (b). By Remark 3.2 $X$ is first countable and by Lemma 3.1 every closed set $A$ is separable. Now, let $A \in CL(X)$, $A \neq X$. The first countability of $(CL(X), \sigma(\Delta U, \delta))$ at $A$ means that there is a countable family $\mathcal{L}_A$ of sets of the form $\bigcap_{j\in J}(K_j)^{++}_\delta \cap \bigcap_{t\in T}(H_t)^{++}_\delta \cap \bigcap_{i\in I} V_i^-$, with $I$, $T$ and $J$ finite subsets of $\mathbb{N}$, $H_t$, $K_j$ and $V_i \in \tau$, $A \ll_\delta H_t$, $A \ll_\delta K_j$, $H_t^c \in \Delta$ and $\mathrm{cl}\, K_j \in \Delta$.

Set $\Delta'_A = \{H^c\colon H$ occurs in the presentation of some element in $\mathcal{L}_A\} \cup \{\mathrm{cl}\, K\colon K$ occurs in the presentation of some element in $\mathcal{L}_A\}$.

Without loss of generality we may assume $\Delta'_A$ is a ring.

It is a routine exercise to verify that $\Delta'_A$ is nonempty and thus countable. We prove that $\Delta'_A$ is a proximal local $\Delta U$ base at $A$.

So, let $U \in \tau$ with $A \ll_\delta U$ and $U^c$ or $\mathrm{cl}\, U \in \Delta$. Hence, there is $L = \bigcap_{j\in J}(K_j)^{++}_\delta \cap \bigcap_{t\in T}(H_t)^{++}_\delta \cap \bigcap_{i\in I} V_i^- \in \mathcal{L}_A$ such that $A \in L \subset U^{++}_\delta$. Since $A \neq X$, we may assume that also $U \neq X$. Clearly, in the expression of $L$, either $T$ or $J$ is nonempty (in fact, if $T = \emptyset$ and $J = \emptyset$, then by choosing $x \in U^c \cap \mathrm{cl}\, U$ we have that $F = (A \cup \{x\}) \in L$ but $F \notin U^{++}_\delta$; a contradiction).

If $T \neq \emptyset$, then the following subcases occur.

(I) $J = \emptyset$. Then

$$L = \bigcap_{t\in T}(H_t)^{++}_\delta \cap \bigcap_{i\in I} V_i^- \subset U^{++}_\delta.$$

Let $S = \bigcup_{t\in T} H_t^c$ and set $V = S^c$. Thus $V^c \in \Delta'_A$ and $A \ll_\delta V$ (because $A \in L$). With a similar argument as in Theorem 5.4 in [11] we have $A \ll_\delta V \subset U$.

(II) $J \neq \emptyset$. Then

$$L = \bigcap_{j\in J}(K_j)^{++}_\delta \cap \bigcap_{t\in T}(H_t)^{++}_\delta \cap \bigcap_{i\in I} V_i^- \subset U^{++}_\delta.$$

Let $B^j = K_j \cap \bigcap_{t\in T} H_t$ for each $j \in J$. Therefore, $\mathrm{cl}\, B^j \in \Delta'_A$ because $\mathrm{cl}\, K_j \in \Delta'_A$, $B^j \subset K_j$ and $\Delta'_A$ is a ring. Clearly, $A \ll_\delta B^j$ (in fact $A \not\delta [K_j^c \cup \bigcup_{t\in T} H_t^c]$ because $A \in L$). Set

$V = \bigcap_{j \in J} B^j$, then $\operatorname{cl} V \in \Delta'_A$ (because $V \subset K_j$, $\operatorname{cl} K_j \in \Delta'_A$ and $\Delta'_A$ is a ring) and $A \ll_\delta V$ (see Theorem (1.18) in [17]). We claim that if either $U^c \in \Delta$ or $\operatorname{cl} U \in \Delta$, then $V \subset U$.

Assume not, then there exists an $x \in V \cap U^c$. Since $V$ is open and $x \in V$, then $x \not\delta V^c$. So $x \not\delta [\bigcup_{j \in J} K_j^c \cup \bigcup_{t \in T} H_t^c]$ because $V = [\bigcap_{j \in J} K_j \cap \bigcap_{t \in T} H_t]$. Set $F = A \cup \{x\}$, then it easy to check that $F \in CL(X)$, $F \in L$ but $F \notin U_\delta^{++}$; a contradiction because $L \subset U_\delta^{++}$.

If $J \neq \emptyset$, then the following two subcases may occur.

(I′) $T \neq \emptyset$. But this is the above subcase (II).

(II′) $T = \emptyset$. Then

$$L = \bigcap_{j \in J} (K_j)_\delta^{++} \cap \bigcap_{i \in I} V_i^- \subset U_\delta^{++}.$$

Set $V = \bigcap_{j \in J} K_j$, then $\operatorname{cl} V \in \Delta'_A$ and $A \ll_\delta V$ (see Theorem (1.18) in [17]). As in case (I), we have that if either $U^c \in \Delta$ or $\operatorname{cl} U \in \Delta$, then $A \ll_\delta V \subset U$.

(b) ⇒ (a). Let $A \in CL(X)$. The case $A = X$ is standard.

So, let $A \neq X$ and $\Delta'_A$ be a proximal local $\Delta U$-base at $A$.

Let $\{a_1, a_2, \ldots, a_n, \ldots\}$ be a countable dense set in $A$, $\mathcal{S}(a_i)$ $(i = 1, 2, \ldots, n, \ldots)$ be a countable base of neighbourhoods at $a_i$. Set $\mathcal{S} = \{\mathcal{S}(a_i)\colon i = 1, 2, \ldots, n, \ldots\}$ and consider the family $\mathcal{L}_A$ of all subsets of the form $\bigcap_{j \in J} (V_j)_\delta^{++} \cap \bigcap_{i \in I} U_i^-$, with $I$, $J$ finite subsets of $\mathbb{N}$, $V_j \in \Delta'_A$ and $U_i \in \mathcal{S}$. We claim that $\mathcal{L}_A$ is a countable local base of open $\sigma(\Delta U, \delta)$-neighbourhoods at $A$. It suffices to show that $\mathcal{L}_A$ is a local base for a subbasic $\sigma(\Delta U, \delta)$-neighbourhoods system at $A$.

Case (1). Let $A \in H_\delta^{++} \cap \bigcap_{i \in I} Q_i^-$ with $I$ finite subset of integers, $Q_i \in \tau$ for each $i \in I$, $H \in \tau$ and $H^c \in \Delta$. Then $A \ll_\delta H$ and for each $i \in I$ let $U_i \in \mathcal{S}$ be such that $U_i \subset Q_i$. By assumption there exists a $V \in \tau$ with $V^c \in \Delta'_A$ and $A \ll_\delta V \subset H$. Set $L = V_\delta^{++} \cap \bigcap_{i \in I} U_i^-$, then it is easy to check that $L \in \mathcal{L}_A$ and $L \subset H_\delta^{++} \cap \bigcap_{i \in I} Q_i^-$.

Case (2). Suppose $A \in K_\delta^{++} \cap \bigcap_{i \in I} Q_i^-$ with $I$ finite subset of integers $Q_i \in \tau$ for each $i \in I$, $K \in \tau$ and $\operatorname{cl} K \in \Delta$. Then $A \ll_\delta K$ and for each $i \in I$ let $U_i \in \mathcal{S}$ be such that $U_i \subset Q_i$. By hypothesis there exists a $V \in \tau$ with $\operatorname{cl} V \in \Delta'_A$ and such that $A \ll_\delta V \subset K$. Set $L' = V_\delta^{++} \cap \bigcap_{i \in I} U_i^-$ and note that $L' \in \mathcal{L}_A$ and $L' \subset K_\delta^{++} \cap \bigcap_{i \in I} Q_i^-$. □

**Corollary 3.5.** *Let $(X, \tau)$ be a $T_1$ space and $\Delta \subseteq CL(X)$ a ring. The following are equivalent*:

(a) *$(CL(X), \tau(\Delta U))$ is first countable*;
(b) *$X$ is first countable, every closed set $A$ is separable and every $A \in CL(X)$, $A \neq X$, has a countable local $\Delta U$-base $\Delta'_A$ at $A$.*

Now, we analyse the second countability.

**Theorem 3.6.** *Let $(X, \tau)$ be a $T_1$ space with a compatible LO-proximity $\delta$ and $\Delta \subseteq CL(X)$ a ring. The following are equivalent*:

(a) *$(CL(X), \sigma(\Delta U, \delta))$ is second countable*;
(b) *$X$ is second countable and there is a countable subring $\Delta' \subset \Delta$ which is a proximal $\Delta U$-base.*

**Proof.** (a) $\Rightarrow$ (b). By Remark 3.2 $X$ is second countable. Let $\mathcal{L}$ be a countable base of $\sigma(\Delta U, \delta)$. Every element $L \in \mathcal{L}$ has the form

$$L = \bigcap_{j \in J} (K_j)_\delta^{++} \cap \bigcap_{t \in T} (H_t)_\delta^{++} \cap \bigcap_{i \in I} V_i^-,$$

with $I$, $T$ and $J$ finite subsets of $\mathbb{N}$, $H_t$, $U_j$ and $V_i \in \tau$, $H_t^c \in \Delta$ and $\operatorname{cl} K_j \in \Delta$.

Set $\Delta' = \{H^c\colon H$ occurs in the presentation of some element in $\mathcal{L}\} \cup \{\operatorname{cl} K\colon K$ occurs in the presentation of some element in $\mathcal{L}\}$.

Clearly, $\Delta' \subset \Delta$ is countable and by using arguments as in above Theorem 3.4 it is a proximal $\Delta U$-base.

(b) $\Rightarrow$ (a). Let $\mathcal{V}$ be a countable base of $X$. It is easy to verify that the family $L = \bigcap_{j \in J} (K_j)_\delta^{++} \cap \bigcap_{t \in T} (H_t)_\delta^{++} \cap \bigcap_{i \in I} V_i^-$, with $J$, $T$ and $I$ finite subsets of $\mathbb{N}$, $K_j$, $H_t$ and $V_i$ open such that $\operatorname{cl} K_j \in \Delta'$, $H_t^c \in \Delta'$ and $V_i \in \mathcal{V}$ respectively, is a countable base for $\sigma(\Delta U, \delta)$. □

**Corollary 3.7.** *Let $(X, \tau)$ be a $T_1$ space and $\Delta \subseteq CL(X)$ a ring. The following are equivalent*:

(a) $(CL(X), \tau(\Delta U))$ *is second countable*;
(b) *$X$ is second countable and there is a countable subring $\Delta' \subset \Delta$ which is a $\Delta U$-base.*

## 4. Metrizability of (proximal) $\Delta U$-topologies

**Definition 4.1.** Let $(X, \tau)$ be a $T_1$ space with a compatible LO-proximity $\delta$ and $\Delta \subseteq CL(X)$ be a nonempty subfamily of $CL(X)$. A subfamily $\Delta'$ of $\Delta$ is called *relatively $\delta$-Urysohn w.r.t. $\Delta$* if the following condition is fulfilled:

($*$) for every $D \in \Delta$ with $D \neq X$ and every $V \in \tau$ with $D \ll_\delta V$, there is an $S \in \Delta'$ with $D \ll_\delta S \ll_\delta V$.

A subfamily $\Delta'$ of $\Delta$ is called *relatively Urysohn w.r.t. $\Delta$* provided:

($**$) for every $D \in \Delta$ with $D \neq X$ and every $V \in \tau$ with $D \subset V$, there is an $S \in \Delta'$ with $D \subset S \subset V$.

**Theorem 4.2.** *Let $(X, \tau)$ be a $T_1$ space with a compatible LO-proximity $\delta$ and $\Delta \subseteq CL(X)$ a cover of $X$. The following are equivalent*:

(a) $(CL(X), \sigma(\Delta U, \delta))$ *is metrizable*;
(b) *$X$ is Tychonoff and second countable and there is a countable subring $\Delta' \subset \Delta$ which is relatively $\delta$-Urysohn w.r.t. $\Delta$*;
(c) $(CL(X), \sigma(\Delta, \delta))$ *is metrizable.*

**Proof.** (a) $\Rightarrow$ (b). If $(CL(X), \sigma(\Delta U, \delta))$ is metrizable, then $\sigma(\Delta U, \delta)$ is second countable and Tychonoff. Thus, $X$ is Tychonoff and second countable. By Theorem 2.2 $\Delta$ is $\delta$-Urysohn. Moreover, second countability assures that there is a countable subring $\Delta'$ of $\Delta$ which is a proximal $\Delta U$ base. We claim $\Delta'$ fulfills $(*)$ of Definition 4.1. Let $D \in \Delta$ with $D \neq X$, $V \in \tau$ and $D \ll_\delta V$. Without loss of generality we may suppose $V \neq X$. Put $A = X \setminus V$. Then $A \in CL(X)$ and $A \not\delta D$. So, using twice the $\delta$-Urysohn condition on the family $\Delta$ and Theorem (1.17) in [17] there are $R$ and $T \in \Delta$ such that $D \ll_\delta R \ll_\delta \operatorname{int} T \subset T \ll_\delta V$. Hence there exists an open set $M$ with $\operatorname{cl} M \in \Delta'$ that $R \ll_\delta M \subset (\operatorname{int} T)$. Set $S = \operatorname{cl} M$. Then $S \in \Delta'$ and $D \ll_\delta S \ll_\delta V$.

(b) $\Rightarrow$ (a). It is clear that $(CL(X), \sigma(\Delta U, \delta))$ is a Tychonoff space (cf. Theorem 2.2). By assumption there is a countable subring $\Delta'$ of $\Delta$ which satisfies condition $(*)$. But clearly $(*)$ implies that $\Delta'$ is a proximal $\Delta U$ base. Thus, by Theorem 3.6 $(CL(X), \sigma(\Delta U, \delta))$ is second countable, too. Therefore, by Urysohn Metrization Theorem $(CL(X), \sigma(\Delta U, \delta))$ is metrizable.

(c) $\Leftrightarrow$ (b). Use an argument similar as in Theorem 5.20 in [11]. □

**Corollary 4.3.** *Let $(X, \tau)$ be a Tychonoff space and $\Delta \subseteq CL(X)$ a cover of $X$. The following are equivalent*:

(a) $(CL(X), \tau(\Delta U))$ *is metrizable*;
(b) $X$ *is Tychonoff and second countable and there is a countable subring $\Delta' \subset \Delta$ which is relatively Urysohn w.r.t. $\Delta$*;
(c) $(CL(X), \tau(\Delta))$ *is metrizable.*

**Corollary 4.4.** *Let $(X, \tau)$ be a Tychonoff space with a compatible LO-proximity $\delta$ and $\Delta \subseteq CL(X)$ a cover of $X$.*

*Then $(CL(X), \sigma(\Delta U, \delta))$ (respectively $(CL(X), \tau(\Delta U))$) is metrizable if and only if $(CL(X), \sigma(\Delta, \delta))$ (respectively $(CL(X), \tau(\Delta))$) is metrizable.*

If we focus our attention on the $U$-topology, we have:

**Corollary 4.5.** *Let $(X, \tau)$ be a Tychonoff space. The following are equivalent*:

(a) $(CL(X), \tau(U))$ *is second countable*;
(b) $X$ *is locally compact and second countable*;
(c) $(CL(X), \tau(U))$ *is metrizable*;
(d) $(CL(X), \tau(F))$ *is metrizable.*

**Proof.** (a) $\Rightarrow$ (b). It follows from (a) $\Rightarrow$ (b) of Corollary 3.7 when $\Delta = K(X)$.

(b) $\Rightarrow$ (c). By assumption, $X$ admits a countable base $\mathcal{B}$ such that for each $W \in \mathcal{B}$, $\operatorname{cl} W$ is compact. Let $\Sigma(\mathcal{B})$ the family of all finite unions and finite intersection of elements in $\mathcal{B}$. Set $\Delta' = \{\operatorname{cl} S\colon S \in \Sigma(\mathcal{B})\}$. Clearly, $\Delta' \subset K(X)$ and $\Delta'$ satisfies $(**)$ of Definition 4.1. By Corollary 4.3 the claim holds.

(c) $\Rightarrow$ (a) it is trivial and (b) $\Leftrightarrow$ (d) is nicely dealt with in Theorem 5.1.5 in [1]. □

In Theorem 5.7 in [12] the authors have shown that whenever $(X,\mathcal{U})$ is a Hausdorff uniform space, then $(CL(X),\sigma(\mathcal{U}))$ is metrizable if and only if there is a totally bounded metric $\varrho$ on $X$ compatible with $\mathcal{U}$.

So, we have a complete and attractive solution to the metrization problem for the proximal $\Delta U$-topology $\sigma(\Delta U,\delta)$ with respect to a given LO-proximity on $X$.

**Theorem 4.6.** *Let $(X,\tau)$ be a $T_1$ space with a compatible LO-proximity $\delta$ and $\Delta \subseteq CL(X)$ a cover of $X$. The following are equivalent*:

(a) $(CL(X),\sigma(\Delta U,\delta))$ *is metrizable*;
(b) *there exists a compatible totally bounded metric $\varrho$ on $X$ such that $\sigma(\Delta U,\delta)=\sigma(\varrho)$.*

**Proof.** (b) ⇒ (a). By a result in [3] it is known that $\sigma(\varrho)$ is metrizable. Hence $\sigma(\Delta U,\delta)$ is metrizable.

(a) ⇒ (b). By Theorems 1.6 and 2.2 there is a compatible EF-proximity $\delta'$ on $X$ such that $\sigma(\Delta U,\delta)=\sigma(\delta')$. Let $\mathcal{U}_w$ be the unique totally bounded uniformity which induces $\delta'$. Then $\sigma(\delta')=\sigma(\mathcal{U}_w)$.

Since $(CL(X),\sigma(\mathcal{U}_w))$ is metrizable, by Theorem 5.7 in [12] there exists a totally bounded metric $\varrho$ compatible with respect to $\mathcal{U}_w$ with $\sigma(\mathcal{U}_w)=\sigma(\varrho)$. But $\sigma(\Delta U,\delta)=\sigma(\delta')=\sigma(\mathcal{U}_w)$ and hence the claim holds. □

**Corollary 4.7.** *Let $(X,\tau)$ be a $T_1$ space and $\Delta \subset CL(X)$ a cover of $X$. The following are equivalent*:

(a) $(CL(X),\tau(\Delta U))$ *is metrizable*;
(b) *there exists a compatible totally bounded metric $\varrho$ on $X$ such that $\tau(\Delta U)=\sigma(\varrho)$.*

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