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RELAXED EXACT CONTROLLABILITY AND ASYMPTOTIC LIMIT FOR THIN SHELLS

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Abstract. This paper deals with a singular perturbation problem related to the relaxed exact controllability of a thin shell and its membrane approximation. We point out the subspaces in which we can construct control functions and which allow us to look at the asymptotic limit. Since the problem depends on the geometry of the shell and the selected boundary control action, specific results for elastic hemispherical shells are given.

1. Introduction

The shell vibrations can be described by the system of differential equations

$$\begin{cases} \mathbf{v}_{tt} + \mathbf{A}^\varepsilon \mathbf{v} = 0 & \text{in} \quad \Omega \times (0,T) \\ \mathbf{B}^\varepsilon \mathbf{v} = 0 & \text{on} \quad \partial\Omega \times (0,T) \end{cases} \tag{1.1}$$

and the initial conditions $\mathbf{v}(0) = \mathbf{v}^0$, $\mathbf{v}_t(0) = \mathbf{v}^1$; where $\mathbf{A}^\varepsilon = \mathbf{A}^0 + \varepsilon \mathbf{A}^1$ is a linear, self-adjoint Douglas-Nirenberg elliptic operator of mixed order which depends on the shell thinness parameter $\varepsilon \geq 0$ and $\mathbf{B}^\varepsilon = \{\mathbf{B}^0, \sqrt{\varepsilon}\,\mathbf{B}^1\}$ is a system of normal boundary conditions. The operator $\mathbf{A}^\varepsilon$ is associated with a symmetric sesquilinear form

$$a^\varepsilon(\mathbf{u},\mathbf{v}) = a^0(\mathbf{u},\mathbf{v}) + \varepsilon a^1(\mathbf{u},\mathbf{v}), \tag{1.2}$$

which we assume continuous and coercive in the Hilbert space $\mathbf{V} \subset \mathbf{H}$.

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We refer to the papers of Grubb and Geymonat (see [5] and [6]) for the spectral analysis of elliptic systems of mixed order.

The exact controllability problem is strictly connected to the spectral properties of $\mathbf{A}^\varepsilon$; in particular, when $\varepsilon > 0$ the spectrum of $\mathbf{A}$ is formed only of eigenvalues of finite multiplicity, and the exact controllability problem can be solved with the techniques of harmonic analysis and multiplier methods (see [2], [3], [10], [12], and [13]).

Moreover it is well known, in the linear thin-shell theory, that the limit as $\varepsilon \to 0$ leads to a *singular perturbation* problem. When ε vanishes $\mathbf{A}^0$ has a nonempty essential spectrum and the system is no more controllable; that is, there exist some initial data such that the system (1.1) is not exactly controllable. In a recent paper [4] we proved, by means of the Weyl sequences, an abstract theorem of noncontrollability of Douglas-Nirenberg operators of mixed order with nonempty essential spectrum. The established result suggests that if we impose boundary control functions on the system (1.1), the asymptotic limit when $\varepsilon \to 0$ can not, in general, be computed. It is quite natural to wonder whether there exist some initial data which we can control also to the limit as $\varepsilon \to 0$. The problem is, then to solve the controllability problem in a relaxed sense and to look for subspaces where displacements and controls converge uniformly with respect to ε.

We assume that the shell is inhibited (i.e., $a^0(\mathbf{u}, \mathbf{u}) > 0$). In this case the vibrations are *medium frequency* vibrations and their limit behavior only involves the form a^0, so the *membrane approximation* and its spectral properties assume an important role in the present situation. We consider the case in which $a^0(\mathbf{u}, \mathbf{u})$ is continuous and coercive (i.e., well-inhibited); moreover, we denote by $\mathbf{V}^0$ the limit space completion of $\mathbf{V}$ with the norm $a^0(\mathbf{u}, \mathbf{u})$.

Since the essential spectrum of $\mathbf{A}^0$, which we denote by $\sigma_{ess}(\mathbf{A}^0)$, is a bounded set and since $\mathbf{A}^0$ is a positive definite, self-adjoint operator, we can consider the inverse operator $(\mathbf{A}^0)^{-1}$ and decompose it into two operators $(\mathbf{A}^0)_0^{-1}$ and $(\mathbf{A}^0)_*^{-1}$ where $(\mathbf{A}^0)_0^{-1}$ is a self-adjoint, compact operator of class C_p. That is, if we denote by σ_j its eigenvalues arranged in decreasing order and repeated according to multiplicity, it happens that

$$\sum_{j=1}^{\infty} \sigma_j^p < +\infty, \qquad 0 < p < +\infty.$$

Assumption I. We assume that there exists a positive value μ^* (which is not an eigenvalue of $(\mathbf{A}^0)^{-1}$) and an index j^* such that $\{\sigma_j\}_{j>j^*}$ are the only eingenvalues of $(\mathbf{A}^0)^{-1}$ in the set $(0, \mu^*)$.

We introduce the sequence $\{\lambda_k^0\}$ of eigenvalues defined by $\lambda_k^0 = 1/\sigma_{j^*+k}$, $k = 1, \ldots$, and denote by $\mathbf{E}_0$ the subspace spanned by the eigenfunctions $\boldsymbol{\varphi}_j^0$ corresponding to the eigenvalues λ_j^0 and by $\mathbf{E}_0^*$ the complementary subspace.

When we disturb the spectrum of $\mathbf{A}^0$ by the small positive parameter ε, $\sigma_{ess}(\mathbf{A}^0)$ generates a sequence of eigenvalues. The operator $\mathbf{A}^\varepsilon$ has only a discrete spectrum with accumulation point $+\infty$; moreover $\mathbf{A}^\varepsilon$ is a C_p operator. Also in this case we can consider the decomposition of $(\mathbf{A}^\varepsilon)^{-1}$ into a couple of compact operators $(\mathbf{A}^\varepsilon)_0^{-1}$ and $(\mathbf{A}^\varepsilon)_*^{-1}$ such that the C_p norm

$$\|(\mathbf{A}^\varepsilon)_0^{-1} - (\mathbf{A}^0)_0^{-1}\|_p \to 0 \quad \text{as } \varepsilon \to 0. \tag{1.3}$$

We denote by $\mathbf{E}_\varepsilon$ the subspace spanned by the eigenfunctions $\boldsymbol{\varphi}_j^\varepsilon$ corresponding to the eigenvalues λ_j^ε (derived from the perturbation of λ_j^0) and by $\mathbf{E}_\varepsilon^*$ the complementary subspace.

We also remark that the power p depends on ε and the dimension of Ω. In this situation we can consider the following *relaxed* exact controllability problems.

$(\mathcal{P}_\varepsilon)$ Let $\widehat{\mathbf{G}}_\varepsilon$ be a closed subspace of $\mathbf{H} \times \mathbf{V}'$. We say that there is *relaxed exact controllability* if given $T > 0$ and an initial state $\{\mathbf{z}^0, \mathbf{z}^1\} \in \mathbf{H} \times \mathbf{V}'$ there exists a control $\mathbf{g}_\varepsilon \in L^2(\Sigma)$ such that the unique solution $\mathbf{z}_\varepsilon$ of

$$\begin{array}{cc}
\ddot{\mathbf{z}}_\varepsilon + \mathbf{A}^0\mathbf{z}_\varepsilon + \varepsilon\mathbf{A}^1\mathbf{z}_\varepsilon = 0 & \text{in } Q = \Omega \times (0,T) \\
\mathbf{B}^0\mathbf{z}_\varepsilon = \mathbf{g}_\varepsilon^1, \quad \sqrt{\varepsilon}\,\mathbf{B}^1\mathbf{z}_\varepsilon = \mathbf{g}_\varepsilon^2 & \text{on } \Sigma = \partial\Omega \times (0,T) \\
\mathbf{z}_\varepsilon(0) = \mathbf{z}^0 \ , \ \dot{\mathbf{z}}_\varepsilon(0) = \mathbf{z}^1 & \text{in } \Omega
\end{array}$$

satisfies the following condition: $\{\mathbf{z}_\varepsilon(T), \dot{\mathbf{z}}_\varepsilon(T)\} \in \widehat{\mathbf{G}}_\varepsilon$.

The aim of this paper is to find sufficient conditions on the subspace $\widehat{\mathbf{G}}_\varepsilon$ in order to pass to the limit as $\varepsilon \to 0$ and obtain uniform convergence to the nonperturbed, relaxed exact controllability problem $(\mathcal{P})$, which reads

$(\mathcal{P})$ Let $\widehat{\mathbf{G}}_0$ be a closed subspace of $\mathbf{H} \times (\mathbf{V}^0)'$. We have relaxed exact controllability if, given $T > 0$ and an initial state $\{\mathbf{z}^0, \mathbf{z}^1\} \in \mathbf{H} \times (\mathbf{V}^0)'$ there exists a control $\mathbf{g}_0 \in L^2(\Sigma)$ such that the unique solution $\mathbf{z}$ of

$$\begin{array}{cc}
\ddot{\mathbf{z}} + \mathbf{A}^0\mathbf{z} = 0 & \text{in } Q = \Omega \times (0,T) \\
\mathbf{B}^0\mathbf{z} = \mathbf{g}_0 & \text{on } \Sigma = \partial\Omega \times (0,T) \\
\mathbf{z}(0) = \mathbf{z}^0 \ , \ \dot{\mathbf{z}}(0) = \mathbf{z}^1 & \text{in } \Omega
\end{array}$$

satisfies the following condition: $\{\mathbf{z}(T), \dot{\mathbf{z}}(T)\} \in \widehat{\mathbf{G}}_0$.

The next section is devoted to recalling and determining some convergence results for the homogeneous problem, also referring, in particular, to some convergence results in the papers of Sanchez-Palencia (see for example [14] and [15]) and Kato [8].

The relaxed exact controllability for the hemispherical shell and the membrane approximation is examined and carried out in Section 3 as an example of our theory.

2. The homogeneous problem

Since we want to attack the controllability problem by the HUM method (see [9] and [11]), we consider the homogeneous problems associated with $(\mathcal{P}_\varepsilon)$.

$(\mathcal{HP}_\varepsilon)$We denote by $\mathbf{y}_\varepsilon$ the solution of the perturbed homogeneous problem

$$\begin{array}{cc} \ddot{\mathbf{y}}_\varepsilon + \mathbf{A}^\varepsilon \mathbf{y}_\varepsilon = 0 & \text{in } Q = \Omega \times (0,T) \\ \mathbf{B}^0 \mathbf{y}_\varepsilon = 0, \quad \sqrt{\varepsilon}\,\mathbf{B}^1 \mathbf{y}_\varepsilon = 0 & \text{on } \Sigma = \partial\Omega \times (0,T) \\ \mathbf{y}_\varepsilon(0) = \mathbf{y}^0_\varepsilon \ , \ \dot{\mathbf{y}}_\varepsilon(0) = \mathbf{y}^1_\varepsilon & \text{in } \Omega \end{array}$$

and define $\mathbf{F}_{\widehat{\mathbf{G}}_\varepsilon} = \{(\mathbf{y}^0, \mathbf{y}^1) : \mathbf{y}^0 \in \mathbf{V}\,, \mathbf{y}^1 \in \mathbf{H}$ such that $(\dot{\mathbf{y}}_\varepsilon(T), -\mathbf{y}_\varepsilon(T)) \in \mathbf{G}^*_\varepsilon\}$, where $\mathbf{G}^*_\varepsilon$ is the polar set of $\widehat{\mathbf{G}}_\varepsilon$ in $\mathbf{H} \times \mathbf{V}$; i.e.,

$$\{\mathbf{f}^0, \mathbf{f}^1\} \in \mathbf{G}^*_\varepsilon \iff (\mathbf{f}^0, \mathbf{g}^0) + (\mathbf{f}^1, \mathbf{g}^1) = 0 \qquad \forall \{\mathbf{g}^0, \mathbf{g}^1\} \in \widehat{\mathbf{G}}_\varepsilon.$$

We assume that $(\mathbf{y}^0, \mathbf{y}^1) \in \mathbf{G}_\varepsilon$ where $\mathbf{G}_\varepsilon$ is the completion of the subspace $(\mathbf{E}_\varepsilon \times \mathbf{E}_\varepsilon)$ with the norm of $\mathbf{V} \times \mathbf{H}$. It follows from the well-known Proposition 1 below that $\mathbf{G}_\varepsilon$ is a subset of $\mathbf{F}_{\widehat{\mathbf{G}}_\varepsilon}$, and in the sequel we identify $\mathbf{G}_\varepsilon$ with $\mathbf{F}_{\widehat{\mathbf{G}}_\varepsilon}$. The first step is to establish some convergence results (as $\varepsilon \to 0$) to the limit homogeneous problem:

$(\mathcal{HP})$ Given $\{\mathbf{y}^0, \mathbf{y}^1\} \in \mathbf{G}_0$ (completion of $(\mathbf{E}_0 \times \mathbf{E}_0)$ with the norm $\mathbf{V}^0 \times \mathbf{H}$), the limit homogeneous problem reads

$$\begin{array}{cc} \ddot{\mathbf{y}} + \mathbf{A}^0 \mathbf{y} = 0 & \text{in } Q = \Omega \times (0,T) \\ \mathbf{B}^0 \mathbf{y} = 0 & \text{on } \Sigma = \partial\Omega \times (0,T) \\ \mathbf{y}(0) = \mathbf{y}^0 \ , \ \dot{\mathbf{y}}(0) = \mathbf{y}^1 & \text{in } \Omega. \end{array}$$

We briefly recall the existence and uniqueness theorem for problems $(\mathcal{HP}_\varepsilon)$ and $(\mathcal{HP})$.

Proposition 1. [Existence and Uniqueness Theorem]. *Let* $\mathbf{V}^0$ (*respectively* $\mathbf{V}$) *and* $\mathbf{H}$ *be two Hilbert spaces,* $\mathbf{V}^0$ (*respectively* $\mathbf{V}$) $\subset \mathbf{H}$ *with dense and continuous imbedding. Let* a^0 (*respectively* a^ε) *be a continuous, symmetric and coercive form in* $\mathbf{V}^0$ (*respectively* $\mathbf{V}$), *then there exists a unique solution of the problem* $(\mathcal{HP})$ (*respectively* $\mathcal{HP}_\varepsilon$) *which belongs to* $\mathbf{G}_0$ (*respectively* $\mathbf{G}_\varepsilon$).

In our situation we have $\mathbf{H} = L^2(\Omega)^3$; $\mathbf{V}$ is the subspace of $H^1(\Omega) \times H^1(\Omega) \times H^2(\Omega)$ of the functions satisfying the boundary condition defined by the operator $\mathbf{B}^\varepsilon$ and $\mathbf{V}^0 = H^1(\Omega) \times H^1(\Omega) \times L^2(\Omega)$, with the boundary condition defined by the operator $\mathbf{B}^0$, is the limit space completion of $\mathbf{V}$ with the norm $a^0(\mathbf{v},\mathbf{v})$. We denote by $\{\mathbf{y}_\varepsilon^0, \mathbf{y}_\varepsilon^1\}$ the initial data in the space $\mathbf{G}_\varepsilon \in \mathbf{V} \times \mathbf{H}$. The unique solution to the problem $\mathcal{HP}_\varepsilon$ satisfies

$$\mathbf{y}_\varepsilon \in L^\infty(0,T;\mathbf{V}), \quad \dot{\mathbf{y}}_\varepsilon \in L^\infty(0,T;\mathbf{H});$$

moreover, $\mathbf{y}_\varepsilon(0) = \mathbf{y}_\varepsilon^0$, and the following integral equality holds true:

$$\int_\Omega \mathbf{y}_\varepsilon^1 \mathbf{u} + \int_0^T \int_\Omega \dot{\mathbf{y}}_\varepsilon \dot{\mathbf{u}} = \int_0^T a^0(\mathbf{y}_\varepsilon, \mathbf{u}) + \varepsilon \int_0^T a^1(\mathbf{y}_\varepsilon, \mathbf{u}) \tag{2.1}$$

for any $\mathbf{u}$ in the space of the test functions with $\mathbf{u}(T) = 0$. We introduce the energy

$$e_\varepsilon(\mathbf{y}_\varepsilon, \dot{\mathbf{y}}_\varepsilon) = \frac{1}{2}\{\|\dot{\mathbf{y}}_\varepsilon\|^2 + a^0(\mathbf{y}_\varepsilon, \mathbf{y}_\varepsilon) + \varepsilon a^1(\mathbf{y}_\varepsilon, \mathbf{y}_\varepsilon)\}, \tag{2.2}$$

and after a formal multiplication by $\dot{\mathbf{y}}_\varepsilon$ we get the energy identity

$$e_\varepsilon(\mathbf{y}_\varepsilon, \dot{\mathbf{y}}_\varepsilon) = e_\varepsilon(\mathbf{y}_\varepsilon^0, \mathbf{y}_\varepsilon^1) \tag{2.3}$$

where

$$e(\mathbf{y}_\varepsilon^0, \mathbf{y}_\varepsilon^1) = \tfrac{1}{2}\{\|\mathbf{y}_\varepsilon^1\|^2 + a^0(\mathbf{y}_\varepsilon^0, \mathbf{y}_\varepsilon^0)\} \text{ and } e_\varepsilon(\mathbf{y}_\varepsilon^0, \mathbf{y}_\varepsilon^1) = e(\mathbf{y}_\varepsilon^0, \mathbf{y}_\varepsilon^1) + \tfrac{1}{2}\varepsilon a^1(\mathbf{y}_\varepsilon^0, \mathbf{y}_\varepsilon^0).$$

Since $a^\varepsilon(\mathbf{u},\mathbf{u}) = a^0(\mathbf{u},\mathbf{u}) + \varepsilon a^1(\mathbf{u},\mathbf{u})$ and $a^0(\mathbf{u},\mathbf{u})$ are continuous and coercive in $\mathbf{V}$ and $\mathbf{V}^0$ respectively, it follows that $e_\varepsilon(\mathbf{u}, \dot{\mathbf{u}})$ is a norm equivalent to the norm in $\mathbf{V} \times \mathbf{H}$ and $e(\mathbf{u}, \dot{\mathbf{u}})$ is a norm equivalent to the norm in $\mathbf{V}^0 \times \mathbf{H}$.

2.1. **Convergence of the initial data.** We prove some preliminary results we shall use later.

Lemma 2.1. *If Assumption I and* (1.3) *are satisfied, then for each* $\{\mathbf{y}_\varepsilon^0, \mathbf{y}_\varepsilon^1\} \in \mathbf{G}_\varepsilon \in \mathbf{V} \times \mathbf{H}$, *we have* $\mathbf{y}_\varepsilon^i \to \mathbf{y}^i$ *strongly in* $\mathbf{H}$, $i = 0, 1$ *and moreover* $\{\mathbf{y}^0, \mathbf{y}^1\} \in \mathbf{G}_0$.

Proof. We consider the spectral families $\mathcal{E}(\mathbf{A}^0, \lambda)$, $\mathcal{E}(\mathbf{A}^\varepsilon, \lambda)$ corresponding to the operators $\mathbf{A}^0$ and $\mathbf{A}^\varepsilon$. The proof of the lemma is a consequence of the Rellich theorem on the convergence of the spectral families related to the bounded inverse operators $(\mathbf{A}^0)^{-1}$ and $(\mathbf{A}^\varepsilon)^{-1}$, in particular to the inverse operators $(\mathbf{A}^0)_0^{-1}$ and $(\mathbf{A}^\varepsilon)_0^{-1}$. Indeed we have (see [15]) that from (1.3) and if μ^* is not an eigenvalue of $(\mathbf{A}^0)^{-1}$ (and hence of $(\mathbf{A}^0)_0^{-1}$), the following convergence holds true

$$\mathcal{E}((\mathbf{A}^\varepsilon)_0^{-1}, \mu^*)\mathbf{u} \to \mathcal{E}((\mathbf{A}^0)_0^{-1}, \mu^*)\mathbf{u} \qquad \text{strongly in } \mathbf{H}. \tag{2.4}$$

Since $(\mathbf{A}^\varepsilon)_0^{-1}$ has a discrete spectrum with eigenfunctions $\boldsymbol{\psi}_n^\varepsilon$, the left-hand side of (2.4) reads

$$\mathcal{E}((\mathbf{A}^\varepsilon)_0^{-1}, \mu^*)\mathbf{u} = \sum_{1/\lambda_n^\varepsilon < \mu^*} (\mathbf{u}, \boldsymbol{\psi}_n^\varepsilon)\boldsymbol{\psi}_n^\varepsilon = \sum_{\lambda_n^\varepsilon > 1/\mu^*} (\mathbf{u}, \boldsymbol{\psi}_n^\varepsilon)\boldsymbol{\psi}_n^\varepsilon. \tag{2.5}$$

From the hypothesis made on μ^*, we have for the right-hand side of (2.4) the equality

$$\mathcal{E}((\mathbf{A}^0)_0^{-1}, \mu^*)\mathbf{u} = \sum_{\lambda_n^0 > 1/\mu^*} (\mathbf{u}, \boldsymbol{\varphi}_n^0)\boldsymbol{\varphi}_n^0. \tag{2.6}$$

Now if we put $\mathbf{u} = \mathbf{y}_\varepsilon^i \in \mathbf{E}_\varepsilon$ with $i = 0, 1$, it follows that

$$\sum_{\lambda_n > 1/\mu^*} (\mathbf{u}, \boldsymbol{\psi}_n^\varepsilon)\boldsymbol{\psi}_n^\varepsilon = \sum_{\lambda_n^\varepsilon > 1/\mu^*} (\mathbf{y}_\varepsilon^i, \boldsymbol{\varphi}_n^\varepsilon)\boldsymbol{\varphi}_n^\varepsilon = \mathbf{y}_\varepsilon^i \to \sum_{\lambda_n^0 > 1/\mu^*} (\mathbf{y}_\varepsilon^i, \boldsymbol{\varphi}_n^0)\boldsymbol{\varphi}_n^0 = \mathbf{y}^i \tag{2.7}$$

strongly in $\mathbf{H}$ and $\mathbf{y}^i \in \mathbf{E}_0$. □

Associated with the boundary condition defined by $\mathbf{B}^\varepsilon$, we consider the system of normal complementary boundary conditions $\mathbf{C}^\varepsilon = \{\mathbf{C}^0, \sqrt{\varepsilon}\,\mathbf{C}^1\}$. The control problem requires us to find estimates for the quantity $\|\mathbf{C}^\varepsilon(\mathbf{y}_\varepsilon)\|_\Sigma^2$. In this section we suppose that we have them, and for the sake of simplicity, in the case in which we have only one control, $\mathbf{C}^\varepsilon$ is identified with one of its components. So $\mathbf{C}^\varepsilon$ stands for the normal boundary condition complementary to $\mathbf{B}^0$ (respectively $\mathbf{B}^1$) if we put $\mathbf{g}_\varepsilon^2 = 0$ (respectively $\mathbf{g}_\varepsilon^1 = 0$).

We introduce the operator $\Lambda_{\widehat{\mathbf{G}}_\varepsilon}$ such that

$$\left\langle \Lambda_{\widehat{\mathbf{G}}_\varepsilon}\{\mathbf{y}_\varepsilon^0, \mathbf{y}_\varepsilon^1\}, \{\mathbf{y}_\varepsilon^0, \mathbf{y}_\varepsilon^1\} \right\rangle = \|\mathbf{C}^\varepsilon \mathbf{y}_\varepsilon\|_\Sigma^2. \tag{2.8}$$

Assumption II. We suppose that there exists a time T^0 independent of ε such that for each $T > T^0$, the quantity (2.8) defines a norm in $\mathbf{F}_{\widehat{\mathbf{G}}_\varepsilon}$ equivalent to the energy norm; i.e., there exist two constants c_1, c_2 independent of ε such that

$$c_1(T)e_\varepsilon(\mathbf{y}_\varepsilon^0, \mathbf{y}_\varepsilon^1) \le \|\mathbf{C}^\varepsilon \mathbf{y}_\varepsilon\|_\Sigma^2 \le c_2(T)e_\varepsilon(\mathbf{y}_\varepsilon^0, \mathbf{y}_\varepsilon^1). \tag{2.9}$$

The estimates (2.9) allow us to solve the problem

$$\left\langle \Lambda_{\widehat{\mathbf{G}}_\varepsilon}\{\mathbf{y}_\varepsilon^0, \mathbf{y}_\varepsilon^1\}, \{\mathbf{y}_\varepsilon^0, \mathbf{y}_\varepsilon^1\} \right\rangle = \int_\Omega \mathbf{z}^1\mathbf{y}_\varepsilon^0 - \mathbf{z}^0\mathbf{y}_\varepsilon^1. \tag{2.10}$$

for any $\mathbf{y}_\varepsilon^0, \mathbf{y}_\varepsilon^1 \in \mathbf{G}_\varepsilon$.

Lemma 2.2. *Under Assumption II, we have that*

$$\mathbf{y}_\varepsilon^1 \to \mathbf{y}^1 \ \text{ strongly in } \mathbf{H}, \qquad \mathbf{y}_\varepsilon^0 \to \mathbf{y}^0 \ \text{ weakly in } \mathbf{V}^0$$

with $\{\mathbf{y}^0, \mathbf{y}^1\} \in \mathbf{G}_0$*. Moreover, the quantity* $\varepsilon a^1(\mathbf{y}^0_\varepsilon, \mathbf{y}^0_\varepsilon)$ *is bounded uniformly in* ε*.*

Proof. From the right-hand side of (2.10) and for any $\{\mathbf{z}^0, \mathbf{z}^1\} \in \mathbf{H} \times (\mathbf{V}^0)'$

$$\Big| \int_\Omega \mathbf{z}^1 \mathbf{y}^0_\varepsilon - \mathbf{z}^0 \mathbf{y}^1_\varepsilon \Big| \leq \hat{c}_1 \|\mathbf{y}^0_\varepsilon\|_{\mathbf{V}^0} + \hat{c}_2 \|\mathbf{y}^1_\varepsilon\|_{\mathbf{H}},$$

and taking into account (2.8), we deduce that

$$\|\mathbf{C}^\varepsilon \mathbf{y}_\varepsilon\|^2_\Sigma \leq \hat{c}\, e(\mathbf{y}^0_\varepsilon, \mathbf{y}^1_\varepsilon)^{1/2}. \tag{2.11}$$

Moreover, from the left inequality in (2.9) and from the above inequality (2.11) we have that

$$c_1(T) e(\mathbf{y}^0_\varepsilon, \mathbf{y}^1_\varepsilon) \leq c_1(T) e_\varepsilon(\mathbf{y}^0_\varepsilon, \mathbf{y}^1_\varepsilon) \leq \|\mathbf{C}^\varepsilon \mathbf{y}_\varepsilon\|^2_\Sigma \leq \hat{c}\, e(\mathbf{y}^0_\varepsilon, \mathbf{y}^1_\varepsilon)^{1/2},$$

which implies

$$e(\mathbf{y}^0_\varepsilon, \mathbf{y}^1_\varepsilon) \leq C \tag{2.12}$$

and also $\varepsilon a^1(\mathbf{y}^0_\varepsilon, \mathbf{y}^0_\varepsilon) \leq C$ where C is independent of ε. So we can obtain weak convergence for a subsequence, which we also denote by $\mathbf{y}^0_\varepsilon, \mathbf{y}^1_\varepsilon$ in $\mathbf{V}^0 \times \mathbf{H}$. Moreover from the strong convergence obtained in Lemma 2.1, we have that the limit functions $\{\mathbf{y}^0, \mathbf{y}^1\} \in \mathbf{G}_0$. □

2.2. **Convergence of the solution of the homogeneous problem.** As a consequence of the Lemma 2.1 and Lemma 2.2, we have the *convergence* of the solution of the problem $(\mathcal{HP}_\varepsilon)$ to the unique solution of the limit problem $(\mathcal{HP})$. Indeed, from the convergence of the initial data (see for example [14], Theorem 3.3 and Remark 3.4) we deduce the following estimates which allow us to pass to the limit as ε goes to zero:

$$\|\dot{\mathbf{y}}_\varepsilon\|^2 \leq \text{const.}, \quad a^0(\mathbf{y}_\varepsilon, \mathbf{y}_\varepsilon) \leq \text{const.}, \quad \varepsilon a^1(\mathbf{y}_\varepsilon, \mathbf{y}_\varepsilon) \leq \text{const.};$$

therefore, we can find a convergent subsequence such that

$$\dot{\mathbf{y}}_\varepsilon \to \dot{\mathbf{y}} \ \text{ in } L^\infty(0,T;\mathbf{H}) \text{ weak} \star, \qquad \mathbf{y}_\varepsilon \to \mathbf{y} \ \text{ in } L^\infty(0,T;\mathbf{V}^0) \text{ weak} \star,$$

and since $a^1(\mathbf{u}, \mathbf{w}) \leq a^1(\mathbf{u}, \mathbf{u})^{1/2} a^1(\mathbf{w}, \mathbf{w})^{1/2}$, the last term in (2.1) vanishes; thanks to the previous lemmas it follows that $\mathbf{y}_\varepsilon$ converges to $\mathbf{y}$, which is the unique solution to the limit problem.

3. Some convergence results for the controllability problem

Taking into account the analysis carried out in the previous section we are in a position to give the following result:

Theorem 3.1. *Given an initial state* $\{\mathbf{z}^0, \mathbf{z}^1\} \in \mathbf{H} \times (\mathbf{V}^0)'$*, if Assumption II is satisified then the unique solution* $\mathbf{z}_\varepsilon$ *of the system*

$$\begin{aligned} &\ddot{\mathbf{z}}_\varepsilon + \mathbf{A}^\varepsilon \mathbf{z}_\varepsilon = 0 && \text{in } Q = \Omega \times (0,T) \\ &\mathbf{B}^\varepsilon \mathbf{z}_\varepsilon = \mathbf{g}_\varepsilon, && \text{on } \Sigma = \partial\Omega \times (0,T) \\ &\mathbf{z}_\varepsilon(0) = \mathbf{z}^0 \ , \ \dot{\mathbf{z}}_\varepsilon(0) = \mathbf{z}^1 && \text{in } \Omega \end{aligned} \tag{3.1}$$

with

$$\mathbf{g}_\varepsilon = \mathbf{C}^\varepsilon \mathbf{y}_\varepsilon \tag{3.2}$$

where $\mathbf{y}_\varepsilon$ *is the unique solution of the homogeneous problem computed by initial data* $\mathbf{y}^0_\varepsilon, \mathbf{y}^1_\varepsilon \in \mathbf{G}_\varepsilon$*, satisfies*

$$\{\mathbf{z}_\varepsilon(T), \dot{\mathbf{z}}_\varepsilon(T)\} \in \widehat{\mathbf{G}}_\varepsilon. \tag{3.3}$$

Moreover,

$$\mathbf{g}_\varepsilon \to \mathbf{q}(t) \qquad \text{weakly in} \quad L^2(\Sigma) \qquad \text{as } \varepsilon \to 0. \tag{3.4}$$

Proof. The first part of the theorem, thanks to Assumption II, follows from the HUM method for relaxed controllability problems (see [11]). The convergence (3.4) follows from Lemma 2.1 and Lemma 2.2. Indeed, from (2.11) and the estimate (2.12) we have that $\|\mathbf{C}^\varepsilon \mathbf{y}_\varepsilon\|^2_\Sigma \leq \text{const.}$; hence, passing to the limit up to subsequences for $\varepsilon \to 0$, we obtain the weak convergence in $L^2(\Sigma)$. □

The asymptotic behavior of the complementary boundary conditions, that is the computation of the limit control function, depends on the geometry of the shell and the specific control problem. In order to characterize the function $\mathbf{q}(t)$ which appears in (3.4) we have to specify the control action on the boundary. In the sequel we shall analyze an example related to the case of elastic hemispherical shells. However we conjecture that, with reference to the subspaces introduced, also in the general case, if the problem $(\mathcal{P}_\varepsilon)$ is exactly controllable in a relaxed sense (as specified above), one can show the convergence of the control function $\mathbf{g}_\varepsilon$ to the control function of the limit problem.

3.1. **A uniqueness result for the relaxed exact controllability of hemispherical shells and asymptotic limits.** As a nontrivial example of our theory we consider the axially symmetric vibrations of a thin hemispherical shell. The main step in the HUM method, to which we intend to refer, is the proof of a uniqueness theorem related to the homogeneous problem.

When the problem, as in this case and in general for a surface of revolution, is set in one space dimension we can use Fourier series expansions to solve the homogeneous problem, and the uniqueness theorem is a consequence of the existence of an asymptotic gap for the eigenvalues according to the Ingham and Ball-Slemrod theorems (see [1] and [7]). The eigenvalues problem and the exact controllability problem for spherical shells has been carried out in some of our previous papers (see for example [2] and [3]); here we recall only some spectral properties in the perturbed and unperturbed case which allow us to get the uniqueness result. The system describing the vibrations of hemispherical shells in terms of the meridianal and radial components of the displacement vector $\mathbf{z} = (z_1, z_2)$ is the following,

$$\begin{cases} \ddot{z}_1 - \mathcal{L}(z_1) + (1+\nu)z_2' - \varepsilon\mathcal{L}(z_1 + z_2') = 0 \\ \ddot{z}_2 + \frac{\varepsilon}{\sin\theta}(\mathcal{L}(z_1 + z_2')\sin\theta)' - \frac{(1+\nu)}{\sin\theta}(z_1 \sin\theta)' + 2(1+\nu)z_2 = 0, \end{cases} \tag{3.5}$$

where $\mathcal{L}(f) = f'' + f'cot\theta - f(\nu + cot^2\theta)$ and the "prime" stands for the first derivative with respect to the opening angle θ varying in the open interval $(0, \theta_o)$. In this section we put $\theta_0 = \pi/2$ and assume control of the vibrations by the action of a unique boundary control $g(t)$. The following boundary conditions are associated with the equations (3.5):

$$\begin{cases} z_1(0,t) = 0, \quad z_2(\theta_o, t) = g_\varepsilon(t) \\ z_1(0,t) + z_2'(0,t) = z_1(\theta_o, t) + z_2'(\theta_o, t) = 0 \\ \mathcal{L}(z_1 + z_2')_{(0,t)} = \mathcal{L}(z_1 + z_2')_{(\theta_o,t)} = 0. \end{cases} \tag{3.6}$$

Let $L^2(0, \theta_0; \sin\theta)$ be the space of the square-integrable functions with respect to the weight $\sin\theta$; we define the following spaces:

$$U = \{u : \frac{\partial u}{\partial\theta}, u\cot\theta \in L^2(0,\theta_0;\sin\theta)\} \quad U_0 = \{u \in U : u(\theta_0) = 0\},$$
$$W = \{w : w \in L^2(0,\theta_0;\sin\theta)\}, \quad W_0 = \{w \in W : w' \in U_0\},$$

and the vector spaces

$$\mathbf{V} = U_0 \times W_0,\ \mathbf{V}^0 = U_0 \times L^2(0,\theta_0;\sin\theta),\ \mathbf{H} = L^2(0,\theta_0;\sin\theta) \times L^2(0,\theta_0;\sin\theta).$$

We recall that for the eigenvalues problem associated with (3.5) we find two subsequences of eigenvalues:

$$\lambda_n^\varepsilon = \frac{1}{2}\{\varepsilon l_n^2 + ((4+\nu)\varepsilon + 1)l_n + 3(1+\nu)(1+\varepsilon)$$
$$+ \sqrt{(\varepsilon l_n^2 + ((4+\nu)\varepsilon+1)l_n + 3(1+\nu)(1+\varepsilon))^2 - 4l_n(\varepsilon l_n^2 + 2\varepsilon l_n + (1+\varepsilon)(1-\nu^2)}\}$$

and

$$\lambda_n^*(\varepsilon) = \frac{1}{2}\{\varepsilon l_n^2 + ((4+\nu)\varepsilon + 1)l_n + 3(1+\nu)(1+\varepsilon)$$
$$- \sqrt{(\varepsilon l_n^2 + ((4+\nu)\varepsilon+1)l_n + 3(1+\nu)(1+\varepsilon))^2 - 4l_n(\varepsilon l_n^2 + 2\varepsilon l_n + (1+\varepsilon)(1-\nu^2)}\}$$

with $\lambda_0 = 2(1+\nu)$, $l_n = 2n(2n+1) - 2$, $n = 1, \ldots$.

Both the sequences go to infinity as $n \to \infty$. For $\varepsilon = 0$ the two subsequences λ_n^0 and $\lambda_n^*(0)$ have different asymptotic behaviors. While the sequence $\lambda_n^0 \to \infty$ as $n \to \infty$, the sequence $\lambda_n^*(0)$ tends to a finite accumulation point; that is, $\lambda_n^*(0) \to (1-\nu^2)$. So the limit operator has a nonempty essential spectrum and the accumulation point $(1-\nu^2)$ is the element of the essential spectrum. Moreover for $\varepsilon = 0$ one has the following eigenvalues arrangement:

$$0 < \lambda_1^*(0) < \lambda_2^*(0) < \cdots < (1-\nu^2) < 2(1+\nu) = \lambda_0 < \lambda_1^0 < \lambda_2^0 < \cdots.$$

We observe that Assumption I is satisfied. Indeed, we can fix a closed curve Γ crossing the point λ^* with $(1-\nu^2) < \lambda^* < 2(1+\nu)$ and containing all the eigenvalues $\lambda_j^*(0)$. We consider the space $\mathbf{E}_0$ spanned by the eigenfuntions $\boldsymbol{\varphi}_n^0$ associated with the eigenvalues λ_n^0 and denote by $\mathbf{E}_\varepsilon$ the space spanned by the eigenfunction $\boldsymbol{\varphi}_n^\varepsilon$ associated with the perturbed eigenvalues λ_n^ε. If we take the initial data $(\mathbf{y}_\varepsilon^0, \mathbf{y}_\varepsilon^1)$ of the homogeneous problem related to (3.5), in the space $\mathbf{G}_\varepsilon$, we obtain the unique solution

$$\mathbf{y}_\varepsilon = \sum_n \{(\mathbf{y}_\varepsilon^0, \boldsymbol{\varphi}_n^\varepsilon)\cos(\mu_n t) + \frac{(\mathbf{y}_\varepsilon^1, \boldsymbol{\varphi}_n^\varepsilon)}{\mu_n}\sin(\mu_n t)\}\boldsymbol{\varphi}_n^\varepsilon \qquad \mu_n = \sqrt{\lambda_n^\varepsilon}.$$

In this situation it is useful to prove the following lemma.

Lemma 3.1. *The eigenvalues $\mu_j(\varepsilon)$ satisfy an asymptotic gap uniformly in ε; that is, there exists a positive costant γ independent of ε such that*

$$|\mu_n(\varepsilon) - \mu_{n-1}(\varepsilon)| = \gamma > 0.$$

Proof. First of all we observe that $\lambda_n^\varepsilon > \lambda_{n-1}^\varepsilon$. We denote by P_n and Q_n the following polynomials in the integer n,

$$P_n = \varepsilon l_n^2 + ((4+\nu)\varepsilon + 1) l_n + 3(1+\nu)(1+\varepsilon),$$
$$Q_n = 4 l_n (\varepsilon l_n^2 + 2\varepsilon l_n + (1+\varepsilon)(1-\nu^2),$$

and consider

$$\mu_n - \mu_{n-1} = \frac{P_n - P_{n-1} + \sqrt{P_n^2 - Q_n} - \sqrt{P_{n-1}^2 - Q_{n-1}}}{\sqrt{P_n + \sqrt{P_n^2 - Q_n}} + \sqrt{P_{n-1} + \sqrt{P_{n-1}^2 - Q_{n-1}}}}$$
$$\geq \frac{P_n - P_{n-1} + \sqrt{P_n^2 - Q_n} - \sqrt{P_{n-1}^2 - Q_{n-1}}}{\sqrt{2P_n} + \sqrt{2P_{n-1}}}.$$

Since $\frac{Q_n}{P_n^2} \to 0$ as $n \to \infty$ $\quad \forall \varepsilon \geq 0$, we have definitively

$$\mu_n - \mu_{n-1} \geq \beta \frac{P_n - P_{n-1}}{\sqrt{2P_n} + \sqrt{2P_{n-1}}}$$

with $1 \leq \beta \leq 2$. Moreover, since $P_n \geq P_{n-1}$ we obtain

$$\mu_n - \mu_{n-1} \geq \beta \frac{P_n - P_{n-1}}{2\sqrt{2P_n}} = \beta \frac{\varepsilon(64n^3 + o(n^3)) + (8n + o(n))}{2\sqrt{\varepsilon(32n^4 + o(n^4)) + (8n^2 + o(n^2))}},$$

and hence $\gamma \geq 2$ for any $\varepsilon \geq 0$. □

We are now in a position to apply the Ball-Slemrod theorem, which assures that in this situation for any $T \geq \frac{2\pi}{\gamma}$

$$c_1(T) \sum_n |a_n|^2 \leq \|\mathbf{C}^\varepsilon \mathbf{y}_\varepsilon\|_\Sigma^2 \leq c_2(T) \sum_n |a_n|^2, \tag{3.7}$$

where $c_1(T)$ and $c_2(T)$ are two constants depending only on T, and

$$|a_n|^2 = (|(\mathbf{y}_\varepsilon^0, \boldsymbol{\varphi}_n^\varepsilon)|^2 + \frac{|(\mathbf{y}_\varepsilon^1, \boldsymbol{\varphi}_n^\varepsilon)|^2}{\lambda_n^\varepsilon})\, |\mathbf{C}^\varepsilon(\boldsymbol{\varphi}_n^\varepsilon)|^2.$$

We recall that in this situation setting $\mathbf{y}_\varepsilon = (u_\varepsilon, w_\varepsilon)$

$$\|\mathbf{C}^\varepsilon \mathbf{y}_\varepsilon\|_\Sigma^2 = \int_\Sigma (\mathbf{C}^\varepsilon \mathbf{y}_\varepsilon)^2 \, ds\, dt = \int_0^T [u_\varepsilon'(\theta_0, t) - (1+\nu) w_\varepsilon(\theta_0, t)]^2 \ \sin\theta_0 \, dt.$$

Since we can verify (see [3], in particular Lemma 2.2) that $\mathbf{C}\boldsymbol{\varphi}_n^\varepsilon \neq 0$ for any n and $\varepsilon \geq 0$, and we can prove the estimates

$$c_3 < \frac{|\mathbf{C}^\varepsilon \boldsymbol{\varphi}_n^\varepsilon|^2}{\lambda_n^\varepsilon} < c_4, \tag{3.8}$$

it follows that $\|\mathbf{C}^\varepsilon \mathbf{y}_\varepsilon\|_\Sigma$ defines a norm equivalent to the energy norm on the set of our initial data. Moreover, the right inequality in (3.7) and the convergence result of the initial data given in Lemma 2.1 and Lemma 2.2 imply the boundness in $L^2(\Sigma)$-norm (independent of ε) of the complementary boundary conditions, so we can find a subsequence which weakly converges; i.e.,

$$\mathbf{C}^\varepsilon \mathbf{y}_\varepsilon \to q(t) \qquad \text{weakly} \quad \text{in} \quad L^2(\Sigma). \tag{3.9}$$

Now we shall prove that $q(t) = \mathbf{C}\mathbf{y}$, where $\mathbf{C}\mathbf{y} = u'(\theta_0, t) - (1+\nu)w(\theta_0, t)$ and $\{u, w\}$ is the solution of the limit homogeneous problem.

Lemma 3.2. *We consider the following boundary value problem:*

$$\begin{cases} -\frac{\sqrt{\varepsilon}}{\sqrt{\sin\theta}}(a_1\sqrt{\sin\theta})'' + a_1 = b_1, \quad a_1(0) = a_1'(\theta_0) = 0, \\ -\sqrt{\varepsilon}\mathcal{L}(a_1 + a_2') + a_1 + a_2' = b_1 + b_2', \\ a_1 + a_2'|_0 = a_1 + a_2'|_{\theta_0} = 0, \quad a_2(0) = b_2(0). \end{cases} \tag{3.10}$$

If $\mathbf{b} = (b_1, b_2) \in U \times W$, *then as* $\varepsilon \to 0$,

$$\mathbf{a} \to \mathbf{b} \ \textit{ strongly in } \ U \times L^2(0, \theta_0; \sin\theta), \qquad a_1(\theta_0) \to b_1(\theta_0)$$
$$a_1 + a_2' \to b_1 + b_2' \ \textit{ strongly in } \ L^2(0, \theta_0; \sin\theta).$$

The proof of the lemma follows from the direct computation of the first equations $(3.10)_1$–$(3.10)_2$, and from a simple analysis of the equations $(3.10)_3$ and $(3.10)_4$.

Theorem 3.2. *We assume that* (3.9) *is satisfied; then*

$$\mathbf{C}^\varepsilon \mathbf{y}_\varepsilon \to \mathbf{C}\mathbf{y} \qquad \text{weakly in } \ L^2(\Sigma).$$

Proof. We introduce the Laplace transform of $\mathbf{y}_\varepsilon$, i.e.,

$$\mathbf{s}^\varepsilon = \int_0^\infty e^{-rt}\,\mathbf{y}_\varepsilon\, dt,$$

which satisfies the system

$$-\mathcal{L}(s_1^\varepsilon) + (1+\nu)s_2^{\varepsilon\prime} - \varepsilon\mathcal{L}(s_1^\varepsilon + s_2^{\varepsilon\prime}) = -r^2 s_1^\varepsilon + r u_\varepsilon^0 + u_\varepsilon^1 \tag{3.11}$$

$$\frac{\varepsilon}{\sin\theta}(\mathcal{L}(s_1^\varepsilon + s_2^{\varepsilon\prime})\sin\theta)' - \frac{(1+\nu)}{\sin\theta}(s_1^\varepsilon \sin\theta)' + 2(1+\nu)s_2^\varepsilon = -r^2 s_2^\varepsilon + r w_\varepsilon^0 + w_\varepsilon^1.$$

From Lemma 3.2, multiplying the first equation of (3.11) by $a_1 \sin\theta$ and the second one by $a_2 \sin\theta$ and taking into account the boundary conditions on $\mathbf{s}^\varepsilon$ and $\mathbf{a} = (a_1, a_2)$, we obtain

$$\mathbf{C}^\varepsilon(\mathbf{s}^\varepsilon)a_1(\theta_0)+a^0(\mathbf{s}^\varepsilon,\mathbf{a})-\varepsilon\int_0^{\theta_0}(s_1^\varepsilon+s_2^{\varepsilon\prime})\mathcal{L}(a_1+a_2')\sin\theta = (-r^2\mathbf{s}^\varepsilon+r\mathbf{y}_\varepsilon^0+\mathbf{y}_\varepsilon^1,\mathbf{a})_{\mathbf{H}},$$

and from the problem (3.10) introduced in Lemma 3.2

$$\begin{aligned}&\mathbf{C}^\varepsilon(\mathbf{s}^\varepsilon)\,a_1(\theta_0) + a^0(\mathbf{s}^\varepsilon,\mathbf{a}) - \sqrt{\varepsilon}\int_0^{\theta_0}(s_1^\varepsilon + s_2^{\varepsilon\prime})((b_1+b_2') - (a_1+a_2'))\sin\theta\\ &= (-r^2\mathbf{s}^\varepsilon + r\mathbf{y}_\varepsilon^0 + \mathbf{y}_\varepsilon^1,\mathbf{a})_{\mathbf{H}}.\end{aligned}$$

The product of the limit system (that is, the system obtained by formally putting $\varepsilon = 0$ in (3.5)) with $\mathbf{b} = (b_1, b_2)$ gives the identity

$$\mathbf{C}(\mathbf{s})\,b_1(\theta_0) + a^0(\mathbf{s},\mathbf{b}) = (-r^2\mathbf{s} + r\mathbf{y}^0 + \mathbf{y}^1,\mathbf{b})_{\mathbf{H}},$$

where $\mathbf{s}$ is the Laplace transform of $\mathbf{y}$. The proof of the theorem follows from the convergence results established in Lemma 3.2, and from the boundedness (uniformly in ε) of $\sqrt{\varepsilon}\,(s_1^\varepsilon + s_2^{\varepsilon\prime})$, and consequently of the energy estimate, for any $\mathbf{y}_\varepsilon^0 \in \mathbf{V}$ established in Lemma 2.2.

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