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Everywhere regularity for a class of vectorial functionals under subquadratic general growth conditions

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Abstract

We consider the integral functional of the calculus of variations

$$\int_{\Omega} f(Du)\,dx,$$

where $f:\mathbb{R}^{nN}\to\mathbb{R}$ satisfies $f(z)=g(|z|)$ and g is an N-function with subquadratic p–q growth. We prove that minimizers $u:\Omega\subset\mathbb{R}^n\to\mathbb{R}^N$ of such a functional are locally Lipschitz continuous, provided g verifies some additional conditions.

Keywords: Calculus of variations; Direct methods; Regularity; Nonstandard growth conditions; General growth conditions

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1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^n$ and let us consider the variational integral

$$\mathcal{I}(u) = \int_\Omega f\big(Du(x)\big)\,dx, \tag{1.1}$$

where $f:\mathbb{R}^{nN} \to \mathbb{R}$ is continuous and nonnegative, $u:\Omega \subset \mathbb{R}^n \to \mathbb{R}^N$ and $Du(x) = (\partial u^\alpha/\partial x_j)_{\alpha=1,\dots,N;\, j=1,\dots,n}$.

We say that a function $u \in W^{1,1}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ is a *local minimizer* of $\mathcal{I}$ if $f(Du) \in L^1_{\mathrm{loc}}(\Omega)$ and, for every $\varphi \in W^{1,1}(\Omega,\mathbb{R}^N)$ with $\operatorname{supp}(\varphi) \Subset \Omega$ we have

$$\int_{\operatorname{supp}(\varphi)} f\big(Du(x)\big)\,dx \leqslant \int_{\operatorname{supp}(\varphi)} f\big(Du(x)+D\varphi(x)\big)\,dx.$$

Let us assume that f satisfies the growth condition

$$|z|^p - m \leqslant f(z) \leqslant M\big(1+|z|^q\big),$$

where m, M are positive constants and $1 < p \leqslant q$. We are going to deal with Lipschitz regularity of vector-valued minimizers, under the special structure assumption

$$f(z) = g\big(|z|\big), \quad \forall z \in \mathbb{R}^{nN}.$$

When handling vector-valued mappings and aiming at Lipschitz continuity, such a special assumption is not surprising: Uhlenbeck [10], Giaquinta and Modica [4] for $p = q \geqslant 2$, Acerbi and Fusco [1] for $1 < p = q < 2$. Recently Marcellini in [7] has proved a $C^{1,\alpha}$-regularity result for local minimizers of functionals when g has a nonoscillating property and, at least, quadratic growth: such a result does not cover the case in which g has sub-quadratic growth. Our present paper is concerned with this case.

We assume that $g:[0,+\infty) \to [0,+\infty)$ is an N-function, i.e., $g(t) = 0$ if and only if $t = 0$,

$$\lim_{t\to\infty} \frac{g(t)}{t} = +\infty, \qquad \lim_{t\to 0} \frac{g(t)}{t} = 0.$$

We assume also that g is strictly convex and the following conditions hold:

(G1) There exist $\Lambda_1, \Lambda_2 > 0$ and $1 < p < q < 2$ such that $g \in C^2((0,+\infty)) \cap C^1([0,+\infty))$, $g'(0) = 0$, $g'(t)/t$ is decreasing and

$$\Lambda_1 t^{p-2} \leqslant \frac{g'(t)}{t} \leqslant \Lambda_2\big(t^{q-2} + t^{p-2}\big); \tag{1.2}$$

(G2) There exists $\gamma > 1$ such that

$$g''(t)t \leqslant g'(t) \leqslant \gamma g''(t)t.$$

We remark that (1.2) implies

$$\frac{\Lambda_1}{p} t^p \leqslant g(t) \leqslant \frac{\Lambda_2}{p}\big(t^q + t^p\big), \quad \forall t \geqslant 0, \tag{1.3}$$

thus we are in the *subquadratic p–q growth*. Let us remark that we do not require p to be close to q; on the contrary, many regularity results assume that p is near q; see [3,5,6,8]. The main result of the paper is the following

Theorem 1.1. *Let g satisfy* (G1), (G2) *and $u \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ be a local minimizer of functional $\mathcal{I}$ in* (1.1). *Then u is locally Lipschitz continuous in Ω. Moreover, for $0 < \rho < R$ with $B_{2R} \Subset \Omega$, there exists a positive constant c such that*

$$\sup_{B_\rho} |Du| \leqslant c \int_{B_R} \bigl(1 + g\bigl(|Du|\bigr)\bigr)\, dx, \tag{1.4}$$

where $c = c(n, N, \gamma, \rho, R, g'(\sqrt{2})) > 0$.

We observe explicitly that the constant c does not depend on Λ_1, Λ_2 of (1.2).

Our result includes energy densities f with slow growth. For instance, it can be proved that the function

$$g(t) = t^p \log^\alpha (a + t)$$

with $1 < p < 2$, $\alpha > 0$ and $a > 0$ large enough is an N-functions satisfying conditions (G1) and (G2). The limit case $g(t) = t \log(1 + t)$ has been studied by Mingione and Siepe in [9].

The proof of our regularity result is splitted into two parts.

First, we consider the standard growth case, i.e., when $f(z) = g(|z|)$ and g satisfies (1.3) with q instead of p in the left-hand side. If $v \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ is a local minimizer for $\mathcal{I}$ in this case, by the results of Acerbi and Fusco in [1] we have that $v \in W^{1,\infty}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$. By our special assumptions we are able to derive an estimate of $\sup |Dv|$ like (1.4), *by using only the properties of the N-function g*, so the constant c does not depend on Λ_1 and Λ_2 in (1.2).

Then we study the case of p–q growth by applying a double approximation procedure as in [2,6,9], combined with some techniques about functionals without explicit polynomial growth. More precisely, we start from a local minimizer u of (1.1), we define $f_\sigma(z) = f(z) + \sigma |z|^q$ with $\sigma > 0$, so that the function f_σ satisfies the standard growth condition of order q.

We regularize the original minimizer u by means of mollifiers, thus obtaining the sequence $\{u_\varepsilon\}$. Then we consider the Dirichlet problem in $B_R \Subset \Omega$,

$$\min\left\{\mathcal{I}_\sigma(v) = \int_{B_R} f_\sigma(Dv)\, dx\colon\ v \in u_\varepsilon + W^{1,q}_0(B_R, \mathbb{R}^N)\right\}. \tag{1.5}$$

Let $v_{\varepsilon,\sigma}$ be the unique solution of (1.5). By the previous results we can estimate

$$\sup_{B_\rho} |Dv_{\varepsilon,\sigma}| \leqslant c\left\{\int_{B_{R+\varepsilon}} \bigl[1 + g\bigl(|Du|\bigr)\bigr]\, dx + \sigma \int_{B_R} |Du_\varepsilon|^q\, dx\right\},$$

where c is independent of σ and ε. Then, by letting first $\sigma \to 0$ and then $\varepsilon \to 0$, estimate (1.4) follows.

2. Regularity under standard growth conditions

In this section we start from Acerbi–Fusco regularity result (see [1]), for the minimizers of subquadratic functionals and we give an estimate for $\sup|Du|$, in which we carefully prove how the constant depends on the assumptions of the energy density. This will allow us to deal with the case of general growth.

Definition 2.1. We say that $h:[0,+\infty)\to[0,+\infty)$ is an N-function if h is convex and increasing, $h(t)=0$ if and only if $t=0$,

$$\lim_{t\to+\infty}\frac{h(t)}{t}=+\infty,\qquad \lim_{t\to 0^+}\frac{h(t)}{t}=0.$$

Moreover we say that an N-function h is of class Δ_2^m if there exists $m>1$ such that

$$h(\lambda t)\leqslant \lambda^m h(t),\quad \forall t\geqslant 0,\ \forall\lambda>1.$$

As it can be easily checked, if h is of class C^1, the latter is equivalent to require that

$$h'(t)t\leqslant mh(t),\quad \forall t\geqslant 0.$$

Let $1<p<q<2$ and let h be an N-function strictly convex in $[0,+\infty)$. We will assume that h satisfies the following assumptions:

(H1) $h\in C^2((0,+\infty))\cap C^1([0,+\infty))$. Moreover $h'(0)=0$ and, for every $t>0$, $h'(t)/t$ is decreasing and two positive constants Λ_1,Λ_2 exist such that

$$\Lambda_1 t^{q-2}\leqslant\frac{h'(t)}{t}\leqslant\Lambda_2\left(t^{q-2}+t^{p-2}\right);\tag{2.1}$$

(H2) There exists $\gamma>1$ such that

$$h''(t)t\leqslant h'(t)\leqslant\gamma h''(t)t,\quad \forall t>0.\tag{2.2}$$

Remark 2.1. We observe that the left inequality in (2.2) implies without other assumptions that $h\in\Delta_2^2$.

Moreover by (2.1) it easily follows that

$$\frac{\Lambda_1}{q}t^q\leqslant h(t)\leqslant\frac{\Lambda_2}{p}\left(t^q+t^p\right).\tag{2.3}$$

Let us consider the integral functional

$$\mathcal{I}(u)=\int_\Omega f(Du)\,dx,\tag{2.4}$$

where $\Omega\subset\mathbb{R}^n$ ($n\geqslant 2$) is a bounded open set, $f:\mathbb{R}^{nN}\to\mathbb{R}$ ($N>1$) is such that

$$f(z)=h\big(|z|\big)\tag{2.5}$$

and $u:\Omega\to\mathbb{R}^N$ is a weakly differentiable function.

We remark also that, under such conditions, f turns out to be strictly convex in $\mathbb{R}^{nN}$.

The main result of this section is the following

Proposition 2.1. *Let u be a local minimizer of functional* (2.4), *where f is as in* (2.5) *and h satisfies conditions* (H1) *and* (H2). *Then $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$. Moreover, for every $0 < \rho < R$ such that $B_R \Subset \Omega$, there exists a positive constant c such that the following inequality holds*:

$$\sup_{B_\rho} |Du| \leqslant c \int_{B_R} \left[1 + h\big(|Du|\big)\right] dx, \tag{2.6}$$

where $c = \tilde{c}[V(h'(\sqrt{2}))]^{2/(2^*-2)}$, $\tilde{c} = \tilde{c}(n, N, \rho, R) > 0$, $V(t) = 1 + t + c_0 t^{-\vartheta}$, $c_0 = c_0(n, \gamma) > 0$, $\vartheta = \vartheta(n) > 0$ and $2^* = 2n/(n-2)$ if $n \geqslant 3$, while $2^* = 3$ if $n = 2$.

Remark 2.2. We note in particular that the constant c in (2.6) does not depend on Λ_1 and Λ_2.

Proof of Proposition 2.1. We divide the proof into three steps.

Step 1 (Approximation by means of nondegenerate densities). Let us fix $\mu \in (0, 1]$ and define

$$H_\mu(t) = h\Big(\sqrt{\mu^2 + t^2}\Big) - h(\mu). \tag{2.7}$$

It is easy to check that H_μ is an N-function of class Δ_2^2. Moreover, by properties (2.1)–(2.3) of h it follows that

$$\begin{aligned} \frac{\Lambda_1}{q}(\mu^2 + t^2)^{q/2} - \frac{\Lambda_1}{q}\mu^q \leqslant H_\mu(t) &\leqslant \frac{\Lambda_2}{p}\big((\mu^2 + t^2)^{q/2} + (\mu^2 + t^2)^{p/2}\big) \\ &\leqslant \frac{2\Lambda_2}{p\mu^q}(\mu^2 + t^2)^{q/2}, \quad \forall t \geqslant 0, \end{aligned} \tag{2.8}$$

$H_\mu \in C^2(\mathbb{R})$, $H'_\mu(0) = 0$, $H'_\mu(t)/t$ is decreasing in $(0, +\infty)$ and

$$\Lambda_1(\mu^2 + t^2)^{(q-2)/2} \leqslant \frac{H'_\mu(t)}{t} \leqslant \Lambda_3(\mu^2 + t^2)^{(q-2)/2}, \quad \forall t > 0, \tag{2.9}$$

where $\Lambda_3 = \Lambda_3(\Lambda_2, p, q, \mu)$ and finally

$$H''_\mu(t)t \leqslant H'_\mu(t) \leqslant \gamma H''_\mu(t)t, \quad \forall t \geqslant 0. \tag{2.10}$$

Let us consider the functionals

$$\mathcal{I}_\mu(v) = \int_\Omega f_\mu(Dv)\, dx, \tag{2.11}$$

where we set $f_\mu(z) = H_\mu(|z|)$ for every $z \in \mathbb{R}^{nN}$. By (2.8) and (2.9) we have

$$\frac{\Lambda_1}{q}\big(\mu^2 + |z|^2\big)^{q/2} - \frac{\Lambda_1}{q}\mu^q \leqslant f_\mu(z) \leqslant \frac{\Lambda_3}{q}\big(\mu^2 + |z|^2\big)^{q/2}. \tag{2.12}$$

Moreover $f_\mu \in C^2(\mathbb{R}^{nN})$,

$$\big|D^2 f_\mu(z)\big| \leqslant \Lambda_3\sqrt{nN}\big(\mu^2 + |z|^2\big)^{(q-2)/2} \tag{2.13}$$

and

$$\left\langle D^2 f_\mu(z)\lambda, \lambda\right\rangle \geqslant \frac{\Lambda_1}{\gamma}\left(\mu^2+|z|^2\right)^{(q-2)/2}|\lambda|^2 \tag{2.14}$$

for every $z, \lambda \in \mathbb{R}^{nN}$. Let us check (2.13) and (2.14).

To simplify our notations, from now on we will write f and H to denote the functions f_μ and H_μ. First we observe that for $z \neq 0$,

$$f_{z_i^\alpha}(z) = H'\left(|z|\right)\frac{z_i^\alpha}{|z|},$$
$$f_{z_i^\alpha z_j^\beta}(z) = \left(\frac{H''(|z|)}{|z|^2} - \frac{H'(|z|)}{|z|^3}\right) z_i^\alpha z_j^\beta + \frac{H'(|z|)}{|z|}\delta_{ij}\delta^{\alpha\beta} \tag{2.15}$$

and then

$$\left|D^2 f(z)\right| = \sqrt{\sum_{\alpha,\beta,i,j}\left(f_{z_i^\alpha z_j^\beta}(z)\right)^2} \leqslant \sqrt{nN}\,\frac{H'(|z|)}{|z|} \leqslant \Lambda_3\sqrt{nN}\left(\mu^2+|z|^2\right)^{(q-2)/2},$$

that is (2.13). Finally we have

$$\sum_{\alpha,\beta,i,j} f_{z_i^\alpha z_j^\beta}(z)\lambda_i^\alpha\lambda_j^\beta = \left(\frac{H''(|z|)}{|z|^2} - \frac{H'(|z|)}{|z|^3}\right)\left|\langle z,\lambda\rangle\right|^2 + \frac{H'(|z|)}{|z|}|\lambda|^2,$$

from which, by (2.10) it follows that

$$\sum_{\alpha,\beta,i,j} f_{z_i^\alpha z_j^\beta}(z)\lambda_i^\alpha\lambda_j^\beta \geqslant \left(H''(|z|) - \frac{H'(|z|)}{|z|}\right)|\lambda|^2 + \frac{H'(|z|)}{|z|}|\lambda|^2$$
$$= H''\left(|z|\right)|\lambda|^2 \geqslant \frac{1}{\gamma}\frac{H'(|z|)}{|z|}|\lambda|^2 \tag{2.16}$$

and then by (2.9), (2.14) follows. Moreover by (2.10) we have

$$\sum_{\alpha,\beta,i,j} f_{z_i^\alpha z_j^\beta}(z)\lambda_i^\alpha\lambda_j^\beta \leqslant \frac{H'(|z|)}{|z|}|\lambda|^2. \tag{2.17}$$

Step 2 (Estimates for minimizers of nondegenerate densities). Let v be a local minimizer of the functional $\mathcal{I}_\mu$ defined in (2.11), with H_μ as in (2.7). By Lemma 2.5 and Proposition 2.7 in [1], taking into account (2.12)–(2.14) we deduce that $v \in W^{2,2}_{\mathrm{loc}}(\Omega,\mathbb{R}^N) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$. Furthermore $f_{\mu_{z_i^\alpha}}(Dv) \in W^{1,2}_{\mathrm{loc}}(\Omega)$ and the chain rule can be used for computing $D_s(f_{\mu_{z_i^\alpha}}(Dv))$.

If $0 < \rho < R$ are such that $B_R \Subset \Omega$, then we claim that there exists a positive constant c such that

$$\sup_{B_\rho}|Dv| \leqslant c\int_{B_R}\left[1 + H_\mu\left(|Dv|\right)\right]dx, \tag{2.18}$$

where $c = c_*[V(h'(\sqrt{2}))]^{2/(2^*-2)}$ with $c_*(n, |\Omega|, \rho, R)$ and $|\Omega|$ is the n-dimensional Lebesgue measure of Ω.

Let us drop again μ from H_μ and f_μ. We will prove our claim starting from the *second variation* of our functional $\mathcal{I}_\mu$,

$$\int_\Omega \sum_{\alpha,\beta,i,j} f_{z_i^\alpha z_j^\beta}(Dv) v^\beta_{x_j x_s} \phi^\alpha_{x_i}\, dx = 0 \tag{2.19}$$

for every function $\phi \in C_0^\infty(\Omega, \mathbb{R}^N)$ and any fixed $s \in \{1, \dots, n\}$. Since $f_{z_i^\alpha z_j^\beta}(Dv) v^\beta_{x_j x_s} \in L^2_{\text{loc}}(\Omega)$, equality (2.19) holds true for every $\phi \in W^{1,2}(\Omega)$ with $\text{supp}(\phi) \Subset \Omega$.

Let $\psi : [0, +\infty) \to [0, +\infty)$ be continuous, bounded, piecewise of class C^1 with only a finite number of corner points, such that ψ' is bounded and $\psi' \geqslant 0$; moreover let $\eta \in C_0^1(\Omega)$ and set $\phi^\alpha = \eta^2 u^\alpha_{x_s} \psi(|Dv|)$ for every $\alpha = 1, \dots, N$. Then $\phi \in W^{1,2}(\Omega, \mathbb{R}^N)$ and

$$\phi^\alpha_{x_i} = 2\eta\eta_{x_i} v^\alpha_{x_s} \psi\big(|Dv|\big) + \eta^2 v^\alpha_{x_i x_s} \psi\big(|Dv|\big) + \chi_{\{|Dv| \notin L\}} \eta^2 v^\alpha_{x_s} \psi'\big(|Dv|\big)\big(|Dv|\big)_{x_i},$$

where we denote by L the set of the corner points of ψ. Now we insert $\phi^\alpha_{x_i}$ in (2.19) and we add up over s,

$$\begin{aligned} &\sum_s \int_\Omega 2\eta\psi\big(|Dv|\big) \sum_{\alpha,\beta,i,j} f_{z_i^\alpha z_j^\beta}(Dv) v^\beta_{x_j x_s} \eta_{x_i} v^\alpha_{x_s}\, dx \\ &\quad + \sum_s \int_\Omega \eta^2 \psi\big(|Dv|\big) \sum_{\alpha,\beta,i,j} f_{z_i^\alpha z_j^\beta}(Dv) v^\beta_{x_j x_s} v^\alpha_{x_i x_s}\, dx \\ &\quad + \int_{\Omega \cap \{|Dv| \notin L\}} \eta^2 \psi'\big(|Dv|\big) \sum_{\alpha,\beta,i,j,s} f_{z_i^\alpha z_j^\beta}(Dv) v^\beta_{x_j x_s} v^\alpha_{x_s} \big(|Dv|\big)_{x_i}\, dx \\ &= \sum_s I_{1,s} + I_2 + I_3 = 0. \end{aligned} \tag{2.20}$$

By Cauchy–Schwartz inequality and since $ab \leqslant (a^2 + b^2)/2$ for every $a, b \geqslant 0$ we have

$$\begin{aligned} |I_{1,s}| &\leqslant 2 \int_\Omega \psi\big(|Dv|\big) \bigg(\eta^2 \sum_{\alpha,\beta,i,j} f_{z_i^\alpha z_j^\beta}(Dv) v^\alpha_{x_i x_s} v^\beta_{x_j x_s}\bigg)^{1/2} \\ &\qquad \times \bigg(\sum_{\alpha,\beta,i,j} f_{z_i^\alpha z_j^\beta}(Dv) \eta_{x_i} v^\alpha_{x_s} \eta_{x_j} v^\beta_{x_s}\bigg)^{1/2} dx \\ &\leqslant \frac{1}{2} \int_\Omega \eta^2 \psi\big(|Dv|\big) \sum_{\alpha,\beta,i,j} f_{z_i^\alpha z_j^\beta}(Dv) v^\alpha_{x_i x_s} v^\beta_{x_j x_s}\, dx \\ &\qquad + 2 \int_\Omega \psi\big(|Dv|\big) \sum_{\alpha,\beta,i,j} f_{z_i^\alpha z_j^\beta}(Dv) \eta_{x_i} v^\alpha_{x_s} \eta_{x_j} v^\beta_{x_s}\, dx. \end{aligned}$$

To give an estimate of I_3 we observe that, by (2.15),

$$A = \sum_{\alpha,\beta,i,j,s} f_{z_i^\alpha z_j^\beta}(Dv) v^\beta_{x_j x_s} v^\alpha_{x_s} \big(|Dv|\big)_{x_i}$$

$$= \left(\frac{H''(|Dv|)}{|Dv|^2} - \frac{H'(|Dv|)}{|Dv|^3} \right) \sum_{\alpha,\beta,i,j,s} v^\alpha_{x_i} v^\beta_{x_j} v^\beta_{x_j x_s} v^\alpha_{x_s} \big(|Dv|\big)_{x_i}$$
$$+ \frac{H'(|Dv|)}{|Dv|} \sum_{\alpha,i,s} v^\alpha_{x_i x_s} v^\alpha_{x_s} \big(|Dv|\big)_{x_i}.$$

Moreover, since

$$\sum_{\beta,j} v^\beta_{x_j} v^\beta_{x_j x_s} = \big(|Dv|\big)_{x_s} |Dv|,$$

we have

$$A = \left(\frac{H''(|Dv|)}{|Dv|} - \frac{H'(|Dv|)}{|Dv|^2} \right) \sum_{\alpha,i,s} v^\alpha_{x_i} \big(|Dv|\big)_{x_i} v^\alpha_{x_s} \big(|Dv|\big)_{x_s}$$
$$+ H'\big(|Dv|\big) \sum_i \Big(\big(|Dv|\big)_{x_i}\Big)^2$$
$$\geqslant \left(\frac{H''(|Dv|)}{|Dv|} - \frac{H'(|Dv|)}{|Dv|^2} \right) |Dv|^2 \big|D\big(|Dv|\big)\big|^2 + H'\big(|Dv|\big) \big|D\big(|Dv|\big)\big|^2$$
$$= H''\big(|Dv|\big)|Dv| \big|D\big(|Dv|\big)\big|^2 \geqslant 0.$$

Since $\psi' \geqslant 0$ and $A \geqslant 0$, it turns out that $I_3 \geqslant 0$, so that (2.20) gives

$$\int_\Omega \eta^2 \psi\big(|Dv|\big) \sum_{\alpha,\beta,i,j,s} f_{z^\alpha_i z^\beta_j}(Dv) v^\alpha_{x_i x_s} v^\beta_{x_j x_s}\, dx$$
$$\leqslant 4 \int_\Omega \psi\big(|Dv|\big) \sum_{\alpha,\beta,i,j,s} f_{z^\alpha_i z^\beta_j}(Dv) \eta_{x_i} v^\alpha_{x_s} \eta_{x_j} v^\beta_{x_s}\, dx.$$

Now since $|D(|Dv|)|^2 \leqslant |D^2 v|^2$, by (2.17) and (2.16) we obtain

$$\int_\Omega \eta^2 \psi\big(|Dv|\big) H''\big(|Dv|\big) \big|D\big(|Dv|\big)\big|^2\, dx$$
$$\leqslant 4 \int_\Omega \psi\big(|Dv|\big) \sum_{\alpha,\beta,i,j,s} f_{z^\alpha_i z^\beta_j}(Dv) \eta_{x_i} v^\alpha_{x_s} \eta_{x_j} v^\beta_{x_s}\, dx$$
$$\leqslant 4 \int_\Omega |D\eta|^2 \psi\big(|Dv|\big) H'\big(|Dv|\big) |Dv|\, dx. \tag{2.21}$$

We use (2.21) with $\psi \equiv 1$,

$$\int_\Omega \eta^2 H''\big(|Dv|\big) \big|D\big(|Dv|\big)\big|^2\, dx \leqslant 4 \int_\Omega |D\eta|^2 H'\big(|Dv|\big) |Dv|\, dx < +\infty.$$

Now let $M > 1$ such that $|Dv| \leqslant M$ on $\operatorname{supp}(\eta)$. For $\delta > 0$ we define

$$\psi(t) = \begin{cases} t^{2\delta} & \text{if } t \in [0, M], \\ M^{2\delta} & \text{if } t \in (M, +\infty). \end{cases}$$

In the case of $\delta \in [1/2, +\infty)$ we can use such a function ψ in (2.21) in order to get

$$\int_\Omega \eta^2 H''(|Dv|)|Dv|^{2\delta}\big|D(|Dv|)\big|^2\,dx \leqslant 4\int_\Omega |D\eta|^2 H'(|Dv|)|Dv|^{2\delta+1}\,dx. \tag{2.22}$$

When $\delta \in (0, 1/2)$, such ψ does not have bounded derivative near 0. So we *linearize* between 0 and $1/k$, for every integer $k \geqslant 1$ and we get the following sequence of functions:

$$\psi_k(t) = \begin{cases} \left(\frac{1}{k}\right)^{2\delta-1} t & \text{if } t \in \left[0, \frac{1}{k}\right], \\ \psi(t) & \text{if } t \in \left(\frac{1}{k}, +\infty\right). \end{cases}$$

It can be easily shown that $0 \leqslant \psi_k \leqslant \psi_{k+1} \leqslant \psi$ and $\psi_k(t) \to \psi(t)$ in $[0, +\infty)$. Then we can use estimate (2.21) with ψ_k and monotone convergence Theorem gives (2.22) for $\delta \in (0, 1/2)$ too.

Let us define

$$G(t) = 1 + \int_0^t \sqrt{s^{2\delta} H''(s)}\,ds.$$

Then, by means of inequality $(a+b)^2 \leqslant 2(a^2+b^2)$ we have

$$\big|D\big(\eta G(|Dv|)\big)\big|^2 \leqslant 2|D\eta|^2\big|G(|Dv|)\big|^2 + 2\eta^2|Dv|^{2\delta} H''(|Dv|)\big|D(|Dv|)\big|^2$$

so that, by (2.22),

$$\int_\Omega \big|D\big(\eta G(|Dv|)\big)\big|^2\,dx \leqslant 2\int_\Omega |D\eta|^2\big|G(|Dv|)\big|^2\,dx + 8\int_\Omega |D\eta|^2 H'(|Dv|)|Dv|^{2\delta+1}\,dx. \tag{2.23}$$

Since H' is increasing, by (2.10),

$$G(t) \leqslant 1 + \int_0^t \sqrt{s^{2\delta-1} H'(s)}\,ds \leqslant 1 + 2\sqrt{H'(t)}\,\frac{t^{(2\delta+1)/2}}{2\delta+1},$$

which implies, since $\delta \geqslant 0$,

$$\big|G(t)\big|^2 \leqslant 2\left(1 + \frac{4H'(t)}{(2\delta+1)^2} t^{2\delta+1}\right) \leqslant 8\big(1 + H'(t)t^{2\delta+1}\big). \tag{2.24}$$

By (2.23) and (2.24) it follows that

$$\int_\Omega \big|D\big(\eta G(|Dv|)\big)\big|^2\,dx \leqslant 24\int_\Omega |D\eta|^2\big[1 + H'(|Dv|)|Dv|^{2\delta+1}\big]\,dx.$$

Set $2^* = 2n/(n-2)$ if $n > 2$ and $2^* = 3$ if $n = 2$. By Sobolev inequality, there exists a positive constant $C_1 = C_1(n, |\Omega|)$ such that

$$\left(\int_\Omega \big[\eta G(|Dv|)\big]^{2^*}\,dx\right)^{2/2^*} \leqslant C_1\int_\Omega |D\eta|^2\big[1 + H'(|Dv|)|Dv|^{2\delta+1}\big]\,dx. \tag{2.25}$$

Now by (2.10), since $H'(t)/t$ is decreasing, we observe that

$$\begin{aligned}\left[G(t)\right]^{2^*} &\geqslant 1+\left(\int_0^t \sqrt{s^{2\delta}H''(s)}\,ds\right)^{2^*}\\ &\geqslant 1+\left(\frac{H'(t)}{t}\right)^{2^*/2}\frac{1}{\gamma^{2^*/2}}\left(\int_0^t s^\delta\,ds\right)^{2^*}\\ &= 1+\left[H'(t)\right]^{2^*/2}\frac{1}{\gamma^{2^*/2}}\frac{t^{2^*(\delta+1/2)}}{(\delta+1)^{2^*}}\end{aligned}$$

for every $t \in [0,+\infty)$.

Let us assume that $t \geqslant 1$. By (2.7) we have

$$H'(t)=\frac{h'(\sqrt{\mu^2+t^2}\,)}{\sqrt{\mu^2+t^2}}t$$

and then, since H' is increasing and $h'(t)/t$ is decreasing, by assuming $\mu \leqslant 1$ we have

$$H'(t)\geqslant H'(1)=\frac{h'(\sqrt{\mu^2+1}\,)}{\sqrt{\mu^2+1}}\geqslant\frac{h'(\sqrt{2}\,)}{\sqrt{2}}.$$

Then

$$\left[G(t)\right]^{2^*}\geqslant\frac{C_2}{(\delta+1)^{2^*}}\left[1+H'(t)t^{(2\delta+1)2^*/2}\right],$$

where

$$C_2=\min\left\{1,\gamma^{-2^*/2}\left(\frac{h'(\sqrt{2}\,)}{\sqrt{2}}\right)^{2^*/2-1}\right\}.$$

For $t \in [0,1)$, since h' is increasing, $G(t) \geqslant 1$ and $\delta \geqslant 0$ we have

$$1+H'(t)t^{(2\delta+1)2^*/2}\leqslant 1+H'(1)\leqslant\left[1+h'(\sqrt{2}\,)\right]\left[G(t)\right]^{2^*}.$$

Then for every $t \geqslant 0$ the inequality

$$\left[G(t)\right]^{2^*}\geqslant\frac{C_3}{(\delta+1)^{2^*}}\left[1+H'(t)t^{(2\delta+1)2^*/2}\right] \tag{2.26}$$

holds with

$$C_3=\min\left\{C_2,\frac{1}{1+h'(\sqrt{2}\,)}\right\}.$$

Let $0<\rho<R$ be such that $B_R \Subset \Omega$ and let us fix η in such a way that $0 \leqslant \eta \leqslant 1$ in $\mathbb{R}^n$, $\eta = 1$ in B_ρ, $\operatorname{supp}(\eta) \subset B_R$ and $|D\eta| \leqslant 2/(R-\rho)$ in $\mathbb{R}^n$. Then by (2.25) and (2.26) we get

$$\left(\int_{B_\rho}\left[1+H'\big(|Dv|\big)|Dv|^{(2\delta+1)2^*/2}\right]dx\right)^{2/2^*}$$
$$\leqslant \frac{C_4(\delta+1)^2}{(R-\rho)^2}\int_{B_R}\left[1+H'\big(|Dv|\big)|Dv|^{(2\delta+1)}\right]dx, \tag{2.27}$$

where $C_4 = 4C_1C_3^{-2/2^*}$.

Let us set $\vartheta = 2\delta + 1$. Then (2.27) becomes

$$\left(\int_{B_\rho}\left[1+H'\big(|Dv|\big)|Dv|^{\vartheta 2^*/2}\right]dx\right)^{2/2^*}$$
$$\leqslant \frac{C_4\vartheta^2}{(R-\rho)^2}\int_{B_R}\left[1+H'\big(|Dv|\big)|Dv|^{\vartheta}\right]dx. \tag{2.28}$$

Now we define a sequence of radii and another one of numbers as follows:

$$\rho_i = \rho + \frac{R-\rho}{2^i}, \qquad \vartheta_i = \left(\frac{2^*}{2}\right)^i$$

for $i = 0, 1, 2, \ldots$. Moreover we set

$$A_i = \left(\int_{B_{\rho_i}}\left[1+H'\big(|Dv|\big)|Dv|^{\vartheta_i}\right]dx\right)^{1/\vartheta_i}.$$

Using this notation, and putting $\rho = \rho_{i+1}$, $R = \rho_i$ and $\vartheta = \vartheta_i$ in (2.28) we easily have

$$A_{i+1} \leqslant \left[\frac{C_4 4^{i+1}\vartheta_i^2}{(R-\rho)^2}\right]^{1/\vartheta_i} A_i,$$

thus, if we iterate this estimate,

$$A_{i+1} \leqslant \left\{\prod_{k=0}^{i}\left[\frac{C_4 4^{k+1}\vartheta_k^2}{(R_0-\rho_0)^2}\right]^{1/\vartheta_k}\right\}A_0 \leqslant C_5 A_0, \tag{2.29}$$

where $C_5 = C_5(n, \gamma, |\Omega|, h'(\sqrt{2}), \rho, R)$ and, in particular,

$$C_5 = \left[\left(1 + \frac{4C_4}{(R-\rho)^2}\right)(2^*)^{4/(2^*-2)}\right]^{2^*/(2^*-2)}.$$

Then (2.29) leads to

$$\left(\int_{B_{\rho_0}}\left[1+H'\big(|Dv|\big)|Dv|^{(2^*/2)^{i+1}}\right]dx\right)^{(2/2^*)^{i+1}}$$
$$\leqslant C_5\int_{B_{R_0}}\left[1+H'\big(|Dv|\big)|Dv|\right]dx < +\infty. \tag{2.30}$$

Now we observe that, since H' is increasing and $H'(t)/t$ is decreasing, for every $\tau > 1$ and every $t \geqslant 1$ we have

$$H'(t)t^\tau \geqslant H'(1)t^\tau \geqslant \frac{h'(\sqrt{2})}{\sqrt{2}}t^\tau.$$

Then we can say that, for every $t > 0$ and every $\tau > 1$,

$$t^\tau \leqslant C_6\bigl(1 + H'(t)t^\tau\bigr),$$

where

$$C_6 = \max\left\{1, \frac{\sqrt{2}}{h'(\sqrt{2})}\right\}.$$

Therefore by (2.30) it follows that

$$\begin{aligned}\sup_{B_\rho}|Dv| &= \lim_{i\to+\infty}\left(\int_{B_\rho}|Dv|^{(2^*/2)^{i+1}}\,dx\right)^{(2/2^*)^{i+1}}\\ &\leqslant \limsup_{i\to+\infty}\left(C_6\int_{B_\rho}\bigl[1 + H'\bigl(|Dv|\bigr)|Dv|^{(2^*/2)^{i+1}}\bigr]\,dx\right)^{(2/2^*)^{i+1}}\\ &\leqslant C_5\int_{B_R}\bigl[1 + H'\bigl(|Dv|\bigr)|Dv|\bigr]\,dx \leqslant 2C_5\int_{B_R}\bigl[1 + H\bigl(|Dv|\bigr)\bigr]\,dx,\end{aligned}$$

where we used the Δ_2^2 property in the last inequality. Thus (2.18) holds true if we check the way C_5 depends on $h'(\sqrt{2})$.

A careful inspection shows that

$$C_5 \leqslant C_7\bigl[V\bigl(h'(\sqrt{2})\bigr)\bigr]^{2/(2^*-2)},$$

where $V(t) = 1 + t + c_0t^{-\vartheta}$, $c_0 = c_0(n,\gamma) > 0$, $\vartheta = \vartheta(n) > 0$ and $C_7 = C_7(n, |\Omega|, \rho, R) > 0$. This ends the second step of the proof.

Step 3 (Let μ go to 0).

We proceed as in Lemma 2.13 of [1].

Let h satisfy conditions (H1) and (H2). We recall that u is a local minimizer of $\mathcal{I}$ defined by (2.4) and (2.5). Let B_R be a ball such that $B_R \Subset \Omega$ and, for every $\mu \in (0,1)$, let us define the function

$$H_\mu(t) = h\bigl(\sqrt{\mu^2 + t^2}\bigr) - h(\mu).$$

We consider the variational problem in B_R,

$$\min\left\{\mathcal{I}_\mu(v) = \int_{B_R} f_\mu(Dv)\,dx\colon\ v \in u + W_0^{1,q}(B_R, \mathbb{R}^N)\right\}, \tag{2.31}$$

where $f_\mu(z) = H_\mu(|z|)$. Because of (2.12) and (2.14), there exists a unique solution v_μ of (2.31). Then we have

$$\begin{aligned}\frac{\Lambda_1}{q}\int_{B_R}|Dv_\mu|^p\,dx &\leqslant \int_{B_R} h\big(|Dv_\mu|\big)\,dx \leqslant \int_{B_R}\big[H_\mu\big(|Dv_\mu|\big)+h(\mu)\big]\,dx\\ &\leqslant \int_{B_R} H_\mu\big(|Du|\big)\,dx + h(\mu)|B_R|\\ &\leqslant \int_{B_R} h\Big(\sqrt{1+|Du|^2}\Big)\,dx < +\infty. \qquad (2.32)\end{aligned}$$

Now, let us consider a sequence $\{\mu_k\}_k \in (0,1)$, with $\mu_k \to 0$. Then, up to a subsequence, $Dv_{\mu_k} \rightharpoonup Du_0$ in $L^q(B_R)$, for some function $u_0 \in u + W_0^{1,q}(B_R,\mathbb{R}^N)$ and eventually, by lower semicontinuity and (2.32),

$$\begin{aligned}\int_{B_R} f(Du_0)\,dx &= \int_{B_R} h\big(|Du_0|\big)\,dx \leqslant \liminf_{k\to+\infty}\int_{B_R} h\big(|Dv_{\mu_k}|\big)\,dx\\ &\leqslant \liminf_{k\to+\infty}\int_{B_R} h\Big(\sqrt{\mu_k^2+|Du|^2}\Big)\,dx = \int_{B_R} h\big(|Du|\big)\,dx\\ &= \int_{B_R} f(Du)\,dx. \qquad (2.33)\end{aligned}$$

Thus u and u_0 are minimizers with the same boundary datum; since f is strictly convex, it follows that $u_0 = u$.

Let $0 < \rho < R$; we use Step 2 with balls B_ρ and $B_{(\rho+R)/2}$, so that the minimality of v_{μ_k} with respect to u gives

$$\begin{aligned}\sup_{B_\rho}|Dv_{\mu_k}| &\leqslant c\int_{B_R}\big[1+H_{\mu_k}\big(|Dv_{\mu_k}|\big)\big]\,dx\\ &\leqslant c\int_{B_R}\Big[1+h\Big(\sqrt{\mu_k^2+|Du|^2}\Big)\Big]\,dx \leqslant c\int_{B_R}\Big[1+h\Big(\sqrt{1+|Du|^2}\Big)\Big]\,dx,\end{aligned}$$

thus

$$\sup_{B_\rho}|Dv_{\mu_k}| \leqslant c, \quad \forall k,$$

for some constant c independent of μ_k. Then, up to a subsequence, $\{Dv_{\mu_k}\}_k$ converges in the weak-* topology of $L^\infty(B_\rho)$, to some function $w \in L^\infty(B_\rho)$ that turns out to be Du.

The lower semicontinuity of the L^∞-norm gives

$$\sup_{B_\rho}|Du| \leqslant c\int_{B_R}\big[1+h\big(|Du|\big)\big]\,dx,$$

where $c = \tilde{c}[V(h'(\sqrt{2}))]^{2/(2^*-2)}$ and $\tilde{c} = \tilde{c}(n,N,\rho,R) > 0$. $\square$

3. Proof of Theorem 1.1

In this section we study the regularity of minimizers of integral functionals assuming this time that the integrand f satisfies nonstandard growth. For convenience of the reader, we recall the assumptions of Theorem 1.1

Let $1 < p < q < 2$ and g be an N-function. We assume that there exist two constants $\Lambda_1, \Lambda_2 > 0$ such that

(G1) $g \in C^2((0,+\infty)) \cap C^1([0,+\infty))$, $g'(0) = 0$, $g'(t)/t$ is decreasing and

$$\Lambda_1 t^{p-2} \leqslant \frac{g'(t)}{t} \leqslant \Lambda_2 (t^{q-2} + t^{p-2}); \tag{3.1}$$

(G2) There exists $\gamma > 1$ such that

$$g''(t)t \leqslant g'(t) \leqslant \gamma g''(t)t. \tag{3.2}$$

As we already observed, by (3.1) and (3.2) it follows that

$$\frac{\Lambda_1}{p} t^p \leqslant g(t) \leqslant \frac{\Lambda_2}{p}(t^q + t^p)$$

and $g \in \Delta_2^2$.

Consider, for $\sigma \in (0,1)$, the functions

$$g_\sigma(t) = g(t) + \sigma t^q.$$

As it can be easily checked, $g_\sigma : [0,+\infty) \to [0,+\infty)$ is an N-function strictly convex, $g_\sigma \in C^2((0,+\infty)) \cap C^1([0,+\infty))$, $g'_\sigma(0) = 0$ and $t \to g'_\sigma(t)/t$ is decreasing in $(0,+\infty)$. Furthermore we have that

$$\sigma q t^{q-2} \leqslant \frac{g'_\sigma(t)}{t} \leqslant (\Lambda_2 + q)(t^{q-2} + t^{p-2}),$$

$$g''_\sigma(t)t \leqslant g'_\sigma(t) \leqslant \max\left\{\gamma, \frac{1}{q-1}\right\} g''_\sigma(t)t.$$

Now let

$$f_\sigma(z) = g_\sigma(|z|)$$

for every $z \in \mathbb{R}^{nN}$ and consider the functional

$$\mathcal{I}_\sigma(v) = \int_\Omega f_\sigma(Dv)\,dx. \tag{3.3}$$

Let $0 < \varepsilon < \min\{1, R\}$, where $R > 0$ is such that $B_{2R} \Subset \Omega$. Moreover let $\{u_\varepsilon\}_\varepsilon$ be a sequence of smooth functions obtained from u by means of standard mollifiers, then $u_\varepsilon \in W^{1,q}(B_R, \mathbb{R}^N)$.

Since $\mathcal{I}_\sigma$ has q-growth, we consider the following variational problem:

$$\min\left\{\mathcal{I}_\sigma(v):\ v \in u_\varepsilon + W_0^{1,q}(B_R, \mathbb{R}^N)\right\} \tag{3.4}$$

and let $v_{\varepsilon,\sigma} \in u_\varepsilon + W_0^{1,q}(B_R, \mathbb{R}^N)$ be the (unique) minimizer. We are now in condition to apply Proposition 2.1 for $0 < \rho < R$,

$$\sup_{B_\rho} |Dv_{\varepsilon,\sigma}| \leqslant \tilde{c}\big[V\big(g'_\sigma(\sqrt{2})\big)\big]^{2/(2^*-2)} \int_{B_R} \big[1 + g_\sigma\big(|Dv_{\varepsilon,\sigma}|\big)\big]\,dx = (I).$$

Let us point out that

$$0 < g'(\sqrt{2}) \leqslant g'_\sigma(\sqrt{2}) \leqslant g'(\sqrt{2}) + 4$$

thus $V(g'_\sigma(\sqrt{2})) \leqslant 5V(g'(\sqrt{2}))$ and

$$(I) \leqslant \tilde{c}\big[5V\big(g'(\sqrt{2})\big)\big]^{2/(2^*-2)} \int_{B_R} \big[1 + g_\sigma\big(|Dv_{\varepsilon,\sigma}|\big)\big]\,dx.$$

Now we use the minimality of $v_{\varepsilon,\sigma}$ with respect to u_ε and Jensen inequality,

$$\begin{aligned} \frac{\Lambda_1}{p}\int_{B_R} |Dv_{\varepsilon,\sigma}|^p\,dx &\leqslant \int_{B_R} g\big(|Dv_{\varepsilon,\sigma}|\big)\,dx \leqslant \int_{B_R} g_\sigma\big(|Dv_{\varepsilon,\sigma}|\big)\,dx \\ &\leqslant \int_{B_R} g_\sigma\big(|Du_\varepsilon|\big)\,dx = \int_{B_R} g\big(|Du_\varepsilon|\big)\,dx + \sigma\int_{B_R} |Du_\varepsilon|^q\,dx \\ &\leqslant \int_{B_{R+\varepsilon}} g\big(|Du|\big)\,dx + \sigma\int_{B_R} |Du_\varepsilon|^q\,dx \leqslant c(\varepsilon) \end{aligned} \tag{3.5}$$

and

$$\sup_{B_\rho} |Dv_{\varepsilon,\sigma}| \leqslant \tilde{c}\big[5V\big(g'(\sqrt{2})\big)\big]^{2/(2^*-2)} \left\{ \int_{B_{R+\varepsilon}} \big[1 + g\big(|Du|\big)\big]\,dx + \sigma\int_{B_R} |Du_\varepsilon|^q\,dx \right\}. \tag{3.6}$$

Then for every fixed ε, (3.5) gives us weak compactness in $L^p(B_R)$ as $\sigma \to 0$. So, up to a subsequence $Dv_{\varepsilon,\sigma} \rightharpoonup Dw_\varepsilon$ in $L^p(B_R)$ as $\sigma \to 0$, for some $w_\varepsilon \in u_\varepsilon + W_0^{1,p}(B_R, \mathbb{R}^N)$.

Moreover, by (3.6), $\sup_{B_\rho} |Dv_{\varepsilon,\sigma}|$ is equibounded with respect to σ. Hence $\{Dv_{\varepsilon,\sigma}\}_\sigma$ converges in the weak-* topology of L^∞ to Dw_ε and

$$\sup_{B_\rho} |Dw_\varepsilon| \leqslant c\big[5V\big(g'(\sqrt{2})\big)\big]^{2/(2^*-2)} \int_{B_{R+\varepsilon}} \big[1 + g\big(|Du|\big)\big]\,dx \tag{3.7}$$

for some $c = c(n, N, \rho, R) > 0$. By lower semicontinuity in (3.5) we get

$$\int_{B_R} g\big(|Dw_\varepsilon|\big)\,dx \leqslant \liminf_{\sigma\to 0} \int_{B_R} g\big(|Dv_{\varepsilon,\sigma}|\big)\,dx \leqslant \int_{B_{R+\varepsilon}} g\big(|Du|\big)\,dx. \tag{3.8}$$

Now, (3.8) gives weak compactness in $L^p(B_R)$ as $\varepsilon \to 0$, thus up to a subsequence, $Dw_\varepsilon \rightharpoonup Dw$ in $L^p(B_R)$ for some function $w \in u + W_0^{1,p}(B_R, \mathbb{R}^N)$.

Lower semicontinuity and (3.8) allow us to write

$$\int_{B_R} g\big(|Dw|\big)\,dx \leqslant \liminf_{\varepsilon\to 0}\int_{B_R} g\big(|Dw_\varepsilon|\big)\,dx \leqslant \int_{B_R} g\big(|Du|\big)\,dx.$$

The minimality of u and the strict convexity of g imply $w \equiv u$.

Finally, using (3.7) we obtain that also Dw_ε converges to $Dw = Du$ as $\varepsilon \to 0$, in the weak-* topology of L^∞ and, letting $\varepsilon \to 0$ in (3.7) we easily get

$$\sup_{B_\rho}|Du| \leqslant c\big[5V\big(g'(\sqrt{2})\big)\big]^{2/(2^*-2)}\int_{B_R}\big[1+g\big(|Du|\big)\big]\,dx$$

for some $c = c(n, N, \rho, R)$. □

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References

[1] E. Acerbi, N. Fusco, Regularity for minimizers of non-quadratic functionals. The case $1 < p < 2$, J. Math. Anal. Appl. 140 (1989) 115–135.

[2] L. Esposito, F. Leonetti, G. Mingione, Higher integrability for minimizers of integral functionals with (p, q) growth, J. Differential Equations 157 (1999) 414–438.

[3] L. Esposito, F. Leonetti, G. Mingione, Regularity results for minimizers of irregular integrals with (p, q) growth, Forum Math. 14 (2002) 245–272.

[4] M. Giaquinta, G. Modica, Remarks on the regularity of the minimizers of certain degenerate functionals, Manuscripta Math. 57 (1986) 55–99.

[5] M.C. Hong, Some remarks on the minimizers of variational integrals with non standard growth conditions, Boll. Un. Mat. Ital. A 6 (1992) 91–101.

[6] F. Leonetti, E. Mascolo, F. Siepe, Gradient regularity for minimizers of functionals under p–q subquadratic growth, Boll. Un. Mat. Ital. B 4 (2001) 571–586.

[7] P. Marcellini, Everywhere regularity for a class of elliptic systems without growth conditions, Ann. Scuola Norm. Sup. Pisa 23 (1996) 1–25.

[8] E. Mascolo, F. Siepe, Functionals of the calculus of variations with non standard growth conditions, in: Conferencias, Seminarios y Trabajos de Matematica, Vol. 1, Universitad Austral, Rosario, 2000, pp. 1–22.

[9] G. Mingione, F. Siepe, Full $C^{1,\alpha}$-regularity for minimizers of integral functionals with $L\log L$-growth, Z. Anal. Anwendungen 18 (1999) 1081–1103.

[10] K. Uhlenbeck, Regularity for a class of non-linear elliptic systems, Acta Math. 138 (1977) 219–240.