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A note on the Voronovskaja theorem for Mellin–Fejer convolution operators

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ABSTRACT

Here, using Mellin derivatives and a different notion of moment, we state a Voronovskaja approximation formula for a class of Mellin–Fejer type convolution operators. This new approach gives direct and simple applications to various important specific examples.

1. Introduction

In [1,2] we studied Voronovskaja theorems for the pointwise convergence of Mellin type convolution operators (see [3]). The main tool is a definition of algebraic moments of the kernel involved, which takes into account the multiplicative structure of the group $\mathbb{R}^+$. Here we use a different and powerful approach which gives direct and immediate applications to specific integral operators, e.g., the Mellin–Gauss–Weierstrass and Mellin–Poisson–Cauchy ones. The main idea is to use a Taylor formula in terms of Mellin derivatives (Proposition 1) and to consider a notion of "logarithmic" moment of the kernel (see also [4]). We treat the case of the Mellin–Fejer type kernels, which are the "multiplicative" counterparts of the classical Fejer type kernels (see [5]). For classical convolution operators on the line group, using usual derivatives, related results can be found in [6].

2. General theory

Let $\mathbb{R}^+$ be the multiplicative topological group endowed with the Haar measure $\mu = \frac{dt}{t}$, dt being the Lebesgue measure. We will denote by $L^p(\mu, \mathbb{R}^+) = L^p(\mu)$, $1 \le p \le +\infty$, the Lebesgue spaces with respect to the measure μ and we will denote by $\|f\|_p$ the corresponding norm of a function $f \in L^p(\mu)$.

In what follows we will say that $f \in C^k$ locally at the point $s \in \mathbb{R}^+$ if there is a neighbourhood U_s of the point s such that f is continuously differentiable $(k-1)$ times in U_s and the derivative of order k exists at the point s.

Let us consider the linear Mellin–Fejer convolution operator

$$(T_w f)(s) = \int_0^{+\infty} K_w(ts^{-1}) f(t) \frac{dt}{t} = \int_0^{+\infty} K_w(t) f(st) \frac{dt}{t}$$

with $K_w(t) = wK(t^w)$ where K is a non-negative kernel function $K : \mathbb{R}^+ \to \mathbb{R}_0^+$ satisfying the following conditions:

(1) we have $K \in L^1(\mu)$, and $\|K\|_1 = 1$,

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(2) there exists $\alpha \geq 1$ such that for every $\delta > 1$ we have

$$\int_{\mathbb{R}^+ \setminus]1/\delta^w, \delta^w[} K(t) \frac{dt}{t} = o(w^{-\alpha}), \quad w \to +\infty.$$

It is easy to see that, for every $w \geq 1$, $\|K_w\|_1 = 1$.

For every $\delta > 1$ and $j \in \mathbb{N}$ we define the local logarithmic moment of order j of the functions K and K_w by

$$m_j(K, \delta) := \int_{\frac{1}{\delta}}^{\delta} K(t)(\log t)^j \frac{dt}{t}, \qquad m_j(K_w, \delta) := \int_{\frac{1}{\delta}}^{\delta} K_w(t)(\log t)^j \frac{dt}{t}.$$

We have

$$m_j(K_w, \delta) = \frac{1}{w^j} m_j(K, \delta^w).$$

Moreover, we define the absolute logarithmic moment of order $j \in \mathbb{N}$ by

$$M_j(K) := \int_0^{+\infty} K(t)|\log t|^j \frac{dt}{t}.$$

In the following we will put

$$\ell_j := \int_0^{+\infty} K(t)(\log t)^j \frac{dt}{t}.$$

In what follows we will use the Mellin differential operator, as introduced by Butzer and Jansche in [3]. The Mellin differential operator Θ or the Mellin derivative Θf of a function $f : \mathbb{R}^+ \to \mathbb{R}$ is defined by

$$\Theta f(s) = sf'(s), s \in \mathbb{R}^+,$$

provided the usual derivative $f'(s)$ exists. The Mellin differential operator of order $r \in \mathbb{N}$ is defined inductively by putting $\Theta^1 = \Theta$, $\Theta^r = \Theta \circ \Theta^{r-1}$, $\Theta^0 = I$, I being the identity operator. From [3] we have the following representation result:

$$\Theta^r f(s) = \sum_{k=0}^{r} S(r, k) f^{(k)}(s) s^k$$

where $S(r, k)$, $r \in \mathbb{N}$, $0 \leq k \leq r$, denotes the Stirling numbers of the second kind.

We begin with the following Taylor type formula (see also [4]).

Proposition 1. *Suppose that $f \in C^n$ locally at a point $s \in \mathbb{R}^+$. Then there exists $\delta > 1$ such that for $t \in]1/\delta, \delta[$*

$$f(st) = f(s) + \Theta f(s) \log t + \frac{\Theta^2 f(s)}{2!} \log^2 t + \cdots + \frac{\Theta^n f(s)}{n!} \log^n t + h(t) \log^n t$$

where $h(t) \to 0$ as $t \to 1$.

Proof. We first prove the proposition when $n = 1$ and $n = 2$. If f is differentiable at the point $s \in \mathbb{R}^+$ we have

$$\lim_{t \to 1} \frac{f(st) - f(s)}{\log t} = \lim_{t \to 1} \frac{f(st) - f(s)}{ts - s} \frac{s(t-1)}{\log t} = sf'(s) = \Theta f(s)$$

and so the assertion follows for $n = 1$. Suppose now that $f \in C^2$ locally at the point $s \in \mathbb{R}^+$. Then, putting $F(t) := f(s) + sf'(s) \log t$ and using Hôpital's rule, we get

$$\begin{aligned} \lim_{t \to 1} \frac{f(st) - F(t)}{\log^2 t} &= \frac{s}{2} \lim_{t \to 1} \frac{t(f'(st) - f'(s))}{\log t} + \frac{sf'(s)}{2} \lim_{t \to 1} \frac{t-1}{\log t} \\ &= \frac{s^2}{2} f''(s) + \frac{s}{2} f'(s) = \frac{\Theta^2 f(s)}{2} \end{aligned}$$

and so the assertion follows for $n = 2$. For the general case, putting $F(t) = f(s) + \Theta f(s) \log t + \frac{\Theta^2 f(s)}{2!} \log^2 t + \cdots + \frac{\Theta^{n-1} f(s)}{(n-1)!} \log^{n-1} t$ we have

$$\lim_{t \to 1} \frac{f(st) - F(t)}{\log^n t} = \lim_{t \to 1} \frac{\Theta^{n-1} f(st) - \Theta^{n-1} f(s)}{n! \log t} = \frac{\Theta^n f(s)}{n!},$$

where we apply the Hôpital rule $(n-1)$ times and the representation of the Mellin derivatives calculated at the point st in terms of the Stirling numbers $S(r, k)$. □

Now we are ready to prove the main theorem.

Theorem 1. *Let $s \in \mathbb{R}^+$ and $f \in L^\infty(\mu)$ be fixed. Then:*

(i) *If $\ell_1 \neq 0$, $M_1(K) < \infty$ and $f \in C^1$ locally at the point s we have*

$$\lim_{w\to+\infty} w[(T_w f)(s) - f(s)] = sf'(s)\ell_1 = \Theta f(s)\ell_1.$$

(ii) *If $\ell_1 = 0$, $M_2(K) < \infty$, $\alpha > 2$ and $f \in C^2$ locally at the point s we have*

$$\lim_{w\to+\infty} w^2[(T_w f)(s) - f(s)] = \frac{s}{2}f'(s)\ell_2 + \frac{s^2}{2}f''(s)\ell_2 = \frac{\Theta^2 f(s)}{2}\ell_2.$$

Proof. We prove only (ii), the proof of (i) being similar. We have

$$\begin{aligned}(T_w f)(s) - f(s) &= \int_0^{+\infty} K_w(t)(f(st) - f(s))\frac{dt}{t} \\ &= \left(\int_0^{1/\delta} + \int_{1/\delta}^{\delta} + \int_\delta^{+\infty}\right) K_w(t)(f(st) - f(s))\frac{dt}{t} = I_1 + I_2 + I_3.\end{aligned}$$

We consider first I_2. By Proposition 1 we obtain

$$I_2 = \Theta f(s) m_1(K_w, \delta) + \frac{\Theta^2 f(s)}{2} m_2(K_w, \delta) + \int_{1/\delta}^{\delta} K_w(t)h(t)\log^2 t\frac{dt}{t}.$$

As to the last integral, for $\varepsilon > 0$ suppose that $\delta > 1$ with $|h(t)| < \varepsilon$ for $t \in]\frac{1}{\delta}, \delta[$. Hence

$$w^2\left|\int_{1/\delta}^{\delta} K_w(t)h(t)\log^2 t\frac{dt}{t}\right| \le \varepsilon \int_0^{+\infty} K(t)\log^2 t\frac{dt}{t} = \varepsilon M_2(K).$$

As regards I_1 (and analogously for I_3) we have

$$|I_1| \le 2\|f\|_\infty \int_0^{1/\delta^w} K(t)\frac{dt}{t} = o(w^{-\alpha})$$

so $\lim_{w\to+\infty} w^2|I_1| = 0$. The proof follows by limsup and liminf arguments. □

Remarks.

(i) The above theorem can be proved also for functions $f \in L^p(\mu)$, for $p \ge 1$ under suitable assumptions on the family of kernels K_w and using the Hölder inequality (for details see [1]).

(ii) When the kernel K is a symmetric function with respect to 1 i.e. $K(t) = K(1/t)$ then we get easily $\ell_j = 0$ for every odd j and $\ell_j = 2\int_0^1 K(t)\log^j t\frac{dt}{t}$ for every even j. Thus for even kernels we get a Voronovskaja formula of order 2.

3. Applications to specific operators

3.1. The Mellin–Gauss–Weierstrass operator (see [3])

Let us consider the function

$$K(t) = \frac{1}{\sqrt{4\pi}}\exp\left(-\left(\frac{1}{2}\log t\right)^2\right), \quad t \in \mathbb{R}^+.$$

This function generates the Mellin–Gauss–Weierstrass operator

$$(\mathcal{G}_w f)(s) = \frac{w}{\sqrt{4\pi}}\int_0^{+\infty} \exp\left(-\left(\frac{w}{2}\log t\right)^2\right) f(ts)\frac{dt}{t}.$$

It is easy to see that property (1) holds and the function K is symmetric with respect to 1, so $\ell_1 = 0$. Moreover property (2) holds for every $\alpha \ge 1$ and

$$\ell_2 = \frac{1}{\sqrt{4\pi}}\int_0^{+\infty} \exp\left(-\left(\frac{1}{2}\log t\right)^2\right)\log^2 t\frac{dt}{t} = \frac{16}{\sqrt{4\pi}}\int_0^{+\infty} e^{-z^2}z^2 dz = 2.$$

So we get the following corollary.

Corollary 1. *Suppose that $f \in L^\infty(\mu)$. Then if $f \in C^2$ locally at the point $s \in \mathbb{R}^+$,*

$$\lim_{w\to+\infty} w^2[(\mathcal{G}_w f)(s) - f(s)] = sf'(s) + s^2 f''(s) = \Theta^2 f(s).$$

3.2. The modified Mellin–Poisson–Cauchy operator

Let us consider the function, for every $p \geq 2$,

$$K(t) = \frac{2^{p-1}(p-1)!}{\pi(2p-3)!!}\frac{1}{(1+\log^2 t)^p}, \quad t \in \mathbb{R}^+.$$

Here $(2n+1)!! = 1 \cdot 3 \cdots (2n+1)$ and $(-1)!! = 1$. This function generates the modified Mellin–Poisson–Cauchy operator

$$(\mathcal{P}_w f)(s) = \frac{2^{p-1}(p-1)!}{\pi(2p-3)!!}\int_0^{+\infty}\frac{w}{(1+w^2\log^2 t)^p}f(ts)\frac{dt}{t}.$$

Using the properties of the Beta function, we obtain that property (1) holds and the function K is symmetric with respect to 1, so $\ell_1 = 0$. Moreover property (2) holds for $\alpha = 2p-2$ and for every $p \geq 2$,

$$\begin{aligned}\ell_2 &= \frac{2^{p-1}(p-1)!}{\pi(2p-3)!!}\int_0^{+\infty}\frac{1}{(1+\log^2 t)^p}\log^2 t\frac{dt}{t}\\ &= \frac{2^{p}(p-1)!}{\pi(2p-3)!!}\int_0^{+\infty}\frac{z^2}{(1+z^2)^p}dz = \frac{1}{2p-3}.\end{aligned}$$

So we get the following corollary:

Corollary 2. *Suppose that $f \in L^\infty(\mu)$. Then if $f \in C^2$ locally at the point $s \in \mathbb{R}^+$,*

$$\lim_{w\to+\infty} w^2[(\mathcal{P}_w f)(s) - f(s)] = \frac{1}{2(2p-3)}(sf'(s) + s^2 f''(s)) = \frac{1}{2(2p-3)}\Theta^2 f(s).$$

A general class of (classical) Poisson–Cauchy singular operators was considered in [7].

3.3

Let us consider the function

$$K(t) = \begin{cases}(4/(4+\pi))t, & 0 < t \leq 1\\ (4/(4+\pi))\dfrac{1}{(1+\log^2 t)^2}, & t > 1.\end{cases}$$

Taking into account the previous example it is easy to see that (1) and (2) hold. Moreover $M_1(K) < +\infty$ and $\ell_1 = -\frac{2}{4+\pi}$. So by Theorem 1, for the corresponding operator $\mathcal{A}_w f$ we obtain the following corollary:

Corollary 3. *Suppose that $f \in L^\infty(\mu)$. Then if $f \in C^1$ locally at the point $s \in \mathbb{R}^+$,*

$$\lim_{w\to+\infty} w[(\mathcal{A}_w f)(s) - f(s)] = -\frac{2}{4+\pi}f'(s)s = -\frac{2}{4+\pi}\Theta f(s).$$

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