ISSN 2590-9770
The Art of Discrete and Applied Mathematics 2 (2019) #P2.08
https://doi.org/10.26493/2590-9770.1324.5b2
(Also available at http://adam-journal.eu)

Dual binary discriminator varieties*

Robert J. Bignall †
Sunway University, 5 Jalan Universiti, Bandar Sunway,
46150 Petaling Jaya, Selangor Darul Ehsan, Malaysia

Matthew Spinks
Dipartimento di Pedagogia, Psicologia, Filosofia,
Università di Cagliari, Via Is Mirrionis 1, 09123 Cagliari, Italy

Received 22 August 2019, accepted 28 November 2019, published online 30 December 2019

Abstract

Left normal bands, strongly distributive skew lattices, implicative BCS-algebras, skew Boolean algebras, skew Boolean intersection algebras, and certain other non-commutative structures occur naturally as term reducts in the study of ternary discriminator algebras and the varieties that they generate, giving rise thereby to various classes of *pointed discriminator varieties*[1] that generalise the class of pointed ternary discriminator varieties. For each such class of varieties there is a corresponding *pointed discriminator function* that generalises the ternary discriminator. In this paper some of the classes of pointed discriminator varieties that are contained in the class of dual binary discriminator varieties are characterised. A key unifying property is that the principal ideals of an algebra in a dual binary discriminator variety are entirely determined by the dual binary discriminator term for that variety.

Keywords: Dual binary discriminator, binary discriminator, ternary discriminator, skew lattice.

Math. Subj. Class.: 06E75, 06B75, 06-06

*The results of this paper were presented at Non-commutative structures 2018: A workshop in honor of Jonathan Leech.

†Corresponding author.

[1] In the literature "discriminator", "discriminator algebra", and "discriminator variety" are normally used to refer to the ternary discriminator function, a ternary discriminator algebra, and a ternary discriminator variety respectively. In this paper these terms are used to refer more generally to various functions, algebras, and varieties that generalise the ternary discriminator, ternary discriminator algebras, and ternary discriminator varieties.

E-mail addresses: rjbignall@gmail.com (Robert J. Bignall), mspinksau@yahoo.com.au (Matthew Spinks)

1 Introduction

Recall that the *ternary discriminator* on a set A is the function $t\colon A^3 \to A$ defined for all elements $a, b, c \in A$ by $t(a, b, c) = a$ if $a \neq b$, and c otherwise. A *ternary discriminator algebra* is an algebra $\mathbf{A}$ for which there exists a term $t(x, y, z)$ in the language of $\mathbf{A}$ that realises the ternary discriminator on A. A *ternary discriminator variety* is a variety generated by a class K of ternary discriminator algebras, for which there exists a term $t(x, y, z)$ realising the ternary discriminator function on each $\mathbf{A} \in \mathsf{K}$. Ternary discriminator varieties generalise Boolean algebras and have been widely studied; see [15, Chapter IV§9].

A key property of the ternary discriminator term $t(x, y, z)$ for a ternary discriminator variety V is that it determines the congruences on each algebra $\mathbf{A} \in \mathsf{V}$, in the sense that the congruences of $\mathbf{A}$ are precisely those of the term reduct $\langle A; t^{\mathbf{A}}\rangle$. This motivates the following definition.

Definition 1.1. The *generic*[2] *ternary discriminator variety* TD is the variety of algebras of similarity type $\langle 3\rangle$ generated by the class of all algebras of the form $\mathbf{A} = \langle A; t\rangle$, where the ternary operation t is the ternary discriminator function on A.

Algebras in ternary discriminator varieties have a number of strong congruence properties. In particular, they are congruence-distributive, congruence-permutable, congruence-regular, and congruence-uniform. Moreover, every compact congruence is a principal factor congruence. Consequently, every algebra in a ternary discriminator variety can be represented as a Boolean product of ternary discriminator algebras; for details see [15, Chapter IV§9].

Examples of ternary discriminator varieties include varieties generated by a primal algebra (and thus the variety of Boolean algebras), monadic algebras, cylindric algebras of dimension n, and skew Boolean intersection algebras. The latter are used as a paradigmatic example of a (pointed) ternary discriminator variety in this paper. Briefly, a skew Boolean intersection algebra (SBIA) is a skew lattice with additional operations such that each principal subalgebra $a \wedge A \wedge a$ is a Boolean lattice, and for which finite meets exist with respect to the natural skew lattice partial order. For a more detailed definition and some key properties of SBIAs see [7].

While the generic ternary discriminator variety is a useful concept (see for example [14] and [29]), it can be somewhat unintuitive to work directly with the ternary discriminator term. In practice, almost all natural examples of ternary discriminator varieties have at least one constant term,[3] which facilitates the definition and use of more familiar binary terms. In particular, it is shown in [7] that every algebra $\mathbf{A}$ in a *pointed* ternary discriminator variety, that is, a ternary discriminator variety with a constant term, has a right handed (and thus also a left handed) SBIA term reduct that has the same congruences as $\mathbf{A}$. This follows from the observation that the variety of left (or right) handed SBIAs is, up to term equivalence, the *generic pointed ternary discriminator variety*, namely the variety TD_0 generated by the class of all algebras of type $\langle 3, 0\rangle$, having the form $\langle A; t, 0\rangle$, where the operation t is the ternary discriminator on A.

More generally, if an algebra $\mathbf{A}$ is a member of a (not necessarily pointed) ternary discriminator variety V with ternary discriminator term $t(x, y, z)$ and c is an arbitrary element

[2] Alternatively called *pure* by some authors; see for example [14].

[3] Two important exceptions [36, Corollary 4.31] are the varieties SA_3 and BN_4 arising respectively as the equivalent quasivariety semantics (in the sense of Blok and Pigozzi [10]) of the 3-valued relevant logic with mingle $\mathbf{RM}_3$ [5, §26.9, §29.12] and its 4-valued cousin, the logic $\mathbf{BN}_4$ of Brady [13].

of its base set A, then the polynomial reduct $\mathbf{A}_c = \langle A; \vee_c, \wedge_c, \backslash_c, \cap_c, c\rangle$ is a left handed skew Boolean intersection algebra[4] such that **Con A** = **Con** $\mathbf{A}_c$, with operations defined by[5]

$$a \vee_c b := t(b, c, a); \qquad a \wedge_c b := t(b, t(b, c, a), a);$$
$$a/_c b := t(a, b, c); \qquad a\backslash_c b := t(c, b, a), \text{ and}$$
$$a \cap_c b := t(c, t(c, b, a), a) = a/_c(a/_c b).$$

The various reducts of $\mathbf{A}_c$ are also familiar structures. For example, $\langle A; \wedge_c, c\rangle$ is a left normal band; $\langle A; \backslash_c, c\rangle$ is an implicative BCS-algebra; $\langle A; /_c, c\rangle$ is an implicative BCK-algebra; $\langle A; \vee_c, \wedge_c, c\rangle$ is a left handed strongly distributive skew lattice with zero c; and $\langle A; \vee_c, \wedge_c, \backslash_c, c\rangle$ is a left handed skew Boolean algebra.

Up to isomorphism the structure of each of these derived algebras is independent of the choice of c. This is shown by the following result from [6]. It can be proved using the Boolean product representations of $\mathbf{A}_c$ and $\mathbf{A}_d$.

Theorem 1.2. *Let* V *be a ternary discriminator variety, with* $\mathbf{A} \in \mathsf{V}$. *Then for all* $c, d \in A$, $\mathbf{A}_c \cong \mathbf{A}_d$.

Every congruence on an algebra must also be a congruence on each of its reducts. Since the congruence lattice of every algebra is always a complete sublattice of the lattice of equivalence relations on its base set, the congruence lattice of $\mathbf{A}_c$, and thus of $\mathbf{A}$, is a sublattice of the congruence lattice of each reduct of $\mathbf{A}_c$. Of course, such reducts do not in general have amenable congruence properties. In particular, their congruence lattices may satisfy no special lattice identities and they need not be congruence n-permutable for any $n \geq 2$. However, it follows from Theorem 2.19 in the next section that whenever $\wedge_c$ is included as one of its operations such a reduct has the same principal ideals as $\mathbf{A}_c$. Moreover, for each such reduct there exists a corresponding function that generalises the ternary discriminator, and each of these generalised discriminator functions gives rise to a class of pointed discriminator varieties that generalises the class of pointed ternary discriminator varieties. The class of dual binary discriminator varieties and its subclass of binary discriminator varieties, which have been studied by a number of authors, are examples.

Section 2 of this paper provides a new characterisation of the class of dual binary discriminator varieties (Theorem 2.19). In subsequent sections a number of its pointed discriminator variety subclasses are described and characterised. These are the classes of binary, skew, skew Boolean, multiplicative, pointed fixedpoint, and pointed ternary discriminator varieties. It is shown that the principal ideals of algebras in such varieties are entirely determined by their dual binary discriminator term. Various characterisations, including some that are purely ideal-theoretic in nature, are obtained for the classes of binary, skew Boolean, pointed fixedpoint, and pointed ternary discriminator varieties; see Theorems 3.2, 6.6, 8.1, and 9.1 respectively.

2 The class of dual binary discriminator varieties

Binary and dual binary discriminator varieties were introduced in [16]. The next three definitions are based on that paper, with some minor differences in the terminology and

[4]Note that $\mathbf{A}_c$ is term equivalent to the algebra $\langle A; \vee_c, \wedge_c, /_c, c\rangle$.

[5]We follow the normal convention of writing $t(a, b, c)$ rather than $t^{\mathbf{A}}(a, b, c)$ for the realisation in an algebra $\mathbf{A}$ of a term $t(x, y, z)$, provided that there is no ambiguity about which algebra is intended.

notation used.

Definition 2.1. Let A be a non-empty set and let $0 \in A$. The *dual binary* 0-*discriminator* on A is the binary function $\wedge$ defined for all $a, b \in A$ by $a \wedge b = a$, if $b \neq 0$, and 0 otherwise. 0 is called the *discriminating element*.

Definition 2.2. A *dual binary discriminator algebra* is an algebra $\mathbf{A}$ for which there exists a binary term $x \wedge y$ and a constant term $\mathbf{0}$ in the language of $\mathbf{A}$ that induce the dual binary 0-discriminator and its discriminating element 0 respectively on the base set A of $\mathbf{A}$.

Definition 2.3. A *dual binary discriminator variety* is a variety V with a binary term $x \wedge y$ and a constant term $\mathbf{0}$ in the language of V such that V is generated by a class K of dual binary discriminator algebras, with the terms $x \wedge y$ and $\mathbf{0}$ inducing the dual binary 0-discriminator and its discriminating element respectively on each $\mathbf{A} \in \mathsf{K}$.

The constant term in Definition 2.3 is referred to as the *discriminating constant* of the variety. A dual binary discriminator variety with discriminating constant $\mathbf{0}$ is called a *dual binary* $\mathbf{0}$-*discriminator variety*. Similarly, a dual binary discriminator algebra with discriminating element 0 is called a *dual binary* 0-*discriminator algebra*.

Natural examples of dual binary discriminator varieties are common and diverse, and include normal bands with zero, semilattices with zero, strongly distributive skew lattices with zero, pseudocomplemented semilattices, bounded distributive lattices, Stone algebras, skew Boolean algebras, skew Boolean intersection algebras, and many others.

Definition 2.4. The *generic dual binary discriminator variety*, denoted by DBD, is the variety of similarity type $\langle 2, 0\rangle$ generated by the class of all dual binary discriminator algebras of the form $\mathbf{A} = \langle A; \wedge, 0\rangle$, with $\wedge$ being the dual binary 0-discriminator on A.

Recall that an idempotent semigroup $\mathbf{A} = \langle A; \cdot\rangle$ (i.e. a *band*) is *normal* if it satisfies the identity $xyzx \approx xzyx$. $\mathbf{A}$ is *left normal* (resp. *right normal*) if it satisfies $xyz \approx xzy$ (resp. $xyz \approx yxz$). A *band with zero* is an algebra $\mathbf{A} = \langle A; \cdot, 0\rangle$ of similarity type $\langle 2, 0\rangle$ with a band operation $\cdot$ and a constant $\mathbf{0}$, satisfying the band identities plus the identities $x\mathbf{0} \approx \mathbf{0}x \approx \mathbf{0}$. By Schein [32] the only subdirectly irreducible normal bands with zero are (up to isomorphism) $\mathbf{S_0}$, the 2-element meet semilattice with zero; $\mathbf{L}$, the three-element left normal band with zero that has no non-trivial two-sided semigroup ideals; and $\mathbf{R}$, the three-element right normal band with zero that has no non-trivial two-sided semigroup ideals. It is easily seen that the term xyx induces the dual binary 0-discriminator $\wedge$ on each of these algebras.

Since the identity $xyx \approx xy$ holds for left normal bands, the semigroup operation realises the dual binary 0-discriminator on $\mathbf{L}$. We denote the variety of left normal bands with zero by $\mathbf{LNB_0}$. Since $\mathbf{S_0}$ and $\mathbf{L}$ are the only subdirectly irreducible members of $\mathbf{LNB_0}$ and $\mathbf{S_0}$ is a subagebra of $\mathbf{L}$, it follows that $\mathsf{LNB}_0 = \mathbf{HSP}(\{\mathbf{L}\})$, the variety generated by $\mathbf{L}$.

Proposition 2.5. $\mathsf{DBD} = \mathsf{LNB}_0$.

Proof. It is straightforward to check that every dual binary 0-discriminator algebra $\langle A; \wedge, 0\rangle$ in DBD is an idempotent semigroup with zero that satisfies the left normal band identity $x \wedge y \wedge z \approx x \wedge z \wedge y$. Let K denote the class of all dual binary discriminator algebras in DBD. Then $\mathsf{K} \subseteq \mathsf{LNB}_0$ and hence $\mathbf{HSP}(\mathsf{K}) = \mathsf{DBD} \subseteq \mathsf{LNB}_0$. But $\mathbf{L} \in \mathsf{K}$, so $\mathbf{HSP}(\{\mathbf{L}\}) = \mathsf{LNB}_0 \subseteq \mathsf{DBD}$. □

It is convenient to use the term $\mathbf{0}$-*band* for a band with a zero element that is the realisation of a constant term $\mathbf{0}$.

Corollary 2.6. *Every algebra* $\mathbf{A}$ *in a dual binary* $\mathbf{0}$*-discriminator variety has a left normal* $\mathbf{0}$*-band term reduct* $\langle A; \wedge, 0\rangle$*, where* $\wedge$ *is the operation induced by the dual binary discriminator term.*

Clearly, the dual binary discriminator term for a given dual binary $\mathbf{0}$-discriminator variety is unique up to identity of terms. However, it is possible for a variety to be a dual binary discriminator variety with respect to more than one constant. For example, the variety L_0^1 of bounded distributive lattices $\langle L; \vee, \wedge, 0, 1\rangle$ is both a dual binary $\mathbf{0}$-discriminator variety, with dual binary discriminator term $x \wedge y$, and a dual binary $\mathbf{1}$-discriminator variety, with dual binary discriminator term $x \vee y$. L_0^1 is generated by the two-element bounded distributive lattice, which is both a dual binary 0-discriminator algebra and a dual binary 1-discriminator algebra.

Let $\mathbf{A}$ be an algebra in a dual binary $\mathbf{0}$-discriminator variety with dual binary $\mathbf{0}$-discriminator term $x \wedge y$. It is well-known from semigroup theory (and straightforward to prove) that the binary relation $\preceq$ defined for all $a, b \in A$ by $a \preceq b$ if $a \wedge b = a$ is a preorder on A, and that the binary relation $\leq$ defined for all a, b by $a \leq b$ if $b \wedge a = a$ is a partial order.[6] Observe that $\leq\ \subseteq\ \preceq$ and $0 \leq a$ for all $a \in A$, where 0 is the element induced by the discriminating constant $\mathbf{0}$. The equivalence relation Ξ given by $a\ \Xi\ b$ if $a \preceq b$ and $b \preceq a$ is referred to as the *Clifford-Mclean* relation on A. It is a congruence on the left normal $\mathbf{0}$-band reduct of $\mathbf{A}$, with $\langle A; \wedge, 0\rangle/\Xi$ being the maximal meet semilattice homomorphic image of $\langle A; \wedge, 0\rangle$.

Definition 2.7. An element $m \in A$ is called *maximal* if $a \preceq m$ for all $a \in A$.

For example, every non-zero element of a dual binary 0-discriminator algebra is maximal. Clearly, when the set M of maximal elements of an algebra $\mathbf{A}$ is non-empty it forms an equivalence class of Ξ.

Recall from [20] that a term $t(\vec{x}, \vec{y})$ is an *ideal term in* $\vec{y}$ for a class K of algebras with respect to a constant term $\mathbf{0}$ if $\mathsf{K} \models t(\vec{x}, \vec{\mathbf{0}}) \approx \mathbf{0}$, where $\vec{x}$ and $\vec{y}$ denote sequences of variables. A non-empty subset I of $\mathbf{A} \in \mathsf{K}$ is a $\mathbf{0}$-*ideal* of $\mathbf{A}$ (or just an *ideal* when there is no ambiguity regarding which constant term is intended) if $0 = \mathbf{0}^{\mathbf{A}} \in I$ and for every $\vec{a} \in A$ and $\vec{b} \in I$, $t^{\mathbf{A}}(\vec{a}, \vec{b}) \in I$ whenever $t(\vec{x}, \vec{y})$ is an ideal term in $\vec{y}$ for K. The ideals of an algebra $\mathbf{A}$ form an algebraic lattice under set inclusion, so for every $X \subseteq A$ the smallest ideal containing X exists. It is denoted by $\langle X\rangle$ and is called the *ideal generated by* X. When $X = \{a\}$ this ideal is called *principal* and is denoted by $\langle a\rangle$. We denote the set of all ideals of $\mathbf{A}$ by $\mathrm{Id}\,\mathbf{A}$, and the lattice of ideals of $\mathbf{A}$ by $\mathbf{Id}\,\mathbf{A}$. Clearly, for every congruence ψ, $[0]\psi = \{a \mid a\ \psi\ 0\}$ is always an ideal. However, it is not always the case that an ideal is a congruence class. If every ideal of $\mathbf{A}$ is the 0-class of a congruence on $\mathbf{A}$, then $\mathbf{A}$ is said to be *normal* or to have normal ideals.

Definition 2.8. Given a language with a constant $\mathbf{0}$, a term $t(x_1, \ldots, x_n)$ is called $\mathbf{0}$-*reflexive* if it satisfies the identity $t(\mathbf{0}, \ldots, \mathbf{0}) \approx \mathbf{0}$. An algebra $\mathbf{A}$ with a constant term $\mathbf{0}$ in its language is said to be *reflexive* if $\{0\} = \{\mathbf{0}^{\mathbf{A}}\}$ is a one-element sub-universe, that is, if $f_\gamma(0, \ldots, 0) = 0$ for each operation f_γ of $\mathbf{A}$. A class K of algebras with a constant term $\mathbf{0}$ is *reflexive* if every member of K is reflexive.

[6] For a detailed discussion of the various order relations (called Green's preorders) on semigroups in general see [33, Section 0].

Clearly, a reflexive algebra must have (up to term equivalence) exactly one constant term in its language. Thus, a reflexive dual binary discriminator variety has, up to term equivalence, exactly one discriminating constant and one dual binary discriminator term. The generic dual binary discriminator variety is an example.

Lemma 2.9. *Let* V *be a dual binary* $\mathbf{0}$*-discriminator variety with dual binary discriminator term* $x \wedge y$ *and let* $t(x_1, \ldots, x_n)$*, where* $n > 0$*, be a term in the language of* V*. Then*

1. V *satisfies every identity of the form*

$$t(x_1, \ldots, x_n) \wedge y \approx t(x_1 \wedge y, \ldots, x_n \wedge y) \wedge y.$$

2. *When* $t(x_1, \ldots, x_n)$ *is* $\mathbf{0}$*-reflexive it satisfies the identity*

$$t(x_1, \ldots, x_n) \wedge y \approx t(x_1 \wedge y, \ldots, x_n \wedge y).$$

Proof. It is straightforward to verify that these identities hold on every member of V that is in the class of dual binary discriminator algebras that generates V. □

Definition 2.10. An algebra $\mathbf{A}$ in a variety with a constant term $\mathbf{0}$ is called $\mathbf{0}$-*ideal simple* if its only $\mathbf{0}$-ideals are $\{\mathbf{0}^{\mathbf{A}}\}$ and A.

For the remainder of this paper, in order to simplify the notation and unless stated otherwise, we regard a dual binary discriminator variety as having just one discriminating constant, which will normally be denoted by $\mathbf{0}$. An ideal term of such a dual binary discriminator variety V means an ideal term with respect to $\mathbf{0}$, while an ideal of an algebra $\mathbf{A} \in \mathsf{V}$ means a $\mathbf{0}$-ideal. In a similar fashion, an ideal simple algebra in V means one that is $\mathbf{0}$-ideal simple.

Lemma 2.11. *Every dual binary* 0*-discriminator algebra* $\mathbf{A}$ *is ideal simple.*

Proof. Clearly $x \wedge y$, the dual binary discriminator term for $\mathbf{A}$, is an ideal term in y. Suppose $I \in A$ is such that $I \neq \{0\}$. Let $b \in I$ be such that $b \neq 0$. Then for all $a \in A$, $a = a \wedge b \in I$. Thus $I = A$. Hence $\mathbf{A}$ is ideal simple. □

Thus a non-trivial dual binary 0-discriminator algebra has exactly two equivalence classes under the relation Ξ, namely $\{0\}$ and $A \setminus \{0\}$. We say that such an algebra is *flat*, since it is order isomorphic to a flat Scott domain.[7]

The universal algebraic notions of a semisimple algebra and a semisimple variety (see [15, Chapter IV§12]) have exact ideal-theoretic analogues.

Definition 2.12. An algebra is said to be *ideal semisimple* if it is isomorphic to a subdirect product of ideal simple algebras. A variety V is *ideal semisimple* if every member of V is ideal semisimple.

The proof of the following lemma is directly analogous to the proof of [15, Lemma IV§12.2] characterising semisimple varieties.

[7] In the literature, a skew lattice having exactly two Clifford-Maclean equivalence classes is said to be *primitive*. In general neither of these classes need be a singleton. However, if $\mathbf{A}$ is a primitive skew lattice with zero then the lower equivalence class is a singleton, and in that case $\mathbf{A}$ is order isomorphic to a flat Scott domain.

Lemma 2.13. *A variety* V *is ideal semisimple if and only if every subdirectly irreducible member of* V *is ideal simple.*

Examples of ideal semisimple dual binary discriminator varieties include normal bands with zero, strongly distributive skew lattices with zero, and skew Boolean algebras; see [26, 27].

In [3] Agliano and Ursini introduce the notion of equationally definable principal ideals for arbitrary varieties with a constant term. A variety V with a constant term $\mathbf{0}$ has *equationally definable principal ideals* (*EDPI*) if there exist pairs of binary terms $p_i, q_i, i = 1, \ldots, n$ such that for every $\mathbf{A} \in \mathsf{V}$ and all $a, b \in A$, $a \in \langle b \rangle$ if and only if $p_i(a, b) = q_i(a, b)$ for $i = 1, \ldots, n$, where $\langle b \rangle$ denotes the principal ideal generated by b. A related notion was subsequently considered by van Alten in [4] as follows:

Definition 2.14. A class K of algebras is said to have *EDPI** if there exists an ideal term $t(x, y)$ in y such that for each $\mathbf{A} \in \mathsf{K}$ and all $a, b \in A$, $a \in \langle b \rangle$ if and only if $a = t^{\mathbf{A}}(a, b)$.

The following key result follows from [4, Theorems 4.2 and 4.3].

Theorem 2.15. *Let* V *be a variety with constant term* $\mathbf{0}$ *generated by a class* K*. The following are equivalent.*

1. V *has EDPI.*
2. V *has EDPI**.
3. K *has EDPI**.

Proposition 2.16. *Every dual binary discriminator variety* V *has EDPI. A term in the language of* V *witnesses EDPI** *if and only if it is, up to term identity, the dual binary discriminator term.*

Proof. If K is a class of dual binary discriminator algebras, then by Lemma 2.11 its members are ideal simple and it is clear that the dual binary discriminator term witnesses *EDPI** for K, so by Theorem 2.15 V has *EDPI*. Suppose that $t(x, y)$ is an ideal term in y witnessing *EDPI** for V. Then for every dual binary discriminator algebra $\mathbf{A} \in \mathsf{V}$ and for all $a, b \in A$, $a \in \langle b \rangle$ if and only if $a = t^{\mathbf{A}}(a, b)$. Since $\mathbf{A}$ is ideal simple, this implies that $t^{\mathbf{A}}(a, b) = a$ when $b \neq 0$, with 0 being the realisation in A of the discriminating constant of V. Also, $t^{\mathbf{A}}(a, 0) = 0$, since $t(x, y)$ is an ideal term in y. Thus $t^{\mathbf{A}}(a, b)$ is the dual binary discriminator on A, and hence $t(x, y)$ is the dual binary discriminator term for V, since V is generated by a class of dual binary discriminator algebras. □

Corollary 2.17. *The principal ideals of every algebra in a dual binary* $\mathbf{0}$*-discriminator variety coincide with those of its left normal* $\mathbf{0}$*-band term reduct.*

In particular, every principal ideal $\langle b \rangle$ of an algebra $\mathbf{A}$ in a dual binary discriminator variety has the form $\langle b \rangle = \{a \in A \mid a \wedge b = a\} = \{a \in A \mid a \preceq b\}$, and so every ideal I is a down set with respect to the natural preorder, that is, if $b \in I$ and $a \preceq b$ then $a \in I$.

Definition 2.18. Let $\mathbf{A}$ be an algebra with a left normal $\mathbf{0}$-band term reduct $\mathbf{A_0} = \langle A; \wedge, 0 \rangle$. $\mathbf{A}$ has *ideal-compatible operations* if the principal $\mathbf{0}$-ideals of $\mathbf{A}$ coincide with those of $\mathbf{A_0}$. A class K of algebras with a left normal $\mathbf{0}$-band term is said to have *ideal-compatible operations* if every $\mathbf{A} \in \mathsf{K}$ has ideal-compatible operations with respect to its left normal $\mathbf{0}$-band term reduct.

Theorem 2.19. *Every dual binary discriminator variety is term equivalent to a variety of left normal bands with ideal-compatible operations. A variety* V *with a constant term* $\mathbf{0}$ *is a dual binary* $\mathbf{0}$*-discriminator variety if and only if it has EDPI and is generated by a class of* $\mathbf{0}$*-ideal simple algebras.*

Proof. The first statement is clear in view of Corollary 2.17. For the second statement, suppose that V is a dual binary $\mathbf{0}$-discriminator variety. Then by Proposition 2.16 and Theorem 2.15 V has *EDPI* and the dual binary $\mathbf{0}$-discriminator term for V witnesses *EDPI**. By Lemma 2.11, every dual binary 0-discriminator algebra is ideal simple, so V is generated by a class of ideal simple algebras. Conversely, if V has *EDPI* and is generated by a family K of ideal simple algebras, then by Theorem 2.15 the members of K have *EDPI**. If $t(x, y)$ is a term in the language of V that witnesses *EDPI** then it follows from the proof of Proposition 2.16 that $t(x, y)$ realises the dual binary 0-discriminator function on each member of K. Hence V is a dual binary $\mathbf{0}$-discriminator variety. □

2.1 Central elements

Let V be a dual binary $\mathbf{0}$-discriminator variety, with $\mathbf{A} \in \mathsf{V}$. Denote the element $\mathbf{0}^{\mathbf{A}}$ by 0. For each $c \in A$ let Ψ_c denote the binary relation on A given by $a\ \Psi_c\ b$ if $a \wedge c = b \wedge c$. It follows from Lemma 2.9 that Ψ_c is a congruence on $\mathbf{A}$; see also [16, Theorem 5.3]. Let Θ_c denote the smallest congruence on $\mathbf{A}$ that identifies the elements 0 and c. Since $a\ \Psi_c\ (a \wedge c)\ \Theta_c\ 0$ for all $a \in A$, it follows that $\Psi_c \vee \Theta_c = \iota_{\mathbf{A}}$, the largest congruence on $\mathbf{A}$.

Definition 2.20. An element c of an algebra $\mathbf{A}$ in a dual binary $\mathbf{0}$-discriminator variety is said to be *central* if Θ_c and Ψ_c are complementary factor congruences of $\mathbf{A}$.

In general, Ψ_c and Θ_c will be complementary factor congruences when $\Theta_c \circ \Psi_c = \Psi_c \circ \Theta_c = \iota_{\mathbf{A}}$, and $\Psi_c \wedge \Theta_c = \omega_{\mathbf{A}}$. Since $\Psi_c \vee \Theta_c = \iota_{\mathbf{A}}$ for every c, a sufficient condition for Ψ_c and Θ_c to be factor congruences is that $\Psi_c \wedge \Theta_c = \omega_{\mathbf{A}}$. When $c = 0$, $\Psi_c = \iota_{\mathbf{A}}$, while $\Theta_c = \omega_{\mathbf{A}}$, the smallest congruence on $\mathbf{A}$. On the other hand, if c is a maximal element, then $\Psi_c = \omega_{\mathbf{A}}$, while $\Theta_c = \iota_{\mathbf{A}}$, so 0, and maximal elements when they exist, are examples of central elements.

Since the concept of a central element considered in this paper does not require algebras to have elements that are residually distinct, it differs from the notion of a central element due to Vaggione [38]. In the case of algebras in dual binary discriminator varieties having a second constant term that is residually distinct from the discriminating constant, the central elements in the sense of Vaggione are the same as the central elements considered in this paper. In view of that, the following definition is useful.[8]

Definition 2.21. A dual binary $\mathbf{0}$-discriminator variety V is said to be *double pointed* if there exists a constant term $\mathbf{1}$ in the language of V that is *residually distinct* from $\mathbf{0}$; that is, $\Theta_1 = \iota_{\mathbf{A}}$ for all $\mathbf{A} \in \mathsf{V}$, where $1 = \mathbf{1}^{\mathbf{A}}$.

Examples of double pointed dual binary discriminator varieties include bounded distributive lattices, pseudocomplemented semilattices, Stone algebras, and Boolean algebras. Many examples that are double pointed ternary discriminator varieties arise in the study

[8] In conformance with our notation, Θ_1 abbreviates $\Theta(0, 1)$, the smallest congruence that identifies the elements 0 and 1.

of discriminator logics, since double pointedness ensures the existence of logical negation. For details, see [35].

Proposition 2.22. *A dual binary $\mathbf{0}$-discriminator variety V is double pointed if and only if there exists a constant term $\mathbf{1}$ in the language of V such that $\mathsf{V} \models x \wedge \mathbf{1} \approx x$. In that case the element $\mathbf{1}^{\mathbf{A}} \in \mathbf{A}$ is both maximal and central for every $\mathbf{A} \in \mathsf{V}$.*

Proof. Let V be a double pointed dual binary $\mathbf{0}$-discriminator variety and let $\mathbf{1}$ be a constant term that is residually distinct from $\mathbf{0}$. Let $\mathbf{A} \in \mathsf{V}$ be a non-trivial dual binary 0-discriminator algebra. Then $\mathbf{A}$ has at least two elements, so $\Theta_1 = \iota_{\mathbf{A}}$ implies that $1 \neq 0$, where $1 = \mathbf{1}^{\mathbf{A}}$ and $0 = \mathbf{0}^{\mathbf{A}}$. But then $a \wedge 1 = a$ for all $a \in \mathbf{A}$, since $\wedge$ is the dual binary 0-discriminator on A. Hence the identity $x \wedge \mathbf{1} \approx x$ is satisfied by a class of algebras that generates V.

Conversely, if V has a constant term $\mathbf{1}$ such that $\mathsf{V} \models x \wedge \mathbf{1} \approx x$, then the element $1 = \mathbf{1}^{\mathbf{A}}$ is maximal for every $\mathbf{A} \in \mathsf{V}$, so $\Theta_1 = \iota_{\mathbf{A}}$ for every $\mathbf{A} \in \mathsf{V}$. The second statement of the proposition follows because maximal elements are always central. □

When a variety V is a ternary discriminator variety every element of an algebra $\mathbf{A} \in \mathsf{V}$ is central. However, the converse does not hold. For example, every element of a skew Boolean algebra is central (see Proposition 6.2), but the variety of skew Boolean algebras is not a ternary discriminator variety. The following lemma identifies a necessary (but not sufficient) condition for every element of every algebra in V to be central.

Lemma 2.23. *Let V be a dual binary $\mathbf{0}$-discriminator variety. If the congruences Θ_c and Ψ_c permute for every $\mathbf{A} \in \mathsf{V}$ and all $c \in A$ then there exists a binary term $s(x, y)$ which satisfies the identities $s(x, \mathbf{0}) \approx x$ and $s(x, x) \approx \mathbf{0}$.*

Proof. Let $\mathbf{F}(x, y)$ denote the free V-algebra on free variables x and y. Assume that $\Theta_y \circ \Psi_y = \iota_{\mathbf{F}(x,y)}$. Then there exists an element $s = s(x, y)$ of $F(x, y)$ such that $x \; \Theta_y \; s(x, y) \; \Psi_y \; 0$. Since $y \equiv 0(\Theta_y)$ this implies that $x = s(x, 0)$. Also, $s(x, y) \; \Psi_y \; 0$ implies that $s(x, y) \wedge y = 0 \wedge y = 0$. Now $\mathbf{F}(x, y)$ is free in x and y, so $s(x, \mathbf{0}) \approx x$ and $s(x, y) \wedge y \approx \mathbf{0}$ are identities of V. Since $s(x, y)$ is a $\mathbf{0}$-reflexive term, $\mathsf{V} \models \mathbf{0} \approx s(x, y) \wedge y \approx s(x \wedge y, y \wedge y)$, by Lemma 2.9. Putting $x = y$ gives $\mathsf{V} \models s(x, x) \approx \mathbf{0}$. Thus, when V has the property that $\Theta_c \circ \Psi_c = \iota_{\mathbf{A}}$ for every $\mathbf{A} \in \mathsf{V}$ and $c \in A$, a binary term satisfying the stated identities must exist. □

A term $s(x, y)$ satisfying the identities of Lemma 2.23 is called $\mathbf{0}$-*subtractive*; see [37]. A variety V with a constant term $\mathbf{0}$ is called *subtractive at* $\mathbf{0}$, or $\mathbf{0}$-*subtractive*, if it has a $\mathbf{0}$-subtractive term. An algebra $\mathbf{A}$ with a constant term $\mathbf{0}$ is $\mathbf{0}$-*subtractive* if the variety $\mathbf{HSP}(\{\mathbf{A}\})$ is $\mathbf{0}$-subtractive. Subtractive algebras have normal $\mathbf{0}$-ideals and are *congruence-permutable at* 0, that is, $[0]\theta \circ \psi = [0]\psi \circ \theta$ for every pair of congruences θ and ψ, where $0 = \mathbf{0}^{\mathbf{A}}$. Conversely, a variety V with a constant $\mathbf{0}$ and the property that every $\mathbf{A} \in \mathsf{V}$ is congruence-permutable at $\mathbf{0}^{\mathbf{A}}$ has a $\mathbf{0}$-subtractive term; see [37, Proposition 1.2]. Such a variety is therefore also called $\mathbf{0}$-*permutable*, or *congruence-permutable at* $\mathbf{0}$. When a dual binary $\mathbf{0}$-discriminator variety is subtractive at $\mathbf{0}$ we simply say that it is subtractive.

3 Binary discriminator varieties

The definitions of the binary discriminator function, a binary discriminator algebra, a binary discriminator variety, and the generic binary discriminator variety mirror Definitions 2.1, 2.2, 2.3, and 2.4 in the previous section. Thus, given a set A with element $0 \in A$, the *binary* 0*-discriminator* on A is the function $\backslash\colon A^2 \to A$ defined for $a, b \in A$ by $a\backslash b = a$ if $b = 0$, and 0 otherwise. A *binary discriminator algebra* is an algebra $\mathbf{A}$ for which there exists a a binary term $x\backslash y$ and a constant term $\mathbf{0}$ that induce the binary 0-discriminator and its discriminating element $0 = \mathbf{0}^{\mathbf{A}}$ on the base set A of $\mathbf{A}$. A *binary discriminator variety* is a variety generated by a class K of binary discriminator algebras, with terms $x\backslash y$ and $\mathbf{0}$ inducing the binary 0-discriminator and its discriminating constant 0 and on each $\mathbf{A} \in \mathsf{K}$. The *generic binary discriminator variety* is the variety of similarity type $\langle 2, 0\rangle$ generated by the class of all algebras of the form $\mathbf{A} = \langle A; \backslash, 0\rangle$, with $\backslash$ being the binary 0-discriminator on A.

Lemma 3.1 (cf. [16, Theorem 2.1])**.** *A variety V is a binary $\mathbf{0}$-discriminator variety if and only if V is a $\mathbf{0}$-subtractive dual binary $\mathbf{0}$-discriminator variety.*

Proof. Suppose V is a binary $\mathbf{0}$-discriminator variety. It is immediate that the binary discriminator term $x\backslash y$ witnesses $\mathbf{0}$-subtractivity. Moreover, $a\backslash(a\backslash b)$ is the dual binary 0-discriminator $a \wedge b$ on every binary 0-discriminator algebra in V. On the other hand, if V is a dual binary $\mathbf{0}$-discriminator variety with a $\mathbf{0}$-subtractive term $s(x, y)$ then it is easily checked that $s(x, x \wedge y)$ realizes the binary 0-discriminator on every dual binary 0-discriminator algebra in V. □

It was shown in [8] that the generic binary discriminator variety is the variety iBCS of *implicative BCS-algebras* of type $\langle 2, 0\rangle$, axiomatised by the identities

$$\mathsf{iBCS}\colon\qquad \begin{array}{cc} x\backslash x \approx \mathbf{0} & (x\backslash y)\backslash z \approx (x\backslash z)\backslash y \\ (x\backslash y)\backslash z \approx (x\backslash z)\backslash(y\backslash z) & x\backslash(y\backslash x) \approx x \end{array}$$

It was also shown there that iBCS is generated as a variety by the three-element binary discriminator algebra $\mathbf{B}_2 = \langle\{0, 1, 2\}; \backslash, 0\rangle$, that is, $\mathsf{iBCS} = \mathbf{HSP}(\{\mathbf{B}_2\})$. Implicative BCS-algebras are precisely the $\langle\backslash, 0\rangle$-subreducts of pseudocomplemented semilattices, where $a\backslash b = a \wedge b^*$ for each pseudocomplemented semilattice $\langle A; \wedge, ^*, 0\rangle$ and all $a, b \in A$. As such, they occur widely as subreducts of algebras such as Stone algebras, linearly ordered Heyting algebras, pseudocomplemented semilattices, skew Boolean algebras, strict basic logic algebras, product logic algebras, and algebras in residually finite varieties of basic logic algebras.

If $\mathbf{A} \in \mathsf{iBCS}$ then the Clifford-McLean equivalence relation Ξ is a congruence on $\mathbf{A}$ and $\mathbf{A}/\Xi \in \mathsf{iBCK}$, the variety of *implicative BCK-algebras*, which is the reflective subvariety of iBCS axiomatised relative to iBCS by the identity $x\backslash(x\backslash y) \approx y\backslash(y\backslash x)$.

Combining Theorem 2.19 and Lemma 3.1 yields the following.

Theorem 3.2. *Every binary discriminator variety is term equivalent to a variety of implicative BCS-algebras with ideal-compatible operations. A variety V with a constant term $\mathbf{0}$ is a binary $\mathbf{0}$-discriminator variety if and only if V is subtractive at $\mathbf{0}$, has EDPI and is generated by a class of $\mathbf{0}$-ideal simple algebras.*

Let $\mathbf{A}$ be an algebra in a binary $\mathbf{0}$-discriminator variety V. Recall that Θ_c denotes the smallest congruence on $\mathbf{A}$ equating the elements 0 and c, where 0 is the realisation of the discriminating constant $\mathbf{0}$ of V. Subtractivity ensures that every $\mathbf{0}$-ideal I of $\mathbf{A}$ is a congruence class, so it is meaningful to let Θ_I denote the smallest congruence θ of $\mathbf{A}$ such that $[0]\theta = I$.[9] It turns out that Θ_I has a simple characterisation that depends only on the binary discriminator term. This means that some key structural properties of algebras in binary discriminator varieties can be conveniently studied by restricting attention to their iBCS-algebra term reducts; see for example [9], where the following result is proved.

Theorem 3.3. *Let* V *be a binary* $\mathbf{0}$*-discriminator variety and let* $\mathbf{A} \in \mathsf{V}$. *For all* $a, b, c \in A$,

1. $a \equiv b \bmod \Theta_c$ *if and only if* $a \backslash c = b \backslash c$.
2. $[0]\Theta_c = \langle c \rangle = \{a \in A \mid a \backslash c = 0\} = \{a \in A \mid a \wedge c = a\}$.
3. *For every ideal* I, $a \equiv b \bmod \Theta_I$ *if and only if* $a \backslash c = b \backslash c$ *for some* $c \in I$.
4. *For all* $\Psi \in \mathrm{Con}\, \mathbf{A}$, $\Theta_I \vee \Psi = \Theta_I \circ \Psi \circ \Theta_I$.

Let $\mathbf{A} \in \mathsf{V}$, where V is a binary $\mathbf{0}$-discriminator variety. Since V is $\mathbf{0}$-subtractive, an element $c \in A$ will be central if and only if $\Psi_c \wedge \Theta_c = \omega_{\mathbf{A}}$. Let QB_2 denote the quasi-identity $x \wedge z \approx y \wedge z \;\&\; x \backslash z \approx y \backslash z \Rightarrow x \approx y$. The next result is immediate.

Theorem 3.4. *Let* V *be a binary* $\mathbf{0}$*-discriminator variety. The following are equivalent.*

1. *For all* $\mathbf{A} \in \mathsf{V}$, *every* $c \in A$ *is central.*
2. *For all* $\mathbf{A} \in \mathsf{V}$ *and* $a, b, c \in A$, $a \wedge c = b \wedge c$ *and* $a \backslash c = b \backslash c$ *implies* $a = b$.
3. $\mathsf{V} \models QB_2$.

We call a binary $\mathbf{0}$-discriminator variety satisfying the equivalent conditions of Theorem 3.4 a QB_2 *variety*. This terminology reflects the fact that the quasi-variety generated by $\mathbf{B}_2$, the three-element binary 0-discriminator algebra, is axiomatised by the iBCS identities together with the QB_2 quasi-identity. Most natural examples of binary discriminator varieties are QB_2 varieties.[10] Such varieties are of interest because their members have weak Boolean product representations. A number of examples of weak Boolean product representations of algebras in QB_2 varieties appear in the literature; see, for example, [18, 23], or [30].

4 Some other pointed discriminator functions

In the following definitions we follow the convention of using infix notation for functions in two variables.

Definition 4.1. Let A be a set and let $0 \in A$. Then the

- *skew* 0*-discriminator* on A is the function s defined for all $a, b, c \in A$ by

$$s(a,b,c) = \begin{cases} c & \text{if } c \neq 0, \\ a & \text{if } c = 0 \text{ and } b \neq 0, \\ 0 & \text{otherwise;} \end{cases}$$

- *multiplicative* 0*-discriminator* on A is the function q defined for all $a, b, c \in A$ by

$$q(a,b,c) = \begin{cases} a & \text{if } c \neq 0 \text{ and } a = b, \\ 0 & \text{otherwise;} \end{cases}$$

[9] Thus Θ_I is I^δ in the terminology of Agliano and Ursini [2].

[10] Two significant exceptions are the varieties of implicative BCS-algebras and pseudocomplemented semilattices.

- *pointed fixedpoint* 0*-discriminator* on A is the function f defined for all $a, b, c \in A$ by

$$f(a,b,c) = \begin{cases} c & \text{if } a = b, \\ 0 & \text{otherwise;} \end{cases}$$

- *skew Boolean* 0*-discriminator* on A is the function w defined for all $a, b, c \in A$ by

$$w(a,b,c) = \begin{cases} 0 & \text{if } c \neq 0, \\ b & \text{if } c = 0 \text{ and } b \neq 0, \\ a & \text{otherwise;} \end{cases}$$

- *meet* 0*-discriminator* on A is the function $\cap$ defined for all $a, b \in A$ by

$$a \cap b = \begin{cases} a & \text{if } a = b, \\ 0 & \text{otherwise;} \end{cases}$$

- *monoidal* 0*-discriminator* on A is the function $\vee$ defined for all $a, b \in A$ by

$$a \vee b = \begin{cases} b & \text{if } b \neq 0, \\ a & \text{otherwise.} \end{cases}$$

- *Implicative BCK difference* (brieflly, *iBCK difference*) is the function $/$ defined for all $a, b \in A$ by

$$a/b = \begin{cases} a & \text{if } a \neq b, \\ 0 & \text{otherwise.} \end{cases}$$

Implicative BCK difference may alternatively be defined in terms of the binary and meet 0-discriminators: $a/b = a\backslash(a \cap b)$. For each of the 0-discriminator functions we have the associated notions of a discriminator algebra, discriminator variety, and generic discriminator variety, with definitions analogous to those for the corresponding dual binary and binary 0-discriminator constructs. Some examples of skew, skew Boolean, multiplicative, and pointed fixedpoint discriminator varieties are provided in the next four sections.

All of these pointed discriminator functions are to some extent interdefinable. For example, each of the cited 0-discriminator functions with two arguments can be written as a composition using just the ternary discriminator and the element 0, while each of the cited 0-discriminator functions with three arguments can be written as a composition of the cited 0-discriminator functions with two arguments. In particular, we have the following.

$$\begin{aligned} a \wedge b &= t(b, t(b,0,a), a) & q(a,b,c) &= (a \cap b) \wedge c \\ a \vee b &= t(b,0,a) & s(a,b,c) &= (a \wedge b) \vee c \\ a \cap b &= t(a, t(a,b,0), 0) & w(a,b,c) &= (a \vee b)\backslash c \\ a \vee b &= w(a,b,0) = s(a,a,b) & a\backslash b &= t(0,b,a) \\ a\backslash b &= w(a,a,b) = f(b,0,a) & a/b &= t(a,b,0) \\ a \cap b &= q(a,b,b) = f(a,b,a) \end{aligned}$$

$$a \wedge b = f(0, f(b,0,a), a) = w(0,a,w(0,a,b)) = s(a,b,0) = q(a,a,b) = a\backslash(a\backslash b)$$

$$f(a,b,c) = (c\backslash(a\backslash(a \cap b)))\backslash(b\backslash(a \cap b)) = (c\backslash(a/b))\backslash(b/a)$$

$$t(a,b,c) = f(a,b,c) \vee (a\backslash(a \cap b)) = ((c\backslash(a/b))\backslash(b/a)) \vee (a/b)$$

In view of these equalities and Theorem 2.19 the following is immediate.

Proposition 4.2. *Every pointed ternary, pointed fixedpoint, skew Boolean, skew, multiplicative, and binary* $\mathbf{0}$*-discriminator variety is also a dual binary* $\mathbf{0}$*-discriminator variety. Hence, every such variety has both ideal-compatible operations and EDPI. Moreover, every pointed ternary discriminator variety is also a monoidal discriminator variety, a binary discriminator variety, a meet discriminator variety, a multiplicative discriminator variety, a skew discriminator variety, a skew Boolean discriminator variety, and a pointed fixedpoint discriminator variety.*

As a consequence of Proposition 4.2 and the displayed equalities, the various classes of pointed discriminator varieties can be ordered by class inclusion, as shown in Figure 1. For each class, the figure also shows which of the pointed discriminator terms with two arguments are definable in the varieties making up that class. Note that there are a number of subclasses of the class of dual binary discriminator varieties that are not included in the diagram; for example, the class of pointed dual discriminator varieties and the class of multiplicative skew discriminator varieties described in Section 7.

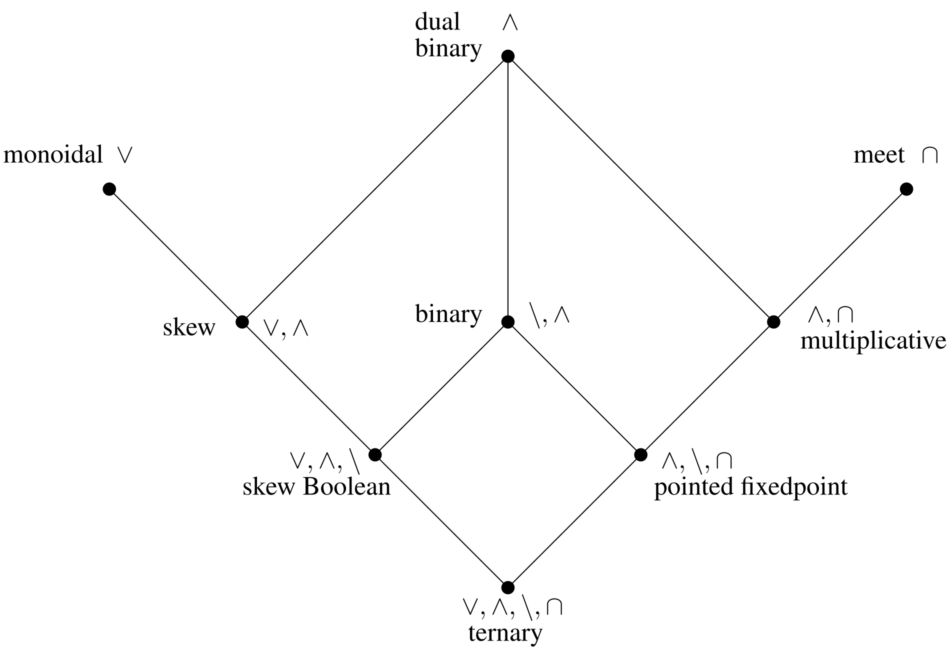

Figure 1: Some classes of pointed discriminator varieties.

4.1 Additivity

Definition 4.3. Let K be a class of algebras with a constant term $\mathbf{0}$ in its language. A binary term $x + y$ is called *additive with respect to* $\mathbf{0}$, or $\mathbf{0}$*-additive*, if $\mathsf{K} \models x + \mathbf{0} \approx x$ and $\mathsf{K} \models \mathbf{0} + x \approx x$. An algebra $\mathbf{A}$ is $\mathbf{0}$-additive if $\{\mathbf{A}\}$ has a $\mathbf{0}$-additive term.[11]

[11]The terminology *additive* is preferred over *monoidal* in this paper in view of the connection with direct summands, and to avoid confusion with the monoidal discriminator.

A variety that is reflexive and additive with respect to a constant term $\mathbf{0}$ has the property that a direct product $\mathbf{A} = \mathbf{B} \times \mathbf{C}$ of two algebras decomposes into the direct sum of two subalgebras $\mathbf{A} = \mathbf{B}_1 \oplus \mathbf{C}_1$, where $\mathbf{B_1} \cong \mathbf{B}$ and $\mathbf{C_1} \cong \mathbf{C}$. (See [28, §2] for the definition of a direct sum and an outline of its properties.) In view of [28, Theorem 2] the converse is also true; a variety in which every product of two algebras decomposes into the direct sum of two subalgebras must have a constant term $\mathbf{0}$ in its language such that it is both reflexive and $\mathbf{0}$-additive.

We omit the prefix and say that a dual binary discriminator variety is additive when there is no ambiguity regarding which discriminating constant is intended. Skew, skew Boolean, and pointed ternary discriminator varieties are additive, so reflexive members of them that are isomorphic to finite direct products can be represented as a direct sum of subalgebras.

5 Skew discriminator varieties

A *skew lattice* $\langle A; \vee, \wedge \rangle$ is an algebra with two associative and idempotent binary operations $\vee$ and $\wedge$, satisfying the dual pair of absorption laws $x \wedge (x \vee y) \approx x \approx (y \vee x) \wedge x$ and $x \vee (x \wedge y) \approx x \approx (y \wedge x) \vee x$. For precise details, see [25]. By a *strongly distributive skew lattice* is meant a skew lattice that is symmetric, normal, and distributive; for details about such algebras see [27, §3].

Proposition 5.1. *The generic skew discriminator variety is term equivalent to the variety of left handed strongly distributive skew lattices with zero. Thus every skew discriminator variety is term equivalent to a variety of strongly distributive skew lattices with zero and with ideal-compatible operations.*

Proof. The skew 0-discriminator on a pointed set $A \supseteq \{0\}$ can be written as a composition of the dual binary and monoidal 0-discriminators: $s(a, b, c) = (a \wedge b) \vee c$. Conversely, we have $a \wedge b = s(a, b, 0)$ and $a \vee b = s(a, a, b)$. Thus any skew 0-discriminator algebra in the generic skew $\mathbf{0}$-discriminator variety is term equivalent to an algebra of the form $\mathbf{A} = \langle A; \vee, \wedge, 0 \rangle$. The remainder of the proof is analogous to the proof of Proposition 2.5. It is straightforward to verify that an algebra such as $\mathbf{A}$ is a primitive, and therefore flat, left handed strongly distributive skew lattice with zero and that every flat left handed strongly distributive skew lattice with zero has this form. On the other hand, it follows from [27, Theorem 3.2] that every subdirectly irreducible strongly distributive skew lattice with a zero is flat. Hence the generic skew discriminator variety is term equivalent to the variety of left handed strongly distributive skew lattices with zero. The second assertion of the proposition now follows from Theorem 2.19. □

More generally, if $\mathbf{A} = \langle A; \vee, \wedge, 0 \rangle$ is a flat strongly distributive skew lattice with a zero, then the skew 0-discriminator can be defined on A by

$$s(a, b, c) = c \vee (a \wedge b \wedge a) \vee c.$$

Examples of skew discriminator varieties that are not ternary discriminator or skew Boolean discriminator varieties thus include strongly distributive skew lattices with zero and hence also distributive lattices with zero, as well as certain varieties in which each member has a bounded distributive lattice term reduct, such as the variety of Q-distributive lattices introduced in [17].

While the term $x \vee y$ witnesses additivity for every skew discriminator variety, an additive dual binary discriminator variety need not be a skew discriminator variety. However, an additive binary discriminator variety is always a skew discriminator variety. In fact, rather more is true.

Proposition 5.2. *A variety* V *with a constant term* $\mathbf{0}$ *is a* $\mathbf{0}$*-additive binary* $\mathbf{0}$*-discriminator variety if and only if it is a skew Boolean* $\mathbf{0}$*-discriminator variety. Hence a subtractive skew discriminator variety is always a skew Boolean discriminator variety.*

Proof. Let V be a binary $\mathbf{0}$-discriminator variety with a $\mathbf{0}$-additive term $x + y$ and a binary $\mathbf{0}$-discriminator term $x\backslash y$. Let $\mathbf{A}$ be a binary 0-discriminator member of V. A straightforward case-splitting argument shows that the term $((x\backslash y) + y)\backslash z$ realises the skew Boolean 0-discriminator on A, with the sub-term $(x\backslash y) + y$ realising the monoidal 0-discriminator on A. Hence V is a skew Boolean $\mathbf{0}$-discriminator variety. Conversely, if V is a skew Boolean $\mathbf{0}$-discriminator variety with skew Boolean $\mathbf{0}$-discriminator term $w(x, y, z)$ then, by considering their realisations on a skew Boolean 0-discriminator algebra in V, it is easy to verify that $w(x, \mathbf{0}, y)$ is a binary $\mathbf{0}$-discriminator term, while $w(x, y, \mathbf{0})$ is a $\mathbf{0}$-additive term. □

6 Skew Boolean discriminator varieties

A *skew Boolean algebra* may be regarded as an algebra $\mathbf{A} = \langle A; \vee, \wedge, \backslash, 0\rangle$, where the reducts $\langle A; \vee, \wedge\rangle$ and $\langle A; \backslash, 0\rangle$ are respectively a strongly distributive skew lattice and an implicative BCS-algebra, such that $\mathbf{A} \models x \wedge y \wedge x \approx x\backslash(x\backslash y)$. This identity ensures that the natural preorders on the two reducts coincide. For an alternative definition, and further details about the variety of skew Boolean algebras, see [26].

By [26, Theorem 1.13], there are, up to isomorphism, just three subdirectly irreducible skew Boolean algebras. Moreover, each of these algebras is flat. Given a flat skew Boolean algebra $\mathbf{A}$, it is straightforward to verify that the ternary function w defined for all $a, b, c \in A$ by $w(a, b, c) = (b \vee a \vee b)\backslash c$ is the skew Boolean 0-discriminator on A. It follows that skew Boolean algebras constitute a skew Boolean discriminator variety.

Proposition 6.1. *The generic skew Boolean discriminator variety is term equivalent to the class of left handed skew Boolean algebras. Thus, every skew Boolean discriminator variety is term equivalent to a variety of skew Boolean algebras with ideal-compatible operations.*

Proof. If $\mathbf{A}$ is a skew Boolean 0-discriminator algebra with a skew Boolean 0-discriminator w, the left handed skew Boolean algebra operations may be defined for all $a, b, c \in A$ by $a \wedge b = w(0, a, w(0, a, b))$, $a \vee b = w(a, b, 0)$ and $a\backslash b = w(0, a, b)$. Conversely, if $\mathbf{A}$ is an ideal simple left handed skew Boolean algebra then the skew Boolean 0-discriminator on A is given for all $a, b, c \in A$ by $w(a, b, c) = (a \vee b)\backslash c$. The result now follows in a similar manner to Proposition 2.5, since every subdirectly irreducible skew Boolean algebra is flat and thus ideal simple. The second statement of the Proposition follows from Theorem 2.19. □

Proposition 6.2. *Every skew Boolean discriminator variety satisfies the* QB_2 *quasi-identity. Thus every element of an algebra in skew Boolean discriminator variety is central.*

Proof. By [26, Theorem 1.13], there are, up isomorphism, just three subdirectly irreducible skew Boolean algebras. Moreover, these algebras are ideal simple and have at most three elements, so their implicative BCS-algebra reducts are isomorphic to either $\mathbf{B}_2$, the three-element implicative BCS-algebra, or to the two-element implicative BCK-algebra, which is a subalgebra of $\mathbf{B}_2$. Therefore, every subdirectly irreducible skew Boolean algebra must satisfy QB_2. Birkhoff's Theorem (see [15, Theorem II§9.6]) ensures that every skew Boolean algebra is isomorphic to a subdirect product of subdirectly irreducible skew Boolean algebras. Thus every skew Boolean algebra must satisfy QB_2, since the satisfaction of quasi-identities is preserved under the taking of subdirect products. The result now follows from Theorem 3.4 and Proposition 6.1. □

Apart from skew Boolean algebras, examples of skew Boolean discriminator varieties include Stone algebras, double Stone algebras, Kleene-Stone algebras, strict basic logic algebras, and many others, including every pointed ternary discriminator variety. Skew Boolean discriminator varieties have a close connection with *Church algebras*, namely algebras that have a ternary term $q(x, y, z)$ and two constant terms $\mathbf{0}$ and $\mathbf{1}$ in their language satisfying the identities $q(\mathbf{1}, x, y) \approx x$ and $q(\mathbf{0}, x, y) \approx y$; see [18]. The next result is inspired by [31, Proposition 3.2].

Proposition 6.3. *Let* V *be a double-pointed skew Boolean* $\mathbf{0}$*-discriminator variety. Then* V *is a variety of Church algebras.*

Proof. Let $\mathbf{1}$ be a constant term that is residually distinct from $\mathbf{0}$. By Proposition 2.22, $\mathsf{V} \models x \wedge \mathbf{1} \approx x$, and for every $\mathbf{A} \in \mathsf{V}$ the element $1 = \mathbf{1}^{\mathbf{A}} \in A$ is maximal. Let x' abbreviate the term $\mathbf{1} \backslash x$ and let $q(x, y, z)$ denote the ternary term $(y \wedge x) \vee (z \wedge x')$. Then for all $a, b \in A$ we have

$$\begin{aligned} q^{\mathbf{A}}(0, a, b) &= (a \wedge 0) \vee (b \wedge 0') = 0 \vee (b \wedge (1 \backslash 0)) \\ &= 0 \vee (b \wedge 1) = 0 \vee b = b, \end{aligned}$$

and

$$\begin{aligned} q^{\mathbf{A}}(1, a, b) &= (a \wedge 1) \vee (b \wedge 1') = a \vee (b \wedge (1 \backslash 1)) \\ &= a \vee (b \wedge 0) = a \vee 0 = a. \end{aligned}$$

Hence V is a variety of Church algebras. □

In the particular case of semicentral right Church algebras there is an even closer correspondence. Briefly, V is a variety of *semicentral right Church algebras* if its language includes a constant term $\mathbf{0}$ and a ternary term $q(x, y, z)$ satisfying $q(0, x, y) \approx y$, such that for every $\mathbf{A} \in \mathsf{V}$, all elements of $\mathbf{A}$ are semicentral. For details see [18].

Proposition 6.4. *The class of skew Boolean discriminator varieties coincides with the class of varieties of semicentral right Church algebras.*[12]

Proof. Let V be a variety of semicentral right Church algebras, with right Church algebra term $q(x, y, z)$. By [18, Lemma 4.5], $\mathbf{A} \in \mathsf{V}$ is directly indecomposable if and only if for all $a, b, c \in A$, $q(a, b, c) = b$ if $a \neq 0$ and c otherwise. Let $w(a, b, c) = q(c, 0, q(b, b, a))$.

[12] The authors are grateful to the referee for pointing out this result.

Then for all $a, b, c \in A$, $w(a, b, c) = 0$ if $c \neq 0$, and $q(b, b, a)$ otherwise. But $q(b, b, a) = b$ if $b \neq 0$, and a otherwise. Hence $w(a, b, c)$ is the skew Boolean 0-discriminator on A. Since every subdirectly irreducible algebra is directly indecomposable it follows that the term $q(z, \mathbf{0}, q(y, y, x))$ realises the skew Boolean 0-discriminator on a class of algebras that generates V. Thus V is a skew Boolean discriminator variety.

Conversely, suppose that V is a skew Boolean discriminator variety with skew Boolean discriminator term $w(x, y, z)$ and discriminating constant term $\mathbf{0}$. Let $q(x, y, z)$ be the term $w(w(y, y, w(y, y, x)), w(z, z, x), \mathbf{0})$ and suppose that $\mathbf{A} \in \mathsf{V}$ is a skew Boolean 0-discriminator algebra. Then for all $a, b, c \in A$,

$$\begin{aligned} q(a, b, c) &= w(w(b, b, w(b, b, a)), w(c, c, a), 0) \\ &= w(w(b, b, 0), 0, 0) = w(b, 0, 0) = b \end{aligned}$$

when $a \neq 0$, while

$$\begin{aligned} q(a, b, c) &= w(w(b, b, w(b, b, 0)), w(c, c, 0), 0) \\ &= w(w(b, b, b), c, 0) = w(0, c, 0) = c \end{aligned}$$

when $a = 0$. Thus $\mathbf{A}$ is a directly indecomposable semicentral right Church algebra. Since V is generated by a class of such algebras, it must be a variety of semicentral right Church algebras. □

In [18] the *variety of pure semicentral right Church algebras* is defined to be the variety of type $\langle 3, 0\rangle$ comprising all semicentral right church algebras of the form $\langle A; q, 0\rangle$, with q being its right Church algebra operation. Combining Propositions 6.1 and 6.4 with [18, Theorem 4.6] yields the following.

Corollary 6.5. *The generic skew Boolean discriminator variety, the variety of pure semicentral right Church algebras, the variety of left handed skew Boolean algebras, and the variety of right handed skew Boolean algebras are all term equivalent.*

Every skew Boolean discriminator variety is additive, as witnessed by the term $x \vee y$. As a consequence, every principal ideal $\langle c\rangle$ of a reflexive algebra $\mathbf{A}$ in a skew Boolean discriminator variety is a direct summand. Its complementary direct summand is the ideal $\mathrm{ann}(c) = \{a \in A \mid a \wedge c = 0\}$. We remark that there is a converse to this result: a reflexive variety V with the property that the principal ideals of every member of V are direct summands must be a skew Boolean discriminator variety.

Theorem 6.6. *Let* V *be a variety with constant* $\mathbf{0}$. *The following are equivalent.*

1. V *is a skew Boolean* $\mathbf{0}$*-discriminator variety.*
2. V *is an additive binary* $\mathbf{0}$*-discriminator variety.*
3. V *is a subtractive skew* $\mathbf{0}$*-discriminator variety.*
4. V *is an additive and subtractive dual binary* $\mathbf{0}$*-discriminator variety.*
5. V *is additive and subtractive at* $\mathbf{0}$*, has EDPI and is generated by a class of* $\mathbf{0}$*-ideal simple algebras.*

Proof. Combine Theorem 3.2 and Proposition 5.2. □

Corollary 6.7. *A congruence-permutable dual binary* $\mathbf{0}$*-discriminator variety is additive and subtractive, and hence is a skew Boolean* $\mathbf{0}$*-discriminator variety.*

Proof. Let V be a congruence-permutable dual binary $\mathbf{0}$-discriminator variety. By a theorem of Mal'cev (see [15, Theorem II§12.2]), there exists a term $p(x, y, z)$ in the language of V such that $\mathsf{V} \models p(x, y, y) \approx p(y, y, x) \approx x$. Let $x + y$ be the term $p(x, \mathbf{0}, y)$ and let $s(x, y)$ be the term $p(x, y, 0)$. Then $x + \mathbf{0} \approx p(x, \mathbf{0}, \mathbf{0}) \approx x$, and $\mathbf{0} + x \approx p(\mathbf{0}, \mathbf{0}, x) \approx x$, so V is additive. Also $s(x, \mathbf{0}) \approx p(x, \mathbf{0}, \mathbf{0}) \approx x$ and $s(x, x) \approx p(x, x, \mathbf{0}) \approx \mathbf{0}$, so V is subtractive. Thus V is a skew Boolean $\mathbf{0}$-discriminator variety. □

7 Multiplicative discriminator varieties

Definition 7.1. An algebra $\mathbf{A}$ in a dual binary discriminator variety V has *intersections* if finite meets exist under the natural dual binary discriminator partial order on A. A dual binary discriminator variety V is said to have *intersections* if every member of V has intersections.

The variety of skew Boolean intersection algebras introduced in [7] is an example of a dual binary discriminator variety with intersections.

Lemma 7.2. *Let* V *be a multiplicative* $\mathbf{0}$*-discriminator variety. Then there exist terms* $x \cap y$ *and* $x \wedge y$ *that induce the meet* 0*-discriminator and the dual binary* 0*-discriminator respectively on the multiplicative* 0*-discriminator algebras in* V*.*

Proof. Let $q(x, y, z)$ be the multiplicative discriminator term for V. Put $x \cap y = q(x, y, x)$ and $x \wedge y = q(x, x, y)$. Let $\mathbf{A} \in \mathsf{V}$ be a multiplicative 0-discriminator algebra with discriminating element $0 = \mathbf{0}^{\mathbf{A}}$. Then for all $a, b \in A$, $q^{\mathbf{A}}(a, b, a) = a$ if $a = b$ and 0 otherwise; while $q^{\mathbf{A}}(a, a, b) = a$ if $b \neq 0$ and 0 otherwise. Hence these functions are respectively the meet and the dual binary 0-discriminators on A. □

Theorem 7.3. *Let* V *be a dual binary* $\mathbf{0}$*-discriminator variety with dual binary discriminator term* $x \wedge y$*. Then* V *is a multiplicative* $\mathbf{0}$*-discriminator variety if and only if there exists a binary term* $x \cap y$ *such that* V *satisfies the following identities:*

$$x \cap \mathbf{0} \approx \mathbf{0} \cap x \approx \mathbf{0} \qquad x \cap y \approx y \cap x \qquad x \cap (y \cap z) \approx (x \cap y) \cap z$$
$$x \cap x \approx x \qquad x \wedge (x \cap y) \approx x \cap y \qquad (x \wedge z) \cap (y \wedge z) \approx (x \cap y) \wedge z$$

Moreover, every dual binary $\mathbf{0}$*-discriminator variety with such a term has intersections and is a meet* $\mathbf{0}$*-discriminator variety.*

Proof. Let V be a multiplicative $\mathbf{0}$-discriminator variety and suppose that $\mathbf{A} \in \mathsf{V}$ is a multiplicative 0-discriminator algebra. By Lemma 7.2 there are terms $x \wedge y$ and $a \cap y$ that realise the dual binary and meet 0-discriminators on $\mathbf{A}$. Straightforward case-splitting arguments show that the displayed identities hold on $\mathbf{A}$ and hence they are identities of V, since it is a variety generated by a family of such algebras.

Conversely, if V is a dual binary $\mathbf{0}$-discriminator variety with dual binary $\mathbf{0}$-discriminator term $x \wedge y$, and a term $x \cap y$ such that the displayed identities are satisfied, then these identities imply that for every $\mathbf{A} \in \mathsf{V}$ the term reduct $\langle A; \cap, 0\rangle$ is a meet semilattice with zero. Moreover, the identities also imply that for all $a, b \in A$, $a \cap b \leq a$ and $a \cap b \leq b$ under the natural dual binary discriminator partial order. Suppose that $c \in A$ is such that

$c \leq a$ and $c \leq b$, so that $a \wedge c = c$ and $b \wedge c = c$. Then $c = (a \wedge c) \cap (b \wedge c) = (a \cap b) \wedge c$, which implies that $c \leq a \cap b$. Thus $a \cap b$ is the greatest lower bound of a and b with respect to the natural dual binary discriminator partial order and hence V has intersections.

To see that V is a meet $\mathbf{0}$-discriminator variety, let $\mathbf{A}$ be a dual binary $\mathbf{0}$-discriminator algebra in V. Let $a, b \in A$. Now $a \cap b$ is the meet of a and b under the natural dual binary discriminator partial order on A. But $\mathbf{A}$ is order isomorphic to a flat domain, which implies that $a \cap b = 0$ when $a \neq b$. Also $a \cap a = a$. Thus the term $x \cap y$ realises the meet $\mathbf{0}$-discriminator on a class of algebras that generates V. □

In view of this result we say that a dual binary discriminator variety V is *multiplicative* if it has a binary term $x \cap y$ such that for every $\mathbf{A} \in \mathsf{V}$ and all $a, b \in A$, $a \cap b$ is the meet of the elements a and b under the natural dual binary discriminator partial order on A.

Example 7.4. The *dual discriminator* on a set A is the ternary function $d\colon A^3 \to A$ given for all $a, b, c \in A$ by $d(a, b, c) = a$ if $a = b$, and c otherwise; see [19]. Let V be a pointed dual discriminator variety, with dual discriminator term $d(x, y, z)$ and a constant term $\mathbf{0}$. If $\mathbf{A} \in \mathsf{V}$ is a dual discriminator algebra then $d^{\mathbf{A}}(a, b, 0) = a$ if $a = b$, and 0 otherwise, while $d^{\mathbf{A}}(0, b, a) = a$ if $b \neq 0$, and 0 otherwise, so $d(x, y, 0)$ and $d(0, y, x)$ are respectively meet and dual binary discriminator terms for V, with the multiplicative discriminator term for V being $d(0, z, d(x, y, 0))$.

Jonathan Leech [24] has shown that a multiplicative skew discriminator variety is a pointed dual discriminator variety (and hence is congruence distributive). The converse does not hold, since the generic pointed dual discriminator variety is not additive.

8 Pointed fixedpoint discriminator varieties

Fixedpoint discriminator varieties arise in algebraic logic and were introduced by W. Blok and D. Pigozzi in [11]. Pointed fixedpoint discriminator varieties were introduced in [1], where they are called *dual fixedpoint discriminator varieties*, and independently in [34]. The generic pointed fixedpoint discriminator variety is (up to term equivalence) the variety iBCSK of *implicative BCSK-algebras*, introduced in [34]. An implicative BCSK-algebra is an algebra $\mathbf{A} = \langle A; /, \backslash, 0 \rangle$ of type $\langle 2, 2, 0 \rangle$, where $\langle A; /, 0 \rangle$ is an implicative BCK-algebra, $\langle A; \backslash, 0 \rangle$ is an implicative BCS-algebra, such that the natural partial orders on each of these term reducts coincide. An equational base for the variety iBCSK may be obtained by taking the iBCS and iBCK identities, together with the identities $(x \backslash y)/x \approx \mathbf{0}$ and $x \wedge (x/y) \approx x/y$. Humberstone [21, 22] has extensively investigated the deductive system canonically associated with the variety iBCSK from the perspective of the normal modal logic $\mathbf{S5}$.

Recall that an algebra with a constant term $\mathbf{0}$ is $\mathbf{0}$-*regular* if for every two congruences θ and ψ, $[0]\theta = [0]\psi$ implies $\theta = \psi$. A variety with a constant term $\mathbf{0}$ is $\mathbf{0}$-regular if every member of it is $\mathbf{0}$-regular. A variety V is said to be *ideal determined at* $\mathbf{0}$ if every ideal of an algebra $\mathbf{A} \in \mathsf{V}$ is the 0-class of a unique congruence relation; see [20, Definition 1.3]. Clearly, every algebra in such a variety has the property that its lattice of $\mathbf{0}$-ideals is isomorphic to its lattice of congruences. By [20, Corollary 1.9] a variety V with a constant term $\mathbf{0}$ is ideal determined at $\mathbf{0}$ if and only if it is both subtractive at $\mathbf{0}$ and $\mathbf{0}$-regular.

Implicative BCSK-algebras are $\mathbf{0}$-regular and, since the iBCS and iBCK operations are both subtractive at $\mathbf{0}$, the variety iBCSK is ideal determined. Moreover, iBCSK is semisimple, that is, every subdirectly irreducible member of iBCSK is simple. Full details appear in [34]. Since every pointed fixedpoint discriminator variety is term equivalent to a variety of iBCSK-algebras with ideal-compatible operations, it follows that such a variety must be ideal determined at its discriminating constant, and thus must also be semi-simple. In summary:

Theorem 8.1. *The following are equivalent for a variety with constant* $\mathbf{0}$.

1. V *is a pointed fixedpoint* $\mathbf{0}$*-discriminator variety.*
2. V *is a subtractive multiplicative* $\mathbf{0}$*-discriminator variety.*
3. V *is a multiplicative binary* $\mathbf{0}$*-discriminator variety.*
4. V *is a* $\mathbf{0}$*-regular binary* $\mathbf{0}$*-discriminator variety.*
5. V *is an ideal determined dual binary* $\mathbf{0}$*-discriminator variety.*
6. V *is ideal determined at* $\mathbf{0}$ *and is semi-simple with EDPI.*

The equivalence of 1 and 6 was shown independently in [1, Theorem 4.8]. We remark that for the double-pointed analogue of Theorem 8.1, further equivalences are possible: in particular, fundamental connections can be established with the pseudo-interior algebras of Blok and Pigozzi [12].

9 Pointed ternary discriminator varieties

Pointed ternary discriminator varieties can be characterised in many different ways. Note that a ternary discriminator variety is a dual binary $\mathbf{0}$-discriminator variety for each constant term $\mathbf{0}$ in its language.

Theorem 9.1. *For each constant term* $\mathbf{0}$ *in the language of a variety* V *the following are equivalent.*

1. V *is a pointed ternary discriminator variety.*
2. V *is term equivalent to a variety of skew Boolean intersection algebras with ideal-compatible operations.*
3. V *is a multiplicative skew Boolean* $\mathbf{0}$*-discriminator variety.*
4. V *is an ideal determined skew* $\mathbf{0}$*-discriminator variety.*
5. V *is a multiplicative and subtractive skew* $\mathbf{0}$*-discriminator variety*
6. V *is an additive and multiplicative binary* $\mathbf{0}$*-discriminator variety.*
7. V *is an additive, subtractive and multiplicative dual binary* $\mathbf{0}$*-discriminator variety.*
8. V *is an additive pointed fixedpoint* $\mathbf{0}$*-discriminator variety.*
9. V *is a congruence-permutable multiplicative* $\mathbf{0}$*-discriminator variety.*

Proof. Let V be a pointed ternary discriminator variety with constant term $\mathbf{0}$. By Theorem 1.2 and Proposition 4.2, V is a dual binary $\mathbf{0}$-discriminator variety. By [7, Theorem 4.7], the generic pointed ternary discriminator variety is term equivalent to the variety of left handed skew Boolean intersection algebras, so it follows from Theorem 2.19 that V must be term equivalent to a variety of skew Boolean intersection algebras with operations that are ideal-compatible with respect to its dual binary $\mathbf{0}$-discriminator term. Thus

1 implies 2. Now 2 implies 3 because every skew Boolean intersection algebra is a multiplicative skew Boolean algebra in view of [7, §4] and Theorem 7.3.

Since every ternary **0**-discriminator variety is also a skew, skew Boolean, binary, dual binary, and pointed fixedpoint **0**-discriminator variety, the equivalence of 3, 4, 5, 6, 7 and 8 follows directly from Theorems 3.2, 6.6, 7.3, and 9.1.

In view of Corollary 6.7, a congruence-permutable dual binary **0**-discriminator variety is skew Boolean **0**-discriminator variety, so 9 implies 3. Ternary discriminator varieties are congruence-permutable by [15, Theorem IV§9.4] and a ternary **0**-discriminator variety is a multiplicative **0**-discriminator variety by Proposition 4.2, so 1 implies 9.

To complete the proof it is sufficient to show that 3 implies 1, so assume that V is a multiplicative skew Boolean **0**-discriminator variety. Then by Proposition 6.1 V is term equivalent to a variety of skew Boolean algebras with ideal-compatible operations. By Theorem 7.3 these algebras have intersections, that are witnessed by a binary meet **0**-discriminator term $x \cap y$. Thus, by [7, Theorem 4.4], when the meet discriminator term is included in their type, they are members of the ternary discriminator variety of skew Boolean intersection algebras. Hence the ideal simple members of V are ternary discriminator algebras, and since it is generated by its ideal simple members, V must be a pointed ternary discriminator variety. □

References

[1] P. Agliano and K. A. Baker, *Idempotent Discriminators*, Technical report, University of Sienna, 1997.

[2] P. Agliano and A. Ursini, On subtractive varieties III: From ideals to congruences, *Algebra Universalis* **37** (1997), 296–333, doi:10.1007/s000120050020.

[3] P. Agliano and A. Ursini, On subtractive varieties IV: Definability of principal ideals, *Algebra Universalis* **38** (1997), 355–389, doi:10.1007/s000120050059.

[4] C. J. van Alten, Congruence properties in congruence permutable and in ideal determined varieties, with applications, *Algebra Universalis* **53** (2005), 433–449, doi:10.1007/s00012-005-1911-7.

[5] A. R. Anderson and N. D. Belnap, Jr., *Entailment. The Logic of Relevance and Necessity, Volume I*, Princeton University Press, Princeton, New Jersey, 1975.

[6] R. J. Bignall, *Quasiprimal Varieties and Components of Universal Algebras*, Ph.D. thesis, The Flinders University of South Australia, Australia, 1976.

[7] R. J. Bignall and J. E. Leech, Skew Boolean algebras and discriminator varieties, *Algebra Universalis* **33** (1995), 387–398, doi:10.1007/bf01190707.

[8] R. J. Bignall and M. Spinks, On binary discriminator varieties I: Implicative BCS-algebras, to appear in *Int. J. Algebra Comput.*

[9] R. J. Bignall and M. Spinks, Principal congruences on pseudocomplemented semilattices, *Algebra Universalis* **65** (2011), 1–7, doi:10.1007/s00012-011-0113-8.

[10] W. J. Blok and D. Pigozzi, *Algebraizable Logics*, Memoirs of the American Mathematical Society, American Mathematical Society, Providence, RI, 1989, doi:10.1090/memo/0396.

[11] W. J. Blok and D. Pigozzi, On the structure of varieties with equationally definable principal congruences III, *Algebra Universalis* **32** (1994), 545–608, doi:10.1007/bf01195727.

[12] W. J. Blok and D. Pigozzi, On the structure of varieties with equationally definable principal congruences IV, *Algebra Universalis* **31** (1994), 1–35, doi:10.1007/bf01188178.

[13] R. T. Brady, Completeness proofs for the systems RM3 and BN4, *Logique et Anal. (N.S.)* **25** (1982), 9–32, https://www.jstor.org/stable/44084001.

[14] S. Burris, Discriminator varieties and symbolic computation, *J. Symbolic Comput.* **13** (1992), 175–207, doi:10.1016/s0747-7171(08)80089-2.

[15] S. Burris and H. P. Sankappanavar, *A Course in Universal Algebra*, volume 78 of *Graduate Texts in Mathematics*, Springer-Verlag, New York, 1981, http://www.math.uwaterloo.ca/~snburris/htdocs/ualg.html.

[16] I. Chajda, R. Halaš and I. G. Rosenberg, Ideals and the binary discriminator in universal algebra, *Algebra Universalis* **42** (1999), 239–251, doi:10.1007/s000120050001.

[17] R. Cignoli, Quantifiers on distributive lattices, *Discrete Math.* **96** (1991), 183–197, doi:10.1016/0012-365x(91)90312-p.

[18] K. Cvetko-Vah and A. Salibra, The connection of skew Boolean algebras and discriminator varieties to Church algebras, *Algebra Universalis* **73** (2015), 369–390, doi:10.1007/s00012-015-0320-9.

[19] E. Fried and A. F. Pixley, The dual discriminator function in universal algebra, *Acta Sci. Math. (Szeged)* **41** (1979), 83–100, http://pub.acta.hu/acta/showCustomerArticle.action?id=10289&dataObjectType=article.

[20] H. P. Gumm and A. Ursini, Ideals in universal algebras, *Algebra Universalis* **19** (1984), 45–54, doi:10.1007/bf01191491.

[21] L. Humberstone, An intriguing logic with two implicational connectives, *Notre Dame J. Formal Logic* **41** (2000), 1–40, doi:10.1305/ndjfl/1027953481.

[22] L. Humberstone, Identical twins, deduction theorems, and pattern functions: exploring the implicative *bcsk* fragment of **S5**, *J. Philos. Logic* **35** (2006), 435–487, doi:10.1007/s10992-005-9023-6.

[23] T. Katriňák and M. Žabka, A weak Boolean representation of double Stone algebras, *Houston J. Math.* **30** (2004), 615–628.

[24] J. Leech, personal communication.

[25] J. Leech, Skew lattices in rings, *Algebra Universalis* **26** (1989), 48–72, doi:10.1007/bf01243872.

[26] J. Leech, Skew Boolean algebras, *Algebra Universalis* **27** (1990), 497–506, doi:10.1007/bf01188995.

[27] J. Leech, Normal skew lattices, *Semigroup Forum* **44** (1992), 1–8, doi:10.1007/bf02574320.

[28] J. Łoś, Direct sums in general algebra, *Colloq. Math.* **14** (1966), 33–38, doi:10.4064/cm-14-1-33-38.

[29] R. McKenzie, On spectra, and the negative solution of the decision problem for identities having a finite nontrivial model, *J. Symbolic Logic* **40** (1975), 186–196, doi:10.2307/2271899.

[30] A. Salibra, A. Ledda and F. Paoli, Boolean product representations of algebras via binary polynomials, in: J. Czelakowski (ed.), *Don Pigozzi on Abstract Algebraic Logic, Universal Algebra, and Computer Science*, Springer, Cham, volume 16 of *Outstanding Contributions to Logic*, pp. 297–321, 2018, doi:https://doi.org/10.1007/978-3-319-74772-9_12.

[31] A. Salibra, A. Ledda, F. Paoli and T. Kowalski, Boolean-like algebras, *Algebra Universalis* **69** (2013), 113–138, doi:10.1007/s00012-013-0223-6.

[32] B. M. Schein, On the theory of the restrictive semigroups (in Russian), *Izv. Vysš. Učebn. Zaved. Matematika* **33** (1963), 152–154, http://mi.mathnet.ru/ivm2166.

[33] B. M. Schein, Bands of semigroups: variations on a Clifford theme, in: K. H. Hofmann and M. W. Mislove (eds.), *Semigroup Theory and Its Applications*, Cambridge University Press, Cambridge, volume 231 of *London Mathematical Society Lecture Note Series*, pp. 53–80, 1996, doi:10.1017/cbo9780511661877.006, proceedings of the conference commemorating the work of Alfred H. Clifford held at Tulane University, New Orleans, LA, March 1994.

[34] M. Spinks, *Contributions to the Theory of Pre-BCK-Algebras*, Ph.D. thesis, Monash University, Australia, 2003, https://monash.figshare.com/articles/Contributions_to_the_theory_of_pre-BCK-algebras/5446369.

[35] M. Spinks, R. J. Bignall and R. Veroff, Discriminator logics (research announcement), *Australas. J. Log.* **11** (2014), 159–171, doi:10.26686/ajl.v11i2.2020.

[36] M. Spinks and R. Veroff, Paraconsistent constructive logic with strong negation as a contraction-free relevant logic, in: J. Czelakowski (ed.), *Don Pigozzi on Abstract Algebraic Logic, Universal Algebra, and Computer Science*, Springer, Cham, volume 16 of *Outstanding Contributions to Logic*, pp. 323–379, 2018, doi:10.1007/978-3-319-74772-9_13.

[37] A. Ursini, On subtractive varieties I, *Algebra Universalis* **31** (1994), 204–222, doi:10.1007/bf01236518.

[38] D. Vaggione, Varieties in which the Pierce stalks are directly indecomposable, *J. Algebra* **184** (1996), 424–434, doi:10.1006/jabr.1996.0268.