ON THE EXISTENCE AND RADIAL SYMMETRY OF MAXIMIZERS FOR FUNCTIONALS WITH CRITICAL EXPONENTIAL GROWTH IN $\mathbb{R}^2$

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Abstract. We investigate the problem of existence and symmetry of maximizers for

$$S(\alpha, 4\pi) = \sup_{\|u\|=1} \int_B \left(e^{4\pi u^2} - 1\right) |x|^\alpha dx,$$

where B is the unit disk in $\mathbb{R}^2$ and $\alpha > 0$, proposed by Secchi and Serra in [11]. Through the notion of spherical symmetrization with respect to a measure, we prove that supremum is attained for α small. Furthermore, we prove that $S(\alpha, 4\pi)$ is attained by a radial function.

1. Introduction

Let $H_0^1(\Omega)$ be the Sobolev space over a bounded domain $\Omega \subset \mathbb{R}^N$, with Dirichlet norm $\|u\|^2 = \int_\Omega |\nabla u|^2 dx$. The Sobolev embedding theorem states that $H_0^1(\Omega) \subset L^p(\Omega)$, for $1 \leq p \leq 2^* = \frac{2N}{N-2}$; equivalently, if we set

$$S_N(p) = \sup_{\|u\| \leq 1} \int_\Omega |u|^p dx,$$

then

$$\begin{aligned} &S_N(p) < \infty, \quad \text{for } 1 < p \leq 2^* = \tfrac{2N}{N-2}; \\ &S_N(p) = \infty, \quad \text{for } p > 2^*; \end{aligned}$$

furthermore, the value of the best Sobolev constant $S_N(2^*)$ is explicit, independent of the domain Ω and it is known that it is never attained in any bounded smooth domain. The maximal growth $|u|^{2^*}$ allowed is called "critical" Sobolev growth. If $N = 2$, every polynomial growth is admitted, but it is easy to show that $H_0^1(\Omega) \not\subseteq L^\infty(\Omega)$: in this case, it is well known that the maximal growth allowed to a function $g : \mathbb{R} \to \mathbb{R}^+$, such that

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$\sup_{\|u\|\leq 1} \int g(u) < \infty$, is of exponential type. More precisely, the Trudinger Moser inequality states that, for bounded domain $\Omega \subset \mathbb{R}^2$

$$\begin{aligned} \sup_{\|u\|\leq 1} \int_\Omega e^{\gamma u^2} dx &\leq C(\gamma)|\Omega| \leq C(4\pi)|\Omega|, \quad \text{for } \gamma \leq 4\pi; \\ \sup_{\|u\|\leq 1} \int_\Omega e^{\gamma u^2} dx &= \infty, \quad \text{for } \gamma > 4\pi, \end{aligned}$$

see [10], [17] and [9]. In contrast with the Sobolev case, the value $C(4\pi)$ is attained when $\Omega = B_1(0)$ is the unit ball in $\mathbb{R}^2$, as proved in an interesting paper by Carleson and Chang [2] (see also [4]). This result was extended to general bounded domains in $\mathbb{R}^2$ by Flucher [5].

In this paper, we consider the maximization problem

$$S(\alpha, \gamma) = \sup_{\|u\|=1} \int_B \left(e^{\gamma u^2} - 1\right) |x|^\alpha dx. \tag{1.1}$$

Problem (1.3) can be seen as a natural two-dimensional extension of the Hénon-type problem

$$\sup_{\|u\|=1} \left(\int_B |u|^p |x|^\alpha dx \right)^{2/p} = \sup_{u\neq 0} \frac{\left(\int_B |u|^p |x|^\alpha dx\right)^{2/p}}{\int_B |\nabla u|^2 dx}, \tag{1.2}$$

in $\mathbb{R}^N$ with $N \geq 3$ and $1 < p < 2^*$, which has been widely investigated in the last few years. It is easy to verify that (1.2) is achieved at least by a positive function; since the quotient in (1.2) is invariant under rotations, it is natural to ask if the supremum is achieved by a radial function. A very interesting result obtained by Smets, Su and Willem ([12]) shows that a symmetry breaking phenomenon occurs for any $p \in (2, 2^*)$: in details, for every p in the subcritical range, the supremum in (1.2) is attained by a non-radial function when $\alpha \to \infty$. This result has generated a line of research on the Hénon-type equations (see references in [11]). On the contrary, the Hénon-type problem in $\mathbb{R}^2$ with exponential non-linearities seems to have been much less studied. Very recently, Calanchi and Terraneo (see [3]) proved some results about the existence of non-radial maximizers for the variational problem

$$\sup_{\|u\|=1} \int_B \left(e^{\gamma u^2} - 1 - \gamma u^2\right) |x|^\alpha dx,$$

where $\alpha > 0$ and $0 < \gamma < 4\pi$; in the same line is the work by Secchi and Serra, [11], where the authors prove a symmetry breaking result for problem (1.1): if the supremum is assumed, it is attained by a non-radial function, for

α large enough. Furthermore, they prove that the supremum is attained in the subcritical case, that is, for $0 < \gamma < 4\pi$, but they left open the problem in the critical case. Here, we give a partial answer: the supremum

$$S(\alpha, 4\pi) = \sup_{\|u\|=1} \int_B \left(e^{4\pi u^2} - 1\right) |x|^\alpha dx, \tag{1.3}$$

is attained, at least if the parameter α is small enough. In the subcritical case, the existence of a maximizer can be proven with standard arguments (see [11], proof of Proposition 1). On the contrary, in the critical case it seems not possible to adapt the proof suggested by Secchi and Serra, which deeply depends on the hypothesis of subcritical growth: due to the presence of the weight $|x|^\alpha$ in front of the non-linearity, problem (1.3) cannot be reduced to a one dimensional problem using the technique of Schwarz symmetrization, as proposed by Carleson and Chang.

Our result depends on a different notion of symmetrization, the so called *spherical symmetrization with respect to a measure*, which is the counterpart of Schwarz symmetrization in the unweighted problem. Although we symmetrize with respect to a measure μ, which is different from the Lebesgue one, a result by Schulz and Vera de Serio (see [13]) states that the gradient norm does not increase, as in the classical case (the result is valid only in $\mathbb{R}^2$ and with suitable assumption on the measure μ). This fact allows us to adapt the proof presented by de Figueiredo, do Ó and Ruf in [4], obtaining the following result.

Theorem 1.1. *There exists $\alpha_* > 0$, such that for every $\alpha \in (0, \alpha_*)$, $S(\alpha, 4\pi)$ is attained.*

We remark that the notion of symmetrization with respect to the measure $\int_B |x|^\alpha dx$ gives also a geometric interpretation of the changes of variable performed by Smets, Su and Willem, and later by Secchi and Serra, when dealing with radial functions: see Remark 1 at the end of Section 3.

Note that when α is small, the symmetry breaking result [11] does not hold, and it is not known if the maximizer is surely non-radial, or not. When $N \geq 3$, Smets, Su and Willem in [12] proved that the maximizers of (1.2) are surely radial if $p \to 2$ and α is small, whereas the maximizers 'tend' to be all non-radial if $p \to 2^*$ (that is, the maximizers are non-radial for every $\alpha > \alpha^*$, with $\alpha^* \to 0$ when $p \to 2^*$). In the second part of this paper, we prove that the symmetry breaking, pointed out in [11], is a 'true' breaking phenomenon. That is, the maximizers are surely radial functions for α small (and they are non-radial for α big), as stated in the following theorem.

Theorem 1.2. *Let $\gamma \in (0, 4\pi]$ and set*

$$S(\alpha, \gamma) = \sup_{\|u\|=1} \int_B \left(e^{\gamma u^2} - 1 \right) |x|^\alpha dx.$$

There exists $\alpha_r = \alpha_r(\gamma) > 0$ such that for every $\alpha \in [0, \alpha_r)$, $S(\alpha, \gamma)$ is attained by a radial function.

2. Symmetrization with respect to a measure

In this section, we recall the main definitions and properties of symmetrization: we refer to [7] or to [1]. We start by a review of the standard definitions. Let Ω be a bounded domain in $\mathbb{R}^2$. We denote by $|\Omega|$ the Lebesgue measure of Ω and by $\mathcal{L}_0(\Omega)$ the set of Lebesgue measurable functions defined in Ω up to a.e. equivalence. For every function $u \in \mathcal{L}_0(\Omega)$, we define the distribution function ϕ_u of u by the formula

$$\phi_u(t) = |\{x \in \Omega : |u(x)| > t\}|.$$

A measurable function u in $\mathbb{R}^n$ is called *radially symmetric*, or radial, for short, if $u(x) = \tilde{u}(r)$, $r = |x|$; it is called *rearranged* if it is non-negative, radially symmetric and $\tilde{u}$ is a non-increasing function of $r > 0$; we also impose that $\tilde{u}(r)$ be right-continuous. We will write $u(x) = u(r)$ by abuse of notation. The *spherical symmetric rearrangement* u^* of u is the unique rearranged function defined in Ω^* which has the same distribution function as u, that is, for every $t > 0$,

$$\phi_u(t) = |\{x \in \Omega : |u(x)| > t\}| = \phi_{u^*}(t) = |\{x \in \Omega^* : |u^*(x)| > t\}|,$$

where $\Omega^* = B_R(0)$ is the ball having the same volume as Ω, i.e., $|\Omega| = \omega_n R^n$ (here ω_n is the volume of the unit sphere in $\mathbb{R}^n$). Then

$$\begin{aligned} u^*(x) &= \inf\{t > 0 : \phi_u(t) \le \omega_n |x|^n\} \\ &= \sup\{t > 0 : \phi_u(t) > \omega_n |x|^n\}. \end{aligned} \tag{2.1}$$

A rearranged function coincides with its spherical rearrangement. Since the distribution functions of u and u^* are identical,

$$\int_\Omega |u|^p dx = \int_{\Omega^*} (u^*)^p dx,$$

for every $p \in [1, +\infty)$; moreover, for every non-negative, increasing and left-continuous real function Φ

$$\int_\Omega \Phi(u) dx = \int_{\Omega^*} \Phi(u^*) dx.$$

Finally, if $u \in W_0^{1,p}(\Omega)$, then $u^* \in W_0^{1,p}(\Omega^*)$ and

$$\int_{\Omega^*} |\nabla u^*|^p dx \leq \int_\Omega |\nabla u|^p dx, \tag{2.2}$$

for $p \in [1, +\infty)$: this is the celebrated Polya-Szegö inequality.

As a natural generalization of the spherical symmetrization (or Schwarz symmetrization), one can introduce the *spherical symmetrization with respect to a measure* μ defined on the domain Ω. We refer to [13]. Let $p : \mathbb{R}^n \to \mathbb{R}^+$ be a non-negative, measurable and locally integrable function, and consider the measure μ given by

$$\mu(A) = \int_A p dx,$$

for any Lebesgue measurable set A in $\mathbb{R}^n$; note that μ is absolutely continuous with respect to the Lebesgue measure. The distribution function $\phi_{\mu,u}$ of u with respect to the measure μ is given by

$$\phi_{\mu,u}(t) = \mu\left(\{x \in \Omega : |u(x)| > t\}\right);$$

as in the classical case, $\phi_{\mu,u}$ is a monotone, non-increasing and right continuous function. The *spherical symmetric rearrangement* u_μ^* of u *with respect to the measure* μ is the unique rearranged function defined in Ω_μ^* whose (classical) distribution function is the same as the distribution function (with respect to the measure μ) of u; that is, for every $t > 0$,

$$\phi_{\mu,u}(t) = \mu\left(\{x \in \Omega : |u(x)| > t\}\right) = \phi_{u^*}(t) = |\{x \in \Omega_\mu^* : |u^*(x)| > t\}|,$$

where $\Omega_\mu^* = B_R(0)$ is the ball centered at the origin with $\mu(\Omega) = |\Omega_\mu^*| = \omega_n R^n$. Then

$$\begin{aligned} u_\mu^*(x) &= \inf\{t > 0 : \phi_{\mu,u}(t) \leq \omega_n |x|^n\} \\ &= \sup\{t > 0 : \phi_{\mu,u}(t) > \omega_n |x|^n\}. \end{aligned} \tag{2.3}$$

Obviously, the spherical symmetric rearrangement $u_{\mathcal{L}}^*$ with respect to the Lebesgue measure is the classical symmetric rearrangement by Schwarz. However, if μ is not the Lebesgue measure, a rearranged function, in the sense defined above, will not coincide with its μ-rearrangement u_μ^*, since an extra contraction/dilation will take place. In particular, if we consider the density function $p_\alpha(x) = |x|^\alpha : \mathbb{R}^n \to \mathbb{R}^+$ with $\alpha > 0$ and the associated measure

$$\mu_\alpha(A) = \int_A |x|^\alpha dx, \tag{2.4}$$

defined on the unit sphere B in $\mathbb{R}^n$, then the μ_α-rearrangement of a rearranged function $u(r)$ (that is, of a non-negative, radial and non-increasing function u) is defined by the formula

$$u^*_\alpha(r) = u\Big(r^{\frac{n}{\alpha+n}} \sqrt[\alpha+n]{\frac{\alpha+n}{n}}\Big), \tag{2.5}$$

where $r \in B\big(0, \sqrt[n]{\frac{\alpha+n}{n}}\big)$. As in the classical case, for every non-negative, increasing and left-continuous real function Φ

$$\int_\Omega \Phi(u)d\mu = \int_{\Omega^*_\mu} \Phi(u^*_\mu)dx,$$

so that $\|u\|_{L^p(\Omega,\mu)} = \|u^*_\mu\|_{L^p(\Omega^*_\mu,\mathcal{L})}$ for every $p \in [1,+\infty)$. Regarding the gradient norm, it is not known (to our knowledge) if the Polya-Szegö inequality can be maintained for all μ-rearrangements and for $p > 0$, $n \geq 1$. With the assumptions stated above on μ, if $u \in W^{1,1}(\mathbb{R}^n)$, then $u^*_\mu \in W^{1,1}(\mathbb{R}^n)$; furthermore, Schulz and Vera de Serio have proved the following result:

Theorem 2.1 (F. Schulz, V. Vera de Serio)**.** *Let $p \in \mathcal{C}^0(\bar{D})$ be a non-negative function on a simply-connected domain D such that $\log p$ is subharmonic where $p > 0$; suppose that $u \in W^{1,2}(\mathbb{R}^2)$ is a non-negative function with compact support in D. Then $u^*_\mu \in W^{1,2}(\mathbb{R}^2)$, and the inequality*

$$\int_{\mathbb{R}^2} |\nabla u^*_\mu|^2 dx \leq \int_{\mathbb{R}^2} |\nabla u|^2 dx, \tag{2.6}$$

holds.

We remark that Theorem 2.1 states that the gradient of the μ- rearrangement does not increase in the $L^2(\mathbb{R}^2)$ norm (that is, considering $\mathbb{R}^2$ endowed with the Lebesgue measure); different results can be found in [15] and in [14] where a similar inequality is obtained for the $L^2(\mathbb{R}^2,\mu)$ norms.

3. Existence of a maximizer for $S(\alpha, 4\pi)$

This section is devoted to the proof of our main result, Theorem 1.1. As well known, in the "unweighted" case $\alpha = 0$ the supremum $S(0, 4\pi)$ is attained: this is the celebrated result due to Carleson and Chang [2]. In the subcritical case $\gamma < 4\pi$, $S(\alpha,\gamma)$ is still attained, as pointed out by Serra and Secchi in [11] (and the proof is quite easy), whereas in the supercritical case $\gamma > 4\pi$, $S(\alpha,\gamma) = +\infty$ for every $\alpha > 0$, as proved by Calanchi and Terraneo [3] testing with a suitable sequence of (radial) functions.

The critical case $\gamma = 4\pi$ is more delicate. If we consider the radial version of the maximization problem (1.3), that is,

$$S^{\mathrm{rad}}(\alpha,\gamma) = \sup_{u \in H^1_{0,\mathrm{rad}}(B)} \int_B \left(e^{\gamma u^2} - 1\right) |x|^\alpha dx, \quad \|u\| = 1, \tag{3.1}$$

it is not hard to prove that the problem is still "subcritical", provided that $\gamma < 4\pi + 2\pi\alpha$, as proved by Secchi and Serra in [11]. More in details, they proved that

$$S^{\mathrm{rad}}(\alpha, 4\pi) = \frac{2}{\alpha+2} S\Big(0, 4\pi\frac{2}{\alpha+2}\Big),$$

and standard arguments show that $S(0, 4\pi\frac{2}{\alpha+2})$ is actually attained by a radial function. See also the remark at the end of Section 3 in [3].

On the contrary, it seems impossible to reduce the problem of maximization of $S(\alpha, 4\pi)$ in the general case, that is, considering also non-radial functions, to a subcritical one. Our proof follows the same idea of the one given by de Figueiredo, do Ó and Ruf in [4] (which differs from the original proof of Carleson and Chang by the use of the concentration-compactness principle). Here is a short outline of the proof:

- if $S(\alpha, 4\pi)$ is not attained, then by the concentration-compactness alternative of P.L. Lions there is a normalized maximizing and concentrating sequence v_n;
- by means of symmetrization with respect to the measure $\mu_\alpha = \int |x|^\alpha dx$, one can prove an upper bound for any normalized concentrating sequence u_n:

$$\overline{\lim}_{n\to+\infty} \int_{B^*_\alpha} \left(e^{4\pi|u_n|^2} - 1\right) dx \le \frac{2}{\alpha+2}\pi e,$$

- give an explicit function $\omega \in H^1_0(B)$ such that $\|\omega\| = 1$ and

$$\int_B \left(e^{4\pi\omega^2} - 1\right) |x|^\alpha dx > \frac{2}{\alpha+2}\pi e.$$

It is clear, then that the notion of spherical symmetrization with respect to a measure is the fundamental tool which allows to reduce the weighted problem $S(\alpha, 4\pi)$ to a one dimensional problem. See also the remarks at the end of the proof.

First of all, let us recall the concentration-compactness result by P.L. Lions [8] (adapted to the 2-dimensional case):

Proposition 3.1 (P.L. Lions). *Let Ω be a bounded domain in $\mathbb{R}^2$, and let u_n be a sequence in $H_0^1(\Omega)$ such that $\|u_n\|_{H_0^1} \leq 1$ for all n. We may suppose that $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$, $|\nabla u_n|^2 \to \nu$ weakly in measure. Then either*

(i) $\nu = \delta_{x_0}$, the Dirac measure of mass 1 concentrated at some $x_0 \in \bar{\Omega}$, and $u \equiv 0$, or

(ii) there exists $\beta > 4\pi$ such that the family $v_n = e^{u_n^2}$ is uniformly bounded in $L^\beta(\Omega)$, and thus

$$\int_\Omega e^{4\pi u_n^2} \to \int_\Omega e^{4\pi u^2} \qquad \text{as } n \to +\infty.$$

In particular, this is the case if u is different from 0.

Proof of Theorem 1.1. We follow [4]. We say that a sequence $u_n \subset H_0^1(B)$ is a *normalized concentrating sequence* if:

i) $\|u_n\|_{H_0^1} = 1$,

ii) $u_n \rightharpoonup 0$ weakly in $H_0^1(B)$,

iii) $\exists\, x_0 \in B$ such that $\forall \rho > 0$,

$$\int_{B \setminus B_\rho(x_0)} |\nabla u_n|^2 dx \to 0.$$

Let us suppose that u_n, $\|u_n\| = 1$, is a maximizing sequence for (1.3), that is,

$$\lim_{n\to+\infty} \int_B (e^{4\pi u_n^2} - 1)|x|^\alpha dx = S(\alpha, 4\pi).$$

Then by the concentration-compactness alternative of P.L. Lions, either u_n is a normalized concentrating (and maximizing) sequence, or $S(\alpha, 4\pi)$ is attained. To conclude the proof, we proceed by the following steps:

1) if u_n is any normalized concentrating sequence in $H_0^1(B)$, then

$$\lim_{n\to+\infty} \int_B \left(e^{4\pi u_n^2} - 1\right) |x|^\alpha dx \leq \frac{2}{\alpha+2}\pi e; \tag{3.2}$$

2) give an explicit function $\omega \in H_0^1(B)$ such that

$$\int_B \left(e^{4\pi \omega^2} - 1\right) |x|^\alpha dx > \frac{2}{\alpha+2}\pi e.$$

1) Upper bound. Using the notion of spherical symmetrization with respect to the measure $\mu_\alpha = \int_B |x|^\alpha$ introduced in Section 2, and Theorem 2.1 of Schulz-Vera de Serio, it suffices to show that

$$\overline{\lim}_{n\to+\infty} \int_{B_\alpha^*} \left(e^{4\pi |u_{\alpha,n}^*|^2} - 1\right) dx \leq \frac{2}{\alpha+2}\pi e,$$

where $u^*_{\alpha,n}$ is the rearranged sequence of u_n, with $\|u^*_{\alpha,n}\| \le \|u_n\| = 1$, and B^*_α is the ball centered in 0 such that $|B^*_\alpha| = \mu_\alpha(B)$, that is,

$$B^*_\alpha = B\Big(0, \sqrt{\frac{2}{\alpha+2}}\Big).$$

Let us set $z_n = \frac{u^*_{\alpha,n}}{\|u^*_{\alpha,n}\|}$; then

$$\int_{B^*_\alpha} \left(e^{4\pi|u^*_{\alpha,n}|^2} - 1\right) dx \le \int_{B^*_\alpha} \left(e^{4\pi z_n^2} - 1\right) dx,$$

so that is suffices to prove that for any radial normalized concentrating sequence in $B(0, \sqrt{\frac{2}{\alpha+2}})$ the upper bound (3.2) holds. First, we perform a change of variable to reduce the domain to the unit ball. Let $R = \sqrt{\frac{2}{\alpha+2}}\rho$, and $y_n(\rho) = z_n(\sqrt{\frac{2}{\alpha+2}}\rho)$; then

$$2\pi \int_0^{\sqrt{\frac{2}{\alpha+2}}} \left(e^{4\pi z_n^2} - 1\right) RdR = 2\pi \frac{2}{\alpha+2} \int_0^1 \left(e^{4\pi y_n^2} - 1\right) \rho d\rho,$$

and

$$1 = \int_{B^*_\alpha} |\nabla z^*_n|^2 dx = 2\pi \int_0^1 |y_n'|^2 \rho d\rho.$$

The proof now reads exactly as in [4] (proof of Theorem 4, step 1), so we can omit it. See also [2].

2) An explicit function. In this step, we exhibit an explicit function $\omega(x)$ such that

$$\int_B \left(e^{4\pi\omega^2} - 1\right) |x|^\alpha dx > \frac{2}{\alpha+2}\pi e;$$

since, by step 1), any normalized concentrating sequence (if exists) must satisfy

$$S(\alpha, 4\pi) = \lim_{n\to+\infty} \int_B (e^{4\pi u_n^2} - 1)|x|^\alpha dx \le \frac{2}{\alpha+2}\pi e,$$

we can conclude that $S(\alpha, 4\pi)$ is attained. From now on we assume that u is a generic radial function, and set

$$\varepsilon = \frac{2}{\alpha+2}. \tag{3.3}$$

As in [11], following an idea of Smets, Su and Willem, define the new function

$$v(\rho) = \frac{1}{\sqrt{\varepsilon}} u(\rho^\varepsilon); \tag{3.4}$$

then

$$\int_B |\nabla u|^2 dx = 2\pi \int_0^1 |u'|^2 r dr = 2\pi \int_0^1 |v'|^2 \rho d\rho, \tag{3.5}$$

and,

$$\int_B \left(e^{4\pi u^2} - 1\right) |x|^\alpha dx = 2\pi\varepsilon \int_0^1 \left(e^{4\pi\varepsilon v^2} - 1\right) \rho d\rho. \tag{3.6}$$

We can now perform the change of variable introduced by Moser [9], which transform the radial integral on $[0,1]$ into an integral on the half-line $[0,+\infty)$,

$$\rho = e^{-t/2} \qquad \text{and} \qquad w(t) = \sqrt{4\pi} v(\rho);$$

we obtain (recalling the definition (3.3))

$$\int_B \left(e^{4\pi u^2} - 1\right) |x|^\alpha dx = \pi \frac{2}{\alpha+2}\Big(\int_0^{+\infty} e^{\frac{2}{\alpha+2} w^2 - t} dt - 1\Big), \tag{3.7}$$

with

$$\int_B |\nabla u|^2 dx = \int_0^{+\infty} |w'(t)|^2 dt.$$

Following [2], take $w : [0,+\infty) \to \mathbb{R}$ to be

$$w(t) = \begin{cases} \frac{1}{2} t & \text{if } 0 \le t \le 2, \\ \sqrt{t-1} & \text{if } 2 \le t \le 1 + e^2, \\ e & \text{if } t \ge 1 + e^2. \end{cases}$$

Then by direct inspection

$$\int_0^{+\infty} |w'(t)|^2 dt = 1,$$

and

$$\int_0^{+\infty} e^{\frac{2}{\alpha+2} w^2 - t} dt = \int_0^{+\infty} e^{w^2 - t} dt + \mathrm{o}(1) = \frac{2}{e} \int_0^1 e^{s^2} ds + e + \mathrm{o}(1);$$

therefore, if ω is the radial function which corresponds to $w(t)$, by (3.7) we have

$$\begin{aligned} \int_B \left(e^{4\pi\omega^2} - 1\right) |x|^\alpha dx &= \pi \frac{2}{\alpha+2}\Big(e + \frac{2}{e}\int_0^1 e^{s^2} ds - 1 + \mathrm{o}(1)\Big) \\ &> \frac{2}{\alpha+2} \pi e \qquad \text{when } \alpha \to 0, \end{aligned}$$

since

$$\frac{2}{e} \int_0^1 e^{s^2} ds > 1,$$

as one can verify estimating the integral with lower Riemann sums, as in [2] (the value obtained is $\frac{2}{e}\int_0^1 e^{s^2}ds \approx \frac{2.723}{e} > 1$), or expanding the integrand in power series, as in [11] (here $\frac{2}{e}\int_0^1 e^{s^2}ds \approx \frac{2.906}{e} > 1$).

Remark 3.2. The notion of symmetrization with respect to the measure $\mu_\alpha = \int |x|^\alpha$ is a fundamental tool in the proof of Theorem 1.1, as remarked in the introduction, since it allows to reduce the variational problem to a one-dimensional problem, as in the unweighted case $\alpha = 0$. Furthermore, it gives a geometric interpretation of the change of variable (3.4), originally introduced by Smets, Su and Willem in [12], which allows to reduce the weighted integral $\int_B (e^{4\pi u^2}-1)|x|^\alpha dx$ to the unweighted integral $\varepsilon\int_B (e^{4\pi\varepsilon u^2}-1)dx$ if u is a radial function. Indeed, let us consider a rearranged function $u(r)$, then by (2.5) of the previous section (and using the notation (3.3) for simplicity)

$$u_\alpha^*(r) = u(r^\varepsilon \varepsilon^{-\frac{\varepsilon}{2}}),$$

so that

$$\int_B |\nabla u|^2 dx = 2\pi \int_0^1 |u'(s)|^2 s ds = \frac{2\pi}{\varepsilon}\int_0^{\sqrt{\varepsilon}} |u_\alpha^{*\prime}(r)|^2 r dr,$$

and

$$\begin{aligned} \int_B \left(e^{4\pi u^2}-1\right)|x|^\alpha dx &= 2\pi \int_0^1 \left(e^{4\pi u^2}-1\right) s^{\alpha+1} ds \\ &= 2\pi \int_0^{\sqrt{\varepsilon}} \left(e^{4\pi |u_\alpha^*|^2}-1\right) r dr. \end{aligned}$$

Now, set $r = \sqrt{\varepsilon}\rho$ and $v(\rho) = u_\alpha^*(\sqrt{\varepsilon}\rho)$; then

$$\frac{2\pi}{\varepsilon}\int_0^{\sqrt{\varepsilon}} |u_\alpha^{*\prime}(r)|^2 r dr = \frac{2\pi}{\varepsilon}\int_0^1 |v'(\rho)|^2 \rho d\rho, \tag{3.8}$$

and

$$2\pi \int_0^{\sqrt{\varepsilon}} \left(e^{4\pi|u_\alpha^*|^2}-1\right) r dr = 2\pi\varepsilon \int_0^1 \left(e^{4\pi v^2}-1\right)\rho d\rho. \tag{3.9}$$

Equalities (3.8) and (3.9) can be restated as

$$\begin{aligned} \int_B |\nabla u|^2 dx &= 2\pi \frac{1}{\varepsilon}\int_0^1 |v'|^2 \rho d\rho, \\ \int_B \left(e^{4\pi u^2}-1\right)|x|^\alpha dx &= 2\pi\varepsilon \int_0^1 \left(e^{4\pi v^2}-1\right)\rho d\rho, \end{aligned}$$

where

$$v(\rho) = u_\alpha^*(\sqrt{\varepsilon}\rho) = u(\rho^\varepsilon);$$

this is exactly the change of variable introduced by Smets, Su and Willem in [12], and differs from (3.4) by a dilation factor.

Remark 3.3. Note that we have proved step 2 testing with a radial function. It easy to show that if $\alpha \to +\infty$, a function $w(x)$ such that

$$\int_B \left(e^{4\pi w^2} - 1\right) |x|^\alpha dx > \frac{2}{\alpha+2}\pi e,$$

if exists, must be *non radial*. Indeed, let us define

$$Z(\varepsilon) = 2\pi \int_0^1 (e^{4\pi\varepsilon v^2} - 1) r dr;$$

then by (3.6), for any u radial function in $H_0^{1,\mathrm{rad}}(B)$,

$$\int_B \left(e^{4\pi u^2} - 1\right) |x|^\alpha dx = \varepsilon Z(\varepsilon);$$

but

$$\frac{dZ(\varepsilon)}{d\varepsilon} > \frac{Z(\varepsilon)}{\varepsilon},$$

since $te^t > e^t - 1$ for every $t > 0$ (note that the inequality is strict, and there is equality if and only if $t = 0$). Integrating the previous inequality over $[\varepsilon, 1]$ leads to $Z(\varepsilon) < \varepsilon Z(1)$, that is,

$$2\pi \int_0^1 (e^{4\pi v^2} - 1) r dr > \frac{1}{\varepsilon^2} \int_B (e^{4\pi u^2} - 1)|x|^\alpha dx.$$

Therefore, if there exists a radial function $w(x)$, with $\|w\| \leq 1$, such that

$$\int_B (e^{4\pi w^2} - 1)|x|^\alpha dx > \frac{2}{\alpha+2}\pi e = \varepsilon\pi e,$$

we have also

$$S(0, 4\pi) \geq \int_B (e^{4\pi w^2} - 1) dx > \frac{1}{\varepsilon^2} \int_B (e^{4\pi w^2} - 1)|x|^\alpha dx > \frac{1}{\varepsilon}\pi e;$$

letting $\varepsilon \to 0$, i.e., $\alpha \to +\infty$, we get a contradiction, since $S(0, 4\pi)$ is finite.

Remark 3.4. It remains an open problem whether the supremum $S(\alpha, 4\pi)$ is attained *for every* $\alpha > 0$.

4. Radial maximizers

When α is large enough, it is known that $S(\alpha,\gamma) > S^{\mathrm{rad}}(\alpha,\gamma)$ (see [11]), that is, the maximizers of $S(\alpha,\gamma)$, if exist, are non-radial functions. Here, we prove a counterpart in the case α small: if $0<\gamma\le 4\pi$ and α is small enough, $S(\alpha,\gamma) = S^{\mathrm{rad}}(\alpha,\gamma)$, that is, $S(\alpha,\gamma)$ is attained by a radial function.

Let us observe that for $\alpha = 0$ the result is trivial: by Schwarz symmetrization, we can reduce problem (1.1) to the radial case, and it is known that the supremum is attained (see [4] for $\gamma < 4\pi$ and [2] for $\gamma = 4\pi$). If $\alpha > 0$, our result will be a consequence of the implicit function theorem. Let us begin with the following lemma.

Lemma 4.1. *When $\alpha \to 0$,*

$$S(\alpha,\gamma) \longrightarrow S(0,\gamma), \qquad S^{rad}(\alpha,\gamma) \longrightarrow S(0,\gamma). \tag{4.1}$$

Proof. Let $\gamma \in (0,4\pi]$, and let $v \in H_0^1(\Omega)$, with $\|v\| = 1$. By the dominated convergence theorem,

$$\int_\Omega \Big(e^{\gamma v^2} - 1\Big)|x|^\alpha dx \longrightarrow \int_\Omega \Big(e^{\gamma v^2} - 1\Big) dx \quad \text{as} \quad \alpha \to 0,$$

so that

$$\lim_{\alpha\to 0} \int_\Omega \Big(e^{\gamma v^2} - 1\Big)|x|^\alpha dx \le S(0,\gamma),$$

and

$$\lim_{\alpha\to 0} S(\alpha,\gamma) \le S(0,\gamma).$$

On the other hand, let $u_{0,\gamma}$ be a maximizer for $S(0,\gamma)$: then

$$S(\alpha,\gamma) \ge \int_\Omega \Big(e^{\gamma u_{0,\gamma}^2} - 1\Big)|x|^\alpha dx \longrightarrow \int_\Omega \Big(e^{\gamma u_{0,\gamma}^2} - 1\Big) dx = S(0,\gamma).$$

The proof for the radial case follows the same argument. □

Let us now denote with $u_{\alpha,\gamma}$ and $u_{\alpha,\gamma}^{\mathrm{rad}}$, respectively, two sequences of positive maximizers for $S(\alpha,\gamma)$ and $S^{\mathrm{rad}}(\alpha,\gamma)$ (if $\gamma = 4\pi$, we suppose that $\alpha \le \alpha_*$). Observe that, by [16], the positive maximizer for $S^{\mathrm{rad}}(\alpha,\gamma)$ is unique, since it correspond to a ground state for the equation

$$\begin{cases} -\Delta u = \lambda u e^{\gamma \varepsilon u^2}, & \Omega \\ u \ge 0, & \Omega \\ u = 0 & \partial\Omega, \end{cases} \tag{4.2}$$

whereas we do not know the multiplicity of the non-radial maximizers. By definition, $\|u_{\alpha,\gamma}\| = \|u_{\alpha,\gamma}^{\mathrm{rad}}\| = 1$, so that, up to a subsequence, $u_{\alpha,\gamma} \rightharpoonup \bar{u}_{0,\gamma}$

and $u^{\rm rad}_{\alpha,\gamma} \rightharpoonup \bar u^{\rm rad}_{0,\gamma}$ as $\alpha \to 0$ (note that, at this point, we don't know if $\bar u_{0,\gamma}$ and $\bar u^{\rm rad}_{0,\gamma}$ are maximizers for $S(0,\gamma)$ or not!). We can now prove the following lemma:

Lemma 4.2. *The sequences $v_{\alpha,\gamma} = e^{u^2_{\alpha,\gamma}}$ and $v^{rad}_{\alpha,\gamma} = e^{(u^{rad}_{\alpha,\gamma})^2}$ are uniformly bounded in $L^\beta(\Omega)$ as $\alpha \to 0$, for some $\beta > 4\pi$.*

Proof. We argue by contradiction. Let us suppose that $v_{\alpha,\gamma}$ is not uniformly bounded in $L^\beta(\Omega)$ as $\alpha \to 0$, for any $\beta > 4\pi$, then by the concentration compactness alternative, Proposition 3.1, $u_{\alpha,\gamma} \rightharpoonup 0$ and $|\nabla u_{\alpha,\gamma}|^2 \to \delta_{x_0}$ weakly in measure. On the other hand,

$$S(\alpha,\gamma) = \int_\Omega \Big(e^{\gamma u^2_{\alpha,\gamma}} - 1\Big)|x|^\alpha dx \le \int_\Omega \Big(e^{\gamma u^2_{\alpha,\gamma}} - 1\Big)dx \le S(0,\gamma),$$

so that, by Lemma 4.1,

$$\int_\Omega \Big(e^{\gamma u^2_{\alpha,\gamma}} - 1\Big)dx \to S(0,\gamma) \quad \text{as } \alpha \to 0.$$

Therefore, $u_{\alpha,\gamma}$ is a maximizing concentrating sequence for $S(0,\gamma)$; furthermore, if we denote with $u^\sharp_{\alpha,\gamma}$ the Schwarz rearrangement of $u_{\alpha,\gamma}$, then

$$\begin{aligned}&\int_\Omega \Big(e^{\gamma (u^\sharp_{\alpha,\gamma})^2} - 1\Big)dx = \int_\Omega \Big(e^{\gamma u^2_{\alpha,\gamma}} - 1\Big)dx \to S(0,\gamma)\\ &\|\nabla u^\sharp_{\alpha,\gamma}\| \le \|\nabla u_{\alpha,\gamma}\| = 1.\end{aligned}$$

Hence, $u^\sharp_{\alpha,\gamma}/\|\nabla u^\sharp_{\alpha,\gamma}\|$ is a maximizing concentrating sequence for $S(0,\gamma)$ (otherwise $e^{(u^\sharp_{\alpha,\gamma}/\|\nabla u^\sharp_{\alpha,\gamma}\|)^2}$ would be uniformly bounded in $L^\beta(\Omega)$ for some $\beta > 4\pi$, and $e^{u^2_{\alpha,\gamma}}$ as well). If $\gamma < 4\pi$,

$$\int \Big(e^{\gamma (u^\sharp_{\alpha,\gamma}/\|\nabla u^\sharp_{\alpha,\gamma}\|)^2} - 1\Big) \to 0,$$

by [4], Theorem 2, that is a contradiction. If $\gamma = 4\pi$, by estimates obtained in [2] (se also [4], Theorem 4),

$$\lim_{\alpha\to 0} \int_\Omega \Big(e^{4\pi (u^\sharp_{\alpha,\gamma}/\|\nabla u^\sharp_{\alpha,\gamma}\|)^2} - 1\Big) \le e\cdot|\Omega| = e\cdot\pi,$$

whereas it is well known that $S(0,4\pi) > e\cdot|\Omega|$. In the same way, one can prove the thesis for the sequence of radial maximizer $u^{\rm rad}_{\alpha,\gamma}$. □

Note that, since $u_{\alpha,\gamma}$ is a positive maximizer for $S(\alpha,\gamma)$, it is a solution of the elliptic problem

$$\begin{cases} -\Delta u = \lambda u e^{\gamma u^2}|x|^\alpha, & \Omega \\ u \geq 0, & \Omega \\ u = 0 & \partial\Omega, \end{cases} \tag{4.3}$$

where

$$\lambda = \lambda_{\alpha,\gamma} = \Big(\int_\Omega u_{\alpha,\gamma}^2 e^{\gamma u_{\alpha,\gamma}^2}|x|^\alpha dx\Big)^{-1}. \tag{4.4}$$

Observe that $\lambda_{\alpha,\gamma}$ *depends on* the sequence of non-radial maximizers $u_{\alpha,\gamma}$ we have chosen (we do not know if the maximizer is unique!), whereas $\lambda_{0,\gamma}$ depends only on γ, since the positive radial maximizer, $u_{\alpha,\gamma}^{\rm rad}$, corresponds to the unique solution to problem (4.2).

Let us now apply the implicit function theorem, in the spirit of [12], Proposition 5.1. Let $(-\Delta)^{-1}$ denote the inverse of the Laplacian operator with Dirichlet boundary conditions. Let $\gamma \in (0, 4\pi]$ be fixed; we define the operator

$$\begin{aligned} P_\gamma : [0,+\infty) \times H_0^1(\Omega) \times \mathbb{R} &\longrightarrow H_0^1(\Omega) \times \mathbb{R} \\ (\alpha, u, \lambda) &\longmapsto \Big((-\Delta)^{-1}\Big(\lambda G_\gamma\big(\tfrac{u}{\|u\|}\big)|x|^\alpha\Big) - u, \|u\|^2 - 1\Big), \end{aligned}$$

where $G_\gamma(t) = \int_0^t e^{\gamma s^2} ds$. Notice that P is well defined: indeed, $\int e^{\gamma(u/\|u\|)^2} \leq C$ only if $\gamma \leq 4\pi$, but $\int e^{\gamma(u/\|u\|)^2}$ is finite for any $\gamma > 0$ (even if not uniformly bounded!). Therefore, $G_\gamma(\frac{u}{\|u\|})|x|^\alpha \in L^p(\Omega)$ for any $p > 0$, so that $(-\Delta)^{-1}\big(\lambda \int G_\gamma(\frac{u}{\|u\|})|x|^\alpha\big) \in H_0^1(\Omega)$.

It is easy to verify that the function P_γ is continuous on $[0,+\infty)\times H_0^1(\Omega)\times \mathbb{R}$ for any $\gamma < 4\pi$. When $\gamma = 4\pi$ we don't know if $P_{4\pi}$ is continuous over *all* the domain; nevertheless, for our purposes it suffices to verify that $P_{4\pi}$ is continuous on $U(0, u_{0,4\pi}, \lambda_{0,4\pi})$ (where $\lambda_{0,4\pi}$ is defined by (4.4)), that is, it suffices to show that

$$\begin{aligned} B(u_{0,4\pi},\delta) \subset H_0^1(\Omega) \quad &\longrightarrow \quad H_0^1(\Omega) \\ u \quad &\longmapsto \quad (-\Delta)^{-1}\Big(\lambda G_{4\pi}\big(\tfrac{u}{\|u\|}\big)|x|^\alpha\Big), \end{aligned}$$

is continuous for some $\delta > 0$. This is a consequence of the concentration compactness principle of Lions. Indeed, let us choose δ such that $B(u_{0,4\pi},\delta) \not\supseteq B(0,\varepsilon)$, for some $\varepsilon > 0$. Let u_n, u such that $u_n, u \subset B(u_{0,4\pi},\delta)$ and $u_n \to u$. Then $\|u\| \gneqq 0$ and, by Proposition 3.1, $\exp(u_n/\|u_n\|)$ is uniformly bounded in $L^\beta(\Omega)$, for some $\beta > 4\pi$. This implies that $L_{u_n} \longrightarrow L_u$

in H^{-1}, where L_u is defined by

$$L_u v = \int v G_{4\pi}\left(\tfrac{u}{\|u\|}\right)|x|^\alpha.$$

Thanks to the continuity of $(-\Delta)^{-1} : H^{-1} \to H_0^1$, we can conclude.

We are now ready to prove the following lemma.

Lemma 4.3. *Let $\gamma \in (0, 4\pi]$ be fixed. Then there exists $\varepsilon > 0$ and continuous function*

$$\Lambda_\gamma : [0, \varepsilon) \to (\lambda_{0,\gamma} - \varepsilon, \lambda_{0,\gamma} + \varepsilon), \quad U_\gamma : [0, \varepsilon) \to B(u_{0,\gamma}, \varepsilon) \tag{4.5}$$

such that, in $[0, \varepsilon) \times (\lambda_{0,\gamma} - \varepsilon, \lambda_{0,\gamma} + \varepsilon) \times B(u_{0,\gamma}, \varepsilon)$,

$$P_\gamma(\alpha, u, \lambda) = 0 \Longleftrightarrow u = U_\gamma(\alpha), \lambda = \Lambda_\gamma(\alpha).$$

Proof. In order to apply the Implicit Function Theorem, we have only to prove that the partial derivative of P_γ with respect to (u, λ) at $(0, u_{0,\gamma}, \lambda_{0,\gamma})$ is an homeomorphism on $H_0^1(\Omega) \times \mathbb{R}$. Clearly (recall that $\|u_{0,\gamma}\| = 1$),

$$\begin{aligned}
& \partial_{(u,\lambda)} P_\gamma(0, u_{0,\gamma}, \lambda_{0,\gamma})[t, v] \\
= \; & \Big((-\Delta)^{-1} \Big(\lambda_{0,\gamma} v e^{\gamma u_{0,\gamma}^2} - 2\lambda_{0,\gamma} \langle u_{0,\gamma}, v \rangle u_{0,\gamma} e^{\gamma u_{0,\gamma}^2} \Big) \\
& -v + t(-\Delta)^{-1} \Big(G_\gamma(u_{0,\gamma}) \Big), 2\langle u_{0,\gamma}, v \rangle \Big),
\end{aligned}$$

so that

$$\begin{aligned}
& \partial_{(u,\lambda)} P_\gamma(0, u_{0,\gamma}, \lambda_{0,\gamma})[t, v] = 0 \\
& \Longrightarrow \begin{cases} \langle u_{0,\gamma}, v \rangle = 0 \\ \\ (-\Delta)^{-1} \Big(\lambda_{0,\gamma} v e^{\gamma u_{0,\gamma}^2} \Big) - v + t(-\Delta)^{-1} \Big(G_\gamma(u_{0,\gamma}) \Big) = 0. \end{cases}
\end{aligned}$$

Hence,

$$\begin{aligned}
0 &= \langle (-\Delta)^{-1}(\lambda_{0,\gamma} v e^{\gamma u_{0,\gamma}^2}) - v + t(-\Delta)^{-1}(G_\gamma(u_{0,\gamma})), u_{0,\gamma} \rangle \\
&= \langle (-\Delta)^{-1}(\lambda_{0,\gamma} v e^{\gamma u_{0,\gamma}^2}), u_{0,\gamma} \rangle + t \langle (-\Delta)^{-1}(G_\gamma(u_{0,\gamma})), u_{0,\gamma} \rangle \\
&= t \langle (-\Delta)^{-1}(G_\gamma(u_{0,\gamma})), u_{0,\gamma} \rangle \Longrightarrow t = 0,
\end{aligned}$$

since

$$\langle (-\Delta)^{-1}(\lambda_{0,\gamma} v e^{\gamma u_{0,\gamma}^2}), u_{0,\gamma} \rangle = \int \lambda_{0,\gamma} u_{0,\gamma} v e^{\gamma u_{0,\gamma}^2} = \int -\Delta u_{0,\gamma} v = 0.$$

Set $m_\gamma(x) = e^{\gamma u_{0,\gamma}^2} > 0$, and consider the linear elliptic problem

$$\begin{cases} -\Delta v = \lambda m_\gamma v \\ \quad\; v = 0. \end{cases} \tag{4.6}$$

It is well known the principle eigenvalue for problem (4.6) is simple, and it is the only eigenvalue whose associated eigenspace contains a positive eigenfunction (see for example [6]). Since $u_{0,\gamma} \geq 0$ is a solution of problem (4.6) with $\lambda = \lambda_{0,\gamma}$, $\lambda_{0,\gamma}$ is the principal eigenvalue of (4.6), and it is simple. Hence, the kernel of the operator $(-\Delta)^{-1}(\lambda_{0,\gamma} m_\gamma I) - I$ is proportional to $u_{0,\gamma}$, so that

$$(-\Delta)^{-1}\big(\lambda_{0,\gamma} v e^{\gamma u_{0,\gamma}^2}\big) - v = 0 \Longrightarrow v = \mu u_{0,\gamma}.$$

But, $v \perp u_{0,\gamma}$, and $v = 0$. Then

$$\partial_{(u,\lambda)} P_\gamma(0, u_{0,\gamma}, \lambda_{0,\gamma})[v,t] = 0 \Longleftrightarrow (v,t) = 0.$$

Let us now prove that $\partial_{(u,\lambda)} P_\gamma(0, u_{0,\gamma}, \lambda_{0,\gamma})$ is surjective. Let $(w,s) \in H_0^1(\Omega) \times \mathbb{R}$, and let $\tilde{u}_{0,\gamma} := (-\Delta)^{-1}(G_\gamma(u_{0,\gamma})) \in H_0^1(\Omega)$. Then

$$\begin{aligned} w &= w^\perp + \langle w, u_{0,\gamma}\rangle u_{0,\gamma} \\ \tilde{u}_{0,\gamma} &= \tilde{u}_{0,\gamma}^\perp + \langle \tilde{u}_{0,\gamma}, u_{0,\gamma}\rangle u_{0,\gamma}, \end{aligned}$$

where $w^\perp, \tilde{u}_{0,\gamma}^\perp \perp u_{0,\gamma}$. Note that $\langle \tilde{u}_{0,\gamma}, u_{0,\gamma}\rangle \gneqq 0$, so that

$$t = \frac{\langle w, u_{0,\gamma}\rangle + s}{\langle \tilde{u}_{0,\gamma}, u_{0,\gamma}\rangle},$$

is well defined. Now, set $w_1 = w^\perp - t\tilde{u}_{0,\gamma}^\perp$. By definition, $w_1 \perp u_{0,\gamma}$; since the operator $(-\Delta)^{-1}(\lambda_{0,\gamma} m_\gamma I) - I$ is self-adjoint and its kernel is proportional to $u_{0,\gamma}$, then there exists a $v_1 \in H_0^1(\Omega)$ such that

$$w_1 = (-\Delta)^{-1}(\lambda_{0,\gamma} m_\gamma v_1) - v_1.$$

Obviously, we can choose v_1 such that $v_1 \perp u_{0,\gamma}$, too. Let us define $v := v_1 + \frac{s}{2} u_{0,\gamma}$. Then

$$\begin{aligned} \partial_{(u,\lambda)} P_\gamma(0, u_{0,\gamma}, \lambda_{0,\gamma})[v,t] &= \Big((-\Delta)^{-1}\Big(\lambda_{0,\gamma} v e^{\gamma u_{0,\gamma}^2} - \lambda_{0,\gamma} s u_{0,\gamma} e^{\gamma u_{0,\gamma}^2}\Big) \\ &\quad -v + t(-\Delta)^{-1}\Big(G_\gamma(u_{0,\gamma})\Big), s\Big) \\ &= \Big((-\Delta)^{-1}\Big(\lambda_{0,\gamma} v e^{\gamma u_{0,\gamma}^2}\Big) - v - s u_{0,\gamma} + t\tilde{u}_{0,\gamma}, s\Big) \\ &= \Big(w_1 + t\tilde{u}_{0,\gamma}^\perp + (t\langle \tilde{u}_{0,\gamma}, u_{0,\gamma}\rangle - s) u_{0,\gamma}, s\Big) \end{aligned}$$

$$= \left(w^{\perp} + \langle w, u_{0,\gamma}\rangle u_{0,\gamma}, s\right) = (w, s).$$

Being continuous and bijective, the partial derivative $\partial_{(u,\lambda)} P_\gamma(0, u_{0,\gamma}, \lambda_{0,\gamma})$ is a homeomorphism. This ends the proof. $\square$

We are now ready to prove Theorem 1.2

Proof of Theorem 1.2. Let $u_{\alpha,\gamma}$ and $u^{\mathrm{rad}}_{\alpha,\gamma}$ be, respectively, two sequences of positive maximizers for $S(\alpha,\gamma)$ and $S^{\mathrm{rad}}(\alpha,\gamma)$. By lemma 4.2, lemma 4.1 and the concentration compactness of Lions, $u_{\alpha,\gamma} \rightharpoonup \bar{u}_{0,\gamma}$, $u^{\mathrm{rad}}_{\alpha,\gamma} \rightharpoonup \bar{u}^{\mathrm{rad}}_{0,\gamma}$ as $\alpha \to 0$ and

$$\begin{aligned} \int_\Omega \left(e^{u^2_{\alpha,\gamma}} - 1)|x|^\alpha \right. &\longrightarrow \int_\Omega \left(e^{\bar{u}^2_{0,\gamma}} - 1\right) = S(0,\gamma) \\ \int_\Omega \left(e^{(u^{\mathrm{rad}}_{\alpha,\gamma})^2} - 1)|x|^\alpha \right. &\longrightarrow \int_\Omega \left(e^{(\bar{u}^{\mathrm{rad}}_{0,\gamma})^2} - 1\right) = S(0,\gamma). \end{aligned}$$

Hence, $\|\bar{u}_{0,\gamma}\| = \|\bar{u}^{\mathrm{rad}}_{0,\gamma}\| = 1$, so that $u_{\alpha,\gamma} \to \bar{u}_{0,\gamma}$, $u^{\mathrm{rad}}_{\alpha,\gamma} \to \bar{u}^{\mathrm{rad}}_{0,\gamma}$ strongly in $H^1_0(\Omega)$; furthermore, $\bar{u}_{0,\gamma} = \bar{u}^{\mathrm{rad}}_{0,\gamma} = u_{0,\gamma}$ since the positive maximizer for $S(0,\gamma)$ is unique, as previously remarked. Furthermore, it is easy to verify, via concentration-compactness, that

$$\begin{aligned} \lambda^{-1}_{\alpha,\gamma} = \int_\Omega u^2_{\alpha,\gamma} e^{u^2_{\alpha,\gamma}} |x|^\alpha &\longrightarrow \int_\Omega u^2_{0,\gamma} e^{u^2_{0,\gamma}} = \lambda^{-1}_{0,\gamma} \\ \left(\lambda^{\mathrm{rad}}_{\alpha,\gamma}\right)^{-1} = \int_\Omega (u^{\mathrm{rad}}_{\alpha,\gamma})^2 e^{(u^{\mathrm{rad}}_{\alpha,\gamma})^2} |x|^\alpha &\longrightarrow \int_\Omega u^2_{0,\gamma} e^{u^2_{0,\gamma}} = \lambda^{-1}_{0,\gamma}, \end{aligned}$$

as $\alpha \to 0$. By the preceding lemma 4.3, this implies that $u_{\alpha,\gamma} = u^{\mathrm{rad}}_{\alpha,\gamma}$ for α small, that is our thesis.

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