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UNIFORM GAUSSIAN ESTIMATES OF THE FUNDAMENTAL SOLUTIONS FOR HEAT OPERATORS ON CARNOT GROUPS†

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Abstract. In this paper, we are concerned with existence, qualitative properties, and *uniform Gaussian estimates* of the global fundamental solutions of a family of heat operators on Carnot groups. As a byproduct, we obtain existence and uniqueness theorems of Thychonov type for the Cauchy problem. Our effort here is also to provide simple and direct proofs relying on few basic tools such as invariant Harnack inequalities and maximum principles. In our study, we thoroughly exploit some structural properties of Carnot groups pointed out in the previous paper [4].

1. Introduction

Let $\mathbb{G} = (\mathbb{R}^N, \circ)$ be a Carnot group and denote by

$$\Delta_{\mathbb{G}} = \sum_{j=1}^m X_j^2$$

its canonical sub-Laplacian. Given a positive definite symmetric matrix $A = (a_{i,j})_{i,j\le m}$, let us consider the following heat-type operator on $\mathbb{R}^{N+1}$

$$\mathcal{H}_A = \mathcal{L}_A - \partial_t = \sum_{i,j=1}^m a_{i,j}\, X_i X_j - \partial_t. \tag{1.1}$$

For a fixed $\Lambda \ge 1$, we denote by $\mathcal{M}_\Lambda$ the set of the symmetric $m\times m$ matrices A such that

$$\Lambda^{-1}|\xi|^2 \le \langle A\,\xi, \xi\rangle \le \Lambda\,|\xi|^2, \qquad \forall\, \xi \in \mathbb{R}^m.$$

In this paper, we are concerned with existence, qualitative properties and *uniform Gaussian estimates* of the global fundamental solutions Γ_A for $\mathcal{H}_A$, with $A \in \mathcal{M}_\Lambda$. We also deal with existence and uniqueness theorems for Cauchy problems related to $\mathcal{H}_A$. Our main results are summarized in Theorem 2.1 and Theorem 2.5 of Section 2. We would like to stress that Theorem

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2.1 contains several results, some of which are somehow already present in literature. We provide complete and elementary proofs of all the results, in order to make the paper self-contained and easier to read. Theorem 2.5 deals with uniform Gaussian estimates for Γ_A and $\Gamma_A - \Gamma_B$ with $A, B \in \mathcal{M}_\Lambda$. Gaussian but not uniform estimates, for heat kernels Γ on Lie groups were proved by Varopoulos, Saloff-Coste and Coulhon [23], via semigroup theory, by Jerison and Sànchez-Calle [13], via Gevrey regularity methods, and by Kusuoka and Stroock [15, 16], via probabilistic techniques. Uniform but not Gaussian estimates, for families of Hörmander operators generalizing (1.1), were proved by Rothschild and Stein [21] and by Bramanti and Brandolini [6] via a technique relying on the subelliptic estimates of Kohn [14].

In this paper, the approach we follow is completely different from the ones quoted above. We start by presenting a direct and simple proof of non-uniform Gaussian estimates of Γ_A from above, which only relies on the maximum principle and the use of barrier functions of exponential type (see Theorem 5.1). Although these estimates have already appeared in the past, we emphasize that our techniques do not make any use of semigroup theory, Gevrey classes or probability arguments which were heavily employed in previous works, see e.g. [13, 15, 16, 23]. The non-uniform estimates from below and the estimates of the derivatives of Γ_A are readily derived, as usual, from the invariant Harnack inequality for $\mathcal{H}_A$. We also give a direct proof of this inequality. Our estimates, in particular, obviously hold for $\Gamma_{\mathbb{G}}$, the fundamental solution for the canonical heat operator

$$\mathcal{H}_{\mathbb{G}} = \Delta_{\mathbb{G}} - \partial_t.$$

To obtain uniform estimates for Γ_A and $\Gamma_A - \Gamma_B$, we follow the naïve idea to look for a diffeomorphism $T_A : \mathbb{R}^N \to \mathbb{R}^N$ such that

$$\Gamma_A(x,t) = J_A(x)\,\Gamma_{\mathbb{G}}(T_A(x),t), \qquad \forall\, x \in \mathbb{R}^N,\ \forall\, t \in \mathbb{R},$$

where J_A is the Jacobian determinant of T_A. Such a diffeomorphism does exist if $\mathbb{G}$ is a free group but may not exist otherwise, as it has been recently proved in [4]. That paper also contains the following result: every Carnot group $\mathbb{G}$ can be *lifted* in the sense of Rothschild-Stein, to a *free* Carnot group $\widetilde{\mathbb{G}}$ in such a way that $\mathcal{L}_A$ and $\Gamma_{\mathbb{G}}$ are lifted to $\widetilde{\mathcal{L}}_A$ and $\Gamma_{\widetilde{\mathbb{G}}}$, respectively. By using these two results and the above mentioned Gaussian bounds for $\Gamma_{\widetilde{\mathbb{G}}}$, we are able to obtain our uniform Gaussian estimates for Γ_A and $\Gamma_A - \Gamma_B$ (see Theorem 7.10 and Theorem 7.12).

These estimates will be crucial tools in a forthcoming paper, in which we shall construct the fundamental solution of the *non-divergence* form operator with "variable" coefficients

$$\sum_{i,j=1}^{m} a_{i,j}(x,t)\, X_i X_j - \partial_t. \tag{1.2}$$

Such a construction, based on the Levi's parametrix method, is one of the main motivation for the present paper. We would like to stress that operators like (1.2) are linearizations of fully non-linear operators that naturally arise in studying motion of surfaces by Levi-curvature (see [22, 19]).

The present paper is organized as follows. In Section 2, we first recall some basic definitions on Carnot groups. Then, we summarize our results in Theorems 2.1-2.5. Section 3 is devoted to a direct analytic construction of the fundamental solutions for $\mathcal{H}_A$ and $\mathcal{L}_A$ previously found in [8, 9, 23]. In Section 4, we give a new proof of the well known invariant Harnack inequality for non-negative solutions to $\mathcal{H}_A u = 0$. Our proof only relies on a weak Harnack inequality of Bony [5], on a general result of abstract potential theory concerning with the support of the harmonic measures and, finally, on the homogeneity of $\mathcal{H}_A$ with respect to a group of dilations. The Gaussian non-uniform estimates of Γ_A are proved in Section 5 and are used in Section 6 to show the solvability of the Cauchy problem for $\mathcal{H}_A$ and uniqueness results of Thychonov-type. Section 7 is devoted to the proof of the uniform Gaussian estimates for Γ_A, γ_A, $\Gamma_A - \Gamma_B$ and $\gamma_A - \gamma_B$. Finally, in the Appendix we recall many basic properties that will be used throughout the paper.

2. Main results

We start by giving the definition of a Carnot group. Our definition, which is the most convenient for our purposes, may seem slightly different from the one usually given in literature. As a matter of fact, as we observe below, the two definitions are equivalent.

Let $\circ$ be an assigned Lie group law on $\mathbb{R}^N$. Suppose $\mathbb{R}^N$ is endowed with a homogeneous structure by a given family of Lie group automorphisms $\{\delta_\lambda\}_{\lambda>0}$ (called *dilations*) of the form

$$\delta_\lambda(x) = \delta_\lambda(x^{(1)}, x^{(2)}, \dots, x^{(r)}) = (\lambda x^{(1)}, \lambda^2 x^{(2)}, \dots, \lambda^r x^{(r)}). \tag{2.1}$$

Here $x^{(i)} \in \mathbb{R}^{N_i}$ for $i = 1, \dots, r$ and $N_1 + \cdots + N_r = N$. We denote by $\mathfrak{g}$ the Lie algebra of $(\mathbb{R}^N, \circ)$, i.e., the Lie algebra of left-invariant vector fields on $\mathbb{R}^N$. For $i = 1, \dots, N_1$, let X_i be the (unique) vector field in $\mathfrak{g}$ that agrees at the origin with $\partial/\partial x_i$. We make the following assumption: the Lie algebra generated by $X_1, \dots, X_{N_1}$ is the whole $\mathfrak{g}$. With the above hypotheses, we call $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_\lambda)$ a *homogeneous Carnot group*. We also say that $\mathbb{G}$ is of *step* r and has $m := N_1$ *generators*. The *canonical sub-Laplacian* on $\mathbb{G}$ is the second order differential operator

$$\Delta_{\mathbb{G}} = \sum_{i=1}^{m} X_i^2.$$

If $Y_1, \dots, Y_m$ is any basis for $\mathrm{span}\{X_1, \dots, X_m\}$, the second order differential operator

$$\mathcal{L} = \sum_{i=1}^{m} Y_i^2$$

will be called a *sub-Laplacian* on $\mathbb{G}$. In literature (see e.g. [8], [21], [23], [12]) a *Carnot group* (or *stratified group*) $\mathbb{H}$ is defined as a connected and simply connected Lie group whose Lie algebra $\mathfrak{h}$ admits a *stratification* $\mathfrak{h} = V_1 \oplus \cdots \oplus V_r$ with $[V_1, V_i] = V_{i+1}$, $[V_1, V_r] = \{0\}$. It is not difficult to recognize that any homogeneous Carnot group is a Carnot group according to the classical definition. On the other hand, up to isomorphism, the opposite implication is also true (see e.g. [4]).

We next give a list of known results about homogeneous Carnot groups. Since $X_1, \dots, X_m$ generate the whole $\mathfrak{g}$, which has rank N at any point, any sub-Laplacian $\mathcal{L}$ satisfies Hörmander's hypoellipticity condition. Moreover, the vector fields $X_1, \dots, X_m$ are homogeneous of degree 1 w.r.t. δ_λ and X_j^* (the adjoint operator of X_j) is $-X_j$. In particular, $\mathcal{L}$ is a self-adjoint operator in divergence form

$$\mathcal{L} = \mathrm{div}\big(M_{\mathcal{L}}(x)\,\nabla\big), \tag{2.2}$$

$M_{\mathcal{L}}(x)$ being a suitable nonnegative-definite symmetric matrix. We denote by $Q = \sum_{j=1}^{r} j\,N_j$ the *homogeneous dimension* of $\mathbb{G}$. Then $\mathrm{meas}(\delta_\lambda(E)) = \lambda^Q \mathrm{meas}(E)$ for any measurable set E. Here $\mathrm{meas}(\cdot)$ denotes the Lebesgue measure on $\mathbb{R}^N$. This measure is invariant w.r.t. the left and right translations on $\mathbb{G}$. If $Q \le 3$, then $\mathbb{G}$ is the ordinary Euclidean group $(\mathbb{R}^Q, +)$ and $\Delta_{\mathbb{G}}$ is the classical Laplace operator. Hence, throughout the paper, we shall always assume $Q \ge 4$. In Section 3, we shall prove that there exists a homogeneous norm $d_{\mathcal{L}}$ on $\mathbb{G}$ such that

$$\gamma(x, y) = \big(d_{\mathcal{L}}(y^{-1} \circ x)\big)^{2-Q} \tag{2.3}$$

is a fundamental solution for $\mathcal{L}$ (see also [8] and [10]). We recall that a *homogeneous norm* on $\mathbb{G}$ is a continuous function $d : \mathbb{R}^N \to [0, \infty)$, smooth away from the origin, such that $d(\delta_\lambda(x)) = \lambda\, d(x)$, $d(x^{-1}) = d(x)$, and $d(x) = 0$ iff $x = 0$. Hereafter, we also denote $d(y^{-1} \circ x)$ by $d(x, y)$. The following *quasi-triangle inequality* holds

$$d_{\mathcal{L}}(x, y) \le \beta_{\mathcal{L}}\,(d_{\mathcal{L}}(x, z) + d_{\mathcal{L}}(z, y)),$$

for a suitable constant $\beta_{\mathcal{L}}$. We finally introduce the *heat operator* $\mathcal{H}$, related to the sub-Laplacian $\mathcal{L}$, on $\mathbb{G} \times \mathbb{R} \equiv \mathbb{R}^{N+1}$:

$$\mathcal{H} = \mathcal{L} - \partial_t.$$

Here we denote by $z = (x, t)$ the point of $\mathbb{R}^{N+1}$ ($x \in \mathbb{G}$, $t \in \mathbb{R}$). The operator $\mathcal{H}$ is hypoelliptic by Hörmander Theorem. When $\mathcal{L} = \Delta_{\mathbb{G}}$, we shall denote by $\mathcal{H}_{\mathbb{G}} = \Delta_{\mathbb{G}} - \partial_t$ the *canonical heat operator* on $\mathbb{G} \times \mathbb{R}$ and by $d_{\mathbb{G}}$ the related homogeneous norm $d_{\Delta_{\mathbb{G}}}$.

We next give a survey of our main results. In what follows, $\mathbb{G}$ will be a fixed homogeneous Carnot group.

The following theorem collects several results that will be proved in Sections 3 and 6.

Theorem 2.1. *There exists a smooth function* Γ *on* $\mathbb{R}^{N+1} \setminus \{0\}$ *such that the fundamental solution for* $\mathcal{H}$ *is given by* $\Gamma(x, t; \xi, \tau) := \Gamma(\xi^{-1} \circ x, t - \tau)$. Γ *has the properties listed below.*

(i) $\Gamma(x, t) \geq 0$ *and* $\Gamma(x, t) = 0$ *iff* $t \leq 0$*; moreover* $\Gamma(x, t) = \Gamma(x^{-1}, t)$.

(ii) $\Gamma(\delta_\lambda(x), \lambda^2 t) = \lambda^{-Q}\Gamma(x, t)$*; in particular* Γ *vanishes at infinity.*

(iii) *For every* $\zeta \in \mathbb{R}^{N+1}$, $\Gamma(\cdot; \zeta)$ *is locally integrable and* $\mathcal{H}\Gamma(\cdot, \zeta) = -\delta_\zeta$ (*the Dirac measure supported at* $\{\zeta\}$).

(iv) *For every test function* $\varphi \in C_0^\infty(\mathbb{R}^{N+1})$ *and* $\zeta \in \mathbb{R}^{N+1}$*, the following identities hold:*

$$\mathcal{H}\Big(\int_{\mathbb{R}^{N+1}} \Gamma(\cdot; \zeta)\, \varphi(\zeta)\, \mathrm{d}\zeta \Big) = \int_{\mathbb{R}^{N+1}} \Gamma(\cdot; \zeta)\, \mathcal{H}\varphi(\zeta)\, \mathrm{d}\zeta = -\varphi.$$

(v) $\Gamma^*(z; \zeta) = \Gamma(\zeta; z)$ *is the fundamental solution for the adjoint operator* $\mathcal{H}^* = \mathcal{L} + \partial_t$ *and it satisfies the dual statements of* (iii) *and* (iv).

(vi) *For every* $t > 0$, $\int_{\mathbb{R}^N} \Gamma(x, t)\, \mathrm{d}x = 1$.

(vii) *Suppose* f *is a continuous function in* $\mathbb{R}^N$ *satisfying the exponential growth condition in Corollary* 6.2*. Then, there exists* $\delta > 0$ *such that the function* $u(x, t) := \int_{\mathbb{R}^N} \Gamma(\xi^{-1} \circ x, t)\, f(\xi)\, \mathrm{d}\xi$ *is well posed for* $0 < t < \delta$ *and is a solution to the Cauchy problem*

$$\mathcal{H}u = 0 \ \text{ in } \mathbb{R}^N \times (0, \delta), \quad u(\cdot, 0) = f.$$

(viii) *Suppose* u *is a classical solution to the Cauchy problem*

$$\mathcal{H}u = 0 \ \text{ in } \mathbb{R}^N \times (0, r), \quad u(\cdot, 0) = 0, \tag{2.4}$$

and assume that either u *is non-negative or there exists* $\mu > 0$ *such that*

$$\int_0^r \int_{\mathbb{R}^N} \exp\big(- \mu\, d_{\mathcal{L}}^2(x)\big)\, |u(x, t)|\, \mathrm{d}x\mathrm{d}t < \infty. \tag{2.5}$$

Then u vanishes identically.

(ix) *For every $x \in \mathbb{R}^N$, $t > 0$ and $\tau > 0$, the following* reproduction property *holds*

$$\Gamma(x, t+\tau) = \int_{\mathbb{R}^N} \Gamma(\xi^{-1} \circ x, t)\, \Gamma(\xi, \tau)\, \mathrm{d}\xi.$$

As a consequence, we obtain in a very natural and simple way the existence of the fundamental solution γ for $\mathcal{L}$, and we prove that γ is the power of a suitable homogeneous norm. The latter result was first proved by Gallardo [10] via probabilistic techniques. The following theorem will be proved in Section 3.

Theorem 2.2. *Setting*

$$\gamma(x) := \int_0^\infty \Gamma(x, t)\, \mathrm{d}t, \quad x \in \mathbb{R}^N,$$

then $\gamma(x, \xi) := \gamma(\xi^{-1} \circ x)$ is the fundamental solution for $\mathcal{L}$. Moreover, there exists a homogeneous norm $d_{\mathcal{L}}$ on $\mathbb{G}$ such that $\gamma = d_{\mathcal{L}}^{2-Q}$.

In Section 5, we prove Gaussian estimates for Γ and its derivatives. The estimate from above is obtained by a direct proof based on a comparison argument. The estimate from below is derived from the Harnack inequality for the $\mathcal{L}$-caloric functions. For this purpose, in Section 4 such Harnack inequality is proved by using some basic properties of the harmonic sheaf related to $\mathcal{H}$ and few general results of abstract Potential Theory.

Our main goal in the paper is to prove that the mentioned Gaussian estimates are *uniform* in the class of the sub-Laplacians

$$\mathcal{L}_A = \textstyle\sum_{i,j=1}^m a_{i,j}\, X_i X_j, \quad A = (a_{i,j})_{i,j} \in \mathcal{M}_\Lambda,$$

where $\mathcal{M}_\Lambda$ is the set of $m \times m$ symmetric matrices A such that $\Lambda^{-1}|\xi|^2 \le \langle A\xi, \xi\rangle \le \Lambda|\xi|^2$ ($\Lambda \ge 1$ being a fixed constant). A natural question is to ask whether the operators $\mathcal{L}_A$'s are all diffeomorphic to the canonical operator $\Delta_{\mathbb{G}}$. This is the case if $\mathbb{G}$ is a *free* Carnot group, i.e., if the Lie algebra of $\mathbb{G}$ is isomorphic to $\mathfrak{f}_{m,r}$ for some m and r ($\mathfrak{f}_{m,r}$ denotes the free nilpotent Lie algebra of step r and m generators).

Theorem A. *Suppose $\mathbb{G}$ is a free homogeneous Carnot group and let A be a given positive-definite symmetric matrix. Then, there exists a Lie group automorphism T_A of $\mathbb{G}$ such that*

$$\begin{aligned} \big(\textstyle\sum_{j=1}^m (A^{1/2})_{i,j}\, X_j\big)(u \circ T_A) &= (X_i u) \circ T_A, \quad i = 1, \dots, m, && (2.6)\\ \mathcal{L}_A(u \circ T_A) &= (\Delta_{\mathbb{G}} u) \circ T_A, && (2.7) \end{aligned}$$

for every smooth function $u : \mathbb{G} \to \mathbb{R}$*. Moreover,* T_A *has polynomial component functions (but in general it is not a linear map) and it commutes with the dilations of* $\mathbb{G}$*.*

This theorem has been recently proved in [4]. In that paper it is also shown that in Theorem A *the hypothesis* $\mathbb{G}$ *free cannot be removed.* In [4] the following estimates of T_A are also given.

Theorem B. *Under the hypotheses and with the notation of the above theorem, we set* $J_A(x) = |\det \mathcal{J}_{T_A}(x)|$*, for* $x \in \mathbb{G}$*. Then,* J_A *turns out to be constant in* x*. Moreover, there exists a positive constant* $\mathbf{c}_\Lambda$ *only depending on* Λ *and on the structure of* $\mathbb{G}$ *such that*

$$(\mathbf{c}_\Lambda)^{-1} \leq J_A \leq \mathbf{c}_\Lambda, \tag{2.8}$$

$$|J_{A_1} - J_{A_2}| \leq \mathbf{c}_\Lambda \, \|A_1 - A_2\|, \tag{2.9}$$

$$(\mathbf{c}_\Lambda)^{-1} \, d_{\mathbb{G}}(x) \leq d_{\mathbb{G}}(T_A(x)) \leq \mathbf{c}_\Lambda \, d_{\mathbb{G}}(x), \tag{2.10}$$

$$d_{\mathbb{G}}(T_{A_1}(x), T_{A_2}(x)) \leq \mathbf{c}_\Lambda \, \|A_1 - A_2\|^{1/r} \, d_{\mathbb{G}}(x), \tag{2.11}$$

$$\|A^{-1/2}\| \leq \mathbf{c}_\Lambda, \quad \|A_1^{1/2} - A_2^{1/2}\| \leq \mathbf{c}_\Lambda \, \|A_1 - A_2\|, \tag{2.12}$$

for every A*,* A_1*,* $A_2 \in \mathcal{M}_\Lambda$ *and* $x \in \mathbb{G}$*. Here,* $\|A\|$ *denotes the matrix norm* $\max_{|\xi|=1} |A\xi|$ *and* $\mathcal{J}_{T_A}$ *the Jacobian matrix of* T_A*. We also recall that* r *is the step of nilpotence of* $\mathbb{G}$ *and* $d_{\mathbb{G}}$ *is defined by* (2.3) *when* $\mathcal{L} = \Delta_{\mathbb{G}}$*.*

Theorem A allows us to obtain the fundamental solution Γ_A for $\mathcal{H}_A = \mathcal{L}_A - \partial_t$ and γ_A for $\mathcal{L}_A$, as the composition of T_A with the fundamental solution $\Gamma_{\mathbb{G}}$ for $\mathcal{H}_{\mathbb{G}}$ and $\gamma_{\mathbb{G}}$ for $\Delta_{\mathbb{G}}$, respectively.

Theorem 2.3. *Suppose* $\mathbb{G}$ *is a free homogeneous Carnot group. For every* $A \in \mathcal{M}_\Lambda$*, we have*

$$\Gamma_A(x,t) = J_A \, \Gamma_{\mathbb{G}}(T_A(x), t), \quad \gamma_A(x) = J_A \, \gamma_{\mathbb{G}}(T_A(x)), \qquad \forall \, x \in \mathbb{R}^N, \; t \in \mathbb{R}.$$

By means of these results, we are able to obtain the desired uniform estimates in the free case. In order to handle the case of an arbitrary Carnot group $\mathbb{G}$, our main tool is the *lifting* of $\mathbb{G}$ to a free group $\widetilde{\mathbb{G}}$, by using the following result also proved in [4].

Theorem C. *Let* $\mathbb{G}$ *be a homogeneous Carnot group on* $\mathbb{R}^N$*. Then, there exists a free homogeneous Carnot group* $\widetilde{\mathbb{G}}$ *on* $\mathbb{R}^H$ (*with* $H \geq N$) *such that, denoting by* $\pi : \mathbb{R}^H \to \mathbb{R}^N$ *the projection on the first* N *coordinates* (*up to a permutation of the coordinates of* $\mathbb{R}^H$)*, we have*

$$\widetilde{X}_i(u \circ \pi) = (X_i u) \circ \pi, \quad \forall \, u \in C^\infty(\mathbb{R}^N),$$

where $\sum_{i=1}^m X_i^2$ *and* $\sum_{i=1}^m \widetilde{X}_i^2$ *are the canonical sub-Laplacians* $\Delta_{\mathbb{G}}$ *and* $\Delta_{\widetilde{\mathbb{G}}}$*, respectively. Moreover* $\pi : \widetilde{\mathbb{G}} \to \mathbb{G}$ *is a Lie group morphism.*

We refer to Theorem 8.3 in the Appendix for a more detailed statement. As a consequence, the following relation holds between the fundamental solutions Γ_A and $\widetilde{\Gamma}_A$ for the heat operators on $\mathbb{G}$ and $\widetilde{\mathbb{G}}$ respectively. A similar relation holds analogously between γ_A and $\widetilde{\gamma}_A$.

Theorem 2.4. *For every* $A \in \mathcal{M}_\Lambda$*, we have*

$$\Gamma_A(x,t) = \int_{\mathbb{R}^{H-N}} \widetilde{\Gamma}_A((x,\widehat{x}),t)\,\mathrm{d}\widehat{x}, \qquad \gamma_A(x) = \int_{\mathbb{R}^{H-N}} \widetilde{\gamma}_A(x,\widehat{x})\,\mathrm{d}\widehat{x},$$

$x \in \mathbb{R}^N$, $t \in \mathbb{R}$, *where* $(x,\widehat{x})$ *denotes the point of* $\mathbb{R}^N \times \mathbb{R}^{H-N}$.

Combining all the above results, we are finally in position to derive our uniform Gaussian estimates in the general case.

Theorem 2.5. *Given any non-negative integers* p, q, *there exist positive constants* $\mathbf{c}_\Lambda$, $\mathbf{c}_{\Lambda,p}$, $\mathbf{c}_{\Lambda,p,q}$ *such that for every* $i_1,\dots,i_p \in \{1,\dots,m\}$ *and for every* A, A_1, $A_2 \in \mathcal{M}_\Lambda$, *we have*

$$\mathbf{c}_\Lambda^{-1}\, t^{-Q/2} \exp\Big(-\frac{\mathbf{c}_\Lambda\, d_{\mathbb{G}}^2(x)}{t}\Big) \le \Gamma_A(x,t) \le \mathbf{c}_\Lambda\, t^{-Q/2} \exp\Big(-\frac{d_{\mathbb{G}}^2(x)}{\mathbf{c}_\Lambda\, t}\Big),$$

$$\big|X_{i_1}\cdots X_{i_p}(\partial_t)^q\,\Gamma_A(x,t)\big| \le \mathbf{c}_{\Lambda,p,q}\, t^{-(Q+p+2q)/2} \exp\Big(-\frac{d_{\mathbb{G}}^2(x)}{\mathbf{c}_\Lambda\, t}\Big),$$

$$\begin{aligned}&\big|X_{i_1}\cdots X_{i_p}(\partial_t)^q\,\Gamma_{A_1}(x,t) - X_{i_1}\cdots X_{i_p}(\partial_t)^q\,\Gamma_{A_2}(x,t)\big|\\ &\qquad \le \mathbf{c}_{\Lambda,p,q}\,\|A_1 - A_2\|^{1/r}\, t^{-(Q+p+2q)/2} \exp\Big(-\frac{d_{\mathbb{G}}^2(x)}{\mathbf{c}_\Lambda\, t}\Big),\end{aligned}$$

for every $x \in \mathbb{R}^N$, $t > 0$. *Moreover,*

$$\mathbf{c}_\Lambda^{-1}\,(d_{\mathbb{G}}(x))^{2-Q} \le \gamma_A(x) \le \mathbf{c}_\Lambda\,(d_{\mathbb{G}}(x))^{2-Q},$$

$$\big|X_{i_1}\cdots X_{i_p}\,\gamma_A(x)\big| \le \mathbf{c}_{\Lambda,p}\,(d_{\mathbb{G}}(x))^{2-Q-p},$$

$$\big|X_{i_1}\cdots X_{i_p}\,\gamma_{A_1}(x) - X_{i_1}\cdots X_{i_p}\,\gamma_{A_2}(x)\big| \le \mathbf{c}_{\Lambda,p}\,\|A_1 - A_2\|^{1/r}\,(d_{\mathbb{G}}(x))^{2-Q-p},$$

for every $x \in \mathbb{R}^N \setminus \{0\}$. *We recall that* r *denotes the step of nilpotence of* $\mathbb{G}$.

Finally, for the reader's convenience, in the Appendix we explicitly recall the main properties of Carnot groups that will be used throughout the paper (for all the proofs and a self contained presentation of Carnot groups, see e.g. [4]).

3. Construction of the fundamental solution

Let $\mathcal{L}$ be a (fixed) sub-Laplacian on $\mathbb{G}$ and let $\mathcal{H} = \mathcal{L} - \partial_t$ be its associated heat operator. In this section, we prove the existence of a fundamental solution Γ for $\mathcal{H}$ vanishing at infinity.

Definition 3.1. We shall say that a function $\Gamma : \mathbb{R}^{N+1} \times \mathbb{R}^{N+1} \to \mathbb{R}$ is *the fundamental solution for* $\mathcal{H}$ if and only if for every $\zeta \in \mathbb{R}^{N+1}$, we have

(i) $\Gamma(\cdot;\zeta) \in L_{\text{loc}}(\mathbb{R}^{N+1})$,

(ii) $\mathcal{H}(\Gamma(\cdot;\zeta)) = -\delta_\zeta$ (the Dirac measure supported at $\{\zeta\}$),

(iii) $\Gamma(z;\zeta) \longrightarrow 0$, as $|z| \to \infty$.

We explicitly remark that such fundamental solution (if it exists) is unique by the weak maximum principle for $\mathcal{H}$.

Following the lines of [17], we shall construct Γ as limit of a sequence of Green functions related to an increasing sequence of regular domains invading $\mathbb{R}^{N+1}$. Finally, at the end of the section, we shall also obtain the fundamental solution γ for $\mathcal{L}$, by saturating the t variable of Γ.

We start our construction by fixing a bounded open neighborhood of the origin $O_1 \subseteq \mathbb{R}^N$, such that $\delta_\lambda O_1 \subseteq O_1$, for every $\lambda \in (0,1]$ (so that $O_\lambda := \delta_\lambda O_1 \nearrow \mathbb{R}^N$, as $\lambda \to \infty$) and such that at any point $x_0 \in \partial O_1$ there exists a $\mathcal{L}$-non-characteristic outer normal to O_1 (i.e., a vector $\nu \neq 0$ such that $\{x \in O_1 : |x_0 + \nu - x| < |\nu|\} = \varnothing$ and $\langle M_{\mathcal{L}}(x_0)\,\nu, \nu\rangle > 0$, where $M_{\mathcal{L}}$ is the matrix defined by (2.2)). The existence of such a set O_1 is clear, observing that $(M_{\mathcal{L}})_{1,1}$ is a positive constant. Indeed, if $\mathcal{L} = \sum_{i=1}^m Y_i^2$ with $Y_i = \sum_{j=1}^m b_{i,j}\, X_j$ (where $B = (b_{i,j})_{i,j}$ is an invertible constant matrix), then it is easy to recognize (see Theorem 8.2) that $(M_{\mathcal{L}})_{1,1} = \sum_{i=1}^m b_{i,1}^2 > 0$. We then define $U_1 = O_1 \times (-1,1)$. Given any cylindrical domain $U = O \times (a,b)$ (O open subset of $\mathbb{R}^N$), we shall use the notation

$$\partial_p U := (O \times \{a\}) \cup (\partial O \times [a,b]), \qquad \widehat{U} := U \cup \partial_p U.$$

The following result is proved in the paper [17, Theorem 2.7] (see also [5]), with direct methods in line with the scope of our presentation. We explicitly remark that the hypothesis (H.3) of [17] is not used in the proof of the cited theorem.

Theorem 3.2.

(i) *For every $f \in C(\widehat{U}_1)$ there exists a unique solution $u \in C(\widehat{U}_1)$ to the Dirichlet problem $\mathcal{H}u = -f$ in U_1, $u|_{\partial_p U_1} = 0$ (in the sense of distributions). We denote by $\mathcal{G}_1 f$ such a solution u.*

(ii) *There exists a non-negative smooth function G_1, defined out of the diagonal of $U_1 \times U_1$, such that*

$$\mathcal{G}_1 f(z) = \int_{U_1} G_1(z;\zeta)\, f(\zeta)\,\mathrm{d}\zeta, \quad \textit{for every } f \in C(\widehat{U}_1) \textit{ and } z \in \widehat{U}_1; \tag{3.1}$$

$$G_1(\cdot;\zeta)|_{\partial_p U_1} = 0, \quad \textit{for every } \zeta \in U_1; \tag{3.2}$$

$$G_1(x,t;\xi,\tau) = 0, \qquad \textit{if } t \le \tau. \tag{3.3}$$

(iii) *Setting $G_1^*(z;\zeta) := G_1(\zeta;z)$, there hold statements analogous to* (i), (ii) *w.r.t. the adjoint operator $\mathcal{H}^*$ (of course, we have to replace $\partial_p U_1$ with $\partial_p^* U_1 = (O_1 \times \{1\}) \cup (\partial O_1 \times [-1,1])$).*

We now set, for any $\lambda > 0$,

$$U_\lambda = \delta_\lambda O_1 \times (-\lambda^2, \lambda^2), \tag{3.4}$$

$$G_\lambda(x,t;\xi,\tau) = \lambda^{-Q} G_1(\delta_{\lambda^{-1}}x, \lambda^{-2}t; \delta_{\lambda^{-1}}\xi, \lambda^{-2}\tau), \tag{3.5}$$

and we prove that G_λ is the Green function of U_λ.

Proposition 3.3. *The assertions in Theorem* 3.2 *hold true replacing U_1, G_1, $\mathcal{G}_1$ with U_λ, G_λ, $\mathcal{G}_\lambda$, respectively.*

Proof. Let $g \in C(\widehat{U}_\lambda)$ and set $f(x,t) = g(\delta_\lambda x, \lambda^2 t)$, $u = \mathcal{G}_1 f$ and $v(x,t) = \lambda^2 u(\delta_{\lambda^{-1}}x, \lambda^{-2}t)$. We claim that $v =: \mathcal{G}_\lambda g$ is the solution to the Dirichlet problem $\mathcal{H}v = -g$ in U_λ, $v|_{\partial_p U_\lambda} = 0$ (in the sense of distributions). Indeed for any test function $\varphi \in C_0^\infty(U_\lambda)$, setting $\varphi_\lambda(x,t) = \varphi(\delta_\lambda x, \lambda^2 t)$ and using the homogeneity properties of $\mathcal{H}^*$, by a change of variable we obtain

$$\int_{U_\lambda} v\,\mathcal{H}^*\varphi = \lambda^{Q+2}\int_{U_1} u\,\mathcal{H}^*\varphi_\lambda = -\lambda^{Q+2}\int_{U_1} f\,\varphi_\lambda = -\int_{U_\lambda} g\,\varphi.$$

This proves part (i), recalling that uniqueness follows from the parabolic maximum principle. We now only need to observe that

$$\begin{aligned}
&\mathcal{G}_\lambda g(x,t) = \lambda^2(\mathcal{G}_1 f)(\delta_{\lambda^{-1}}x, \lambda^{-2}t) \\
&= \lambda^2 \int_{U_1} G_1(\delta_{\lambda^{-1}}x, \lambda^{-2}t; y, s) g(\delta_\lambda y, \lambda^2 s)\,\mathrm{d}y\mathrm{d}s = \int_{U_\lambda} G_\lambda(x,t;\xi,\tau) g(\xi,\tau)\,\mathrm{d}\xi\mathrm{d}\tau.
\end{aligned}$$

This ends the proof. □

In the following, we agree to extend any function to be zero outside of its domain of definition.

Lemma 3.4. *We have $G_{\lambda_1}(z;\zeta) \le G_{\lambda_2}(z;\zeta)$, if $0 < \lambda_1 \le \lambda_2$.*

Proof. Since $G_\lambda \geq 0$, we only have to consider $z,\, \zeta \in U_{\lambda_1}$. We fix ζ and set $w = (G_{\lambda_2} - G_{\lambda_1})(\cdot;\zeta)$. By means of Proposition 3.3, for any test function $\varphi \in C_0^\infty(U_{\lambda_1})$, we have, for $i = 1,\, 2$,

$$-\varphi(\zeta) = \mathcal{G}^*_{\lambda_i}(\mathcal{H}^*\varphi)(\zeta) = \int_{U_{\lambda_i}} G^*_{\lambda_i}(\zeta;z)\,\mathcal{H}^*\varphi(z)\,\mathrm{d}z = \int_{U_{\lambda_1}} G_{\lambda_i}(z;\zeta)\,\mathcal{H}^*\varphi(z)\,\mathrm{d}z,$$

and then

$$\int_{U_{\lambda_1}} w\,\mathcal{H}^*\varphi = 0.$$

This proves that $\mathcal{H}w = 0$ in U_{λ_1}. Moreover, $G_{\lambda_1}(\cdot;\zeta)|_{\partial_p U_{\lambda_1}} = 0$ yields $w|_{\partial_p U_{\lambda_1}} \geq 0$. Hence, from the parabolic maximum principle, we infer that $w \geq 0$ in U_{λ_1}. □

We define

$$\Gamma(z;\zeta) := \sup_{\lambda>0} G_\lambda(z;\zeta) \quad (= \lim_{\lambda\to\infty} G_\lambda(z;\zeta)), \qquad z,\, \zeta \in \mathbb{R}^{N+1}.$$

Theorem 3.5.

(i) Γ *is a non-negative function which is smooth away from the diagonal of* $\mathbb{R}^{N+1} \times \mathbb{R}^{N+1}$.
(ii) *For every fixed* $z \in \mathbb{R}^{N+1}$, $\Gamma(\cdot;z)$, $\Gamma(z;\cdot)$ *are locally integrable.*
(iii) $\mathcal{G}_\infty : C_0^\infty(\mathbb{R}^{N+1}) \to C^\infty(\mathbb{R}^{N+1})$, $\mathcal{G}_\infty\varphi(z) = \int_{\mathbb{R}^{N+1}} \Gamma(z;\zeta)\,\varphi(\zeta)\,\mathrm{d}\zeta$, *is well posed and*
$$\mathcal{H}(\mathcal{G}_\infty\varphi) = \mathcal{G}_\infty(\mathcal{H}\varphi) = -\varphi.$$
(iv) $\Gamma(x,t;\xi,\tau) = 0$ *if and only if* $t \leq \tau$.
(v) *For every* $\zeta \in \mathbb{R}^{N+1}$, $\mathcal{H}(\Gamma(\cdot;\zeta)) = -\delta_\zeta$ *(the Dirac measure supported at* $\{\zeta\}$*).*
(vi) $\Gamma^*(z;\zeta) := \Gamma(\zeta;z)$ *satisfies the dual statements of* (iii) *and* (v) *(w.r.t.* $\mathcal{H}^*$*).*

Proof. The key point of the proof is Lemma 3.6 below. Once proved that lemma, Theorem 3.5 will follow by using the same arguments of the proof of [17, Theorem 1.1]. □

Lemma 3.6. *Let* $0 < \lambda_0 < \lambda$. *For every* $\varphi \in C_0^\infty(\mathbb{R}^N \times (-\lambda_0^2, \lambda_0^2))$, *we have*

$$\sup_{U_\lambda} |\mathcal{G}_\lambda\varphi| \leq 2\,\lambda_0^2 \sup|\varphi|.$$

Proof. We set $u = \mathcal{G}_\lambda\varphi$, $V_0 = O_\lambda \times (-\lambda^2, -\lambda_0^2)$, $V_1 = O_\lambda \times (-\lambda_0^2, \lambda_0^2)$, $V_2 = O_\lambda \times (\lambda_0^2, \lambda^2)$. We shall make use of the parabolic maximum principle on the domains V_0, V_1, V_2. We explicitly remark that $\partial_p V_0 \subseteq \partial_p U_\lambda$. From

$\mathcal{H}u = -\varphi \equiv 0$ in V_0, $u|_{\partial_p V_0} = 0$ we infer $u \equiv 0$ in V_0 and in particular $u|_{\partial_p V_1} = 0$. Hence, setting $w(x,t) := -(t+\lambda_0^2)\sup_{U_\lambda}|\varphi|$, $\omega_1 = u + w$ and $\omega_2 = u - w$, we have $\mathcal{H}\omega_1 \geq 0$, $\mathcal{H}\omega_2 \leq 0$ in V_1, $\omega_1|_{\partial_p V_1} \leq 0$, $\omega_2|_{\partial_p V_1} \geq 0$. As a consequence $\omega_1 \leq 0$, $\omega_2 \geq 0$ in V_1. In this way we have proved that $|u| \leq -w \leq 2\,\lambda_0^2 \sup_{U_\lambda}|\varphi| =: M$ in V_1. Finally, setting $f_1 = M + u$, $f_2 = M - u$ we have $\mathcal{H}f_i = 0$ in V_2, $f_i|_{\partial_p V_2} \geq 0$, which yield $f_i \geq 0$ in V_2 (i=$1,2$). Therefore $|u| \leq M$ also in V_2. □

From the definitions of Γ and of G_λ, we immediately obtain that Γ is homogeneous w.r.t. the parabolic dilations of $\mathbb{G} \times \mathbb{R}$. As a consequence, we are able to prove that Γ vanishes at infinity.

Theorem 3.7. Γ *has the following properties:*

$$\Gamma(\delta_\lambda x, \lambda^2 t; \delta_\lambda \xi, \lambda^2 \tau) = \lambda^{-Q}\,\Gamma(x,t;\xi,\tau), \tag{3.6}$$

$$\Gamma(z;\zeta) \longrightarrow 0, \quad \text{as } |z| \to \infty, \quad \text{for every } \zeta \in \mathbb{R}^{N+1}, \tag{3.7}$$

$$\Gamma(z;\zeta) \longrightarrow 0, \quad \text{as } |\zeta| \to \infty, \quad \text{for every } z \in \mathbb{R}^{N+1}, \tag{3.8}$$

$$\limsup_{z\to\zeta} \Gamma(z;\zeta) = \infty, \quad \text{for every } \zeta \in \mathbb{R}^{N+1}, \tag{3.9}$$

$$\Gamma(\xi,-\tau;x,-t) = \Gamma(x,t;\xi,\tau) = \Gamma(\xi^{-1}\circ x, t-\tau; 0,0). \tag{3.10}$$

In particular, Γ *is the fundamental solution for* $\mathcal{H}$ *according to Definition* 3.1. *Moreover, given any homogeneous norm* d *on* $\mathbb{G}$, Γ *satisfies the estimate*

$$\Gamma(x,t;\xi,\tau) \leq \mathbf{c}\left(d(x,\xi) + |t-\tau|^{1/2}\right)^{-Q}, \tag{3.11}$$

for a suitable positive constant $\mathbf{c}$.

Proof. From (3.5) it follows that

$$G_n(\delta_\lambda x, \lambda^2 t; \delta_\lambda \xi, \lambda^2\tau) = n^{-Q} G_1(\delta_{\lambda/n} x, \tfrac{\lambda^2}{n^2} t; \delta_{\lambda/n}\xi, \tfrac{\lambda^2}{n^2}\tau) = \lambda^{-Q} G_{n/\lambda}(x,t;\xi,\tau).$$

Letting n go to infinity, we obtain (3.6). We now fix a homogeneous norm d on $\mathbb{G}$ (which always exists) and set $\rho(x,t;\xi,\tau) = d(x,\xi) + |t-\tau|^{1/2}$, $\rho(x,t) = \rho(x,t;0,0)$. In order to prove (3.7), it is now sufficient to observe that $\rho(z) \longrightarrow \infty$ as $|z| \to \infty$ and to notice that (3.6) gives ($z = (x,t)$, $\zeta = (\xi,\tau)$)

$$\Gamma(z;\zeta) = \rho^{-Q}(z)\,\Gamma(\delta_{\rho^{-1}(z)}\,x, \rho^{-2}(z)\,t; \delta_{\rho^{-1}(z)}\,\xi, \rho^{-2}(z)\,\tau), \tag{3.12}$$

and then, by the continuity of Γ away from the diagonal,

$$0 \leq \Gamma(z;\zeta) \leq \rho^{-Q}(z) \max_{\rho(z')=1,\,\rho(\zeta')\leq\frac{1}{2}} \Gamma(z';\zeta'),$$

for large $|z|$. The proof of (3.8) is analogous. Let us now prove (3.10). We first want to show that $\Gamma(\cdot;\zeta) = w$, where we have set $w(x,t) = \Gamma(\xi^{-1}\circ x, t-$

$\tau; 0, 0)$, once $\zeta = (\xi, \tau) \in \mathbb{R}^{N+1}$ is fixed. Recalling Theorem 3.5-(v) and (3.7), we only need to prove that $\mathcal{H}w = -\delta_\zeta$ and to use the maximum principle. On the other hand, $\mathcal{H}w = -\delta_\zeta$ easily follows from Theorem 3.5-(v) and the left-invariance of $\mathcal{L}$. We now set $v(x,t) = \Gamma^*(x, -t; 0, 0)$ $(= \Gamma(0, 0; x, -t))$. We want to show that $v = \Gamma(\cdot; 0)$. This would complete the proof of (3.10); indeed we would have $\Gamma(x, t; \xi, \tau) = \Gamma(\xi^{-1} \circ x, t - \tau; 0, 0) = v(\xi^{-1} \circ x, t - \tau) = \Gamma(0, 0; \xi^{-1} \circ x, \tau - t) = \Gamma(x^{-1} \circ \xi, t - \tau; 0, 0) = \Gamma(\xi, -\tau; x, -t)$. In order to prove that $v = \Gamma(\cdot; 0)$, we make again use of the maximum principle. We only need to observe that both v and $\Gamma(\cdot; 0)$ vanish at infinity by (3.8)-(3.7), and that $\mathcal{H}\Gamma(\cdot; 0) = -\delta_0 = \mathcal{H}v$ by Theorem 3.5-(v)-(vi). Finally, we prove (3.9) and (3.11). (3.9) is an easy consequence of (3.6). From (3.10) and (3.12) we derive

$$\Gamma(x, t; \xi, \tau) = \Gamma(\xi^{-1} \circ x, t - \tau; 0, 0) \leq \rho^{-Q}(x, t; \xi, \tau) \max_{\rho(z')=1} \Gamma(z'; 0).$$

This completes the proof. $\square$

For the sake of brevity, in what follows we shall often use the notation

$$\Gamma(x, t) := \Gamma(x, t; 0, 0).$$

We explicitly note that, by (3.10), we have $\Gamma(x, t; \xi, \tau) = \Gamma(\xi^{-1} \circ x, t - \tau)$ and

$$\Gamma(x, t) = \Gamma(x^{-1}, t). \tag{3.13}$$

We now turn to the construction of the fundamental solution for the sub-Laplacian $\mathcal{L}$.

Definition 3.8. We shall say that a function $\gamma : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is *the fundamental solution for* $\mathcal{L}$ if and only if for every $\xi \in \mathbb{R}^N$, we have

(i) $\gamma(\cdot, \xi) \in L_{\text{loc}}(\mathbb{R}^N)$,

(ii) $\mathcal{L}(\gamma(\cdot, \xi)) = -\delta_\xi$ (the Dirac measure supported at $\{\xi\}$),

(iii) $\gamma(x, \xi) \longrightarrow 0$, as $|x| \to \infty$.

We explicitly remark that such a fundamental solution (if it exists) is unique by the weak maximum principle for $\mathcal{L}$.

We prove the existence of the fundamental solution γ for $\mathcal{L}$ by a saturation argument. Then, from the properties of Γ, we are able to prove that γ is a power of a homogeneous norm (see also [8] and [10]).

Theorem 3.9. *Let us set*

$$\gamma(x, \xi) := \int_0^\infty \Gamma(x, t; \xi, 0)\, \mathrm{d}t, \qquad x,\, \xi \in \mathbb{R}^N.$$

Then γ is the fundamental solution for $\mathcal{L}$. Moreover, there exists a homogeneous norm $d_{\mathcal{L}}$ on $\mathbb{G}$ such that

$$\gamma(x,\xi) = (d_{\mathcal{L}}(\xi^{-1} \circ x))^{2-Q}, \qquad \text{for every } x \neq \xi. \tag{3.14}$$

In particular $\gamma(x,\xi) = \gamma(\xi,x)$. As a consequence, setting

$$(\gamma * \varphi)(x) = \int_{\mathbb{R}^N} \gamma(x,\xi)\,\varphi(\xi)\,\mathrm{d}\xi$$

for every $\varphi \in C_0^\infty(\mathbb{R}^N)$, we have

$$\mathcal{L}(\gamma * \varphi) = \gamma * (\mathcal{L}\varphi) = -\varphi.$$

Proof. We first observe that $\gamma_0 := \gamma(\cdot, 0)$ is δ_λ-homogeneous of degree $2-Q$. Indeed, from (3.6) we get

$$\begin{aligned}\gamma_0(\delta_\lambda x) &= \int_0^\infty \Gamma(\delta_\lambda x, t)\,\mathrm{d}t = \lambda^{-Q}\int_0^\infty \Gamma(x, t/\lambda^2)\,\mathrm{d}t \\ &= \lambda^{-Q}\int_0^\infty \Gamma(x,s)\,\lambda^2\,\mathrm{d}s = \lambda^{2-Q}\gamma_0(x).\end{aligned} \tag{3.15}$$

Moreover, by dominated convergence (see (3.11)), γ_0 is continuous on $\mathbb{R}^N \setminus \{0\}$. Since $\gamma(x,\xi) = \gamma_0(\xi^{-1} \circ x)$ (by (3.10)), this is sufficient to get (i), (iii) of Definition 3.8. We now prove (ii). Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ be a test function. We set $\Phi_k(x,t) = \varphi(x)\psi(t/k)$, where $\psi \in C_0^\infty(\mathbb{R})$ is a fixed cut-off function such that $\psi(t) = 1$ if $|t| \le 1$, $\psi(t) = 0$ if $|t| \ge 2$. Since Γ is the fundamental solution for $\mathcal{H}$ and $\Phi_k \in C_0^\infty(\mathbb{R}^{N+1})$, we obtain

$$\begin{aligned}-\varphi(\xi) = -\Phi_k(\xi,0) &= \int_{\mathbb{R}^{N+1}} \Gamma(x,t;\xi,0)\,\mathcal{H}^*\Phi_k(x,t)\,\mathrm{d}x\mathrm{d}t \\ &\longrightarrow \int_{\mathbb{R}^{N+1}} \Gamma(x,t;\xi,0)\,\mathcal{L}\varphi(x)\,\mathrm{d}x\mathrm{d}t,\end{aligned}$$

as $k \to \infty$ (by dominated convergence), being

$$\mathcal{H}^*\Phi_k(x,t) = \psi(t/k)\,\mathcal{L}\varphi(x) + \tfrac{1}{k}\,\varphi(x)\,\psi'(t/k) \longrightarrow \mathcal{L}\varphi(x),$$

as $k \to \infty$, and

$$\begin{aligned}\big|\Gamma(x,t;\xi,0)\big(\mathcal{H}^*\Phi_k(x,t) - \mathcal{L}\varphi(x)\big)\big| &\le \mathbf{c}\,\Gamma(x,t;\xi,0)\,\big(|\mathcal{L}\varphi(x)| + |\varphi(x)|\big) \\ &\in L^1_{(x,t)}(\mathbb{R}^{N+1}),\end{aligned}$$

recalling that $\gamma(\cdot,\xi) \in L_{\mathrm{loc}}(\mathbb{R}^N)$. On the other hand,

$$\int_{\mathbb{R}^{N+1}} \Gamma(x,t;\xi,0)\,\mathcal{L}\varphi(x)\,\mathrm{d}x\mathrm{d}t = \int_{\mathbb{R}^N} \gamma(x,\xi)\,\mathcal{L}\varphi(x)\,\mathrm{d}x.$$

Since $\mathcal{L}$ is self-adjoint, this gives $\mathcal{L}(\gamma(\cdot,\xi)) = -\delta_\xi$. Therefore, we have proved that γ is the fundamental solution for $\mathcal{L}$.

We now set $d_{\mathcal{L}}(x) := (\gamma_0(x))^{1/(2-Q)}$, if $x \neq 0$, $d_{\mathcal{L}}(0) := 0$. We remark that $\gamma_0 > 0$ in $\mathbb{R}^N \setminus \{0\}$, by Bony's strong maximum principle for $\mathcal{L}$ (recalling that $\Gamma \geq 0$). It is easy to recognize that $d_{\mathcal{L}}$ is a homogeneous norm on $\mathbb{G}$. Indeed, the smoothness of $d_{\mathcal{L}}$ away from zero follows from the hypoellipticity of $\mathcal{L}$; the homogeneity of $d_{\mathcal{L}}$ follows from (3.15) which also gives the continuity at zero; finally $d_{\mathcal{L}}(x^{-1}) = d_{\mathcal{L}}(x)$ is a direct consequence of (3.13). The last statement of the theorem straightforwardly follows, recalling that $\mathcal{L}$ is self-adjoint. $\square$

We would like to end this section with some remarks on the heat operators related to sums of sub-Laplacians. Suppose we are given two homogeneous Carnot groups $\mathbb{X} = (\mathbb{R}^N, \circ^{(1)})$, $\mathbb{Y} = (\mathbb{R}^M, \circ^{(2)})$ with dilations

$$\delta_\lambda^{(1)}(x) = (\lambda\, x^{(1)}, \dots, \lambda^r x^{(r)}),\ x \in \mathbb{X}; \quad \delta_\lambda^{(2)}(y) = (\lambda\, y^{(1)}, \dots, \lambda^s y^{(s)}),\ y \in \mathbb{Y}$$

($x^{(i)} \in \mathbb{R}^{N_i}$, $i \leq r$, $N_1 + \cdots + N_r = N$; $y^{(i)} \in \mathbb{R}^{M_i}$, $i \leq s$, $M_1 + \cdots + M_s = M$) and canonical sub-Laplacians $\Delta_{\mathbb{X}} = \sum_{j=1}^{N_1} X_j^2$ and $\Delta_{\mathbb{Y}} = \sum_{j=1}^{M_1} Y_j^2$, respectively. We define a suitable homogeneous Carnot group $\mathbb{G}$ on $\mathbb{R}^{N+M}$ in the way described below. We shall suppose $r \leq s$. If $(x,y) \in \mathbb{R}^N \times \mathbb{R}^M$, we consider the following permutation of the coordinates

$$P(x,y) = (x^{(1)}, y^{(1)}, \dots, x^{(r)}, y^{(r)}, y^{(r+1)}, \dots, y^{(s)}).$$

We then denote the point of $\mathbb{G} \equiv \mathbb{R}^{N+M}$ by $z = P(x,y)$. We finally define the group law $\circ$ and the dilations δ_λ on $\mathbb{G}$ in the natural way: for every $z = P(x,y)$, $\zeta = P(\xi,\eta) \in \mathbb{G}$, we set

$$z \circ \zeta = P(x \circ^{(1)} \xi, y \circ^{(2)} \eta), \quad \delta_\lambda z = P(\delta_\lambda^{(1)} x, \delta_\lambda^{(2)} y).$$

It is then easily proved that $(\mathbb{G}, \circ, \delta_\lambda)$ is a homogeneous Carnot group on $\mathbb{R}^{N+M}$ of step s and $N_1 + M_1$ generators. In other words, the direct product of two homogeneous Carnot groups $\mathbb{X}$, $\mathbb{Y}$ is, up to the permutation P, a homogeneous Carnot group $\mathbb{G}$. Moreover, the canonical sub-Laplacian on $\mathbb{G}$ is the sum of the sub-Laplacians on $\mathbb{X}$ and $\mathbb{Y}$:

$$\Delta_{\mathbb{G}} = \Delta_{\mathbb{X}} + \Delta_{\mathbb{Y}} = \textstyle\sum_{j=1}^{N_1} X_j^2 + \sum_{j=1}^{M_1} Y_j^2.$$

There is a remarkable relationship between the fundamental solution for the heat equation on $\mathbb{G}$ and the fundamental solutions for the heat equations on $\mathbb{X}$ and $\mathbb{Y}$, as stated in the following result.

Proposition 3.10. *Let $\Gamma_{\mathbb{G}}$, $\Gamma_{\mathbb{X}}$ and $\Gamma_{\mathbb{Y}}$ denote the fundamental solutions for the operators $\mathcal{H}_{\mathbb{G}} = \Delta_{\mathbb{G}} - \partial_t$, $\mathcal{H}_{\mathbb{X}} = \Delta_{\mathbb{X}} - \partial_t$ and $\mathcal{H}_{\mathbb{Y}} = \Delta_{\mathbb{Y}} - \partial_t$, respectively. Then, we have*

$$\Gamma_{\mathbb{G}}(P(x,y),t) = \Gamma_{\mathbb{X}}(x,t)\,\Gamma_{\mathbb{Y}}(y,t), \qquad \forall\, x \in \mathbb{R}^N,\ y \in \mathbb{R}^M,\ t \in \mathbb{R}.$$

Proof. Setting $\Gamma(P(x,y),t) = \Gamma_{\mathbb{X}}(x,t)\,\Gamma_{\mathbb{Y}}(y,t)$, we only have to prove that Γ satisfies (i)-(ii)-(iii) of Definition 3.1 with $\mathcal{H} = \mathcal{H}_{\mathbb{G}}$. Applying Theorem 6.1 to $\Gamma_{\mathbb{X}}$, $\Gamma_{\mathbb{Y}}$, (i) easily follows. Let us prove (ii). Let $\varphi \in C_0^\infty(\mathbb{G}\times\mathbb{R})$ be a test function and let us denote $\psi = \varphi \circ P$. We have

$$\int_{\mathbb{G}\times\mathbb{R}} \Gamma\mathcal{H}^*\varphi = \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^N}\int_{\mathbb{R}^M}\int_\varepsilon^\infty \Gamma_{\mathbb{X}}(x,t)\Gamma_{\mathbb{Y}}(y,t)\big(\Delta_{\mathbb{X}}+\Delta_{\mathbb{Y}}+\partial_t\big)\psi(x,y,t)\mathrm{d}t\mathrm{d}y\mathrm{d}x.$$

Moreover, the integral in the right hand-side is equal to

$$-\int_{\mathbb{R}^N}\int_{\mathbb{R}^M} \Gamma_{\mathbb{X}}(x,\varepsilon)\,\Gamma_{\mathbb{Y}}(y,\varepsilon)\,\psi(x,y,\varepsilon)\,\mathrm{d}y\mathrm{d}x.$$

This follows integrating by parts, since $(\Delta_{\mathbb{X}} + \Delta_{\mathbb{Y}} - \partial_t)\Gamma_{\mathbb{X}}\,\Gamma_{\mathbb{Y}} = \Gamma_{\mathbb{Y}}\,\mathcal{H}_{\mathbb{X}}\Gamma_{\mathbb{X}} + \Gamma_{\mathbb{X}}\,\mathcal{H}_{\mathbb{Y}}\Gamma_{\mathbb{Y}} = 0$ in $\mathbb{R}^N \times \mathbb{R}^M \times (\varepsilon,\infty)$. Finally, using Theorem 5.1 and Theorem 6.1, it is a standard argument to prove that such integral goes to $-\psi(0,0,0) = -\varphi(0)$, as $\varepsilon \to 0^+$. This proves (ii). In order to prove (iii), it is now sufficient to observe that $\Gamma \in C^\infty(\mathbb{G}\times\mathbb{R}\setminus\{0\})$ (by the hypoellipticity of $\mathcal{H}$) and to use the homogeneity properties of Γ, which follow directly from the definition of Γ and from the homogeneity of $\Gamma_{\mathbb{X}}$ and $\Gamma_{\mathbb{Y}}$ (see (3.6)). $\square$

As a particular case of Proposition 3.10, one can obtain the formula already found by Gaveau [11] when $\Delta_{\mathbb{X}} = \Delta$ (the Laplace operator) and $\Delta_{\mathbb{Y}} = \Delta_{\mathbb{H}^n}$ (the Kohn Laplacian on the Heisenberg group). We also remark that explicit formulas for some heat kernels on two step nilpotent Lie groups have been shown in the recent papers [2, 3].

4. Harnack inequality for $\mathcal{H}$

Given an open set $\Omega \subseteq \mathbb{R}^{N+1}$, we denote by $H(\Omega)$ the linear space of the $\mathcal{L}$-caloric functions in Ω, i.e., of the smooth functions $u : \Omega \to \mathbb{R}$ such that $\mathcal{H}u = (\mathcal{L} - \partial_t)u = 0$ in Ω. The map $\Omega \mapsto H(\Omega)$ is a harmonic sheaf in $\mathbb{R}^{N+1}$ which we shall denote by H. The aim of this section is to show the Harnack inequality for the $\mathcal{L}$-caloric functions by using some basic properties of H and few general results of abstract Potential Theory. Throughout the section, $\mathcal{L}$ is a fixed sub-Laplacian on $\mathbb{G}$ and all the constants may depend on $\mathcal{L}$. To begin with, we prove a "parabolic" maximum principle for $\mathcal{H}$.

Proposition 4.1. *Let Ω be a bounded open subset of $\mathbb{R}^{N+1}$ and let $z_0 = (x_0, t_0)$ be a point of Ω. Set*

$$\Omega_{z_0} := \{(x,t) \in \Omega \,|\, t < t_0\}, \quad \big(\partial\Omega\big)_{z_0} := \{(x,t) \in \partial\Omega \,|\, t \le t_0\}.$$

Suppose $u \in H(\Omega)$ is such that

$$\limsup_{\Omega_{z_0} \ni z \to \zeta} u(z) \le 0, \quad \forall\, \zeta \in \big(\partial\Omega\big)_{z_0}. \tag{4.1}$$

Then $u \le 0$ in Ω_{z_0}.

Proof. For every $\varepsilon > 0$, define $u_\varepsilon(z) = u(z) - \varepsilon/(t_0 - t)$, for $z = (x,t) \in \Omega_{z_0}$. Then $\mathcal{H}u_\varepsilon = \varepsilon\, \partial_t (t_0 - t)^{-1} > 0$ in Ω_{z_0}. As a consequence, u_ε cannot have a local maximum in Ω_{z_0}. On the other hand, by the boundary condition (4.1),

$$\limsup_{\Omega_{z_0} \ni z \to \zeta} u_\varepsilon(z) < 0, \quad \forall\, \zeta \in \partial(\Omega_{z_0}).$$

It follows that $u_\varepsilon \le 0$ in Ω_{z_0}. Letting ε go to zero, we obtain $u \le 0$ in Ω_{z_0}. □

We call *H-regular* any bounded set $V \subset \mathbb{R}^{N+1}$ such that: for every real function $\varphi \in C(\partial V)$, there exists a function $H^V_\varphi \in H(V) \cap C(\overline{V})$ satisfying $H^V_\varphi = \varphi$ on ∂V. From the maximum principle of Proposition 4.1, it follows that H^V_φ is unique and the map $\varphi \mapsto H^V_\varphi$ is linear and monotone non-decreasing. Then, for every $z \in V$,

$$H^V_\varphi(z) = \int_{\partial V} \varphi \, \mathrm{d}\mu^V_z$$

for a suitable non-negative Radon measure μ^V_z on ∂V. μ^V_z is the *H-harmonic measure* of V at z. The family of the H-regular set is a basis for the Euclidean topology [5, Corollaire 5.2].

We call *$\mathcal{L}$-super-caloric* in an open set $\Omega \subseteq \mathbb{R}^{N+1}$ any lower semicontinuous function $u : \Omega \to (-\infty, \infty]$ such that $u < \infty$ in a dense subset of Ω and $u(z) \ge \int_{\partial V} u \, \mathrm{d}\mu^V_z$, for every H-regular set $V \subset \overline{V} \subset \Omega$ and for every $z \in V$.

It is a standard matter to show that, for any fixed $\zeta \in \mathbb{R}^{N+1}$, the fundamental solution Γ for $\mathcal{H}$ with pole at ζ, i.e., the function $z \mapsto \Gamma(\zeta^{-1} \circ z)$ is $\mathcal{L}$-super-caloric in $\mathbb{R}^{N+1}$. Moreover, the families $\{z \mapsto \Gamma(\zeta^{-1} \circ z) \,|\, \zeta \in \mathbb{R}^{N+1}\}$, $\{(x,t) \mapsto \gamma(\xi^{-1} \circ x) \,|\, \xi \in \mathbb{R}^N\}$ separate the points of $\mathbb{R}^{N+1}$. Then, by Théorème 8.2 and Remarque 8.4 in [5], $(\mathbb{R}^{N+1}, H)$ is a Bauer-harmonic space satisfying the Doob convergence property. This enables us to apply an abstract result by Bauer [1, Satz 1.4.4] in order to get the Harnack inequality for the $\mathcal{L}$-caloric functions. To this end, we fix the standard notations for the parabolic Harnack inequality. Given $z_0 = (x_0, t_0) \in \mathbb{R}^{N+1}$ and $r > 0$, define

$$C(z_0, r) := \{(x,t) \in \mathbb{R}^{N+1} \mid d_{\mathcal{L}}(x, x_0) < r,\ |t - t_0| < r^2\},$$

and, for $0 < \lambda < 1/2$,

$$S_\lambda(z_0, r) := \{(x,t) \in \mathbb{R}^{N+1} \mid d_{\mathcal{L}}(x, x_0) < (1-\lambda)\, r,\ \lambda\, r^2 < t_0 - t < (1-\lambda)\, r^2\}.$$

We recall that $d_{\mathcal{L}}$ is defined by (3.14). Then the following result holds.

Theorem 4.2. *For every $\lambda \in (0, 1/2)$ there exists a positive constant $\mathbf{c} = \mathbf{c}(\lambda)$ such that*

$$\sup_{S_\lambda(z_0,r)} u \le \mathbf{c}\, u(z_0), \tag{4.2}$$

for every non-negative $\mathcal{L}$-caloric function u in $C(z_0, r)$ and for every $r > 0$ and $z_0 \in \mathbb{R}^{N+1}$.

Proof. Since $\mathcal{H}$ is homogeneous w.r.t. the δ_λ-parabolic dilations and invariant w.r.t. the left translations on $\mathbb{G} \times \mathbb{R}$, it is enough to prove (4.2) in the case $r = 1$ and $z_0 = 0$. We split the proof in three steps.

Step I. For brevity of notation, we denote by C the unit cylinder $C(0, 1)$. Following Bauer, we call *absorbent set* any closed set $F \subseteq C$ such that the support of the H-harmonic measure μ_z^V is contained in F for every $z \in F$ and for every regular set $V \subset \overline{V} \subset C$, $V \ni z$. We denote by A_0 the smallest absorbent set containing 0. In Step II and III, we shall prove that

$$A_0 = \{(x,t) \in C \mid t \le 0\}. \tag{4.3}$$

Then, since $\overline{S_\lambda(0,1)}$ is a compact set contained in the interior of A_0, by Satz 1.4.4 in [1] (see also [7, Proposition 6.1.5]) we have $\sup_{S_\lambda(0,1)} u \le \mathbf{c}\, u(0)$, for every non-negative $\mathcal{L}$-caloric function u in C. The constant $\mathbf{c}$ only depends on λ. Then, to complete the proof of the theorem, we only have to prove (4.3).

Step II. The fundamental solution Γ restricted to C is $\mathcal{L}$-super-caloric and non-negative and its support is equal to $\{(x,t) \in C \mid t \ge 0\}$. Then, by Satz 1.4.1 in [1] (see also [7, Proposition 6.1.1])

$$A_0 \subseteq \{(x,t) \in C \mid t \le 0\}. \tag{4.4}$$

Step III. In this step we shall prove the opposite inclusion of (4.4). For this we need two lemmas.

Lemma 4.3. *For every $r < 1$, we denote by $\partial_p C(0, r)$ the parabolic boundary of $C(0, r)$:*

$$\partial_p C(0, r) = \overline{\partial C(0, r) \setminus \{t = r^2\}}.$$

Then, for every $\varphi \in C(\partial_p C(0,r))$, the boundary value problem

$$\begin{cases} \mathcal{H}u = 0 & \text{in } C(0,r) \\ u = \varphi & \text{on } \partial_p C(0,r), \end{cases} \tag{4.5}$$

has a unique solution $H_\varphi \in H(C(0,r))$ satisfying the boundary condition in the following sense

$$\lim_{z \to \zeta} u(z) = \varphi(\zeta), \quad \forall\, \zeta \in \partial_p C(0,r).$$

Proof. The uniqueness of H_φ follows from Proposition 4.1. In order to prove the existence, we first choose a continuous continuation ψ of φ to the complete boundary $\partial C(0,r)$. Let us denote by h the Perron-Wiener solution to the generalized Dirichlet problem $h \in H(C(0,r))$, $h|_{\partial C(0,r)} = \psi$. Then, h is $\mathcal{L}$-caloric in $C(0,r)$ and $\lim_{z\to\zeta} h(z) = \psi(\zeta)$ for any H-regular boundary point of $C(0,r)$. On the other hand, by a result of Negrini [20], every point of the parabolic boundary $\partial_p C(0,r)$ is H-regular. Then, since $\psi = \varphi$ on $\partial_p C(0,r)$, the function h solves problem (4.5). □

Remark. Obviously, again by Proposition 4.1, $\varphi \mapsto H_\varphi(0)$ is linear and monotone non-decreasing. Then,

$$H_\varphi(0) = \int_{\partial_p C(0,r)} \varphi \, \mathrm{d}\mu_0^r,$$

where μ_0^r is a suitable positive Radon measure on $\partial_p C(0,r)$. By Proposition 6.1.1 in [7],

$$\operatorname{spt}(\mu_0^r) \subseteq A_0, \quad \text{for } 0 < r < 1. \tag{4.6}$$

The second lemma we need is the following one.

Lemma 4.4. *For every $r \in (0,1)$ we have*

$$\partial_p C(0,r) \cap \{t \le 0\} \subseteq \operatorname{spt}(\mu_0^r). \tag{4.7}$$

Proof. Suppose, by contradiction, the inclusion (4.7) is false. Then, there exists a boundary continuous function φ such that: $\varphi \ge 0$, φ is not identically 0, $\operatorname{spt}(\varphi) \subseteq \partial_p C(0,r) \cap \{t \le 0\}$ and $H_\varphi(0) = 0$. Since $\varphi \ge 0$ we also have $H_\varphi \ge 0$. Thus, H_φ is a $\mathcal{L}$-caloric function in $C(0,r)$ attaining its minimum value at 0. As a consequence, by Bony's Minimum Propagation Principle [5, Théorème 3.2], $H_\varphi(x,t) = H_\varphi(0,0) = 0$ for every $(x,t) \in C(0,r)$, $t \le 0$. It follows that

$$\lim_{C(0,r) \ni z \to \zeta} H_\varphi(z) = 0, \quad \text{for every } \zeta = (\xi,\tau) \in \partial C(0,r),\ \tau < 0.$$

On the other hand (see Lemma 4.3) $\lim_{z\to\zeta} H_\varphi(z) = \varphi(\zeta)$, for every $\zeta \in \partial_p C(0,r)$. Then, since $\mathrm{spt}(\varphi) \subseteq \partial_p C(0,r) \cap \{t \le 0\}$, $\varphi \equiv 0$. This contradicts our assumption on φ and completes the proof of the lemma. $\square$

Together with (4.6), this lemma implies $\{(x,t) \in C \,|\, t \le 0\} \subseteq A_0$, so that, by the reverse inclusion (4.4), (4.3) holds. This ends the proof of Theorem 4.2. $\square$

If we combine Theorem 4.2 with Théorème 7.1 in [5], we immediately get the following corollary.

Corollary 4.5. *For every $p, q \in \mathbb{N} \cup \{0\}$ and for every $\lambda \in (0, 1/2)$, there exists a positive constant $\mathbf{c} = \mathbf{c}(p,q,\lambda)$ such that*

$$\sup_{S_\lambda(z_0,r)} |X_{i_1} \cdots X_{i_p} (\partial_t)^q\, u| \le \mathbf{c}\, r^{-(p+2q)}\, u(z_0), \tag{4.8}$$

for any non-negative $\mathcal{L}$-caloric function u in $C(z_0,r)$ and for any $z_0 \in \mathbb{R}^{N+1}$, $r > 0$, $i_1, \ldots, i_p \in \{1, \ldots, m\}$.

Proof. Since the vector fields X_j's and ∂_t are homogeneous of degree 1 and 2 respectively w.r.t. the δ_λ-parabolic dilations and invariant w.r.t. the left translations of $\mathbb{G} \times \mathbb{R}$, it is enough to prove (4.8) in the case $r = 1$ and $z_0 = 0$. It is also non-restrictive to assume $u > 0$. By Théorème 7.1 in [5], there exist $z_1, \ldots, z_k \in C(0,1) \cap \{t < 0\}$ such that

$$\sup_{S_\lambda(0,1)} |X_{i_1} \cdots X_{i_p} (\partial_t)^q\, u| \le \mathbf{c}'\, (u(z_1) + \cdots + u(z_k)), \tag{4.9}$$

for every positive $\mathcal{L}$-caloric function u in $C(0,1)$. On the other hand, there exists $\mu \in (0, 1/2)$ such that $\{z_1, \ldots, z_k\} \subseteq S_\mu(0,1)$. Then, by Theorem 4.2, $u(z_1) + \cdots + u(z_k) \le \mathbf{c}''\, u(0)$. This inequality and (4.9) imply (4.8). $\square$

5. Estimates of Γ

In this section we prove the Gaussian estimates of the fundamental solution Γ for $\mathcal{H} = \mathcal{L} - \partial_t$, constructed in Section 3. $\mathcal{L}$ is any fixed sub-Laplacian on $\mathbb{G}$ and all the constants here may depend on $\mathcal{L}$. The estimate from above is obtained by a direct proof based on a comparison argument. The estimate from below follows from the Harnack inequality for $\mathcal{H}$ by means of a classical argument. Since (3.10) holds, we only need to study the function $\Gamma(x,t)$ $(= \Gamma(x,t;0,0))$.

Theorem 5.1. *There exists a positive constant $\mathbf{c}$ such that*

$$\Gamma(x,t) \le \mathbf{c}\, t^{-Q/2} \exp\Big(- \frac{d_{\mathcal{L}}^2(x)}{\mathbf{c}\, t} \Big)$$

for every $x \in \mathbb{R}^N$ *and* $t > 0$. *We recall that* $d_{\mathcal{L}}$ *is the homogeneous norm defined by* (3.14).

Proof. We set $A = \{x \in \mathbb{R}^N | d_{\mathcal{L}}(x) > 1\}$ and $\Omega = A \times (0, 1)$. We want to compare $\Gamma(x,t)$ with the function $w(x,t) = \exp(-\sigma\,(1-t) d_{\mathcal{L}}^2(x))$ in Ω. Here, σ is a positive constant to be chosen in the sequel. The following formula holds for radial functions $f(x) = F(d_{\mathcal{L}}(x))$:

$$\mathcal{L} f(x) = |\nabla_{\mathcal{L}} d_{\mathcal{L}}(x)|^2 \Big(\frac{Q-1}{d_{\mathcal{L}}(x)}\, F'(d_{\mathcal{L}}(x)) + F''(d_{\mathcal{L}}(x)) \Big) \tag{5.1}$$

(we have denoted by $\nabla_{\mathcal{L}}$ the subelliptic gradient $(Y_1, ..., Y_m)$, where $\mathcal{L} = \sum_{j=1}^m Y_j^2$ (see Section 2)). Hence, a direct computation shows that $\mathcal{H} w(x,t)$ is equal to $w(x,t) |\nabla_{\mathcal{L}} d_{\mathcal{L}}(x)|^2 \{-2\sigma(Q-1)(1-t) + (4\sigma^2 d_{\mathcal{L}}^2(x)(1-t)^2 - 2\sigma(1-t))\} - \sigma d_{\mathcal{L}}^2(x) w(x,t)$. For $(x,t) \in \Omega$, we obtain

$$\mathcal{H} w(x,t) \le \big(4\,\sigma^2\,|\nabla_{\mathcal{L}} d_{\mathcal{L}}(x)|^2 - \sigma\big)\, d_{\mathcal{L}}^2(x)\, w(x,t) \le 0$$

if σ is chosen small enough (note that $|\nabla_{\mathcal{L}} d_{\mathcal{L}}(x)|$ is bounded since it is δ_λ-homogeneous of degree zero). Recalling that $\mathcal{H}\Gamma = 0$ in Ω, that Γ is continuous on $\overline{\Omega}$ and that Γ vanishes at infinity and on $A \times \{0\}$ (see Theorem 3.5 and (3.7)), from the maximum principle we infer that

$$\Gamma \le \mathbf{c}\, w \qquad \text{in } \overline{\Omega}$$

for a suitable constant $\mathbf{c} > 0$. In particular, chosen $t = 1/2$, we obtain

$$\Gamma(x, \tfrac{1}{2}) \le \mathbf{c}\, \exp\big(-\sigma\, d_{\mathcal{L}}^2(x)/2\big), \qquad \text{if } d_{\mathcal{L}}(x) \ge 1.$$

By the homogeneity (3.6) of Γ, we then deduce

$$\Gamma(x,t) = (2\,t)^{-Q/2} \Gamma(\delta_{1/\sqrt{2t}} x, \tfrac{1}{2}) \le \mathbf{c} t^{-Q/2} \exp\big(-\frac{\sigma d_{\mathcal{L}}^2(x)}{4t}\big), \text{ if } 0 < 2t \le d_{\mathcal{L}}^2(x).$$

On the other hand, if $d_{\mathcal{L}}^2(x) < 2\,t$, then (3.11) directly yields

$$\Gamma(x,t) \le \mathbf{c}\, t^{-Q/2} \le \mathbf{c}\, t^{-Q/2} \exp\big(-\frac{d_{\mathcal{L}}^2(x)}{\mathbf{c}\, t}\big).$$

This ends the proof. □

Theorem 5.2. *There exists a positive constant* $\mathbf{c}$ *such that*

$$\Gamma(x,t) \ge \mathbf{c}^{-1}\, t^{-Q/2}\, \exp\big(-\mathbf{c}\, \frac{d_{\mathcal{L}}^2(x)}{t}\big)$$

for every $x \in \mathbb{R}^N$ *and* $t > 0$.

Proof. From Theorem 4.2, it follows that there exists a positive constant $\mathbf{c}$ such that

$$u(x,t) \le \mathbf{c}\, u(y,2t)\, \exp\big(\mathbf{c}\,\frac{d_{\mathcal{L}}^2(x,y)}{t}\big), \qquad x,\, y\in\mathbb{R}^N,\ t>0, \tag{5.2}$$

for every function $u>0$ such that $\mathcal{H}u=0$ in $\mathbb{R}^N\times(0,\infty)$. The proof of this assertion is standard; however, for the reader's convenience, we give it below. We first observe that it is sufficient to prove (5.2) with $d_{\mathcal{L}}$ replaced by d_X (see (8.1) in the Appendix), since such distances are equivalent. By means of Theorem 4.2, we have $u(x,s)\le \mathbf{c}\,u(y,2s)$ whenever u is in the above class, $d_{\mathcal{L}}(x,y)\le\sqrt{2s}$ and $s>0$. If we apply this result to the functions $u_r(x,\cdot)=u(x,\cdot+r)$, choosing $r=t-s\ge 0$, we get

$$u(x,t)=u_r(x,t-r)\le \mathbf{c}\,u_r(y,2(t-r))=\mathbf{c}\,u(y,t+s), \tag{5.3}$$

if $d_{\mathcal{L}}(x,y)\le\sqrt{2s}$ and $t\ge s>0$. This proves (5.2) when $d_{\mathcal{L}}(x,y)\le\sqrt{2\,t}$ (we only have to choose $s=t$). On the other hand, if $d_{\mathcal{L}}(x,y)>\sqrt{2\,t}$, one can find a chain of points $x_0=x,x_1,\dots,x_p=y\in\mathbb{R}^N$ (laying on a X-subunit path connecting x and y), with p “proportional” to $d_X^2(x,y)/t$, such that $d_{\mathcal{L}}(x_i,x_{i+1})\le\sqrt{2\,s}$ for $s=t/p$ (proportional to $(d_X(x,y)/p)^2$). Now, applying (5.3), we obtain $u(x_i,t+i\,s)\le\mathbf{c}\,u(x_{i+1},t+(i+1)\,s)$, whence $u(x,t)\le\mathbf{c}^p\,u(y,t+p\,s)\le\mathbf{c}\,u(y,2\,t)\,\exp(\mathbf{c}\,d_X^2(x,y)/t)$. This completes the proof of (5.2).

Finally, applying (5.2) to $u=\Gamma$, we get

$$\Gamma(y,2\,t)\ge\mathbf{c}^{-1}\,\Gamma(0,t)\,\exp\big(-\mathbf{c}\,\frac{d_{\mathcal{L}}^2(y)}{t}\big)=\mathbf{c}^{-1}\,t^{-Q/2}\,\Gamma(0,1)\,\exp\big(-\mathbf{c}\,\frac{d_{\mathcal{L}}^2(y)}{t}\big),$$

by the homogeneity (3.6) of Γ. □

Theorem 5.3. *Given any non-negative integers p, q, there exist positive constants $\mathbf{c}$, $\mathbf{c}_{p,q}$ such that for every $i_1,\dots,i_p\in\{1,\dots,m\}$ we have*

$$\big|X_{i_1}\cdots X_{i_p}(\partial_t)^q\,\Gamma(x,t)\big|\le\mathbf{c}_{p,q}\,t^{-(Q+p+2q)/2}\,\exp\big(-\frac{d_{\mathcal{L}}^2(x)}{\mathbf{c}\,t}\big),$$

for every $x\in\mathbb{R}^N$ and $t>0$.

Proof. The assertion is an easy consequence of Theorem 5.1 and the Harnack inequality in Corollary 4.5. □

6. THE CAUCHY PROBLEM

In this section we establish some more properties of the fundamental solution Γ for $\mathcal{H}$, constructed in Section 3, and we obtain some existence and uniqueness results for the Cauchy problem related to $\mathcal{H}$.

Theorem 6.1. *For every $x \in \mathbb{R}^N$ and $t > \tau$, we have*

$$\int_{\mathbb{R}^N} \Gamma(x,t;\xi,\tau)\,\mathrm{d}\xi = 1.$$

Proof. We first observe that the above integral is finite by Theorem 5.1. Moreover, using (3.6) and (3.10), it is easy to see that it does not depend on x, t, τ, i.e., there exists a positive constant α such that

$$\int_{\mathbb{R}^N} \Gamma(x,t;\xi,\tau)\,\mathrm{d}\xi = \alpha, \quad \text{for every } x \in \mathbb{R}^N \text{ and } t > \tau \tag{6.1}$$

($\alpha \neq 0$ since Γ is not identically zero in $\mathbb{R}^{N+1}$). In order to prove that $\alpha = 1$, we shall test the identity

$$\mathcal{G}_\infty(\mathcal{H}\Phi) = -\Phi$$

(see Theorem 3.5-(iii)) on some suitable $\Phi \in C_0^\infty(\mathbb{R}^{N+1})$. Let us choose $\varphi \in C_0^\infty((-1,1))$ such that $\varphi(0) = 1$ and $g \in C_0^\infty(\mathbb{R})$ such that $0 \le g \le 1$, $g(r) = 1$ if $|r| \le 1$, $g(r) = 0$ if $|r| \ge 2$. We define $\Phi_k(x,t) = \varphi(t)\,\psi_k(x)$ where $\psi_k(x) = g(d_{\mathcal{L}}(x)/k)$. We have $\mathcal{H}\Phi_k(x,t) = \varphi(t)\,\mathcal{L}\psi_k(x) - \psi_k(x)\,\varphi'(t)$. Hence

$$\begin{aligned} 1 &= \Phi_k(0) = -\mathcal{G}_\infty(\mathcal{H}\Phi_k)(0) \\ &= \int_{\mathbb{R}^{N+1}} \Gamma(0,0;\xi,\tau)\psi_k(\xi)\varphi'(\tau)\,\mathrm{d}\xi\mathrm{d}\tau - \int_{\mathbb{R}^{N+1}} \Gamma(0,0;\xi,\tau)\varphi(\tau)\mathcal{L}\psi_k(\xi)\,\mathrm{d}\xi\mathrm{d}\tau. \end{aligned}$$

Moreover, using (5.1) it is easy to see that $|\mathcal{L}\psi_k(x)| \le \mathbf{c}/k^2$. Therefore, letting $k \to \infty$ in the above identity and recalling Theorem 3.5-(iv) and (6.1), we obtain

$$1 = \int_{\mathbb{R}^{N+1}} \Gamma(0,0;\xi,\tau)\,\varphi'(\tau)\,\mathrm{d}\xi\mathrm{d}\tau = \alpha \int_{-1}^0 \varphi'(\tau)\,\mathrm{d}\tau = \alpha\,\varphi(0) = \alpha.$$

This completes the proof. □

Corollary 6.2. *Let f be a continuous function on $\mathbb{R}^N$ satisfying the growth condition*

$$|f(x)| \le c\,\exp\left(\mu\, d_{\mathcal{L}}^2(x)\right)$$

for some constants c, $\mu \ge 0$. Then the function

$$u(x,t) = \int_{\mathbb{R}^N} \Gamma(x,t;\xi,0)\,f(\xi)\,\mathrm{d}\xi, \qquad x \in \mathbb{R}^N,\ t \in (0, T/\mu)$$

is well posed (being $T > 0$ an absolute constant) and is a classical solution to the Cauchy problem

$$\begin{cases} \mathcal{H}u(x,t) = 0, & (x,t) \in \mathbb{R}^N \times (0, T/\mu), \\ u(x,0) = f(x), & x \in \mathbb{R}^N. \end{cases}$$

Proof. The function u is well posed by means of Theorem 5.1. Moreover, from Theorem 3.5-(v) and the estimates in Theorem 5.3, it follows that $\mathcal{H}u = 0$. Finally, using Theorem 5.1 and Theorem 6.1, it is a standard argument to prove that $u(x,t) \longrightarrow f(x_0)$ as $(x,t) \to (x_0, 0)$. □

Corollary 6.3. *For every $x \in \mathbb{R}^N$, $t > 0$ and $\tau > 0$, we have the following* reproduction property

$$\Gamma(x, t+\tau) = \int_{\mathbb{R}^N} \Gamma(\xi^{-1} \circ x, t)\, \Gamma(\xi, \tau)\, \mathrm{d}\xi.$$

Proof. We fix $\tau > 0$ and set $v(x,t) = \Gamma(x, t+\tau)$, $u(x,t) = \int_{\mathbb{R}^N} \Gamma(\xi^{-1} \circ x, t)\, \Gamma(\xi, \tau)\, \mathrm{d}\xi$ for $(x,t) \in \Omega = \mathbb{R}^N \times (0,\infty)$. We have $\mathcal{H}v = 0$ in Ω, $v \longrightarrow 0$ at infinity (see Theorem 3.5 and (3.7)). Moreover, from Corollary 6.2 (see also (3.10)) it follows that $\mathcal{H}u = 0$ in Ω, $u|_{\partial\Omega} = v|_{\partial\Omega}$. Hence, by the maximum principle, in order to prove that $u \equiv v$ in Ω, we only need to show that $u \longrightarrow 0$ at infinity. This statement can be easily proved if we note that, by Theorems 5.1 and 6.1, the following inequality holds:

$$0 \le u(x,t) \le \mathbf{c}_\tau \int_{d(\xi)\le k} \Gamma(\xi^{-1} \circ x, t)\, \mathrm{d}\xi + \mathbf{c}_\tau \exp(-\widetilde{\mathbf{c}}_\tau\, k^2),$$

for every $k > 0$. □

We now want to establish a uniqueness result for the solutions to the Cauchy problem related to $\mathcal{H}$. To this purpose, we need the following lemma.

Lemma 6.4. *Let $r > 0$ be fixed. Let u be a classical solution to the Cauchy problem*

$$\mathcal{H}u = 0 \ \text{ in } \mathbb{R}^N \times (0,r), \quad u(\cdot, 0) = 0. \tag{6.2}$$

Then, extending $u(x,t)$ to be zero for $t < 0$, we have $u \in C^\infty(\mathbb{R}^N \times (-\infty, r))$.

Proof. By the hypoellipticity of $\mathcal{H}$, it is sufficient to prove $\mathcal{H}u = 0$ on $\mathbb{R}^N \times (-\infty, r)$, in the weak sense of distributions. Let $\varphi \in C_0^\infty(\mathbb{R}^N \times (-\infty, r))$ be a fixed test function. We introduce the family of cut-off functions $\psi_\sigma(t) = \psi(|t|/\sigma)$, $(\sigma > 0)$ with $\psi \in C^\infty([0,\infty))$, $0 \le \psi \le 1$, $\psi \equiv 0$ on $[2,\infty)$, $\psi \equiv 1$ on $[0,1]$. We have

$$\Big|\int u\mathcal{H}^*\varphi\Big| \le \int_{|t|\le 2\sigma}\int_{x\in\,\mathrm{spt}(\varphi)} |u(x,t)||\mathcal{H}^*(\varphi\psi_\sigma)(x,t)|\,\mathrm{d}x\mathrm{d}t + \Big|\int u\mathcal{H}^*(\varphi(1-\psi_\sigma))\Big|.$$

The second integral in the right hand-side vanishes since $\mathcal{H}u = 0$ in the support of $\varphi\,(1-\psi_\sigma)$. On the other hand, $|\mathcal{H}^*(\varphi\,\psi_\sigma)|(x,t) \le |\mathcal{H}^*(\varphi)\,\psi_\sigma|(x,t) + 1/\sigma\,|\psi'(|t|/\sigma)|\,|\varphi|(x,t) \le \mathbf{c}/\sigma$, for small σ. Hence, by the continuity of u and since $u(\cdot,0) = 0$, the first integral in the right hand-side vanishes as σ goes to zero. This completes the proof. □

Theorem 6.5. *Let $r > 0$ be fixed. Let u be a classical solution to the Cauchy problem*

$$\mathcal{H}u = 0 \ \textit{ in } \mathbb{R}^N \times (0, r), \quad u(\cdot, 0) = 0. \tag{6.3}$$

Suppose that one of the following conditions holds: either u is non-negative or there exists $\mu > 0$ such that

$$\int_0^r \int_{\mathbb{R}^N} \exp\big(-\mu\, d_{\mathcal{L}}^2(x)\big)\, |u(x,t)|\, \mathrm{d}x\mathrm{d}t < \infty. \tag{6.4}$$

Then u vanishes identically.

Proof. We first prove that $u \equiv 0$ if (6.4) holds. It is sufficient to prove that there exists $\nu = \nu(\mu) > 0$ such that $u = 0$ on $\mathbb{R}^N \times (0, \nu)$ and to repeat the argument finitely many times. Let $\nu > 0$ be fixed as we shall specify in the sequel and let $\bar{z} = (\bar{x}, \bar{t}) \in \mathbb{R}^N \times (0, \nu)$. We also set $B_\rho(\bar{x}) = \{x \in \mathbb{R}^N \,|\, d_{\mathcal{L}}(\bar{x}, x) < \rho\}$, for any $\rho > 0$. Let ψ_ρ be a smooth function on $[0, \infty)$ such that $0 \le \psi_\rho \le 1$, $\psi_\rho \equiv 0$ on $[\rho + 1, \infty)$, $\psi_\rho \equiv 1$ on $[0, \rho]$ and such that ψ'_ρ, ψ''_ρ are bounded by a constant independent of ρ. We now define the cut-off functions $h_\rho(x) = \psi_\rho(d_{\mathcal{L}}(\overline{x}, x))$ so that $X_i h_\rho$, $X_i X_j h_\rho$ are bounded by a constant independent of ρ (when $\rho \ge 1$) for every i, $j \in \{1, \dots, m\}$.

We set $v = h_\rho\, \Gamma(\bar{z}; \cdot)$ and we integrate the Green's identity $v\, \mathcal{H}u - u\, \mathcal{H}^* v = \operatorname{div}(v\, M_{\mathcal{L}} \nabla u - u\, M_{\mathcal{L}} \nabla v) - \partial_t(u\, v)$ on the domain $B_{\rho+1}(\bar{x}) \times (0, \bar{t} - \varepsilon)$, where $M_{\mathcal{L}}$ is defined by (2.2) and $\varepsilon > 0$ is small. We explicitly remark that, by Lemma 6.4, u is smooth up to the boundary of that domain. As $\varepsilon \to 0$ we obtain

$$u(\bar{z}) = \lim_{\varepsilon \to 0} \int_{B_{\rho+1}(\bar{x})} u(x, \bar{t} - \varepsilon) h_\rho(x) \Gamma(\bar{x}^{-1} \circ x, \varepsilon)\, \mathrm{d}x = \int_0^{\bar{t}} \int_{B_{\rho+1}(\bar{x})} u(x,t) \mathcal{H}^* v(x,t)\, \mathrm{d}x\mathrm{d}t.$$

The first equality follows by a standard argument using Theorem 6.1 and Theorem 5.1. Since $\mathcal{H}^* \Gamma(\bar{z}; \cdot) = 0$ away from $\bar{z}$ (see Theorem 3.5-(vi)), then

$$u(\bar{z}) = \int_0^{\bar{t}} \int_{B_{\rho+1}(\bar{x}) \setminus B_\rho(\bar{x})} u(x,t)\, \mathcal{H}^* v(x,t)\, \mathrm{d}x\mathrm{d}t. \tag{6.5}$$

On the other hand, by (3.13) and Theorems 5.1 and 5.3,

$$\begin{aligned} |\mathcal{H}^* v(z)| &= \Big| \Gamma(\bar{z}; z)\, \mathcal{L} h_\rho + \sum_{i=1}^m X_i h_\rho\, X_i \Gamma(\bar{x}^{-1} \circ x, \bar{t} - t) \Big| \\ &\le \mathbf{c}\, (\bar{t} - t)^{-(Q+1)/2} \exp\Big(- \frac{d_{\mathcal{L}}^2(\bar{x}, x)}{\mathbf{c}\, (\bar{t} - t)} \Big). \end{aligned}$$

Hence,

$$
\begin{aligned}
|u(\bar z)| &\le \mathbf{c}\int_0^{\bar t}\int_{B_{\mathbf{c}\,\rho}(0)\setminus B_{\rho/\mathbf{c}}(0)} |u(x,t)|\,(\bar t-t)^{-(Q+1)/2}\,\exp\Big(-\frac{d_{\mathcal{L}}^2(\bar x,x)}{\mathbf{c}\,(\bar t-t)}\Big)\,\mathrm{d}x\mathrm{d}t\\
&\le \mathbf{c}\int_0^{\bar t}\int_{B_{\mathbf{c}\,\rho}(0)\setminus B_{\rho/\mathbf{c}}(0)} |u(x,t)|\,\exp\big(-\mu\, d_{\mathcal{L}}^2(\bar x,x)\big)\,\mathrm{d}x\mathrm{d}t,
\end{aligned}
$$

if $\nu(\mu)$ is suitably chosen (recall that $\bar t \le \nu(\mu)$). Finally, letting ρ go to infinity, and using (6.4), we obtain $u(\bar z)=0$. This proves that $u\equiv 0$ if (6.4) holds.

We now consider the case when u is a non-negative solution to (6.3). We set, for fixed $\tau\in(0,r)$, $\rho>0$,

$$
w_\rho^\tau(x,t) := \int_{\mathbb{R}^N} \Gamma(x,t;\xi,\tau)\, f_\rho^\tau(\xi)\,\mathrm{d}\xi, \qquad x\in\mathbb{R}^N,\ t\in(\tau,r),
$$

where $f_\rho^\tau(\xi)=u(\xi,\tau)$ if $d_{\mathcal{L}}(\xi)\le\rho$, $f_\rho^\tau(\xi)=0$ otherwise. It is a standard argument (see also Corollary 6.2) to prove that

$$
\mathcal{H} w_\rho^\tau = 0 \ \text{ in } \mathbb{R}^N\times(\tau,r); \qquad \lim_{(x,t)\to(y,\tau)} w_\rho^\tau(x,t) = f_\rho^\tau(y), \quad \text{for } d_{\mathcal{L}}(y)\neq\rho.
$$

Since $u\ge 0$ this readily implies that

$$
\liminf_{(x,t)\to(y,\tau)} \big(u(y,\tau)-w_\rho^\tau(x,t)\big)\ge 0, \qquad y\in\mathbb{R}^N.
$$

Here, we have used the fact that, if $d_{\mathcal{L}}(y)=\rho$, then $w_\rho^\tau(y)\le w_{2\rho}^\tau(y)\longrightarrow u(y,\tau)$. Moreover, it is easy to see that $\liminf_{d_{\mathcal{L}}(x)\to\infty}\big(u(y,\tau)-w_\rho^\tau(x,t)\big)\ge 0$, uniformly in $t\in(\tau,r)$. Applying the parabolic maximum principle, we obtain $w_\rho^\tau\le u$ in $\mathbb{R}^N\times(\tau,r)$. Now, we take $x=0$ and we fix $t\in(0,r)$. Letting ρ go to infinity in $w_\rho^\tau\le u$ and integrating in τ, we get

$$
\begin{aligned}
(t-\varepsilon)\,u(0,t) &\ge \int_0^{t-\varepsilon}\int_{\mathbb{R}^N}\Gamma(0,t;\xi,\tau)\,u(\xi,\tau)\,\mathrm{d}\xi\mathrm{d}\tau\\
&\ge \mathbf{c}\int_0^{t-\varepsilon}\int_{\mathbb{R}^N}\exp\Big(-\frac{\mathbf{c}\,d_{\mathcal{L}}^2(\xi)}{\varepsilon}\Big)\,u(\xi,\tau)\,\mathrm{d}\xi\mathrm{d}\tau
\end{aligned}
$$

by Theorem 5.2. Using the first part of the proof, this implies that u vanishes on $\mathbb{R}^N\times(0,t-\varepsilon)$. Since $t\in(0,r)$ is arbitrary, the theorem is completely proved. $\square$

7. Uniform estimates for a family of fundamental solutions

In the first part of this section, we restrict our attention to the special case of free Carnot groups and we derive the uniform estimates using Theorem A and Theorem B. In the second part of the section, we shall prove the uniform estimates for general Carnot groups: using the lifting procedure recalled in Theorem C, we shall reduce to the free case.

Throughout the section, we shall use the notations of Section 2. In particular $\Lambda \geq 1$ will be a fixed constant and $\mathcal{M}_\Lambda$ will denote the set of $m \times m$ symmetric matrices A such that $\Lambda^{-1}|\xi|^2 \leq \langle A\xi, \xi\rangle \leq \Lambda|\xi|^2$. We shall denote by $\mathbf{c}_\Lambda$, $\widetilde{\mathbf{c}}_\Lambda, \dots$ any positive constant depending only on Λ and on the structure of $\mathbb{G}$.

I - The free case. In this first part of the section, we shall suppose that $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_\lambda)$ is a *free* Carnot group.

Theorem 7.1. *Suppose $\mathbb{G}$ is free. For every $A \in \mathcal{M}_\Lambda$ we have*

$$\Gamma_A(x,t;\xi,\tau) = J_A \cdot \Gamma_{\mathbb{G}}(T_A(x),t;T_A(\xi),\tau), \qquad x,\, \xi \in \mathbb{R}^N,\ t,\, \tau \in \mathbb{R}.$$

We recall that Γ_A denotes the fundamental solution for $\mathcal{H}_A = \mathcal{L}_A - \partial_t$ and $\Gamma_{\mathbb{G}}$ the fundamental solution for $\mathcal{H}_{\mathbb{G}} = \Delta_{\mathbb{G}} - \partial_t$, while J_A and T_A are defined in Theorem A and Theorem B.

Proof. Setting $\Gamma(x,t;\xi,\tau) = J_A \cdot \Gamma_{\mathbb{G}}(T_A(x),t;T_A(\xi),\tau)$, we only have to prove that Γ satisfies (i)-(ii)-(iii) of Definition 3.1 with $\mathcal{L} = \mathcal{L}_A$. Recalling that T_A is a diffeomorphism, (i) and (iii) follow immediately from the analogous properties of $\Gamma_{\mathbb{G}}$. Let us prove (ii). Let $\varphi \in C_0^\infty(\mathbb{R}^{N+1})$ be a test function and set $\psi(x,t) = \varphi(T_A^{-1}(x),t)$. By (2.7) we have $\Delta_{\mathbb{G}}\psi(x,t) = (\mathcal{L}_A\varphi)(T_A^{-1}(x),t)$. Since also $\psi \in C_0^\infty(\mathbb{R}^{N+1})$ we have

$$\begin{aligned}
&\int_{\mathbb{R}^{N+1}} \Gamma(x,t;\xi,\tau)\, \mathcal{H}_A^*\varphi(x,t)\, \mathrm{d}x\mathrm{d}t \\
&= \int_{\mathbb{R}^{N+1}} \Gamma_{\mathbb{G}}(T_A(x),t;T_A(\xi),\tau)\, \mathcal{H}_A^*\varphi(x,t)\, J_A(x)\, \mathrm{d}x\mathrm{d}t \\
&= \int_{\mathbb{R}^{N+1}} \Gamma_{\mathbb{G}}(x',t;T_A(\xi),\tau)\, \mathcal{H}_{\mathbb{G}}^*\psi(x',t)\, \mathrm{d}x'\mathrm{d}t = -\psi(T_A(\xi),\tau) = -\varphi(\xi,\tau).
\end{aligned}$$

This completes the proof. □

From Theorem 7.1 and Theorem 3.9, we also immediately get the following relation between the fundamental solutions for the sub-Laplacians on $\mathbb{G}$.

Corollary 7.2. *Suppose $\mathbb{G}$ is free. For every $A \in \mathcal{M}_\Lambda$ we have*

$$\gamma_A(x) = J_A \cdot \gamma_{\mathbb{G}}(T_A(x)), \qquad x \in \mathbb{R}^N,$$

where $\gamma_A, \gamma_{\mathbb{G}}$ denote the fundamental solutions for $\mathcal{L}_A, \Delta_{\mathbb{G}}$, respectively.

Theorem 7.3. *Suppose $\mathbb{G}$ is free. There exists a positive constant $\mathbf{c}_\Lambda$ such that*

$$\mathbf{c}_\Lambda^{-1}\, t^{-Q/2} \exp\big(-\frac{\mathbf{c}_\Lambda\, d_{\mathbb{G}}^2(x)}{t}\big) \le \Gamma_A(x,t) \le \mathbf{c}_\Lambda t^{-Q/2} \exp\big(-\frac{d_{\mathbb{G}}^2(x)}{\mathbf{c}_\Lambda\, t}\big),$$

$x \in \mathbb{R}^N$, $t > 0$, for every $A \in \mathcal{M}_\Lambda$. We recall that $d_{\mathbb{G}}$ is defined by (3.14) *when $\mathcal{L} = \Delta_{\mathbb{G}}$.*

Proof. The assertion follows from Theorem 7.1, from the estimates of $\Gamma_{\mathbb{G}}$ (see Theorem 5.1 and Theorem 5.2) and from the good properties of T_A established in Theorem B. Indeed, recalling that $T_A(0) = 0$, we have

$$\begin{aligned}\Gamma_A(x,t) &= J_A \cdot \Gamma_{\mathbb{G}}(T_A(x),t) \le \mathbf{c}_\Lambda\, \Gamma_{\mathbb{G}}(T_A(x),t) \qquad \text{(by (2.8))}\\ &\le \mathbf{c}_\Lambda\, t^{-Q/2} \exp\big(-\frac{d_{\mathbb{G}}^2(T_A(x))}{\mathbf{c}\, t}\big) \qquad \text{(by Theorem 5.1)}\\ &\le \mathbf{c}_\Lambda\, t^{-Q/2} \exp\big(-\frac{d_{\mathbb{G}}^2(x)}{\mathbf{c}_\Lambda\, t}\big) \qquad \text{(by (2.10)).}\end{aligned}$$

The estimate from below is analogous. □

Theorem 7.4. *Suppose $\mathbb{G}$ is free. Given any non-negative integers p, q, there exist positive constants $\mathbf{c}_\Lambda$ and $\mathbf{c}_{\Lambda,p,q}$ such that*

$$\big|X_{i_1}\cdots X_{i_p}(\partial_t)^q \Gamma_A(x,t)\big| \le \mathbf{c}_{\Lambda,p,q} t^{-(Q+p+2q)/2} \exp\big(-\frac{d_{\mathbb{G}}^2(x)}{\mathbf{c}_\Lambda\, t}\big),\ x \in \mathbb{R}^N, t > 0,$$

for every $i_1, \ldots, i_p \in \{1, \ldots, m\}$ and for every $A \in \mathcal{M}_\Lambda$.

Proof. We set $Y_i = \sum_{j=1}^m (A^{1/2})_{i,j}\, X_j$ and we recall that (2.6) holds. Setting for brevity $B = A^{-1/2}$, we have $X_i = \sum_{k=1}^m b_{i,k} Y_k$. Hence, from Theorem 7.1, we get

$$\begin{aligned}&\big|X_{i_1}\cdots X_{i_p}(\partial_t)^q\, \Gamma_A(x,t)\big|\\ &= J_A \Big| \sum_{k_1,\ldots,k_p=1}^m b_{i_1,k_1}\cdots b_{i_p,k_p} Y_{k_1}\cdots Y_{k_p}\big((\partial_t)^q \Gamma_{\mathbb{G}}(\cdot,t)\circ T_A\big)(x)\Big|\\ &= J_A \Big| \sum_{k_1,\ldots,k_p=1}^m b_{i_1,k_1}\cdots b_{i_p,k_p}\big(X_{k_1}\cdots X_{k_p}(\partial_t)^q \Gamma_{\mathbb{G}}\big)(T_A(x),t)\Big| \qquad \text{(by (2.6))}\\ &\le \mathbf{c}_{\Lambda,p,q}\, t^{-(Q+p+2q)/2} \exp\big(-\frac{d_{\mathbb{G}}^2(T_A(x))}{\mathbf{c}\, t}\big). \qquad (7.1)\end{aligned}$$

The last inequality follows from Theorem 5.3, (2.8) and (2.12). Recalling (2.10), the proof is complete. □

Theorem 7.5. *Suppose $\mathbb{G}$ is free. There exists a positive constant $\mathbf{c}_\Lambda$ such that*

$$\left|\Gamma_{A_1}(x,t)-\Gamma_{A_2}(x,t)\right| \le \mathbf{c}_\Lambda \|A_1-A_2\|^{1/r} t^{-Q/2} \exp\big(-\frac{d_{\mathbb{G}}^2(x)}{\mathbf{c}_\Lambda\, t}\big),$$

$x \in \mathbb{R}^N$, $t>0$, *for every* $A_1, A_2 \in \mathcal{M}_\Lambda$.

In the proof of Theorem 7.5, we shall use the following result.

Lemma 7.6. *There exists a positive constant* $\mathbf{c}$*, such that*

$$|u(x_1)-u(x_2)| \le \mathbf{c}\, d_{\mathbb{G}}(x_1,x_2) \sup_{d_{\mathbb{G}}(y,x_1)\le \mathbf{c}\, d_{\mathbb{G}}(x_1,x_2)} \big(|X_1u(y)|+\cdots+|X_mu(y)|\big)$$

for every x_1, $x_2 \in \mathbb{R}^N$ *and for every function* $u:\mathbb{R}^N\to\mathbb{R}$ *regular enough.*

Proof. Since $d_{\mathbb{G}}$ is equivalent to the Carnot-Carathéodory distance d_X (see (8.1)), we may prove the lemma with d_X instead of $d_{\mathbb{G}}$. The assertion then easily follows arguing with X-subunit paths $\alpha:[0,\delta]\to\mathbb{R}^N$ connecting x_1 and x_2 and writing $u(x_2)-u(x_1)=\int_0^\delta \frac{\mathrm{d}}{\mathrm{d}s}(u(\alpha(s)))\,\mathrm{d}s$. □

Proof of Theorem 7.5. Applying Lemma 7.6 to the function $u=\Gamma_{\mathbb{G}}(\cdot,t)$ and using Theorem 5.3, we obtain

$$\begin{aligned}
&\left|\Gamma_{\mathbb{G}}(T_{A_1}(x),t)-\Gamma_{\mathbb{G}}(T_{A_2}(x),t)\right| && (7.2)\\
&\le \mathbf{c} d_{\mathbb{G}}(T_{A_1}(x),T_{A_2}(x)) t^{-(Q+1)/2} \sup_{d_{\mathbb{G}}(y,T_{A_1}(x))\le \mathbf{c} d_{\mathbb{G}}(T_{A_1}(x),T_{A_2}(x))} \exp\big(-\frac{d_{\mathbb{G}}^2(y)}{\mathbf{c}t}\big)\\
&\le \mathbf{c}_\Lambda \|A_1-A_2\|^{1/r} d_{\mathbb{G}}(x) t^{-(Q+1)/2} \sup_{d_{\mathbb{G}}(y,T_{A_1}(x))\le \mathbf{c}_\Lambda\|A_1-A_2\|^{1/r} d_{\mathbb{G}}(x)} \exp\big(-\frac{d_{\mathbb{G}}^2(y)}{\mathbf{c}t}\big).
\end{aligned}$$

The last inequality follows from (2.11). Since $d_{\mathbb{G}}(T_{A_1(x)}) \ge {\mathbf{c}_\Lambda}^{-1} d_{\mathbb{G}}(x)$ by (2.10), there exists a positive constant $\sigma_\Lambda \le 1$ such that, if $\|A_1-A_2\|<\sigma_\Lambda$, then the supremum in the far right hand-side of (7.2) is smaller than $\exp(-d_{\mathbb{G}}^2(x)/(\mathbf{c}_\Lambda\, t))$. Thus, for $\|A_1-A_2\|<\sigma_\Lambda$, from (7.2), we get

$$\begin{aligned}
&\left|\Gamma_{\mathbb{G}}(T_{A_1}(x),t)-\Gamma_{\mathbb{G}}(T_{A_2}(x),t)\right| && (7.3)\\
&\le \mathbf{c}_\Lambda \,\|A_1-A_2\|^{1/r} t^{-Q/2} \frac{d_{\mathbb{G}}(x)}{\sqrt{t}} \exp\big(-\frac{d_{\mathbb{G}}^2(x)}{\mathbf{c}_\Lambda\, t}\big)\\
&\le \widetilde{\mathbf{c}}_\Lambda \,\|A_1-A_2\|^{1/r}\, t^{-Q/2} \exp\big(-\frac{d_{\mathbb{G}}^2(x)}{\mathbf{c}_\Lambda\, t}\big).
\end{aligned}$$

On the other hand, by Theorem 7.1,

$$|\Gamma_{A_1}(x,t) - \Gamma_{A_2}(x,t)| \\ \leq |J_{A_1} - J_{A_2}|\, \Gamma_{\mathbb{G}}(T_{A_1}(x),t) + J_{A_2}\, |\Gamma_{\mathbb{G}}(T_{A_1}(x),t) - \Gamma_{\mathbb{G}}(T_{A_2}(x),t)|.$$

Therefore, when $\|A_1 - A_2\| < \sigma_\Lambda$, the thesis follows using (7.3), Theorem 5.1, (2.8), (2.9) and (2.10). If on the contrary $\|A_1 - A_2\| \geq \sigma_\Lambda$, then the thesis straightforwardly follows from Theorem 7.3, writing $|\Gamma_{A_1}(x,t) - \Gamma_{A_2}(x,t)| \leq |\Gamma_{A_1}(x,t)| + |\Gamma_{A_2}(x,t)|$. □

Theorem 7.7. *Suppose $\mathbb{G}$ is free. Given any non-negative integers p, q, there exist positive constants $\mathbf{c}_\Lambda$ and $\mathbf{c}_{\Lambda,p,q}$ such that*

$$\left|X_{i_1} \cdots X_{i_p} (\partial_t)^q\, \Gamma_{A_1}(x,t) - X_{i_1} \cdots X_{i_p} (\partial_t)^q\, \Gamma_{A_2}(x,t)\right| \\ \leq \mathbf{c}_{\Lambda,p,q}\, \|A_1 - A_2\|^{1/r}\, t^{-(Q+p+2q)/2} \exp\big(-\frac{d_{\mathbb{G}}^2(x)}{\mathbf{c}_\Lambda\, t}\big), \qquad x \in \mathbb{R}^N, t > 0,$$

for every $i_1, \ldots, i_p \in \{1, \ldots, m\}$ and for every $A_1, A_2 \in \mathcal{M}_\Lambda$.

Proof. Following the notation in the proof of Theorem 7.4, we set

$$(Y^{(h)})_i = \sum_{j=1}^m (A_h^{1/2})_{i,j}\, X_j \quad (h = 1,\, 2),\ \ i = 1, ..., m.$$

From Theorem 7.1 and (2.6), arguing as in (7.1) and using the identity $a_1 \cdots a_n - b_1 \cdots b_n = \sum_{j=1}^n a_1 \cdots a_{j-1}(a_j - b_j) b_{j+1} \cdots b_n$, we get

$$|X_{i_1} \cdots X_{i_p} (\partial_t)^q \Gamma_{A_1}(x,t) - X_{i_1} \cdots X_{i_p} (\partial_t)^q \Gamma_{A_2}(x,t)| = \Big| \sum_{k_1,\ldots,k_p=1}^m \Big(J_{A_1} (A_1^{-\frac{1}{2}})_{i_1,k_1} \\ \cdots (A_1^{-\frac{1}{2}})_{i_p,k_p} (Y^{(1)})_{k_1} \cdots (Y^{(1)})_{k_p} \big((\partial_t)^q \Gamma_{\mathbb{G}}(\cdot,t) \circ T_{A_1}\big)(x) \\ - J_{A_2} (A_2^{-\frac{1}{2}})_{i_1,k_1} \cdots (A_2^{-\frac{1}{2}})_{i_p,k_p} (Y^{(2)})_{k_1} \cdots (Y^{(2)})_{k_p} \big((\partial_t)^q \Gamma_{\mathbb{G}}(\cdot,t) \circ T_{A_2}\big)(x)\Big)\Big| \\ \leq \sum_{k_1,\ldots,k_p=1}^m \Big(|J_{A_1} - J_{A_2}| \big|(A_2^{-\frac{1}{2}})_{i_1,k_1} \cdots (A_2^{-\frac{1}{2}})_{i_p,k_p}\big| |X_{k_1} \cdots X_{k_p} (\partial_t)^q \Gamma_{\mathbb{G}}(T_{A_2}(x),t)| \\ + \Big(\sum_{j=1}^p J_{A_1} \big|(A_1^{-\frac{1}{2}})_{i_1,k_1} \cdots (A_1^{-\frac{1}{2}})_{i_{j-1},k_{j-1}}\big| \big|(A_1^{-\frac{1}{2}})_{i_j,k_j} - (A_2^{-\frac{1}{2}})_{i_j,k_j}\big| \cdot \\ \big|(A_2^{-\frac{1}{2}})_{i_{j+1},k_{j+1}} \cdots (A_2^{-\frac{1}{2}})_{i_p,k_p}\big| \, |X_{k_1} \cdots X_{k_p} (\partial_t)^q \Gamma_{\mathbb{G}}(T_{A_2}(x),t)|\Big) \\ + J_{A_1} \big|(A_1^{-\frac{1}{2}})_{i_1,k_1} \cdots (A_1^{-\frac{1}{2}})_{i_p,k_p}\big| \cdot \\ \big|X_{k_1} \cdots X_{k_p} (\partial_t)^q \Gamma_{\mathbb{G}}(T_{A_1}(x),t) - X_{k_1} \cdots X_{k_p} (\partial_t)^q \Gamma_{\mathbb{G}}(T_{A_2}(x),t)\big|\Big).$$

The thesis now follows from Theorem 5.3 and from (2.8), (2.9), (2.10), (2.12). We apply Lemma 7.6 to the function $X_{k_1} \cdots X_{k_p} (\partial_t)^q \Gamma_{\mathbb{G}}(\cdot, t)$ and we use the same arguments as in the proof of Theorem 7.5. Theorem 7.4 is also used, to treat the case when $\|A_1 - A_2\|$ is not small. $\square$

II - The general case. From now on, we shall consider the case when the Carnot group $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_\lambda)$ is *not* free. By Theorem 8.3, we know that there exists a free homogeneous Carnot group $\widetilde{\mathbb{G}} = (\mathbb{R}^{\widetilde{N}}, \widetilde{\circ}, \widetilde{\delta_\lambda})$ which lifts $\mathbb{G}$ in the following sense. Denoting by $x = (x^{(1)}, \dots, x^{(r)})$ the point of $\mathbb{G} = \mathbb{R}^N = \mathbb{R}^{N_1} \times \cdots \times \mathbb{R}^{N_r}$, by

$$\widetilde{x} = (\widetilde{x}^{(1)}, \dots, \widetilde{x}^{(r)}) = (x^{(1)}, \dots, x^{(i_0)}, (x^{(i_0+1)}, \widehat{x}^{(i_0+1)}), \dots, (x^{(r)}, \widehat{x}^{(r)})) \equiv (x, \widehat{x})$$

the point of $\widetilde{\mathbb{G}} = \mathbb{R}^{\widetilde{N}} = \mathbb{R}^{\widetilde{N}_1} \times \cdots \times \mathbb{R}^{\widetilde{N}_r} = \mathbb{R}^{N_1} \times \cdots \times \mathbb{R}^{N_{i_0}} \times (\mathbb{R}^{N_{i_0+1}} \times \mathbb{R}^{\widehat{N}_{i_0+1}}) \times \cdots \times (\mathbb{R}^{N_r} \times \mathbb{R}^{\widehat{N}_r}) \equiv \mathbb{R}^N \times \mathbb{R}^{\widehat{N}}$, and by $\pi : \widetilde{\mathbb{G}} \to \mathbb{G}$, $\widetilde{x} = (x, \widehat{x}) \mapsto x$ the natural projection, we have (setting $m = N_1$)

$$\widetilde{X}_i (u \circ \pi) = (X_i u) \circ \pi, \quad \forall\, u \in C^\infty(\mathbb{R}^N),\ i \in \{1, ..., m\}, \tag{7.4}$$

where $\Delta_{\mathbb{G}} = \sum_{i=1}^m X_i^2$ and $\Delta_{\widetilde{\mathbb{G}}} = \sum_{i=1}^m \widetilde{X}_i^2$ are the canonical sub-Laplacians on $\mathbb{G}$, $\widetilde{\mathbb{G}}$. Moreover π is a Lie group morphism.

Given $A \in \mathcal{M}_\Lambda$, we shall use the notation

$$\mathcal{L}_A = \textstyle\sum_{i,j=1}^m a_{i,j}\, X_i X_j = \sum_{i=1}^m Y_i^2, \qquad Y_i = \sum_{j=1}^m (A^{1/2})_{i,j}\, X_j,$$

$$\widetilde{\mathcal{L}}_A = \textstyle\sum_{i,j=1}^m a_{i,j}\, \widetilde{X}_i \widetilde{X}_j = \sum_{i=1}^m \widetilde{Y}_i^2, \qquad \widetilde{Y}_i = \sum_{j=1}^m (A^{1/2})_{i,j}\, \widetilde{X}_j,$$

for the related sub-Laplacians on $\mathbb{G}$, $\widetilde{\mathbb{G}}$. Moreover we shall denote by Γ_A, $\widetilde{\Gamma}_A$ the fundamental solutions for the associated heat operators $\mathcal{H}_A = \mathcal{L}_A - \partial_t$, $\widetilde{\mathcal{H}}_A = \widetilde{\mathcal{L}}_A - \partial_t$ on $\mathbb{G} \times \mathbb{R}$, $\widetilde{\mathbb{G}} \times \mathbb{R}$.

Theorem 7.8. *We have, for every* $(x, t) \in \mathbb{R}^{N+1}$, $A \in \mathcal{M}_\Lambda$,

$$\Gamma_A(x, t) = \int_{\mathbb{R}^{\widehat{N}}} \widetilde{\Gamma}_A\big((x, \widehat{x}), t\big)\, \mathrm{d}\widehat{x},$$

Proof. Setting $\Gamma(x,t) = \int_{\mathbb{R}^{\widehat{N}}} \widetilde{\Gamma}_A((x, \widehat{x}), t)\, \mathrm{d}\widehat{x}$, we only have to prove that Γ satisfies (i)-(ii)-(iii) of Definition 3.1, with $\mathcal{H} = \mathcal{H}_A$ and $\zeta = 0$. (i) easily follows from Theorem 6.1 and Theorem 3.5-(iv) (applied to $\widetilde{\Gamma}_A$) observing that

$$\int_{-R}^{R} \int_{\mathbb{R}^N} \Gamma(x,t)\, \mathrm{d}x \mathrm{d}t = \int_0^R \int_{\mathbb{R}^{\widetilde{N}}} \widetilde{\Gamma}_A(\widetilde{x}, t)\, \mathrm{d}\widetilde{x} \mathrm{d}t = R.$$

Let us prove (ii). Let $\varphi \in C_0^\infty(\mathbb{R}^{N+1})$ be a test function. We set $\widetilde{\varphi}_k(\widetilde{x}, t) = \widetilde{\varphi}_k((x, \widehat{x}), t) = \varphi(x,t)\, \psi_k(\widetilde{x})$, where $\psi_k(\widetilde{x}) = \psi(d_{\widetilde{\mathbb{G}}}(\widetilde{x})/k)$ and $\psi \in C_0^\infty(\mathbb{R})$ is a fixed cut-off function such that $\psi(t)=1$ if $|t|\le 1$, $\psi(t)=0$ if $|t|\ge 2$. Since (7.4) holds, we have

$$\widetilde{\mathcal{H}}_A^*\widetilde{\varphi}_k(\widetilde{x},t)=\psi_k(\widetilde{x})\mathcal{H}_A^*\varphi(x,t)+\varphi(x,t)\widetilde{\mathcal{L}}_A\psi_k(\widetilde{x})+2\sum_{i=1}^m Y_i\varphi(x,t)\widetilde{Y}_i\psi_k(\widetilde{x}). \tag{7.5}$$

We have to prove that $\int_{\mathbb{R}^{N+1}}\Gamma\,\mathcal{H}_A^*\varphi=-\varphi(0)$. By the definition of Γ, we have

$$\begin{aligned}\int_{\mathbb{R}^{N+1}}\Gamma(x,t)\,\mathcal{H}_A^*\varphi(x,t)\,\mathrm{d}x\mathrm{d}t&=\int_{\mathbb{R}^{\widetilde{N}+1}}\widetilde{\Gamma}_A(\widetilde{x},t)\,\mathcal{H}_A^*\varphi(x,t)\,\mathrm{d}\widetilde{x}\mathrm{d}t\\&=\lim_{k\to\infty}\int_{\mathbb{R}^{\widetilde{N}+1}}\widetilde{\Gamma}_A(\widetilde{x},t)\,\psi_k(\widetilde{x})\,\mathcal{H}_A^*\varphi(x,t)\,\mathrm{d}\widetilde{x}\mathrm{d}t,\end{aligned}$$

by dominated convergence. On the other hand,

$$\int_{\mathbb{R}^{\widetilde{N}+1}}\widetilde{\Gamma}_A(\widetilde{x},t)\,\widetilde{\mathcal{H}}_A^*\widetilde{\varphi}_k(\widetilde{x},t)\,\mathrm{d}\widetilde{x}\mathrm{d}t=-\widetilde{\varphi}_k(0)=-\varphi(0),$$

since $\widetilde{\Gamma}_A$ is the fundamental solution for $\widetilde{\mathcal{H}}_A$ and $\widetilde{\varphi}_k\in C_0^\infty(\mathbb{R}^{\widetilde{N}+1})$. Thus, recalling (7.5), we only have to prove that

$$\int_{\mathbb{R}^{\widetilde{N}+1}}\widetilde{\Gamma}_A(\widetilde{x},t)\Big(\varphi(x,t)\,\widetilde{\mathcal{L}}_A\psi_k(\widetilde{x})+2\sum_{i=1}^m Y_i\varphi(x,t)\,\widetilde{Y}_i\psi_k(\widetilde{x})\Big)\mathrm{d}\widetilde{x}\mathrm{d}t\longrightarrow 0$$

as $k\to\infty$. This follows by dominated convergence. Indeed, setting for brevity $\widetilde{d}=d_{\widetilde{\mathbb{G}}}$, we have $\widetilde{Y}_i\psi_k(\widetilde{x})=\frac{1}{k}\psi'(\widetilde{d}(\widetilde{x})/k)\,\widetilde{Y}_i\widetilde{d}(\widetilde{x})$ and

$$\widetilde{\mathcal{L}}_A\psi_k(\widetilde{x})=\sum_{i=1}^m\Big(\frac{1}{k^2}\,\psi''(\widetilde{d}(\widetilde{x})/k)\,(\widetilde{Y}_i\widetilde{d}(\widetilde{x}))^2+\frac{1}{k}\,\psi'(\widetilde{d}(\widetilde{x})/k)\,\widetilde{Y}_i^2\widetilde{d}(\widetilde{x})\Big),$$

where $\widetilde{Y}_i\widetilde{d}$ and $\widetilde{Y}_i^2\widetilde{d}$ are $\widetilde{\delta}_\lambda$-homogeneous of degree 0 and of degree -1 respectively, thus bounded for large $\widetilde{x}$. Whence

$$\Big|\widetilde{\Gamma}_A\Big(\varphi\widetilde{\mathcal{L}}_A\psi_k+2\sum_{i=1}^m(Y_i\varphi)(\widetilde{Y}_i\psi_k)\Big)\Big|\le\frac{c}{k}\Big(|\varphi|+\sum_{i=1}^m|Y_i\varphi|\Big)\widetilde{\Gamma}_A\in L^1(\mathbb{R}^{\widetilde{N}+1}),$$

being $\varphi\in C_0^\infty(\mathbb{R}^{N+1})$. This concludes the proof of (ii). Finally, in order to prove (iii) it is sufficient to observe that Γ is smooth away from the origin (by the hypoellipticity of $\mathcal{H}_A$) and it is homogeneous of degree $-Q$ (w.r.t. the parabolic dilations). Indeed the change of variable $\widehat{y}=(\widehat{y}^{(i_0+1)},\dots,\widehat{y}^{(r)})=$

$(\lambda^{i_0+1}\widehat{x}^{(i_0+1)}, \dots, \lambda^r \widehat{x}^{(r)})$ gives

$$\Gamma(\delta_\lambda x, \lambda^2 t) = \int_{\mathbb{R}^{\widehat{N}}} \widetilde{\Gamma}_A((\delta_\lambda x, \widehat{y}), \lambda^2 t)\, \mathrm{d}\widehat{y} = \lambda^{\widetilde{Q}-Q} \int_{\mathbb{R}^{\widehat{N}}} \widetilde{\Gamma}_A(\widetilde{\delta_\lambda}\widetilde{x}, \lambda^2 t)\, \mathrm{d}\widehat{x}$$
$$= \lambda^{-Q} \int_{\mathbb{R}^{\widehat{N}}} \widetilde{\Gamma}_A(\widetilde{x}, t)\, \mathrm{d}\widehat{x} = \lambda^{-Q}\, \Gamma(x,t),$$

being $\widetilde{\Gamma}_A$ homogeneous of degree $-\widetilde{Q}$ (see 3.6). □

Remark 7.9. From Theorem 7.8 and Theorem 3.9, we also get

$$\gamma_A(x) = \int_{\mathbb{R}^{\widehat{N}}} \widetilde{\gamma}_A(x, \widehat{x})\, \mathrm{d}\widehat{x}, \qquad \forall\ x \in \mathbb{R}^N.$$

Theorem 7.10. *There exists a positive constant* $\mathbf{c}_\Lambda$ *such that*

$$\mathbf{c}_\Lambda^{-1}\, t^{-Q/2} \exp\big(-\frac{\mathbf{c}_\Lambda\, d_{\mathbb{G}}^2(x)}{t}\big) \le \Gamma_A(x,t) \le \mathbf{c}_\Lambda\, t^{-Q/2} \exp\big(-\frac{d_{\mathbb{G}}^2(x)}{\mathbf{c}_\Lambda\, t}\big),$$

$x \in \mathbb{R}^N$, $t > 0$, *for every* $A \in \mathcal{M}_\Lambda$.

Proof. The statement will directly follow from Theorem 7.8 and Theorem 7.3 if we prove that

$$\mathbf{c}_\Lambda t^{-Q/2} \exp\big(-\frac{\mathbf{c}_\Lambda\, d_{\mathbb{G}}^2(x)}{t}\big) \le \int_{\mathbb{R}^{\widehat{N}}} t^{-\frac{\widetilde{Q}}{2}} \exp\big(-\frac{d_{\widetilde{\mathbb{G}}}^2(x,\widehat{x})}{\mathbf{c}_\Lambda t}\big) \mathrm{d}\widehat{x}$$
$$\le \mathbf{c}_\Lambda t^{-Q/2} \exp\big(-\frac{\mathbf{c}_\Lambda\, d_{\mathbb{G}}^2(x)}{t}\big) \tag{7.6}$$

(where $\mathbf{c}_\Lambda$ is used to denote different constants). In order to prove (7.6), we introduce the functions

$$\rho(x) = \rho(x^{(1)}, \dots, x^{(r)}) = \Big(\sum_{i=1}^{r} \sum_{j=1}^{N_i} |x_j^{(i)}|^{\frac{2}{i}}\Big)^{\frac{1}{2}},$$

$$\widetilde{\rho}(\widetilde{x}) = \widetilde{\rho}(\widetilde{x}^{(1)}, \dots, \widetilde{x}^{(r)}) = \Big(\sum_{i=1}^{r} \sum_{j=1}^{\widetilde{N}_i} |\widetilde{x}_j^{(i)}|^{\frac{2}{i}}\Big)^{\frac{1}{2}},$$

$$\widehat{\rho}(\widehat{x}) = \widehat{\rho}(\widehat{x}^{(i_0+1)}, \dots, \widehat{x}^{(r)}) = \Big(\sum_{i=i_0+1}^{r} \sum_{j=1}^{\widehat{N}_i} |\widehat{x}_j^{(i)}|^{2/i}\Big)^{1/2}.$$

These functions have been defined in such a way to give

$$\widetilde{\rho}^{\,2}(x, \widehat{x}) = \rho^2(x) + \widehat{\rho}^{\,2}(\widehat{x}). \tag{7.7}$$

Moreover, since both ρ and $d_{\mathbb{G}}$ are continuous, positive away from zero, and δ_λ-homogeneous of degree 1, they are equivalent in the sense that $\mathbf{c}^{-1} d_{\mathbb{G}}(x) \le \rho(x) \le \mathbf{c}\, d_{\mathbb{G}}(x)$, $x \in \mathbb{G}$, holds for a suitable positive constant $\mathbf{c}$. Analogously $\widetilde{\rho}$ and $d_{\widetilde{\mathbb{G}}}$ are equivalent on $\widetilde{\mathbb{G}}$. As a consequence, it is sufficient to prove (7.6) with $\widetilde{\rho}$, ρ instead of $d_{\widetilde{\mathbb{G}}}$, $d_{\mathbb{G}}$, respectively. Now, exploiting (7.7), the change of variable $\widehat{x} = (\widehat{x}^{(i_0+1)}, \dots, \widehat{x}^{(r)}) = (t^{(i_0+1)/2}\, \widehat{y}^{(i_0+1)}, \dots, t^{r/2}\, \widehat{y}^{(r)})$ gives

$$\begin{aligned}
&\int_{\mathbb{R}^{\widehat{N}}} t^{-\widetilde{Q}/2} \exp\Big(-\frac{\widetilde{\rho}^2(x,\widehat{x})}{\mathbf{c}_\Lambda\, t}\Big)\,\mathrm{d}\widehat{x}\\
&= t^{-Q/2} \exp\Big(-\frac{\rho^2(x)}{\mathbf{c}_\Lambda\, t}\Big) \int_{\mathbb{R}^{\widehat{N}}} t^{\frac{Q-\widetilde{Q}}{2}} \exp\Big(-\frac{\widehat{\rho}^2(\widehat{x})}{\mathbf{c}_\Lambda\, t}\Big)\,\mathrm{d}\widehat{x}\\
&= t^{-Q/2} \exp\Big(-\frac{\rho^2(x)}{\mathbf{c}_\Lambda\, t}\Big) \int_{\mathbb{R}^{\widehat{N}}} \exp\Big(-\frac{\widehat{\rho}^2(\widehat{y})}{\mathbf{c}_\Lambda}\Big)\,\mathrm{d}\widehat{y} = \mathbf{c}_\Lambda\, t^{-Q/2} \exp\Big(-\frac{\rho^2(x)}{\mathbf{c}_\Lambda\, t}\Big).
\end{aligned}$$

This completes the proof. □

Lemma 7.11. *For every non-negative integers p, q and for every $i_1, \dots, i_p \in \{1, \dots, m\}$, $A \in \mathcal{M}_\Lambda$, we have, for $x \in \mathbb{R}^N$, $t > 0$*

$$X_{i_1} \cdots X_{i_p} (\partial_t)^q\, \Gamma_A(x,t) = \int_{\mathbb{R}^{\widehat{N}}} \widetilde{X}_{i_1} \cdots \widetilde{X}_{i_p} (\partial_t)^q\, \widetilde{\Gamma}_A((x,\widehat{x}),t)\,\mathrm{d}\widehat{x}.$$

Proof. We shall prove the lemma in the case $p = 1$, $q = 0$. The general case can be proved with the same arguments. Let us fix $t > 0$ and set

$$\begin{aligned}
&\Gamma(x) = \Gamma_A(x,t), \quad \Gamma_k(x) = \int_{|\widehat{x}| \le k} \widetilde{\Gamma}_A((x,\widehat{x}),t)\,\mathrm{d}\widehat{x},\\
&\Gamma^{(i)}(x) = \int_{\mathbb{R}^{\widehat{N}}} \widetilde{X}_i \widetilde{\Gamma}_A((x,\widehat{x}),t)\,\mathrm{d}\widehat{x}.
\end{aligned}$$

Since Theorem 7.8 holds, $\Gamma_k(x) \longrightarrow \Gamma(x)$ point-wise, for every $x \in \mathbb{R}^N$. If we prove that

$$X_i \Gamma_k(x) \longrightarrow \Gamma^{(i)}(x), \qquad \text{as } k \to \infty, \text{ uniformly in } x \in \mathbb{R}^N, \tag{7.8}$$

we shall get $X_i \Gamma(x) = \Gamma^{(i)}(x)$ for every $x \in \mathbb{R}^N$, which is our thesis. Indeed, taking any integral path α of the vector field X_i, $\alpha(0) = x_0$, $\alpha'(s) = X_i(\alpha(s))$, we have $\Gamma_k(\alpha(s)) - \Gamma_k(\alpha(0)) = \int_0^s X_i \Gamma_k(\alpha(\sigma))\,\mathrm{d}\sigma$ and $(\Gamma(\alpha(s)) - \Gamma(\alpha(0)))/s \longrightarrow X_i \Gamma(x_0)$ as $s \to 0$. Therefore we only have to prove (7.8). Since (7.4) holds, we can write $\widetilde{X}_i = X_i + \widehat{X}_i$ where $\widehat{X}_i$ is a vector field operating only in the $\widehat{x}$-variables, i.e., in the form $\widehat{X}_i = \sum_{j=i_0+1}^r \sum_{h=1}^{\widehat{N}_j} p_h^{(j)}(x,\widehat{x}) \partial_{\widehat{x}_h^{(j)}}$

(moreover $p_h^{(j)}$ are polynomials $\widetilde{\delta}_\lambda$-homogeneous of degree $j-1$, see the Appendix; in particular $\widehat{X}_i^* = -\widehat{X}_i$). Then, we have

$$X_i\Gamma_k(x) = \int_{|\widehat{x}|\le k} X_i\widetilde{\Gamma}_A(\widetilde{x},t)\,\mathrm{d}\widehat{x} = \int_{|\widehat{x}|\le k} \widetilde{X}_i\widetilde{\Gamma}_A(\widetilde{x},t)\,\mathrm{d}\widehat{x} - \int_{|\widehat{x}|\le k} \widehat{X}_i\widetilde{\Gamma}_A(\widetilde{x},t)\,\mathrm{d}\widehat{x}.$$

The first integral in the far right hand-side goes to $\Gamma^{(i)}(x)$, as $k\to\infty$, uniformly in $x\in\mathbb{R}^N$. This can be easily proved using the Gaussian estimates of $\widetilde{X}_i\widetilde{\Gamma}_A$ and the arguments in the proof of Theorem 7.10. Thus, we only have to prove that

$$\Big|\int_{|\widehat{x}|\le k} \widehat{X}_i\widetilde{\Gamma}_A(\widetilde{x},t)\,\mathrm{d}\widehat{x}\Big| \longrightarrow 0, \qquad \text{as } k\to\infty,\ \text{uniformly in } x\in\mathbb{R}^N. \tag{7.9}$$

On the other hand, since $\widehat{X}_i^* = -\widehat{X}_i$, the divergence theorem yields ($H_{\widehat{N}-1}$ denotes the $(\widehat{N}-1)$-dimensional Hausdorff measure)

$$\Big|\int_{|\widehat{x}|\le k} \widehat{X}_i\widetilde{\Gamma}_A(\widetilde{x},t)\,\mathrm{d}\widehat{x}\Big| = \Big|\int_{|\widehat{x}|=k} \widetilde{\Gamma}_A(\widetilde{x},t)\,\big\langle \widehat{X}_i(\widetilde{x}), \widehat{x}/|\widehat{x}|\big\rangle\,\mathrm{d}H_{\widehat{N}-1}(\widehat{x})\Big| \le$$

(arguing as in the proof of Theorem 7.10 and with the notation introduced in that proof)

$$\le \mathbf{c}(\Lambda,t)\,\exp\Big(-\frac{\rho^2(x)}{\mathbf{c}(\Lambda,t)}\Big)\int_{|\widehat{x}|=k} |\widehat{X}_i(x,\widehat{x})|\,\exp\Big(-\frac{\widehat{\rho}^{\,2}(\widehat{x})}{\mathbf{c}(\Lambda,t)}\Big)\,\mathrm{d}H_{\widehat{N}-1}(\widehat{x}).$$

Recalling that $\widehat{X}_i$ has polynomial component functions, (7.9) easily follows.

Theorem 7.12. *Given any non-negative integers p, q, there exist positive constants $\mathbf{c}_\Lambda$ and $\mathbf{c}_{\Lambda,p,q}$ such that for $x\in\mathbb{R}^N$, $t>0$,*

$$\begin{aligned}
&\big|X_{i_1}\cdots X_{i_p}(\partial_t)^q\,\Gamma_A(x,t)\big| \le \mathbf{c}_{\Lambda,p,q}\,t^{-(Q+p+2q)/2}\,\exp\big(-\frac{d_{\mathbb{G}}^2(x)}{\mathbf{c}_\Lambda\,t}\big),\\
&\big|X_{i_1}\cdots X_{i_p}(\partial_t)^q\,\Gamma_{A_1}(x,t) - X_{i_1}\cdots X_{i_p}(\partial_t)^q\,\Gamma_{A_2}(x,t)\big|\\
&\qquad\qquad \le \mathbf{c}_{\Lambda,p,q}\,\|A_1-A_2\|^{1/r}\,t^{-(Q+p+2q)/2}\,\exp\big(-\frac{d_{\mathbb{G}}^2(x)}{\mathbf{c}_\Lambda\,t}\big),
\end{aligned}$$

for every $i_1,\dots,i_p\in\{1,\dots,m\}$ and for every $A, A_1, A_2\in\mathcal{M}_\Lambda$.

Proof. It follows from Lemma 7.11, Theorem 7.4 and Theorem 7.7, arguing as in the proof of Theorem 7.10. □

From the above estimates and from Theorem 3.9, we can also easily deduce uniform estimates for the fundamental solutions for the sub-Laplacians on $\mathbb{G}$.

Corollary 7.13. *Given any non-negative integer p, there exist positive constants $\mathbf{c}_\Lambda$, $\mathbf{c}_{\Lambda,p}$ such that*

$$\mathbf{c}_\Lambda^{-1}\,(d_{\mathbb{G}}(x))^{2-Q} \le \gamma_A(x) \le \mathbf{c}_\Lambda\,(d_{\mathbb{G}}(x))^{2-Q},$$
$$\left|X_{i_1}\cdots X_{i_p}\,\gamma_A(x)\right| \le \mathbf{c}_{\Lambda,p}\,(d_{\mathbb{G}}(x))^{2-Q-p},$$
$$\left|X_{i_1}\cdots X_{i_p}\,\gamma_{A_1}(x) - X_{i_1}\cdots X_{i_p}\,\gamma_{A_2}(x)\right| \le \mathbf{c}_{\Lambda,p}\,\|A_1 - A_2\|^{1/r}\,(d_{\mathbb{G}}(x))^{2-Q-p},$$

for every $i_1,\dots,i_p \in \{1,\dots,m\}$, for every A, A_1, $A_2 \in \mathcal{M}_\Lambda$ and for every $x \in \mathbb{R}^N \setminus \{0\}$.

8. Appendix.

Throughout this Appendix, $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_\lambda)$ is a fixed homogeneous Carnot group with Lie algebra $\mathfrak{g}$. Moreover, we suppose $\mathbb{G}$ is of step r and has N_1 generators. For the proofs of all the results we are going to recall, see e.g. [4].

If τ_x denotes the left-translation by x on $\mathbb{G}$, then a vector field X belongs to $\mathfrak{g}$ if and only if $X(x) = \mathcal{J}_{\tau_x}(0)\,X(0)$, for every $x \in \mathbb{G}$ ($\mathcal{J}_{\tau_x}$ denotes the Jacobian matrix of τ_x). Furthermore, the map $J : \mathbb{R}^N \to \mathfrak{g}$, $\eta \mapsto X$ defined by $X(x) = \mathcal{J}_{\tau_x}(0)\,\eta$ is an isomorphism of vector spaces. As a consequence, any basis for $\mathfrak{g}$ is the image via J of a basis of $\mathbb{R}^N$. We call the *Jacobian basis* of $\mathfrak{g}$ the one resulting from the canonical basis of $\mathbb{R}^N$, i.e.,the basis of vector fields in $\mathfrak{g}$ agreeing at the origin with the coordinate partial derivatives. It is significant to notice that the Jacobian basis is simply obtained by the N columns of the matrix $\mathcal{J}_{\tau_x}(0)$. We also remark that, if $X_1,\dots,X_m$ belong to $\mathfrak{g}$, then they are linearly independent if and only if they are linearly independent at every point or equivalently at one point at least.

We now recall the definition of the exponential map on $\mathfrak{g}$. If $X \in \mathfrak{g}$, then, for every fixed $x \in \mathbb{G}$, the system of ODE's

$$\dot\gamma(t) = (XI)(\gamma(t)), \quad \gamma(0) = x,$$

has a unique C^∞ solution defined on the whole $\mathbb{R}$. If γ is such solution, we denote $\exp[X](x) = \gamma(1)$. The *exponential map* is defined as

$$\mathrm{Exp}\,:\mathfrak{g} \to \mathbb{G}, \quad \mathrm{Exp}\,(X) = \exp[X](0).$$

When $\mathfrak{g}$ is equipped with Jacobian coordinates, the Jacobian matrix of $\exp[\,\cdot\,](x)$ at the origin is $\mathcal{J}_{\tau_x}(0)$. In particular, the Jacobian matrix of Exp at the origin is the identity matrix of order N, whence Exp is a diffeomorphism from a neighborhood of $0 \in \mathfrak{g}$ onto a neighborhood of $0 \in \mathbb{G}$. Where defined, we denote by Log the inverse map of Exp.

A real function $a(x)$ defined on $\mathbb{R}^N$ is called δ_λ-*homogeneous* of degree $\beta \in \mathbb{R}$ if, for every $x \in \mathbb{R}^N$ and $\lambda > 0$, it holds $a(\delta_\lambda(x)) = \lambda^\beta a(x)$. A linear differential operator X is called δ_λ-*homogeneous* of degree $\beta \in \mathbb{R}$ if, for every $\varphi \in C^\infty(\mathbb{G})$ and $\lambda > 0$, it holds $X(\varphi \circ \delta_\lambda) = \lambda^\beta (X\varphi) \circ \delta_\lambda$. With reference to the form (2.1) of the dilation δ_λ, we define a *homogeneous weight* of a multi-index $\gamma \in (\mathbb{N} \cup \{0\})^N$, $|\gamma|_{\mathbb{G}} := \sum_{i=1}^r \sum_{j=1}^{N_i} i\, \gamma_j^{(i)}$. With this notation, it is easy to see that the only smooth δ_λ-homogeneous functions of degree β are the polynomial functions of the form $\sum_{|\gamma|_{\mathbb{G}}=\beta} c_\gamma\, x^\gamma$, $c_\gamma \in \mathbb{R}$. In particular, a smooth vector field δ_λ-homogeneous of degree $k \le r$ ($k \in \mathbb{N}$) has the following form

$$\sum_{j=1}^{N_k} a_j^{(k)}\, (\partial/\partial x_j^{(k)}) + \sum_{i=k+1}^r \sum_{j=1}^{N_i} a_j^{(i)} (x^{(1)}, \dots, x^{(i-k)})\, (\partial/\partial x_j^{(i)})$$

where $a_j^{(i)}$ is a δ_λ-homogeneous polynomial of degree $i - k$.

Remark 8.1. If X is a smooth vector field δ_λ-homogeneous of non-negative degree, then its adjoint X^* is $-X$ and X^2 is a divergence form operator.

The following result describes in an "explicit" form the composition law of a homogeneous group.

Theorem 8.2. *Following the notation in* (2.1) $x \circ y = ((x \circ y)^{(1)}, \dots, (x \circ y)^{(r)})$, *we have*

$$(x\circ y)^{(1)} = x^{(1)} + y^{(1)}, \quad (x \circ y)^{(i)} = x^{(i)} + y^{(i)} + Q^{(i)}(x, y), \quad 2 \le i \le r, \quad \textit{where}$$

(1) $Q^{(i)}$ *depends only on* $x^{(1)}, \dots, x^{(i-1)}$ *and* $y^{(1)}, \dots, y^{(i-1)}$;
(2) *the component functions of* $Q^{(i)}$ *are sums of mixed monomials in* x *and* y;
(3) $Q^{(i)}(\delta_\lambda x, \delta_\lambda y) = \lambda^i Q^{(i)}(x, y)$.

Moreover, we have

$$\mathcal{J}_{\tau_x}(0) = \begin{pmatrix} \mathbb{I}_{N_1} & 0 & \cdots & 0 \\ J_2^{(1)}(x) & \mathbb{I}_{N_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ J_r^{(1)}(x) & \cdots & J_r^{(r-1)}(x) & \mathbb{I}_{N_r} \end{pmatrix}$$

where $\mathbb{I}_n$ *is the* $n \times n$ *identity matrix, whereas* $J_j^{(i)}(x)$ *is a* $N_j \times N_i$ *matrix whose entries are* δ_λ-*homogeneous polynomials of degree* $j - i$. *In particular, if we let* $\mathcal{J}_{\tau_x}(0) = \left(Z^{(1)}(x) \cdots Z^{(r)}(x)\right)$ (*where* $Z^{(i)}(x)$ *is a* $N \times N_i$ *matrix*), *then the column vectors of* $Z^{(i)}(x)$ (*the Jacobian basis for* $\mathfrak{g}$) *define* δ_λ-*homogeneous vector fields of degree* i.

Finally, Exp : $\mathfrak{g} \to \mathbb{G}$ *and* Log : $\mathbb{G} \to \mathfrak{g}$ *are globally defined diffeomorphisms with polynomial components and they commute with the dilations* δ_λ (*we suppose* $\mathfrak{g}$ *is identified with* $\mathbb{R}^N$ *via the Jacobian coordinates*).

Moreover, the Jacobian matrices of Exp, Log, right and left translations on $\mathbb{G}$ are lower triangular matrices with entries in the main diagonal all equal to 1. The Jacobian matrix of the map $x \mapsto x^{-1}$ is a lower triangular matrix with entries in the main diagonal all equal to -1. In particular, right and left translations and the inversion on $\mathbb{G}$ preserve the Lebesgue measure.

Let $\sum_{j=1}^m X_j^2$ be the canonical sub-Laplacian on $\mathbb{G}$. An absolutely continuous curve $\gamma : [0,T] \to \mathbb{R}^N$ is called X*-subunit* if and only if there exist measurable functions $c_1, \dots, c_m$ such that $\dot\gamma(t) = \sum_{j=1}^m c_j(t)\, X_j(\gamma(t))$ and $\sum_{j=1}^m c_j^2(t) \le 1$ for almost every $t \in [0,T]$. For x, $y \in \mathbb{R}^N$ we set

$$d_X(x,y) := \inf\{T > 0 \mid \exists\, \gamma : [0,T] \to \mathbb{R}^N \ X\text{-subunit},\ \gamma(0) = x,\ \gamma(T) = y\}. \tag{8.1}$$

Since $X_1, \dots, X_m$ satisfy Hörmander's hypoellipticity condition, then by the Carathéodory-Chow Theorem, $\mathbb{R}^N$ is X-connected (i.e., any pair of points in $\mathbb{R}^N$ is joined by a X-subunit curve) and d_X defines a continuous distance on $\mathbb{R}^N$ which is called the *Carnot-Carathéodory distance on* $\mathbb{R}^N$ *related to* X (see, for example, [12]). We remark that $d_X(\cdot, 0)$ is a continuous homogeneous norm. This follows since, for every x, $y \in \mathbb{G}$, we have

$$\begin{aligned} &d_X(\delta_\lambda(x), \delta_\lambda(y)) = \lambda\, d_X(x,y), \quad \forall\, \lambda > 0,\\ &d_X(\alpha \circ x, \alpha \circ y) = d_X(x,y), \quad \forall\, \alpha \in \mathbb{G}. \end{aligned}$$

To end the Appendix, we recall the following Lifting Theorem proved in [4]. Here, $\mathfrak{f}_{m,r}$ denotes the free nilpotent Lie algebra of step r and with m generators.

Theorem 8.3. *Let* $\mathbb{G}$ *be a homogeneous Carnot group on* $\mathbb{R}^N$ *of step* r *and* m $(= N_1)$ *generators. Then, there exists a free homogeneous Carnot group* $\widetilde{\mathbb{G}}$ *on* $\mathbb{R}^H$ $(H = \dim \mathfrak{f}_{m,r})$ *with the properties* (i) *and* (ii) *stated below.*

We fix the following notations:

$$\begin{aligned} &\delta_\lambda(x) = \delta_\lambda(x^{(1)}, x^{(2)}, \dots, x^{(r)}) = (\lambda x^{(1)}, \lambda^2 x^{(2)}, \dots, \lambda^r x^{(r)}),\\ &\widetilde{\delta}_\lambda(\widetilde{x}) = \widetilde{\delta}_\lambda(\widetilde{x}^{(1)}, \widetilde{x}^{(2)}, \dots, \widetilde{x}^{(r)}) = (\lambda \widetilde{x}^{(1)}, \lambda^2 \widetilde{x}^{(2)}, \dots, \lambda^r \widetilde{x}^{(r)}) \end{aligned}$$

denote the dilations on $\mathbb{G}$ *and* $\widetilde{\mathbb{G}}$, *respectively* ($x^{(i)} \in \mathbb{R}^{N_i}$ $i = 1, \dots, r$, $N_1 + \dots + N_r = N$; $\widetilde{x}^{(i)} \in \mathbb{R}^{\widetilde{N}_i}$, $i = 1, \dots, r$, $\widetilde{N}_1 + \dots + \widetilde{N}_r = H$); $Z_j^{(i)}$ $(i \le r,$

$j \le N_i$) *denote the Jacobian basis of the Lie algebra* $\mathfrak{g}$ *of* $\mathbb{G}$ *and* $\widetilde{Z}_j^{(i)}$ ($i \le r$, $j \le \widetilde{N}_i$) *denote the Jacobian basis of the Lie algebra* $\widetilde{\mathfrak{g}}$ *of* $\widetilde{\mathbb{G}}$.

(i) $\widetilde{\mathbb{G}}$ *has step* r *and* m *generators and its Lie algebra is isomorphic to* $\mathfrak{f}_{m,r}$.

(ii) *For a certain* $i_0 \in \{1, \dots, r\}$, *we have*

$$\widetilde{N}_i = N_i \quad (i = 1, \dots, i_0) \quad \text{and} \quad \widetilde{N}_i > N_i \quad (i = i_0 + 1, \dots, r);$$

moreover, if $\pi^{(i)} : \mathbb{R}^{\widetilde{N}_i} \to \mathbb{R}^{N_i}$ *denotes the projection on the first* N_i *coordinates and*
$\pi : \mathbb{R}^H \to \mathbb{R}^N$ *is defined by* $\pi(\widetilde{x}) = (\pi^{(1)}(\widetilde{x}^{(1)}), \dots, \pi^{(r)}(\widetilde{x}^{(r)}))$, *then*

$$\widetilde{Z}_j^{(i)}(u \circ \pi) = \big(Z_j^{(i)} u\big) \circ \pi, \qquad \forall\, u \in C^\infty(\mathbb{R}^N), \quad i \le r,\ j \le N_i, \tag{8.2}$$

i.e., $\widetilde{Z}_j^{(i)}$ *lifts* $Z_j^{(i)}$. *Moreover,* π *is a Lie group morphism.*

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