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A MODEL FOR INTERFACES AND ITS MESOSCOPIC LIMIT

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We study a system of N layers with a Kac horizontal interaction of parameter $\gamma > 0$ and a Kac vertical interaction of parameter $\gamma^{1/2}$. We shall prove that the limit free energy functional is the rate function of the large deviations of the Gibbs measure (of a canonical constrained magnetization). The limit free energy functional is achieved as a Γ-limit for $\gamma \to 0$ for magnetizations with fixed average. Among all such magnetizations there exists a quasiconstant magnetization that minimizes the energy.

1. Introduction

Equilibrium and dynamics of interfaces is a very well studied issue both in physics and mathematics. In several instances to simplify the problem it is supposed that the interface is a graph, an assumption which is not at all unrealistic if the interface is studied locally. In the SOS models of statistical mechanics the interface is a graph over a lattice $\mathbb{Z}^d$; namely for each site $i \in \mathbb{Z}^d$ we draw a vertical line and the position of the interface on the line (its height) is represented by a real-valued spin S_i. One then introduces a Hamiltonian which describes the interactions among the spins so that the equilibrium properties of the interface are derived from the Gibbs properties of the Hamiltonian. The difficulty in this approach arises from the fact that the Hamiltonian is massless, which corresponds to the fact that vertical translations of the interface do not cost energy. The theory of DLR states is then quite more involved than in the classical Ising model; a breakthrough was achieved in [Funaki and Spohn 1997], followed by many other papers.

In this paper we take a step back towards microscopic scalar; namely we suppose that on each horizontal line there is an Ising system so that instead of a real-valued spin S_i we have a configuration $\sigma(x, i)$, $x \in \mathbb{Z}$, of ± 1-valued spins. We actually consider a finite system with $i = 1, \dots, N$ and $x \in [0, L] \cap \mathbb{Z}$, $L = \gamma^{-1}\ell$, $\gamma > 0$, $\ell > 0$ (L an integer). To simulate a phase transition the spins on each horizontal line interact via a Kac potential $J_\gamma(x, y)$ (the same on each line), whose strength is 1 and whose range is γ^{-1} (see Section 2 for a precise definition). The spins

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between nearest neighbor horizontal lines (say (x, i) and $(y, i + 1)$) interact via the Kac potential $\lambda J_{\gamma^{1/2}}(x, y)$, $\lambda > 0$; that is, the vertical interaction is much more local than the horizontal one.

We study the mesoscopic limit $\gamma \to 0$. The mesoscopic state of the system is a collection $m \equiv \{m(r, i) : r \in [0, \ell], i = 1, \dots, N\}$ of measurable functions with values in $[-1, 1]$. Its statistical properties are then described by a free energy functional $F(m)$. According to the Gibbs theory such a functional is the limit as $\gamma \to 0$ of $-1/\beta$ times the log of a constrained partition function where the spin configurations are required to be "close" to the mesoscopic state m (this involves a coarse grain procedure which is specified in Section 2). This is not as in the classical Lebowitz–Penrose [1966; Penrose and Lebowitz 1971] procedure because there are two scales, γ^{-1} for the horizontal interaction and $\gamma^{-1/2}$ for the vertical one. Thus, there could be oscillations on the scale $\gamma^{-1/2}$ which do not appear in m because the latter is defined by averages over $\approx \gamma^{-1}$ but which could affect the free energy of m. These oscillations actually do not occur if λ is small; indeed by Theorem 4.1 the optimal profile is quasiconstant on the scale $\gamma^{-\alpha}$ with $\alpha \in (0, 1)$. However, if λ is large enough we can provide an example where such a phenomena occurs.

The paper is organized as follows. In Section 2 we introduce the microscopic and mesoscopic models and enunciate the main results. In Section 3 we introduce the coarse graining procedure used to prove the Lebowitz–Penrose limit. In Section 4 we prove a key result, that is, Theorem 4.1, in which we provide a technique to minimize the free energy. This theorem is needed to prove the main results in Section 2. In Section 5 we prove the Lebowitz–Penrose limit for our model. In Section 6 we prove the Γ-limit result. The proofs of Theorem 2.4 and Proposition 3.1 are deferred to Appendix A. In Appendix B, finally, we illustrate the case in which Theorem 2.3 fails for the parameter λ large enough.

Similar model have been studied in [Cassandro et al. 2016; Fontes et al. 2014; 2015]. A numerical investigation of the mesoscopic limit for lattice gas model was also recently tackled in [Colangeli et al. 2016; 2017].

This work is the first step of a research program pointed towards the characterization of the surface tension associated to free energy in the thermodynamic limit.

2. Model and main results

We consider an Ising spin system in a rectangle $T_{L,N} = \{(x, i) \in \mathbb{Z}^2 : x \in [0, L-1], i \in [1, N]\}$, $L = \gamma^{-1}\ell$, with $\gamma^{-1} \in \{2^n : n \in \mathbb{N}\}$ and $\ell \in \{2^k : k \in \mathbb{Z}\}$. We will eventually take the limit $\gamma \to 0$ keeping ℓ and N fixed. We denote by σ a spin configuration $\sigma = \{\sigma(x, i) \in \{-1, 1\} : (x, i) \in T_{L,N}\} \in \{-1, 1\}^{T_{L,N}}$, and since we will consider periodic boundary conditions we extend periodically σ to a configuration on $\mathbb{Z}^2$ (denoted by the same symbol) by setting $\sigma(x,i)=\sigma(y,j)$ if $(x,i)\sim(y,j)$ where

$$(x,i)\sim(y,j) \quad \text{if } y=x+kL \text{ and } j=i+k'N,\ k,k'\in\mathbb{Z}. \tag{2-1}$$

The interaction among spins is given by a highly anisotropic Kac potential which will be defined in terms of a function $J(r)$, $r\in\mathbb{R}$: we suppose that $J(r)$ is a nonnegative C^2 function with $\int J(r)\,dr=1$ supported by $|r|\le 1$. We then define for any x, y in $\mathbb{R}$

$$J_{\gamma^{1/2}}(x,y):=\gamma^{1/2}J(\gamma^{1/2}|x-y|), \qquad J_\gamma(x,y):=\gamma J(\gamma|x-y|). \tag{2-2}$$

The Hamiltonian of the system (with periodic boundary conditions) is then defined as

$$\begin{aligned} H_{\gamma,\lambda}(\sigma)=\sum_{i=1}^{N}\Bigg[-\frac{1}{2}\sum_{x,y\in[0,L-1]\cap\mathbb{Z}} &\Big\{\mathbf{1}_{\{x\neq y\}}J_\gamma(x,y)\sigma(x,i)\sigma(y,i)\\ &\quad -\lambda J_{\gamma^{1/2}}(x,y)\sigma(x,i)(\sigma(y,i-1)+\sigma(y,i+1))\Big\}\\ -\sum_{\substack{x\in[0,L-1]\cap\mathbb{Z}\\ y\notin[0,L-1]\cap\mathbb{Z}}} &\Big\{J_\gamma(x,y)\sigma(x,i)\sigma(y,i)\\ &\quad -\lambda J_{\gamma^{1/2}}(x,y)(\sigma(y,i-1)+\sigma(y,i+1))\Big\}\Bigg]. \end{aligned} \tag{2-3}$$

Thus, the range of the vertical interaction is much shorter than the range of the horizontal one.

We denote by $\mu_{\beta,\gamma,\lambda}$ the Gibbs measure at inverse temperature β:

$$\mu_{\beta,\gamma,\lambda}(\sigma)=\frac{e^{-\beta H_{\gamma,\lambda}(\sigma)}}{Z_{\beta,\gamma,\lambda}}$$

with

$$Z_{\beta,\gamma,\lambda}=\sum_{\sigma}e^{-\beta H_{\gamma,\lambda}(\sigma)},$$

being interested in the mesoscopic limit $\gamma\to 0$. The aim is to compute the limiting free energy and the probability of mesoscopic states.

A mesoscopic state is a measurable function m on $T_{\ell,N}=[0,\ell]\times\{1,\dots,N\}$ with values in $[-1,1]$. We extend m periodically by setting $m(r,i)=m(r',j)$ if $(r,i)\sim(r',j)$, which means $r'=r+k\ell$ and $j=i+k'N$, $k,k'\in\mathbb{Z}$. The correspondence between spin configurations σ and mesoscopic states m is via coarse graining, namely by comparing averages. The "microscopic length" used for averaging is $\gamma^{-\alpha}$, $\alpha\in(0,1)$, and to avoid taking integer parts we suppose α a rational number. We tacitly suppose that γ is small enough so that γ^{-1} and therefore also L are integer multiples of $\gamma^{-\alpha}$.

Definition 2.1 (partition and empirical averages). Let α and γ as above. We define for any $k \in \mathbb{Z}$

$$C_{k,i}^{(\alpha)} := \{(x, i) \in \mathbb{R} \times \mathbb{Z} : k\gamma^{-\alpha} \leq x < (k+1)\gamma^{-\alpha}\}.$$

The collection $\mathscr{C}^{(\alpha)}$ of all $C_{k,i}^{(\alpha)}$ defines a partition of $\mathbb{R} \times \mathbb{Z}$. Moreover, $\mathscr{C}^{(\alpha)} \cap \mathbb{Z}^2$ paves exactly $T_{L,N}$: namely any $C_{k,i}^{(\alpha)} \cap \mathbb{Z}^2$ is either contained in $T_{L,N}$ or in its complement.

Given a spin configuration σ we then define

$$\sigma^{(\alpha)}(x, i) := \gamma^{\alpha} \sum_{y \in C_{k,i}^{(\alpha)} \cap \mathbb{Z}^2} \sigma(y, i), \quad \text{where } k \text{ is such that } (x, i) \in C_{k,i}^{(\alpha)} \tag{2-4}$$

and $\sigma^{(\alpha)}$ is a function with values in $M^{(\alpha)}$ where

$$M^{(\alpha)} := \left\{-1, -1 + \frac{2}{\gamma^{-\alpha}}, \dots, 1 - \frac{2}{\gamma^{-\alpha}}, 1\right\}. \tag{2-5}$$

Analogously, given a mesoscopic state $m \in L^{\infty}(T_{\ell,N}; [-1, 1])$ we set

$$m^{(\alpha)}(x, i) := \gamma^{\alpha} \int_{k\gamma^{-\alpha}}^{(k+1)\gamma^{-\alpha}} m(\gamma r, i)\, dr \tag{2-6}$$

where k is such that $(x, i) \in C_{k,i}^{(\alpha)}$ and $m^{(\alpha)}$ is a function with values in $[-1, 1]$.

We next specify in which sense a spin configuration σ "recognizes" a mesoscopic state m and use this notion to define the free energy and the probability associated to a mesoscopic state.

Definition 2.2. σ "recognizes" m, and we write $\sigma \approx^{\alpha} m$ if

$$|\sigma^{(\alpha)}(x, i) - m^{(\alpha)}(x, i)| \leq 2\gamma^{\alpha} \quad \text{for all } (x, i) \in T_{L,N} \tag{2-7}$$

(recall that, by flipping a spin, $\sigma^{(\alpha)}(x, i)$ changes by $2\gamma^{\alpha}$). We then define the finite volume free energy of the mesoscopic state m as

$$F_{\beta,\gamma,\lambda}^{(\alpha)}(m) := -\frac{1}{\beta\gamma^{-1}} \log Z_{\beta,\gamma,\lambda}^{(\alpha)}(m), \tag{2-8}$$

where

$$Z_{\beta,\gamma,\lambda}^{(\alpha)}(m) := Z_{\beta,\gamma,\lambda}(\{\sigma \approx^{\alpha} m\}) = \sum_{\sigma : \sigma \approx^{\alpha} m} e^{-\beta H_{\gamma,\lambda}(\sigma)}.$$

Analogously we define the Gibbs probability of the mesoscopic state m as

$$\mu_{\beta,\lambda,\gamma}[\sigma \approx^{\alpha} m] = \frac{Z_{\beta,\gamma,\lambda}^{(\alpha)}(m)}{Z_{\beta,\gamma,\lambda}}.$$

The main result in this paper is this:

Theorem 2.3. *For any* $\alpha \in (0, 1)$, *any* $\lambda \in (0, 1/(8\beta))$, *and any mesoscopic state* $m \in L^\infty(T_{\ell,N}, [-1, 1])$,

$$\lim_{\gamma\to 0} F^{(\alpha)}_{\beta,\gamma,\lambda}(m) = F_{\beta,\lambda}(m) \tag{2-9}$$

where

$$\begin{aligned}F_{\beta,\lambda}(m) = -\frac{1}{2}\sum_{i=1}^{N}\int_0^\ell\int_0^\ell J(r,r')m(r,i)m(r',i)\,dr\,dr' \\ -\frac{\lambda}{2}\sum_{i=1}^{N}\int_0^\ell m(r,i)(m(r,i+1)+m(r,i-1))\,dr \\ -\frac{1}{\beta}\sum_{i=1}^{N}\int_0^\ell I(m(r,i))\,dr\end{aligned} \tag{2-10}$$

and

$$I(m) = -\frac{1+m}{2}\log\frac{1+m}{2} - \frac{1-m}{2}\log\frac{1-m}{2}. \tag{2-11}$$

The following two theorems are essentially a corollary of Theorem 2.3. The first one is about free energy.

Theorem 2.4. *Let* $0 < \lambda < 1/(8\beta)$ *and* $\alpha \in (0, 1)$. *Then*

$$-\lim_{\gamma\to 0}\frac{1}{\beta\gamma^{-1}}\log Z_{\beta,\gamma,\lambda} = \inf_{m\in L^\infty(T_{\ell,N};[-1,1])} F_{\beta,\lambda}(m). \tag{2-12}$$

Moreover, *if* $\beta(1+2\lambda) > 1$, *then* (*recalling* (2-11) *for notation*)

$$\inf_m F_{\beta,\lambda}(m) = N\ell\left(-\frac{b}{2}m^2_{b\beta} - \frac{I(m_{b\beta})}{\beta}\right), \quad b = 1+2\lambda, \tag{2-13}$$

where $m_{b\beta}$ *is the positive solution of the equation*

$$m_{b\beta} = \tanh\{\beta\lambda m_{b\beta}\}. \tag{2-14}$$

If instead $\beta(1+2\lambda) \le 1$, *then*

$$\inf_m F_{\beta,\lambda}(m) = \frac{N\ell}{\beta}\log(\tfrac{1}{2}). \tag{2-15}$$

The next theorem is about large deviations; on the general issue see for instance [Ellis 2006].

Theorem 2.5. *Let* $0 < \lambda < 1/(8\beta)$, $\alpha \in (0, 1)$, *and* $m \in L^\infty(T_{\ell,N}; [-1, 1])$ *be a mesoscopic state*; *then*

$$\lim_{\gamma\to 0}\gamma\log\mu_{\beta,\lambda,\gamma}[\sigma \approx^{(\alpha)} m] = -(F_{\beta,\lambda}(m) - \inf_{m'} F_{\beta,\lambda}(m')). \tag{2-16}$$

The theorems are proved in the next sections; here we make some remarks on Theorem 2.3. We note in particular that the limit free energy of a mesoscopic state is independent of the coarse graining parameter α, a fact to some extent unexpected.

The point is that the partition function $Z^{(\alpha)}_{\beta,\gamma,\lambda}(m)$ is clearly an increasing function of α because the constraint $\sigma \approx^{\alpha} m$ is weakened when increasing α. In particular the result contained in Theorem 2.3 shows that this effect is negligible in the limit $\gamma \to 0$. The basic idea in the proofs goes back to Lebowitz and Penrose, and it is based on a coarse graining with grain lengths which must be large with respect to the lattice spacing but small with respect to the range of the interaction. Following Lebowitz and Penrose we use a coarse graining with grain length $\gamma^{-\alpha'}$ with $\alpha' < \frac{1}{2}$ and $\gamma^{-\alpha'} \le \gamma^{-\alpha}$. We then obtain an estimate for the logarithm of the partition function characterized to the leading orders (as $\gamma \to 0$) by a nonrescaled functional

$$\begin{aligned}\bar{F}_{\beta,\gamma,\lambda}(\bar{m}) &= -\frac{1}{2}\sum_{i=1}^{N}\int_0^{\gamma^{-1}\ell}\int_0^{\gamma^{-1}\ell} J_\gamma(r,r')\bar{m}(r,i)\bar{m}(r',i)\,dr\,dr' \\ &\quad -\frac{\lambda}{2}\sum_{i=1}^{N}\int_0^{\gamma^{-1}\ell}\int_0^{\gamma^{-1}\ell} J_{\gamma^{1/2}}(r,r')\bar{m}(r,i)(\bar{m}(r',i-1)+\bar{m}(r',i+1))\,dr\,dr' \\ &\quad -\frac{1}{\beta}\sum_{i=1}^{N}\int_0^{\gamma^{-1}\ell} I(\bar{m}(r,i))\,dr, \end{aligned} \tag{2-17}$$

where $\bar{m}$ is constant on the scale $\gamma^{-\alpha'}$ used in the coarse graining.

To simplify the argument, let us assume that the mesoscopic profile $m(r,i) = 0$ for all r and i. If the constraint $\sigma \approx^{(\alpha)} m$ with $\alpha < \frac{1}{2}$, then by letting $\alpha' = \alpha$ (so that $\bar{m} \equiv 0$) the functional $\bar{F}$ becomes F after rescaling.

If instead $\alpha > \frac{1}{2}$, we cannot take $\alpha' = \alpha$ and there may be vertical energy gains via suitable oscillations of the magnetization within the constraint $\sigma \approx^{(\alpha)} m$. This is not just a theoretical possibility as it may indeed occur when λ is large. Let $\beta\lambda > 1$; then

$$\bar{F}_{\beta,\gamma,\lambda}(m \equiv 0) = -N\ell\frac{I(0)}{\beta}.$$

Fix $\bar{m} = +m_{\beta\lambda}$ (the positive solution of (2-14)) in the left half of each interval of length $\gamma^{-\alpha}$ and equal to $-m_{\beta\lambda}$ in the right half. Note that $\bar{m}$ satisfies the constraint $\{\sigma \approx^{\alpha} m\}$. In Appendix B we prove that the rescaled free energy of $\bar{m}$ in the limit of $\gamma \to 0$ is equal to

$$N\ell\left(-\lambda m^2_{\beta\lambda} - \frac{I(m_{\beta\lambda})}{\beta}\right),$$

which is smaller than $\bar{F}_{\beta,\gamma,\lambda}(0)$.

Instead when λ is small as in Theorem 2.3, then the optimal $\bar{m}$ is constant on the scale $\gamma^{-\alpha}$ (when $\gamma \to 0$). The proof of Theorem 2.3 is then reduced to prove that the functional in (2-17) Γ-converges [Braides 2002] to the functional in (2-10).

3. Coarse graining procedure

In this section we prove some estimates for the logarithm of the partition function $\log Z^{(\alpha)}_{\beta,\lambda,\gamma}(m)$ in terms of $\bar{F}_{\beta,\lambda,\gamma}$ defined in (2-17). These estimates will be used in the Lebowitz–Penrose limit discussed in the Section 5. A different coarse graining procedure from the classical Lebowitz–Penrose result will be used. This is needed due to the presence of two different scales of interaction along the horizontal and vertical directions.

The partition function $Z_{\beta,\lambda,\gamma}(\,\cdot\,)$ is defined on the space of the configurations while $\bar{F}_{\beta,\lambda,\gamma}(\,\cdot\,)$ is defined on the space of measurable functions. Recalling Definition 2.2, we consider $\mathcal{M}^{(\alpha)}$ the space of all functions which are constant on $\{C^{(\alpha)}_{i,k}\}_{i,k\in\mathbb{Z}}$ with values in $M^{(\alpha)}$. For each empirical average $m^{(\alpha)}(\,\cdot\,)$ there exists a function $\bar{m} \in \mathcal{M}^{(\alpha)}$ such that $|m^{(\alpha)}(x,i) - \bar{m}(x,i)| \le 2\gamma^{-\alpha}$ for all $(x,i) \in T_{\ell,N}$. Furthermore, given a function $\bar{m} \in \mathcal{M}^{(\alpha)}$ we define the set

$$\{\sigma^{(\alpha)} := \bar{m}\} = \{\sigma \in \{-1,1\}^{T_{L,N}} : \sigma^{(\alpha)}(x,i) = \bar{m}(x,i) \text{ for all } (x,i) \in T_{L,N}\}.$$

The next results are the basic steps in establishing the Lebowitz–Penrose limit.

Proposition 3.1. *For any $\alpha \in (0, \frac{1}{2})$, there is a constant $c > 0$ such that for any $\bar{m} \in \mathcal{M}^{(\alpha)}$*

$$\log Z_{\beta,\gamma,\lambda}(\{\sigma^{(\alpha)} = \bar{m}\}) \le -\beta \bar{F}_{\beta,\gamma,\lambda}(\bar{m}) + \beta c \epsilon(\gamma,\lambda)|T_{L,N}|, \tag{3-1}$$

$$\log Z_{\beta,\gamma,\lambda}(\{\sigma^{(\alpha)} = \bar{m}\}) \ge -\beta \bar{F}_{\beta,\gamma,\lambda}(\bar{m}) - \beta c \epsilon(\gamma,\lambda)|T_{L,N}|, \tag{3-2}$$

where $\bar{F}_{\beta,\gamma,\lambda}$ is defined in (2-17) and

$$\epsilon(\gamma,\lambda) := \lambda\gamma^{1/2-\alpha} + \gamma^{\alpha}\log\gamma^{-\alpha}. \tag{3-3}$$

The proof, which follows the standard techniques, we postpone to Appendix A. For such choice of $\bar{m}$ the set $\{\sigma \approx^{\alpha} m\} = \{\sigma^{(\alpha)} = \bar{m}\}$, and for any $\mathcal{A} \subseteq \mathcal{M}^{(\alpha)}$ we define

$$Z^{(\alpha)}_{\beta,\gamma,\lambda}(\mathcal{A}) = \sum_{m\in\mathcal{A}} Z_{\beta,\gamma,\lambda}(\{\sigma^{(\alpha)} = m\}).$$

Proposition 3.2. *For any $\alpha \in (0, \frac{1}{2})$, there is a constant $c > 0$ such that for any $\mathcal{A} \subseteq \mathcal{M}^{(\alpha)}$*

$$\log Z^{(\alpha)}_{\beta,\gamma,\lambda}(\mathcal{A}) \le -\beta \inf_{\bar{m}\in\mathcal{A}} \bar{F}_{\beta,\gamma,\lambda}(\bar{m}) + \beta c \epsilon(\gamma,\lambda)|T_{L,N}|, \tag{3-4}$$

$$\log Z^{(\alpha)}_{\beta,\gamma,\lambda}(\mathcal{A}) \ge -\beta \inf_{\bar{m}\in\mathcal{A}} \bar{F}_{\beta,\gamma,\lambda}(\bar{m}) - \beta c \epsilon(\gamma,\lambda)|T_{L,N}|. \tag{3-5}$$

Proof. The proof is the same as that of Theorem 4.2.2.2 in [Presutti 2009]. □

Now we consider the case $\alpha > \frac{1}{2}$. We cannot directly apply Proposition 3.1 since the length of the vertical interaction is less than the length of the coarse graining. The idea is to write the fixed average of m on the scale of $\gamma^{-\alpha}$ as an average after the coarse graining of scale $\gamma^{-\alpha'}$, $\alpha' \in (0, \frac{1}{2})$.

For $\bar{m}_\alpha \in \mathcal{M}^{(\alpha)}$ we define the set

$$\mathcal{A}_{\bar{m}_\alpha} = \left\{ \bar{m}_{\alpha'} \in \mathcal{M}^{(\alpha')} : \frac{1}{\gamma^{-\alpha}} \int_{C_{k,i}^{(\alpha)}} \bar{m}_{\alpha'}(r', i)\, dr' = \bar{m}_\alpha(r, i) \text{ for all } (r, i) \in T_{\ell,N} \right\}. \tag{3-6}$$

Using the above definition we prove a same result as in Proposition 3.1:

Proposition 3.3. *For any $\alpha \in (\frac{1}{2}, 1)$, there is a constant $c > 0$ such that for any $\bar{m}_\alpha \in \mathcal{M}^{(\alpha)}$*

$$\log Z_{\beta,\gamma,\lambda}(\{\sigma^{(\alpha)} = \bar{m}_\alpha\}) \le -\beta \inf_{\bar{m}_{\alpha'} \in \mathcal{A}_{\bar{m}_\alpha}} \bar{F}_{\beta,\gamma,\lambda}(\bar{m}_{\alpha'}) + \beta c \epsilon(\gamma, \lambda) |T_{L,N}|, \tag{3-7}$$

$$\log Z_{\beta,\gamma,\lambda}(\{\sigma^{(\alpha)} = \bar{m}_\alpha\}) \ge -\beta \inf_{\bar{m}_{\alpha'} \in \mathcal{A}_{\bar{m}_\alpha}} \bar{F}_{\beta,\gamma,\lambda}(\bar{m}_{\alpha'}) - \beta c \epsilon(\gamma, \lambda) |T_{L,N}|, \tag{3-8}$$

where $\bar{F}_{\beta,\gamma,\lambda}$ is defined in (2-17) and $\epsilon(\gamma, \lambda)$ in (3-3).

Proof. The proof follows by Propositions 3.1 and 3.2 □

4. Minimizer of the free energy functional

In this section we prove a technical result needed to prove Theorem 2.3. This key theorem tells us that a minimizer of the free energy functional under the constraint $⨍_\Lambda m^\Lambda = s$ is a "quasiconstant" function in a subset of $\Lambda \subset T_{\ell,N}$ (see (4-1)). So there are not oscillations that can affect the minimum of the free energy.

Fix $k \in \eta\mathbb{Z} \cap [0, \ell]$, with $\eta = \gamma \lceil \gamma^{-\alpha} \rceil$. We define the set $\Lambda_k = [k\eta, (k+1)\eta] \times \{1, \dots, N\}$, its complement $\Lambda_k^c = T_{\ell,N} \setminus \Lambda_k$, and the set $\Lambda_{k,i}$ as the restriction of Λ_k to the i-th column.

Let $m^{\Lambda_k} \in L^\infty(\Lambda_k, [-1, 1])$; we define the *free energy functional* restricted to Λ_k

$$\begin{aligned} F^{\Lambda_k}_{\beta,\gamma,\lambda}(m^{\Lambda_k}) = -\frac{1}{2} \sum_{i=1}^N \int_{\Lambda_{k,i}} m^{\Lambda_k}(r, i) \bigg[\int_{\Lambda_{k,i}} & J(r, r') m^{\Lambda_k}(r', i)\, dr' \\ & + \lambda \int_{\Lambda_{k,i}} J_{\gamma^{-1/2}}(r, r')(m^{\Lambda_k}(r', i-1) + m^{\Lambda_k}(r', i+1))\, dr' \bigg] dr \\ & - \frac{1}{\beta} \sum_{i=1}^N \int_{\Lambda_{k,i}} I(m^{\Lambda_k}(r, i))\, dr. \end{aligned}$$

Let $m^{\Lambda_k^c} \in L^\infty(\Lambda_k^c, [-1, 1])$; we define the *conditioned free energy functional*

$$
\begin{aligned}
F^{\Lambda_k}_{\beta,\gamma,\lambda}&(m^{\Lambda_k}\,|\,m^{\Lambda_k^c})\\
&= F^{\Lambda_k}_{\beta,\gamma,\lambda}(m^{\Lambda_k}) - \sum_{i=1}^{N}\int_{\Lambda_{k,i}} m^{\Lambda_k}(r,i)\int_{\Lambda^c_{k,i}} J(r,r')m^{\Lambda_k^c}(r',i)\,dr'\,dr\\
&\qquad\qquad - \lambda\sum_{i=1}^{N}\int_{\Lambda_{k,i}} m^{\Lambda_k}(r,i)\biggl[\int_{\Lambda^c_{k,i-1}} J_{\gamma^{-1/2}}(r,r')m^{\Lambda_k^c}(r',i-1)\,dr'\\
&\qquad\qquad\qquad\qquad + \int_{\Lambda^c_{k,i+1}} J_{\gamma^{-1/2}}(r,r')m^{\Lambda_k^c}(r',i+1)\,dr'\biggr]dr,
\end{aligned}
$$

where the set $\Lambda^c_{k,i}$ is the set Λ^c_k restricted to the i-th column.

The following theorem is the most relevant contribution of this work.

Theorem 4.1. *Take $\gamma > 0$, and define $\eta := \gamma[\gamma^{-\alpha}] = \gamma^{1/2}(1+\gamma^{-\varepsilon}\gamma^{-\delta}\zeta)$ where $\varepsilon \in (0, \frac{1}{2})$, $\delta \in (0, \frac{1}{2}-\varepsilon]$, and $\zeta > 0$ is small enough.*

If $\beta(\eta\|J\|_\infty + 2\lambda) \le \frac{1}{4}$, then for all $k \in \eta\mathbb{Z}\cap[0,\ell]$, $s \in [-1,1]^N$, and $m^{\Lambda_k^c} \in L^\infty(\Lambda_k^c, [-1,1])$ there exists a unique $\phi^{\Lambda_k} \in L^\infty(\Lambda_k, [-1,1])$ such that $⨍_{\Lambda_{k,i}}\phi^{\Lambda_k} = s_i$ for all i, and

$$F^{\Lambda_k}_{\beta,\gamma,\lambda}(m^{\Lambda_k}\,|\,m^{\Lambda_k^c}) \ge F^{\Lambda_k}_{\beta,\gamma,\lambda}(\phi^{\Lambda_k}\,|\,m^{\Lambda_k^c}),$$

for any $m^{\Lambda_k} \in L^\infty(\Lambda_k, [-1,1])$ such that $⨍_{\Lambda_{k,i}} m_{\Lambda_k} = s_i$ for all i.

Moreover, there exists a constant $C > 0$ such that, for any $r \in \bar\Lambda_{k,i}$,

$$|\phi^{\Lambda_k}(r,i) - s_i| \le C\|\nabla_r\phi\|_{\infty,\bar\Lambda_{k,i}}\,\eta \tag{4-1}$$

where $\bar\Lambda_{k,i} = [k\eta + \gamma^{1/2}(1+\gamma^{-\varepsilon}), (k+1)\eta - \gamma^{1/2}(1+\gamma^{-\varepsilon})]$.

Proof. If $s_i = \pm 1$ for all i, we have $m^{\Lambda_k} = \pm 1$ almost everywhere and the theorem follows easily. Now we take $|s_i| < 1$ for all i and we use Lagrange multipliers. In the following we omit the dependence on k and we keep only the dependence on the column i; then we take $\Lambda = \Lambda_k$ and $\Lambda_i = \Lambda_{k,i}$.

For all $h \in \mathbb{R}^N$ we define

$$F^{\Lambda,h}_{\beta,\gamma,\lambda}(m^\Lambda, m^{\Lambda^c}) = F^{\Lambda}_{\beta,\gamma,\lambda}(m^\Lambda\,|\,m^{\Lambda^c}) - \sum_{i=1}^{N} h_i\int_{\Lambda_i} m^\Lambda(r,i)\,dr.$$

For all $r \in [k\eta, (k+1)\eta)$ we define the vectors

$$\underline{m}^\Lambda(r) = (m_i^\Lambda(r))_{i=1}^N = (m^\Lambda(r,1), \dots, m^\Lambda(r,N))$$

and

$$(J * \underline{m}^\Lambda)(r) = ((J * m_i^\Lambda)(r))_{i=1}^N = \left(\int_{\Lambda_i} J(r,r')m^\Lambda(r,i)\,dr\right)_{i=1}^N.$$

In this notation the free energy becomes

$$
\begin{aligned}
&F^{\Lambda,h}_{\beta,\gamma,\lambda}(m^\Lambda \,|\, m^{\Lambda^c})\\
&= -\frac{1}{2}\sum_{i=1}^N \int_{\Lambda_i} m_i^\Lambda(r)((J * m_i^\Lambda)(r) + \lambda(J_{\gamma^{-1/2}} * (m_{i-1}^\Lambda + m_{i+1}^\Lambda))(r))\,dr\\
&\quad - \sum_{i=1}^N \int_{\Lambda_i} m_i^\Lambda(r)[(J * m_i^{\Lambda^c})(r) - \lambda((J_{\gamma^{-1/2}} * m_{i+1}^{\Lambda^c})(r) + (J_{\gamma^{-1/2}} * m_{i-1}^{\Lambda^c})(r))]\,dr\\
&\quad - \sum_{i=1}^N h_i \int_{\Lambda_i} m_i^\Lambda(r)\,dr - \frac{1}{\beta}\sum_{i=1}^N \int_{\Lambda_i} I(m_i^\Lambda(r))\,dr.
\end{aligned}
$$

Let $A_h(\underline{m}^\Lambda) = (A_i(\underline{m}^\Lambda))_{i=1}^N$, where

$$
\begin{aligned}
A_i(\underline{m}^\Lambda) &= \tanh(\beta[J * (m_i^\Lambda + m_i^{\Lambda^c}) + \lambda J_{\gamma^{-1/2}} * (m_{i-1}^\Lambda + m_{i-1}^{\Lambda^c} + m_{i+1}^\Lambda + m_{i+1}^{\Lambda^c}) + h_i])\\
&= A_i(m_i, m_{i+1}, m_{i-1}).
\end{aligned}
$$

From general results[1] the infimum of $F^{\Lambda,h}_{\beta,\gamma,\lambda}(\,\cdot\,|\,m^{\Lambda^c})$ is a minimum attained on functions such that $A_h(\underline{\psi}^\Lambda) = \underline{\psi}^\Lambda$. Thus, the set

$$
\mathscr{G}_{h,m^{\Lambda^c}} = \{\underline{\psi}^\Lambda \in L^\infty(\Lambda, [-1,1]^N) : \underline{\psi}^\Lambda = A_h(\underline{\psi}^\Lambda)\}
$$

is nonempty. We want to show that $\mathscr{G}$ is actually a singleton.

Step 1. *A_h is a contraction.*

Proof. We define the norm $\|A_h(\underline{m}^\Lambda)\|_{\infty,N} = \max_{\{i=1,\dots,N\}} \|A_i(\underline{m}^\Lambda)\|_\infty$. Given $\underline{m}^\Lambda$, $\underline{m}'^\Lambda$ we have, by the triangle inequality, the Lagrange theorem, and properties of J,

$$
\|A_i(\underline{m}^\Lambda) - A_i(\underline{m}'^\Lambda)\|_\infty \le \beta(\eta\|J\|_\infty + 2\lambda)\|\underline{m}^\Lambda - \underline{m}'^\Lambda\|_{\infty,N}.
$$

We observe that in this framework we can identify the set Λ_{i+1} with the set Λ_i, and with an abuse of notation we call it Λ. Then A is a contraction and there exists a unique fixed point $\underline{\phi}^{\Lambda,h}$ such that

$$
\underline{\phi}^{\Lambda,h} = \lim_{n\to\infty} A_h(\underline{u}_n) \quad \text{with } \underline{u}_n = A_h(\underline{u}_{n-1}) \text{ and } \underline{u}_0 = s\mathbf{1}_\Lambda.
$$

The convergence is in the sup norm, and then it is uniform in h. □

Step 2. *$\underline{\phi}^{\Lambda,h}$ is differentiable in h.*

Proof. We prove by induction on n that $\underline{u}_n$ is differentiable in h with derivative

$$
\frac{\partial}{\partial h_j} u_i^{n,\Lambda} = p^{i,n}\left[J * \frac{\partial}{\partial h_j} u_i^{n-1,\Lambda} + \lambda J_{\gamma^{-1/2}} * \left(\frac{\partial}{\partial h_j} u_{i-1}^{n-1,\Lambda} + \frac{\partial}{\partial h_j} u_{i+1}^{n-1,\Lambda}\right) + \delta_{i,j}\right] \tag{4-2}
$$

[1] See [Presutti 2009, Theorem 6.2.6.2].

where

$$p^{i,n} = \beta \cosh^{-2}\big[\beta\big(J * (u_i^{\Lambda,n-1} + m_i^{\Lambda^c}) + \lambda J_{\gamma^{-1/2}} * (u_{i-1}^{\Lambda,n-1} + u_{i+1}^{\Lambda,n-1} + m_{i-1}^{\Lambda^c} + m_{i+1}^{\Lambda^c}) + h_i\big)\big].$$

Indeed $D\underline{u}_0 = 0$ and if $\underline{u}_{n-1}$ is differentiable, $D\underline{u}_n$ exists and it is given by (4-2). Suppose $\|\frac{\partial}{\partial h_j} u_i^{n-1,\Lambda}\|_\infty \le 2\beta$; then

$$\left\| \frac{\partial}{\partial h_j} u_i^{n,\Lambda} \right\|_\infty \le \beta(\|J\|_\infty \eta 2\beta + 4\lambda\beta + 1)$$

by hypothesis $2\beta(\|J\|_\infty \eta + 2\lambda) + 1 \le 2$.

Then $\underline{\phi}^{\Lambda,h}$ is differentiable on h and

$$\nabla_h \underline{\phi}^{\Lambda,h} = \lim_{n\to\infty} \nabla_h \underline{u}_n. \qquad \square$$

Step 3. *For all λ small enough, there exists exactly one function $h(\lambda)$ such that*

- *$\underline{\phi}^{\Lambda,h}$ is the minimum of $F^{\Lambda,h}_{\beta,\gamma,\lambda}$ and*
- *$H(\lambda, h(\lambda)) = ⨍_\Lambda \underline{\phi}^{\Lambda,h}\, dr - s = 0$.*

Proof. If $\lambda = 0$, every column is independent of the other columns; then for each column we can find h_i^0 such that $⨍_\Lambda \phi^{\lambda,h_i^0}\, dr = s_i$.[2] This implies that $H(0, h^0) = 0$. In order to apply the implicit function theorem, we prove the invertibility of $\frac{\partial H(\lambda, h(\lambda))}{\partial h}$. We start by explicitly writing the derivative

$$\begin{aligned}\frac{\partial H}{\partial h} &= ⨍_\Lambda \frac{\partial}{\partial h} A_h(\underline{\phi}^{\Lambda,h})\, dr \\ &= ⨍_\Lambda \frac{\partial}{\partial h}\big(\tanh\big\{\beta\big[J * (\phi_i^\Lambda + m_i^{\Lambda^c}) + \lambda J_{\gamma^{-1/2}} * (\phi_{i-1}^\Lambda + m_{i-1}^{\Lambda^c} + \phi_{i+1}^\Lambda + m_{i+1}^{\Lambda^c}) + h_i\big]\big\}\big)_{i=1}^N\, dr.\end{aligned}$$

We define the square matrices $P, K \in M_N(\mathbb{R})$:

$$P_{i,j} = \begin{cases} 0 & \text{if } i \ne j, \\ p_i & \text{if } i = j, \end{cases} \qquad K_{i,j} = p_i[J * b_{i,j} + \lambda J_{\gamma^{-1/2}} * (b_{i+1,j} + b_{i-1,j})]$$

where $b_{i,j} = \frac{\partial \phi_i^{\Lambda,h}}{\partial h_j}$ and

$$p_i = \beta \cosh^{-2}[\beta(J * (\phi_i^{\Lambda,h} + m_i^{\Lambda^c}) + \lambda J_{\gamma^{-1/2}} * (\phi_{i-1}^{\Lambda,h} + \phi_{i+1}^{\Lambda,h} + m_{i-1}^{\Lambda^c} + m_{i+1}^{\Lambda^c}) + h_i)].$$

We write the derivative with respect to h of H in terms of P and K:

$$\frac{\partial}{\partial h} H = ⨍_\Lambda (K + P)\, dr = \bar{K} + \bar{P} = \bar{P}(\bar{P}^{-1}\bar{K} + I).$$

[2] This follows from [Presutti 2009, Theorem 6.4.1.1].

We observe that

$$
\begin{aligned}
|(\bar{P}^{-1}\bar{K})_{i,j}| &\le \frac{1}{\fint_\Lambda p_i\,dr}\int_\Lambda |K_{i,j}|\,dr \\
&\le \frac{1}{\fint_\Lambda p_i\,dr}\int_\Lambda p_i\|(P^{-1}K)_{i,j}\|_\infty\,dr \le \|(P^{-1}K)_{i,j}\|_\infty.
\end{aligned}
$$

To prove the existence of the matrix $(\bar{P}^{-1}\bar{K}+I)^{-1}$ we show that $\sum_{n=0}^{\infty}(-P^{-1}K)^n<\infty$, proving that $\sup_i\sum_{j=0}^{N}(P^{-1}K)_{i,j}\le c<1$.

We give an estimate for $\|b_{i,j}\|_\infty$. We recall that

$$
b_{i,j}=p_i(J*b_{i,j}+\lambda J_{\gamma^{-1/2}}*(b_{i-1,j}+b_{i+1,j})+\delta_{i,j}).
$$

Then

$$
\|b_{i,j}\|_\infty\le\frac{\|p_i\|_\infty}{1-\|p_i\|_\infty\eta\|J\|_\infty}(\lambda(\|b_{i-1,j}\|_\infty+\|b_{i+1,j}\|_\infty)+\delta_{i,j}). \tag{4-3}
$$

We define for all $i\in\{1,\dots,N\}$ and $a\in\mathbb{N}$

$$
\begin{aligned}
q_i &= \frac{\|p_i\|_\infty}{1-\|p_i\|_\infty\eta\|J\|_\infty}, \\
\Omega_a^i &= \{\sigma\in\mathbb{Z}^a:\sigma(0)=i,\ \sigma(k)\equiv\sigma(k-1)\pm 1 \bmod N\}.
\end{aligned}
$$

We observe that

$$
q_i\le\frac{\beta}{1-\beta\eta\|J\|_\infty},\qquad |\Omega_a^i|=2^a.
$$

Iterating the inequality (4-3) a times we obtain

$$
\begin{aligned}
\|b_{i,j}\|_\infty &\le \sum_{\sigma\in\Omega_a^i} q_{\sigma(0)}\cdots q_{\sigma(a)}\|b_{\sigma(a),j}\|_\infty\lambda^a+\sum_{n=0}^{a-1}\sum_{\sigma\in\Omega_n^i} q_{\sigma(0)}\cdots q_{\sigma(n)}\lambda^n\delta_{\sigma(n),j} \\
&\le \left(\frac{2\beta\lambda}{1-\eta\|J\|_\infty\beta}\right)^a\|b_{\sigma(a),j}\|_\infty+q_{\sigma(0)}\sum_{n=0}^{a-1}\left(\frac{2\beta\lambda}{1-\eta\|J\|_\infty\beta}\right)^n\delta_{\sigma(n),j}.
\end{aligned}
$$

If the number of iterations is big enough, then the Dirac delta is 1. We define $n:=n(i-j)$ where

$$
n(i-j)=\begin{cases}|i-j| & \text{if } |i-j|\le\lceil N/2\rceil,\\ N-|i-j| & \text{otherwise.}\end{cases} \tag{4-4}
$$

Then $\delta_{\sigma(n),j}=1$ if the number of iterations is at least $n(i-j)$. Taking the limit $a\to\infty$,

$$
\|b_{i,j}\|_\infty\le q_{\sigma(0)}\sum_{n=[i-j]}^{\infty}\left(\frac{2\beta\lambda}{1-\eta\|J\|_\infty\beta}\right)^n\delta_{\sigma(n),j}\le q_i\frac{\theta^{n(i-j)}}{1-\theta}, \tag{4-5}
$$

with $\theta=2\beta\lambda/(1-\eta\|J\|_\infty\beta)<1$.

Let $r \in \Lambda$, and consider

$$(P^{-1}K)_{i,j}(r) = (J * b_{i,j})(r) + \lambda(J_{\gamma^{-1/2}} * (b_{i+1,j} + b_{i-1,j}))(r).$$

Using (4-5) we have

$$\sum_{j=1}^{N} \|(P^{-1}K)_{i,j}\|_\infty \leq \frac{\alpha}{1-\theta} \sum_{j=1}^{N} \frac{\theta^{(\min_{r=i-1,i,i+1} n(r-j))}}{1-\theta} \leq \frac{2\alpha}{(1-\theta)^2}$$

where $\alpha = (\eta\|J\|_\infty + 2\lambda)\beta/(1-\eta\|J\|_\infty\beta)$.

Now keeping in mind that $\beta(\eta\|J\|_\infty + 2\lambda) \leq \frac{1}{4}$ we obtain

$$\frac{2\alpha}{(1-\theta)^2} \leq \tfrac{8}{9}(1-\eta\|J\|_\infty\beta).$$

For η small enough the matrix $(\bar{P}^{-1}\bar{K} + I)$ can be inverted and we find the function $h(\lambda) = h$ such that $H(\lambda, h(\lambda)) = 0$. For $\underline{m}^\Lambda$ such that $⨍_\Lambda \underline{m}^\Lambda = s$ the conditioned free energy

$$\begin{aligned} F^{\Lambda,h}_{\beta,\gamma,\lambda}(m^\Lambda \,|\, m^{\Lambda^c}) &= F^{\Lambda,h}_{\beta,\gamma,\lambda}(m^\Lambda \,|\, m^{\Lambda^c}) + h\sum_{i=1}^{N} |\Lambda_i| s_i \\ &\geq F^{\Lambda,h}_{\beta,\gamma,\lambda}(\underline{\phi}^{\Lambda,h} \,|\, m^{\Lambda^c}) + h\sum_{i=1}^{N} |\Lambda_i| s_i \\ &= F^{\Lambda,h}_{\beta,\gamma,\lambda}(\underline{\phi}^{\Lambda,h} \,|\, m^{\Lambda^c}). \qquad \square \end{aligned}$$

We now prove the last part of the theorem. Let $\bar{\Lambda}_i = (k\eta + \gamma^{1/2}(1+\gamma^{-\varepsilon}), (k+1)\eta - \gamma^{1/2}(1+\gamma^{-\varepsilon}))$, and define

$$\bar{s}_i = ⨍_{\bar{\Lambda}_i} \phi_i^{\Lambda,h}(r)\,dr.$$

We observe that, for such a constant $c > 0$,

$$\begin{aligned} |s_i - \bar{s}_i| &\leq \left(\frac{|\bar{\Lambda}_i|}{|\Lambda|} - 1\right) ⨍_{\bar{\Lambda}_i} |\phi_i^{\Lambda,h}(r)|\,dr + \frac{|\Lambda \setminus \bar{\Lambda}_i|}{|\Lambda|} ⨍_{\Lambda\setminus\bar{\Lambda}_i} |\phi_i^{\Lambda,h}(r)|\,dr \\ &\leq c\gamma^\delta \leq c\eta \end{aligned} \tag{4-6}$$

because of the choice of η. Fix $r' \in \bar{\Lambda}_i$:

$$|\bar{s}_i - \phi_i^{\Lambda,h}(r')| \leq C \left\| \frac{\partial}{\partial r}\phi_i^{\Lambda,h} \right\|_\infty \eta,$$

where $\left\| \frac{\partial}{\partial r}\phi_i^{\Lambda,h} \right\|_\infty = \sup_{r\in\bar{\Lambda}_i} \left| \frac{\partial}{\partial r}\phi_i^{\Lambda,h}(r) \right|$.

It remains to prove that $\|\frac{\partial}{\partial r}\phi_i^{\Lambda,h}\|_\infty < \infty$. We shall use the recursive formula

$$\frac{\partial}{\partial r}\phi_i^{\Lambda,h}(r) = p_i\left[\frac{\partial}{\partial r}J*(\phi_i^{\Lambda,h}+m_i^{\Lambda^c})+\lambda\frac{\partial}{\partial r}J_{\gamma^{-1/2}}*(\phi_{i-1}^{\Lambda,h}+\phi_{i+1}^{\Lambda,h}+m_{i-1}^{\Lambda^c}+m_{i+1}^{\Lambda^c})\right]. \tag{4-7}$$

If we iterate (4-7) a times we obtain

$$\frac{\partial}{\partial r}\phi_i^{\Lambda,h}(r) = \sum_{\sigma\in\Omega_a^i} p_{\sigma(0)}\cdots p_{\sigma(a)}\lambda^a J_{\gamma^{-1/2}} *^{(a-1)}\cdots * \frac{\partial J}{\partial r} * (\phi_{\sigma(a)}^{\Lambda,h}+m_{\sigma(a)}^{\Lambda^c}) + \sum_{n=0}^{a-1}\sum_{\sigma\in\Omega_n^i} p_{\sigma(0)}\cdots p_{\sigma(n)}\lambda^{n+1} J_{\gamma^{-1/2}} *^{(n)}\cdots * \frac{\partial}{\partial r}\phi_{\sigma(n)}^{\Lambda,h}.$$

Observing that, at each iteration n, if $n<\gamma^{-\varepsilon}$, then $(J_{\gamma^{-1/2}} *^{(n)}\cdots * m_{\sigma(n)}^{\Lambda^c})(r)=0$ by the choice of the set $\bar{\Lambda}$. Taking the norm,

$$\left\|\frac{\partial}{\partial r}\phi_i^{\Lambda,h}\right\|_\infty \le \eta\|\nabla J\|_\infty\beta\sum_{n=0}^{a}(2\beta\lambda)^n + (2\beta\lambda)^a\|\nabla J\|_\infty 2\gamma^{-1/2}$$

where we took the derivative of $J_{\gamma^{-1/2}}$ in the last term indexed by a. If $a=\lceil\gamma^{-\varepsilon}\rceil$, then

$$\lim_{\gamma\to 0}(2\beta\lambda)^a\|\nabla J\|_\infty 2\gamma^{-1/2}=0$$

and

$$\left\|\frac{\partial}{\partial r}\phi_i^{\Lambda,h}\right\|_\infty \le c'\eta\|\nabla J\|_\infty\beta<\infty,$$

for some constant $c'>0$. Equation (4-6) gives

$$|\phi_i^{\Lambda}(r)-s_i|\le C\eta \quad \text{for all } r\in\bar{\Lambda}_i,$$

and the theorem is proved. □

5. The Lebowitz–Penrose limit

Proof of Theorem 2.3. The proof is divided into two parts: $\alpha<\frac{1}{2}$ and $\alpha>\frac{1}{2}$. We will use the results of the previous sections in both cases. While for $\alpha<\frac{1}{2}$ we can use them straightforwardly in the Lebowitz–Penrose procedure, for $\alpha>\frac{1}{2}$ the technical Theorem 4.1 is needed in order to control the fluctuations.

Case 1 ($\alpha\in(0,\frac{1}{2})$). *Let $m\in L^\infty(T_{\ell,N};[-1,1])$; we prove that*

$$\lim_{\gamma\to 0} F_{\beta,\gamma,\lambda}^{(\alpha)}(m) = F_{\beta,\lambda}(m). \tag{5-1}$$

Proof. Given a mesoscopic state m we choose a function $\overline{m}_\alpha \in \mathcal{M}^{(\alpha)}$ that "recognizes" m (see Definition 2.2):

$$\lim_{\gamma\to 0} F^{(\alpha)}_{\beta,\gamma,\lambda}(m) = \lim_{\gamma\to 0} -\frac{1}{\beta\gamma^{-1}} \log Z_{\beta,\gamma,\lambda}(\{\sigma^{(\alpha)} = \overline{m}_\alpha\}).$$

We apply Proposition 3.1 and the change of coordinates $m_\alpha(r, i) := \overline{m}_\alpha(\gamma^{-1}r, i)$. We shall show that $|F_{\beta,\gamma,\lambda}(m^{(\alpha)}) - F_{\beta,\lambda}(m)| \to 0$ as $\gamma \to 0$ where

$$F_{\beta,\gamma,\lambda}(m) = -\frac{1}{2}\sum_{i=1}^{N}\int_0^\ell m(r,i)\Big(\int_0^\ell J(r,r')m(r',i)\,dr' + \lambda\int_0^\ell J_{\gamma^{-1/2}}(r,r')(m(r',i-1)+m(r',i+1))\,dr'\Big)\,dr - \frac{1}{\beta}\sum_{i=1}^{N}\int_0^\ell I(m(r,i))\,dr. \tag{5-2}$$

By the Lebesgue differentiation theorem [Rudin 1987] we know that $m^{(\alpha)} \xrightarrow{L^1} m$ (see (2-6)); thus, by the triangle inequality the limit can be divided into three parts. The *first term* is

$$\begin{aligned}\Big|\int_0^\ell\int_0^\ell J_{\gamma^{-1/2}}(r,r')m^{(\alpha)}(r,i)m^{(\alpha)}(r',i-1)\,dr\,dr' - \int_0^\ell m(r,i)m(r,i-1)\,dr\Big| \\ \le \Big|\int_0^\ell m^{(\alpha)}(r,i)\Big[\int_0^\ell J_{\gamma^{-1/2}}(r,r')m^{(\alpha)}(r',i-1)\,dr' - m(r,i-1)\Big]\,dr\Big| \\ + \Big|\int_0^\ell m(r,i-1)[m^{(\alpha)}(r,i)-m(r,i)]\,dr\Big|.\end{aligned}$$

We observe that

$$J_{\gamma^{-1/2}} * m(r) = \int_0^\ell J_{\gamma^{-1/2}}(r,r')m(r')\,dr'$$

converges to the Dirac delta as $\gamma \to 0$; then by the dominated convergence theorem

$$\Big|\int_0^\ell\int_0^\ell J_{\gamma^{-1/2}}(r,r')m^{(\alpha)}(r,i)m^{(\alpha)}(r',i-1)\,dr\,dr' - \int_0^\ell m(r,i)m(r,i-1)\,dr\Big| \to 0.$$

The other two terms converge to 0 by the dominated convergence theorem. And (5-1) is proved. □

Case 2 ($\alpha \in (\frac{1}{2}, 1)$). *By Proposition 3.3 and using the same notations introduced in the case $\alpha \in (0, \frac{1}{2})$, we have*

$$\lim_{\gamma\to 0} F^{(\alpha)}_{\beta,\gamma,\lambda}(m) = \lim_{\gamma\to 0} \gamma \inf_{\overline{m}_{\alpha'}\in\mathcal{A}_{\overline{m}_\alpha}} \overline{F}_{\beta,\gamma,\lambda}(\overline{m}_{\alpha'}) = \lim_{\gamma\to 0} \inf_{\substack{m_{\alpha'}(\cdot)=\overline{m}_{\alpha'}(\gamma^{-1}\cdot)\\ \overline{m}_{\alpha'}\in\mathcal{A}_{\overline{m}_\alpha}}} F_{\beta,\gamma,\lambda}(m_{\alpha'}).$$

In order to pass the limit through the infimum, we need to prove a result of Γ-convergence. Let us start defining a notion of convergence:

Definition 5.1 ($\star$-convergence). Set $\eta := \gamma[\gamma^{-\alpha}]$; then for all sequences $\{m_\gamma\}$ and m in $L^\infty(T_{\ell,N}, [-1,1])$ we say that $m_\gamma \overset{\star}{\to} m$ if

$$\lim_{\gamma\to 0} \sum_{i=1}^{N} \sum_{k=1}^{\lceil \ell/\eta \rceil} \left| \fint_{D_{k,i}^{(\alpha)}} m_\gamma(r', i)\, dr' - \fint_{D_{k,i}^{(\alpha)}} m(r, i)\, dr \right| = 0 \tag{5-3}$$

where $D_{k,i}^{(\alpha)} = \{(x, i) \in \mathbb{R} \times \mathbb{Z} : k\eta \le x \le (k+1)\eta\}$.

We can write (5-3) as $\lim_\gamma d(m_\gamma, m) = 0$, since d is actually a distance.

Remark. Following the same notation of the case $\alpha < \frac{1}{2}$, we observe that the sequence $\{m_\alpha\}_\gamma \overset{\star}{\to} m$ as $\gamma \to 0$.

The following Γ-convergence result will be proved in the next section.

Proposition 5.2. *Let $F_{\beta,\gamma,\lambda}$ be as in (5-2) and $F_{\beta,\lambda}$ as in (2-10). Then*

$$F_{\beta,\lambda} = \Gamma \lim_{\gamma\to 0} F_{\beta,\gamma,\lambda}$$

according to the $\star$-convergence.

We move to the last part of proof of Theorem 2.3.

We start by proving the *lower bound*. For each $\delta > 0$ we can take γ small enough such that there exists a function $m_\alpha(\cdot) = \overline{m}_\alpha(\gamma^{-1}\cdot)$ such that $d(m_\alpha, m) < \delta$; then

$$\inf_{\substack{m_{\alpha'}(\cdot) = \overline{m}_{\alpha'}(\gamma^{-1}\cdot) \\ \overline{m}_{\alpha'} \in \mathcal{A}_{\overline{m}_\alpha}}} F_{\beta,\gamma,\lambda}(m_{\alpha'}) \ge \inf_{\substack{m' \in L^\infty(T_{\ell,N}, [-1,+1]) \\ d(m',m) < \delta}} F_{\beta,\gamma,\lambda}(m').$$

Taking the infimum limit with respect to γ and the supremum with respect to δ,

$$\begin{aligned} \liminf_{\gamma\to 0} F^{(\alpha)}_{\beta,\gamma,\lambda}(m) &\ge \sup_{\delta>0} \liminf_{\gamma\to 0} \inf_{\substack{m' \in L^\infty(T_{\ell,N}, [-1,+1]) \\ d(m',m) < \delta}} F_{\beta,\gamma,\lambda}(m') \\ &\ge F_{\beta,\lambda}(m). \end{aligned}$$

The last inequality follows by definition of the Γ-limit[3] and Proposition 5.2.

Now we consider the *upper bound*. Let $\tilde{\overline{m}}_{\alpha'}$ be the closest element in $\mathcal{A}_{\overline{m}_\alpha}$ to $\overline{m}_\alpha$, namely

$$|\overline{m}_\alpha(r, i) - \tilde{\overline{m}}_{\alpha'}(r, i)| \le \frac{2}{\gamma^{-\alpha'}} \quad \text{for all } (r, i) \in T_{L,N}. \tag{5-4}$$

[3] See [Braides 2002].

Such a magnetization exists by the definition of the sets $M^{(\alpha')}$ and $M^{(\alpha)}$. We define $\tilde{m}_{\alpha'}(r,i)=\tilde{\bar{m}}_{\alpha'}(\gamma^{-1}r,i)$, for all (r,i); then

$$\begin{aligned}\limsup_{\gamma\to 0} F^{(\alpha)}_{\beta,\gamma,\lambda}(m) &= \limsup_{\gamma\to 0} \inf_{\substack{m_{\alpha'}(\,\cdot\,)=\bar{m}_{\alpha'}(\gamma^{-1}\,\cdot\,)\\ \bar{m}_{\alpha'}\in\mathcal{A}_{\bar{m}_\alpha}}} F_{\beta,\gamma,\lambda}(m_{\alpha'})\\ &\le \limsup_{\gamma\to 0} F_{\beta,\gamma,\lambda}(\tilde{m}_{\alpha'})\\ &\le F_{\beta,\lambda}(m).\end{aligned}$$

The last inequality follows from Proposition 5.2 and (5-4). □

6. Γ-limit

In this section we shall prove the existence of the Γ-limit of $F_{\beta,\gamma,\lambda}$. Definition 5.1 of ⋆-convergence involves the average of m on sets of length $\gamma[\gamma^{-\alpha}]=\eta$. This implies a constraint on the minimizer of the free energy functional, and at this level we use the Theorem 4.1.

Proof of Proposition 5.2. We start with the *lower bound*: for all $\{m_\gamma\}$ such that $m_\gamma \overset{\star}{\to} m$,

$$\liminf_{\gamma\to 0} F_{\beta,\gamma,\lambda}(m_\gamma)\ge F_{\beta,\lambda}(m).$$

Fix η, Λ_k as in Theorem 4.1; we set $n=\ell/\eta$ and observe that $T_{\ell,N}=\bigcup_{k=1}^n \Lambda_k$. We define $m_\gamma^{\Lambda_k} := m_\gamma|_{\Lambda_k}$, the restriction of m_γ to the set Λ_k. Fix Λ_1; then by Theorem 4.1 there exists $\phi_\gamma^{\Lambda_1}$ such that

$$\begin{aligned}F_{\beta,\gamma,\lambda}(m_\gamma) &= F^{\Lambda_1}_{\beta,\gamma,\lambda}(m_\gamma^{\Lambda_1}\,|\,m_\gamma^{\Lambda_1^c}) + F^{\Lambda_1^c}_{\beta,\gamma,\lambda}(m_\gamma^{\Lambda_1^c})\\ &\ge F^{\Lambda_1}_{\beta,\gamma,\lambda}(\phi_\gamma^{\Lambda_1}\,|\,m_\gamma^{\Lambda_1^c}) + F^{\Lambda_1^c}_{\beta,\gamma,\lambda}(m_\gamma^{\Lambda_1^c})\\ &= F^{\Lambda_1}_{\beta,\gamma,\lambda}(m_\gamma^1)\end{aligned}$$

where $m_\gamma^1=m_\gamma \mathbb{1}_{\Lambda_1^c}+\phi_\gamma^{\Lambda_1}\mathbb{1}_{\Lambda_1}$ and $\mathbb{1}_A$ is the indicator function of the set A.

We iterate this procedure for each k, and we define $m_\gamma^{1,\dots,n}$; then

$$F_\gamma(m_\gamma)\ge F_\gamma(m_\gamma^{1,\dots,n}).$$

Lemma 6.1. *Let $m_\gamma^{1,\dots,n}$ be as above. We have*

$$\lim_{\gamma\to 0}\sum_{i=1}^N \int_0^\ell |m_\gamma^{1,\dots,n}(r,i)-m(r,i)|\,dr=0.$$

Proof. For any $i \in \{1, \dots, N\}$ we split the integral following the partition given by $\Lambda_{k,i}$, and we consider

$$\sum_{k=1}^{n} \int_{\Lambda_{k,i}} \left| m_\gamma^{1,\dots,n}(r,i) \pm \fint_{\Lambda_{k,i}} m_\gamma^{1,\dots,n}(r',i)\,dr' - m(r,i) \right| dr.$$

Applying the triangle inequality, the fist term converges by definition of $m_\gamma^{1,\dots,n}$ and the Lebesgue differentiation theorem. The second term can be estimated as

$$\begin{aligned} &\sum_{k=1}^{n} \int_{\Lambda_{k,i}} \left| m_\gamma^{1,\dots,n}(r,i) - \fint_{\Lambda_{k,i}} m_\gamma^{1,\dots,n}(r',i)\,dr' \right| dr \\ &\quad \le \sum_{k=1}^{n} \int_{\bar\Lambda_{k,i}} |m_\gamma^{1,\dots,n}(r,i) - s|\,dr + \int_{\bar E_k} \left| m_\gamma^{1,\dots,n}(r,i) - \fint_{\Lambda_{k,i}} m_\gamma^{1,\dots,n}(r',i)\,dr' \right| dr \\ &\quad \le \sum_{k=1}^{n} |\Lambda_{k,i}|\eta c + |\bar E_k| c' \end{aligned}$$

where

$$\bar\Lambda_{k,i} = (k\eta + \gamma^{1/2}(1+\gamma^{-\varepsilon}),\, (k+1)\eta - \gamma^{1/2}(1+\gamma^{-\varepsilon}))$$

and $\bar E_k = \Lambda_{k,i} \setminus \bar\Lambda_{k,i}$. To finish the proof we just observe that the size of $\bar E_k$ is of the order of $\gamma^{1/2}(1+\gamma^{-\varepsilon})$. □

To prove the lower bound we separately consider the convergence of the three terms of $F_{\beta,\gamma,\lambda}$.

The *first term* is

$$\sum_{i=1}^{N} \frac{1}{2} \left| \int_0^\ell \int_0^\ell J(r,r')(m_\gamma^{1,\dots,n}(r,i) m_\gamma^{1,\dots,n}(r',i) - m(r,i)m(r',i))\,dr\,dr' \right|.$$

Using the triangle inequality with $m_\gamma^{1,\dots,n}(r,i)m(r,i)$, the convergence follows from Lemma 6.1.

The *second term* is

$$\begin{aligned} \sum_{i=1}^{N} \frac{\lambda}{2} \bigg| \int_0^\ell \int_0^\ell m_\gamma^{1,\dots,n}(r,i) J_{\gamma^{-1/2}}(r,r')[m_\gamma^{1,\dots,n}(r',i+1) + m_\gamma^{1,\dots,n}(r',i-1)]\,dr\,dr' \\ - \int_0^\ell m(r,i)[m(r,i+1) + m(r,i-1)]\,dr \bigg|. \end{aligned}$$

We only discuss the term $m(\cdot,i)m(\cdot,i+1)$ because for the other term the proof is analogous. We sum and subtract for each i the term $m_\gamma^{1,\dots,n}(r,i+1)$; then

$$\left|\int_0^\ell m_\gamma^{1,\dots,n}(r,i)\left[\int_0^\ell J_{\gamma^{-1/2}}(r,r')m_\gamma^{1,\dots,n}(r',i+1)\,dr' \pm m_\gamma^{1,\dots,n}(r,i+1)\right]dr \right.$$
$$\left. - \int_0^\ell m(r,i)m(r,i+1)\,dr\right|.$$

We split the first integral in the sum of integrals $\sum_{k=1}^n \int_{\bar{\bar{\Lambda}}_{k,i}} + \int_{\bar{\bar{E}}_k}$ where

$$\bar{\bar{\Lambda}}_{k,i} = (k\eta + 2\gamma^{1/2}(1+\gamma^{-\varepsilon}), (k+1)\eta - 2\gamma^{1/2}(1+\gamma^{-\varepsilon}))$$

and $\bar{\bar{E}}_k = \Lambda_{k,i} \setminus \bar{\bar{\Lambda}}_{k,i}$. If $r \in \bar{\bar{\Lambda}}_{k,i}$, we have that $\int_{\bar{\Lambda}_{k,i}} J_{\gamma^{-1/2}}(r,r')\,dr' = 1$; then

$$\sum_{k=1}^n \left|\int_{\bar{\bar{\Lambda}}_{k,i}} m_\gamma^{1,\dots,n}(r,i)\int_{\bar{\Lambda}_{k,i}} J_{\gamma^{-1/2}}(r,r')(m_\gamma^{1,\dots,n}(r',i+1) - m_\gamma^{1,\dots,n}(r,i+1))\,dr'\,dr\right|$$
$$\le \sum_{k=1}^n \int_{\bar{\bar{\Lambda}}_{k,i}}\int_{\bar{\Lambda}_{k,i}} J_{\gamma^{-1/2}}(r,r')|m_\gamma^{1,\dots,n}(r',i+1) - m_\gamma^{1,\dots,n}(r,i+1)|\,dr'\,dr$$
$$\le \sum_{k=1}^n |\bar{\bar{\Lambda}}_{k,i}|\eta c$$

because $m_\gamma^{1,\dots,n}$ is almost constant in $\bar{\Lambda}_{k,i}$. While integrating over $\bar{\bar{E}}_k$,

$$\sum_{k=1}^n \left|\int_{\bar{\bar{E}}_k} m_\gamma^{1,\dots,n}(r,i)\int_{\Lambda_{k,i}} J_{\gamma^{-1/2}}(r,r')(m_\gamma^{1,\dots,n}(r',i+1) - m_\gamma^{1,\dots,n}(r,i+1))\,dr'\,dr\right|$$
$$\le \sum_{k=1}^n \int_{\bar{\bar{E}}_k}\int_{\overline{\overline{E^\star}}_k} J_{\gamma^{-1/2}}(r,r')|m_\gamma^{1,\dots,n}(r',i+1) - m_\gamma^{1,\dots,n}(r,i+1)|\,dr'\,dr$$
$$\le c\sum_{k=1}^n |\bar{\bar{E}}_k||\overline{\overline{E^\star}}_k|\gamma^{-1/2}\|J\|_\infty$$

with $c > 0$ a constant. The size of $|\overline{\overline{E^\star}}_k|$ is of the same order as $|\bar{\bar{E}}_k| + 2\gamma^{1/2}$. In the end the term

$$\sum_{i=1}^N \frac{\lambda}{2}\left|\int_0^\ell (m_\gamma^{1,\dots,n}(r,i)m_\gamma^{1,\dots,n}(r,i+1) - m(r,i)m(r,i+1))\,dr\right|$$

can be estimated using Lemma 6.1. All the other terms that we did not consider can be estimated in the same way.

For the *third term*, we consider $\liminf_{\gamma} F_{\beta,\gamma,\lambda}(m_\gamma^{1,\dots,n})$. Then

$$\begin{aligned}
\liminf_{\gamma\to 0}\sum_{i=1}^{N} -\frac{1}{\beta}\int_0^{\ell} I(m_\gamma^{1,\dots,n})(r,i)\,dr
&= \liminf_{\gamma\to 0}\sum_{i=1}^{N} -\frac{1}{\beta}\sum_{k=1}^{n}\int_{\Lambda_{k,i}} I(m_\gamma^{1,\dots,n}(r,i))\,dr \\
&\geq \liminf_{\gamma\to 0}\sum_{i=1}^{N} -\frac{1}{\beta}\sum_{k=1}^{n}|\Lambda_{k,i}|\, I\Big(\fint_{\Lambda_{k,i}} m_\gamma^{1,\dots,n}(r,i)\,dr\Big) \\
&= \liminf_{\gamma\to 0}\sum_{i=1}^{N} -\frac{1}{\beta}\int_0^{\ell} I\Big(\fint_{\Lambda_{k(r),i}} m_\gamma^{1,\dots,n}(r',i)\,dr'\Big)\,dr
\end{aligned}$$

by Jensen's inequality. We write the sum over k as an integral over r observing that the function I is constant for all r in the set $\Lambda_{k,i}$. Moreover, there exists a subsequence $\{m_{\gamma_j}^{1,\dots,n}\}$ that achieves the infimum limit:

$$\begin{aligned}
\liminf_{\gamma\to 0}\sum_{i=1}^{N} -\frac{1}{\beta}\int_0^{\ell} I\Big(\fint_{\Lambda_{k(r),i}} m_\gamma^{1,\dots,n}(r',i)\,dr'\Big)\,dr
&= \lim_{\gamma_j\to 0}\sum_{i=1}^{N} -\frac{1}{\beta}\int_0^{\ell} I\Big(\fint_{\Lambda_{k_j(r),i}} m_{\gamma_j}^{1,\dots,n}(r',i)\,dr'\Big)\,dr.
\end{aligned}$$

Let $\tilde m_\gamma(r,i) = \fint_{\Lambda_{k_j(r),i}} m_{\gamma_j}^{1,\dots,n}(r',i)\,dr'$; then by Lemma 6.1 $\tilde m_\gamma \xrightarrow{L^1} m$ and there exists a subsubsequence, which we denote again $\{m_{\gamma_j}^{1,\dots,n}\}$, that converges to m almost everywhere. Then by the dominated convergence theorem

$$\lim_{\gamma\to 0} -\frac{1}{\beta}\int_0^{\ell} I\Big(\fint_{\Lambda_{k_j(r),i}} m_{\gamma_j}^{1,\dots,n}(r',i)\,dr'\Big)\,dr = -\frac{1}{\beta}\int_0^{\ell} I(m(r,i))\,dr$$

and

$$\begin{aligned}
\liminf_{\gamma\to 0} F_{\beta,\gamma,\lambda}(m_\gamma) &\geq \liminf_{\gamma\to 0} F_{\beta,\gamma,\lambda}(m_\gamma^{1,\dots,n}) \\
&= \lim_{\gamma_j\to 0} F_{\beta,\gamma,\lambda}(m_{\gamma_j}^{1,\dots,n}) \\
&= F_{\beta,\lambda}(m).
\end{aligned}$$

Now we prove the *upper bound*. There exists a sequence $\{m_\gamma\}$ such that $m_\gamma \xrightarrow{\star} m$ and

$$\lim_{\gamma\to 0} F_{\beta,\gamma,\lambda}(m_\gamma) = F_{\beta,\lambda}(m).$$

We take $m_\gamma = s = ⨍_{\Lambda_{k,i}} m(r,i)\,dr$ on $\Lambda_{k,i} \subset [0,\ell]$ for all k. Then from the dominated convergence theorem

$$\lim_{\gamma\to 0} |F_{\beta,\gamma,\lambda}(m_\gamma) - F_{\beta,\lambda}(m)| = 0.$$

And $F_{\beta,\lambda}(m)$ is the Γ-limit of $F_{\beta,\gamma,\lambda}(m_\gamma)$. □

Appendix A: Proofs of Proposition 3.1 and Theorem 2.4

Proof of Proposition 3.1. The proof follows the guidelines of Section 4.2.2 of [Presutti 2009] taking care of the two different scales of interaction γ^{-1} and $\gamma^{-1/2}$.

We define

$$U_{\gamma,\lambda}(\bar{m}) = \bar{F}_{\beta,\gamma,\lambda}(\bar{m}) + \sum_{i=1}^{N} \frac{1}{\beta} \int_0^{\gamma^{-1}\ell} I(\bar{m}(r,i))\,dr$$

where $\bar{F}_{\beta,\gamma,\lambda}$ is defined as in (2-17). We want to estimate $|H_{\gamma,\lambda}(\sigma) - U_{\gamma,\lambda}(\sigma^{(\alpha)})|$, and we start taking $(x,i), (y,j) \in T_{L,N}$ and defining

$$\hat{J}_\gamma((x,i),(y,j)) = J_\gamma(x,y)1_{i=j} + J_{\gamma^{1/2}}(x,y)1_{i\neq j}.$$

Recall that for each point (x,i) there is an integer k such that $(x,i) \in C^{(\alpha)}_{k,i}$. Let

$$\hat{J}^{(\alpha)}_\gamma((x,i),(y,j)) = \int_{C^{(\alpha)}_{k,i}\times C^{(\alpha)}_{h,j}} \hat{J}_\gamma((r,i),(r',j))\,dr\,dr'.$$

We want to give a bound of $|\hat{J}_\gamma((x,i),(y,j)) - \hat{J}^{(\alpha)}_\gamma((x,i),(y,j))|$. We consider only the worst case, namely the vertical interaction, $i \neq j$. In this case

$$\begin{aligned}|J_{\gamma^{1/2}}(x,y) - \hat{J}^{(\alpha)}_\gamma((x,i),(y,j))| &\le ⨍_{C^{(\alpha)}_{k,i}\times C^{(\alpha)}_{h,j}} |J_{\gamma^{1/2}}(x,y) - J_{\gamma^{1/2}}(r,r')|\,dr\,dr' \\ &\le c\gamma^{1-\alpha} 1_{|x-y|\le 2\gamma^{-1/2}}.\end{aligned}$$

Let C and C' be two elements in the partition $\mathscr{C}^{(\alpha)}$, and consider two points $(r,i)\in C$ and $(r',i')\in C'$. As in the previous estimate, we consider the worst case. If $i\neq j$, by the estimate above

$$\begin{aligned}\Bigg| \sum_{(x,i)\in C}\sum_{(y,j)\in C'} 1_{|(x,i)\neq(y,j)|} \hat{J}_\gamma((x,i),(y,j))\sigma(x,i)\sigma(y,j) \\ - \sum_{(x,i)\in C}\sum_{(y,j)\in C'} 1_{|(x,i)\neq(y,j)|} \hat{J}^{(\alpha)}_\gamma((x,i),(y,j))\sigma(x,i)\sigma(y,j)\Bigg| \\ \le c'|C|^2\gamma^{1-\alpha} 1_{|r-r'|\le 3\gamma^{-1/2}}. \quad \text{(A-1)}\end{aligned}$$

Then

$$|H_{\gamma,\lambda}(\sigma)-U_{\gamma,\lambda}(\sigma^{(\alpha)})|\le c'|T_{L,N}|\lambda\gamma^{1/2-\alpha}. \tag{A-2}$$

We prove (3-1) writing the definition of the partition function

$$\log(Z_{\beta,\gamma,\lambda}(\{\sigma^{(\alpha)}=\overline{m}\}))\le\beta\lambda\gamma^{1/2-\alpha}|T_{L,N}|c-\beta U_{\gamma,\lambda}(\overline{m})+\log(\mathrm{card}\{\sigma^{(\alpha)}=\overline{m}\}).$$

We can observe that

$$\begin{aligned}\log(\mathrm{card}\{\sigma^{(\alpha)}=\overline{m}\})&=\log\biggl(\prod_{C_{i,k}}\mathrm{card}\biggl\{\sigma\in\{-1,1\}^{C_{k,i}}:\sum_{(x,i)\in C_{k,i}}\sigma(x,i)=\overline{m}(r,i)\gamma^{-\alpha}\text{ for all }(r,i)\biggr\}\biggr)\\&=\log\biggl(\prod_{C_{k,i}}e^{|C_{k,i}|I_{C_{k,i}}(\overline{m}(r,i))}\biggr)\end{aligned}$$

and[4]

$$|I_{C_{k,i}}(\overline{m}(r,i))-I(\overline{m}(r,i))|\le c\gamma^{\alpha}\log\gamma^{-\alpha}. \tag{A-3}$$

At the end collecting the previous inequalities we have

$$\log(Z_{\beta,\gamma,\lambda}(\{\sigma^{(\alpha)}=\overline{m}\}))\le-\beta\overline{F}_{\beta,\gamma,\lambda}(\overline{m})+\beta c|T_{L,N}|(\lambda\gamma^{1/2-\alpha}+\gamma^{\alpha}\log(\gamma^{-\alpha})).$$

The inequality (3-2) is proved in a similar way, so the proposition is proved. □

Proof of Theorem 2.4. We start by introducing the following proposition.

Proposition A.1 (Lebowitz–Penrose limit). *Let* $Z_{\beta,\gamma,\lambda}:=Z_{\beta,\gamma,\lambda}(\mathcal{M}^{(\alpha')})$ *with* $\alpha'\in(0,\frac{1}{2})$; *then*

$$\lim_{\gamma\to0}\frac{1}{\beta|T_{L,N}|}\log Z_{\beta,\gamma,\lambda}=p_{\beta,\lambda}$$

where $p_{\beta,\lambda}=\sup_{m\in[-1,+1]}\{-\phi_{\beta,\lambda}(m)\}$ *and*

$$\phi_{\beta,\lambda}(m)=-\frac{1+2\lambda}{2}m^2-\frac{1}{\beta}I(m). \tag{A-4}$$

Proof. For the proof see Theorem 4.2.1.1 in [Presutti 2009]. □

We consider

$$\begin{aligned}\gamma\log\mu_{\beta,\gamma,\lambda}[\sigma\approx^{\alpha}m]&=\gamma\log\biggl(\frac{Z^{(\alpha)}_{\beta,\gamma,\lambda}(m)}{\sum_{m'\in M^{(\alpha)}}Z^{(\alpha)}_{\beta,\gamma,\lambda}(m')}\biggr)\\&=\gamma\log(Z^{(\alpha)}_{\beta,\gamma,\lambda}(m))-\gamma\log\biggl(\sum_{m'\in\mathcal{M}^{(\alpha)}}Z^{(\alpha)}_{\beta,\gamma,\lambda}(m')\biggr).\end{aligned}$$

[4] The definition of $I_{C_{k,i}}$ and the inequality (A-3) can be found in Appendix A of [Presutti 2009]

By Theorem 2.3 for $\alpha \in (0, 1)$

$$\lim_{\gamma \to 0} \gamma \log(Z^{(\alpha)}_{\beta,\gamma,\lambda}(m)) = -F_{\beta,\lambda}(m).$$

If $\alpha < \frac{1}{2}$ by Propositions 3.1 and A.1, we have that

$$\lim_{\gamma \to 0} -\frac{\gamma}{\beta} \log Z_{\beta,\gamma,\lambda} = \inf_{m' \in \mathcal{M}^{(\alpha)}} F_{\beta,\lambda}(m').$$

For $\alpha > \frac{1}{2}$, instead,

$$\begin{aligned} -\gamma \log \sum_{m' \in \mathcal{M}^{(\alpha)}} Z^{(\alpha)}_{\beta,\gamma,\lambda}(m') &= -\gamma \log \sum_{m' \in \mathcal{M}^{(\alpha)}} \sum_{\sigma : \sigma \approx^{\alpha} m'} e^{-\beta H_{\gamma,\lambda}(\sigma)} \\ &= -\gamma \log \sum_{m' \in \mathcal{M}^{(\alpha)}} \sum_{m_{\alpha'} \in \mathcal{A}_{m'_\alpha}} \sum_{\sigma : \sigma \approx^{\alpha'} m_{\alpha'}} e^{-\beta H_{\gamma,\lambda}(\sigma)} \\ &= -\gamma \log(Z_{\beta,\gamma,\lambda}) \end{aligned}$$

observing that

$$\inf_{m'} F_{\beta,\lambda}(m') = \sup_{h \in [-1,+1]} \{-\phi_{\beta,\lambda}(h)\} \cdot \ell N = p_{\beta,\lambda} \ell N.$$

Then

$$\lim_{\gamma \to 0} \gamma \log \mu_{\beta,\lambda,\gamma}[\sigma \approx^{(\alpha)} m] = -(F_{\beta,\lambda}(m) - \inf_{m'} F_{\beta,\lambda}(m')). \qquad \square$$

Appendix B: A counterexample

In this appendix we shall show that Theorem 2.3 cannot be extended to the case $\beta\lambda > 1$; indeed for the mesoscopic state $m \equiv 0$

$$\liminf_{\gamma \to 0} F^{(\alpha)}_{\beta,\gamma,\lambda}(0) < F_{\beta,\lambda}(0).$$

If $\alpha > \frac{1}{2}$ we can take a sequence m_α where m_α is equal, in the first half of each interval $D^{(\alpha)}_r$, to $m_{\beta\lambda}$ and in the second half to $-m_{\beta\lambda}$; we obtain $m^{(\alpha)}_\alpha \equiv 0$. Recalling Definition 5.1, we have that $m_\alpha \overset{\star}{\to} m \equiv 0$. By the definition of $F^{(\alpha)}_{\beta,\gamma,\lambda}$ and Proposition 3.2,

$$F^{(\alpha)}_{\beta,\gamma,\lambda}(0) \le \frac{1}{\gamma^{-1}} \bar{F}_{\beta,\gamma,\lambda}(\bar{m}_\alpha) + \epsilon(\gamma, \lambda).$$

Now we observe that

$$\bar F_{\beta,\gamma,\lambda}(\overline{m}_\alpha)=\frac{1}{4}\sum_{i=1}^{N}\int_0^{\gamma^{-1}\ell}\int_0^{\gamma^{-1}\ell} J_\gamma(r,r')[\overline{m}_\alpha(r,i)-\overline{m}_\alpha(r',i)]^2\,dr'\,dr$$
$$+\frac{\lambda}{4}\sum_{i=1}^{N}\int_0^{\gamma^{-1}\ell}\int_0^{\gamma^{-1}\ell} J_{\gamma^{1/2}}(r,r')\big([\overline{m}_\alpha(r,i)-\overline{m}_\alpha(r',i-1)]^2$$
$$+[\overline{m}_\alpha(r,i)-\overline{m}_\alpha(r',i+1)]^2\big)\,dr'\,dr+\sum_{i=1}^{N}\int_0^{\gamma^{-1}\ell}\phi_{\beta,\lambda}(\overline{m}_\alpha(r,i))\,dr$$

where $\phi_{\beta,\lambda}$ as in (A-4). Since $\overline{m}_\alpha$ is the same on each line, we have

$$\bar F_{\beta,\gamma,\lambda}(\overline{m}_\alpha)=\frac{N}{4}\int_0^{\gamma^{-1}\ell}\int_{r-\gamma^{-1}}^{r+\gamma^{-1}} J_\gamma(r,r')[\overline{m}_\alpha(r,1)-\overline{m}_\alpha(r',1)]^2\,dr'\,dr$$
$$+\frac{\lambda N}{2}\int_0^{\gamma^{-1}\ell}\int_{r-\gamma^{-1/2}}^{r+\gamma^{-1/2}} J_{\gamma^{1/2}}(r,r')[\overline{m}_\alpha(r,1)-\overline{m}_\alpha(r',1)]^2\,dr'\,dr$$
$$+N\int_0^{\gamma^{-1}\ell}\phi_{\beta,\lambda}(\overline{m}_\alpha(r,1))\,dr.$$

By the symmetry of J and the definition of $\overline{m}_\alpha$ we have

$$\frac{1}{\gamma^{-1}}\bar F_{\beta,\gamma,\lambda}(\overline{m}_\alpha)\le\frac{N}{2}\ell m_{\beta\lambda}^2+N8\lambda\gamma^{\alpha-1/2}m_{\beta\lambda}^2-N\ell\frac{1+2\lambda}{2}\overline{m}_{\beta\lambda}^2-\frac{N\ell}{\beta}I(\overline{m}_{\beta\lambda}).$$

Then

$$\liminf_{\gamma\to0}\frac{1}{\gamma^{-1}}\bar F_{\beta,\gamma,\lambda}(0)\le N\ell\Big(-\lambda\overline{m}_{\beta\lambda}^2-\frac{I(\overline{m}_{\beta\lambda})}{\beta}\Big)$$
$$<-N\ell\frac{I(0)}{\beta}=F_{\beta,\lambda}(0).$$

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