Available at
www.ElsevierMathematics.com
POWERED BY SCIENCE DIRECT
European Journal of Combinatorics 25 (2004) 1–22

European Journal of Combinatorics

www.elsevier.com/locate/ejc

Construction of BLT-sets over small fields

Maska Law[a], Tim Penttila[b]

[a] *Dipartimento di Matematica, Istituto "Guido Castelnuovo", Università di Roma "La Sapienza", Piazzale Aldo Moro 2, 00185 Roma, Italy*

[b] *School of Mathematics and Statistics, The University of Western Australia, 35 Stirling Hwy, Crawley, WA 6009, Australia*

Received 18 April 2003; received in revised form 20 September 2003; accepted 23 September 2003

Abstract

The computer algebra system MAGMA is used to search for BLT-sets of the nonsingular parabolic quadric $Q(4, q)$. In total, 28 new BLT-sets for $27 \leq q \leq 125$ are presented, these giving rise to 158 new flocks of the quadratic cone in $PG(3, q)$.

1. Introduction

Due to their connections with a myriad of other geometric structures, BLT-sets have become important objects of study in finite projective spaces. The present paper is intended as a sequel to the paper of Penttila and Royle entitled "BLT-sets over small fields" [30]. In that paper, the authors extended the computer searches of De Clerck and Herssens [10] for flocks of the quadratic cone in $PG(3, q)$ to $q \leq 25$, by placing them in the more general context of finding BLT-sets rather than individual flocks. Earlier computer-free results had appeared in Thas [32] and De Clerck et al. [9]. The purpose of the present paper is to continue the more general theme of searching for BLT-sets, and we present here results of computer searches yielding 28 new BLT-sets. The techniques for searching are different to those of the prequel paper, and have been implemented using the computer algebra system MAGMA [8]. The techniques used by Penttila and Royle involved computing the automorphism groups of the generalised quadrangles (GQs) $Q(4, q)$ for $q \leq 25$, even though these are well known to be $P\Gamma O(5, q)$. The implementation of $PGO(5, q)$ in MAGMA for $q \leq 97$ removes the need for this computation, which in terms of computer time, was by far the most expensive part of the work in [30]. Theoretical developments have also provided an inexpensive and easily implemented check for determining if a set of points of the quadric $Q(4, q)$ is a partial BLT-set. Details of the new searches will be given

E-mail address: law@mat.uniromal.it (T. Penttila).

0195-6698/$ - see front matter
doi:10.1016/j.ejc.2003.09.006

later in the paper. For background and motivation to the study of BLT-sets, we refer to the aforementioned paper of Penttila and Royle, as well as the papers of Law and Penttila [20] and Johnson and Payne [15].

The construction of infinite families of BLT-sets is a difficult task, and can require detailed analysis of existing data. Indeed, since the prequel paper, only two infinite families have been described, and these were generalised from two examples in the first case and just one example in the second (see [28], [20]). The wealth of existing data suggests the possibility of more infinite families, and yet any new family discovered may only contain one or two of these existing examples. This observation displays the value of continuing these sorts of experimental searches, especially the value of continuing past the field of order 25, where Penttila and Royle left off.

It should be noted, however, that this paper makes no attempt at classification of BLT-sets for any particular values of q. The prequel paper classified BLT-sets for $q \leq 17$; the current paper simply presents new examples for some values of $q \leq 125$. A forthcoming paper by the present authors classifies BLT-sets by computer for $q \leq 29$ [21].

The theme of our searches is the use of nontrivial symmetry hypotheses. Since isomorph rejection is expensive, we make no claim of completeness even under a particular symmetry hypothesis. To paraphrase a good friend and colleague of ours, the searches are not complete, only thorough.

The use of symmetry hypotheses involves the choice of a subgroup H of $P\Gamma O(5, q)$ and the determination of all BLT-sets that are H-invariant. Since we are only interested in BLT-sets up to equivalence, we are only interested in conjugacy classes of subgroups. Such a search determines, up to equivalence, all BLT-sets with stabiliser containing a conjugate of H. The weakest nontrivial symmetry hypothesis is to use a minimal subgroup of $P\Gamma O(5, q)$. The minimal subgroups of a finite group—the atoms of the lattice of subgroups—are the subgroups of prime order. So our basic philosophy is to loop through the primes p dividing $|P\Gamma O(5, q)|$ in descending order of magnitude, take one representative H of each conjugacy class of subgroups of $P\Gamma O(5, q)$ of order p, and determine all BLT-sets that are H-invariant. This "prime-at-a-time" technique has previously been used for hyperovals [22], [29].

Several caveats are necessary here. The first has been mentioned earlier: that we do not determine the BLT-sets up to equivalence, but only up to certain invariants, such as the order of the stabiliser, that we find appealing and that are inexpensive to compute. The second is in the spirit of the first: since conjugacy is expensive to compute, we also do not determine the conjugacy classes, but again work with computationally cheaper invariants. The third is that small primes, particularly 2 and 3, are often beyond our means, so a selection of subgroups properly containing such subgroups is chosen as an alternative. It is worth remarking that, by Sylow's theorem, the conjugacy classes of subgroups of prime order p are all represented within a single Sylow p-subgroup.

The determination of all BLT-sets admitting a subgroup H is achieved by enlarging BLT-sets to the hereditary class of partial BLT-sets and then stitching together H-orbits. Namely a graph with vertices H-orbits that are partial BLT-sets and edges pairs of vertices with union a partial BLT-set is (perhaps only conceptually) formed. Every BLT-set is a clique of this graph with union of the vertices containing $q + 1$ points of $Q(4, q)$, although not every such clique needs to be a BLT-set.

The availability of a computationally inexpensive check for being a partial BLT-set is a key to our success.

In the following sections, we give a brief survey of the theory of BLT-sets and related structures: flocks of the quadratic cone, GQs, translation planes, and hyperbolic fibrations. We include an exposition of the theory underlying our cheap method of testing that a set of points is a partial BLT-set. We then go on to describe our techniques for searching, based upon the computer algebra system MAGMA, and present our results explicitly using coordinates. Finally, we briefly analyse some of the examples obtained.

2. Theory and background

As mentioned earlier, BLT-sets are connected with many varied geometric structures. They originally arose through the study of **flocks** of the quadratic cone in $PG(3, q)$, partitions of the points of the cone minus the vertex into q disjoint conics, and hence are connected with the many structures arising from such flocks.

A **linespread** of $PG(3, q)$ is a partition of the points into lines, namely, a set of $q^2 + 1$ lines, no two intersecting. Given a linespread of $PG(3, q)$, one can construct a translation plane of order q^2 via the André/Bruck-Bose construction. An **ovoid** of the Klein quadric $Q^+(5, q)$ is a set of $q^2 + 1$ points, no two collinear. In 1976, both Walker [33] and Thas independently described the construction of an ovoid of $Q^+(5, q)$ from a flock of the quadratic cone. By the Klein correspondence, an ovoid of $Q^+(5, q)$ is equivalent to a linespread of $PG(3, q)$, and so for each flock of the quadratic cone, we can construct a translation plane of order q^2. In 1987, Hiramine et al. [12] gave an amazing construction of a linespread of $PG(3, q^2)$ from a linespread of $PG(3, q)$, for q odd. The following year, Johnson [13] extended the construction to q even. Thus for each linespread of $PG(3, q)$, one can construct an infinite family of linespreads, and hence an infinite family of translation planes of orders q^{2^n}, $n = 1, 2, \ldots$. Furthermore, inequivalent linespreads of $PG(3, q^2)$ can arise from the same equivalence class of linespreads of $PG(3, q)$. Hence to each flock of the quadratic cone corresponds many infinite families of translation planes, and indeed, application of this construction to a plane (linespread) arising from a flock need not give a plane (linespread) arising from a flock.

A (finite) GQ of order (s, t) is an incidence structure of points and lines with a symmetric incidence relation $\mathtt{I}$, such that each point (line) is incident with $t + 1$ lines ($s + 1$ points), two distinct points (lines) are incident with at most one line (point), and for a nonincident point-line pair (P, l) there is a unique point Q and a unique line m such that $P \mathtt{I} m \mathtt{I} Q \mathtt{I} l$. The standard reference is [27]. In the early 1980's, a great deal of work was done by both Kantor [16] and Payne [23] on constructing particular types of GQs, namely, **elation generalised quadrangles** (EGQ). A GQ S is an EGQ if there is a point P of S and a subgroup G of the automorphism group Aut$S$ of S which fixes every line on P and acts regularly on the set of points not collinear with P. This work described a group coset geometry construction of EGQs of order (q^2, q) via the notion of a **q-clan**, a family of q, 2×2 matrices over $GF(q)$ such that the quadratic form described by the difference of any two distinct matrices is anisotropic. In 1987, Thas [32] connected flocks of the quadratic cone with EGQs of order (q^2, q) using q-clans, showing that to every flock there arises a corresponding EGQ, and conversely.

These connections motivated the concentrated study of flocks of the quadratic cone, and in 1990, Bader et al. [3] described a process of constructing q flocks from a given flock when q is odd. This process was called *derivation*, and gave rise to the concept of a **BLT-set** of the nonsingular quadric $Q(4, q)$, namely, a set of $q + 1$ points of $Q(4, q)$, q odd, such that no point of $Q(4, q)$ is collinear with more than 2 points of the set. The name is due to Kantor [17]. Given a flock of a quadratic cone of $PG(3, q)$ embedded as a hyperplane of $PG(4, q)$, one constructs a BLT-set of $Q(4, q)$ with a distinguished point P (it is the vertex of the cone). For each choice Q of the remaining q points of the BLT-set, one constructs a flock of the quadratic cone $Q^\perp \cap Q(4, q)$ with vertex Q. These q flocks are said to be *derived* from the original flock.

Two flocks $\mathcal{F}_1$ and $\mathcal{F}_2$ of the quadratic cone K of $PG(3, q)$ are said to be isomorphic if and only if there exists an element of $P\Gamma L(4, q)$ which fixes K and maps $\mathcal{F}_1$ to $\mathcal{F}_2$. Bader, Lunardon and Thas showed that two points P_1 and P_2 of a BLT-set $\mathcal{P}$ are in the same orbit of the stabiliser $P\Gamma O(5, q)_{\mathcal{P}}$ of $\mathcal{P}$ if and only if the flocks $\mathcal{F}_1$ and $\mathcal{F}_2$ arising from P_1 and P_2 respectively are isomorphic. Thus the concept of a BLT-set provides a greater unified approach to the study of flocks in odd characteristic, and this result displays the power in searching for BLT-sets in order to construct examples of flocks.

Furthermore, in 1990, Payne and Rogers [26] showed that to each BLT-set there corresponds just one GQ, and in 1992, Knarr [18] gave a beautiful geometric construction of the GQ directly from the associated BLT-set (see [7]). In 1996, Payne [25] showed that the number of orbits of lines through the base point of the GQ arising from a BLT-set $\mathcal{P}$ is equal to the number of orbits of $P\Gamma O(5, q)_{\mathcal{P}}$ on $\mathcal{P}$. In particular, the group of the GQ is determined by the group of the BLT-set.

To add further motivation to the study of BLT-sets, a very recent connection has been made with yet another geometric structure. A **hyperbolic fibration** of $PG(3, q)$ is a partition of the points into two lines and $q - 1$ hyperbolic quadrics. Each of the choices of one regulus for each quadric gives rise to a linespread of $PG(3, q)$, and so 2^{q-1} linespreads arise. In Baker et al. [5, 6], it is shown that each choice of an ordered pair of points on a BLT-set gives rise to a hyperbolic fibration. The linespreads arising in general differ from those arising via the Thas–Walker construction from the corresponding flocks.

Discriminants

Let $F = GF(q)$, q odd, and $\square = \{t^2 \mid t \in F^*\}$. Since q is odd, $\square$ is a subgroup of F^* of index 2. Let $\boxtimes$ be the other coset of $\square$ in F^*, so that $F^*/\square = \{\square, \boxtimes\}$.

For a quadratic form Q with polar form f, define the **discriminant of** Q to be $\det(B)\square \in F^*/\square$, where B is the matrix of f with respect to some basis. So if $Q(x) = x^{\mathrm{T}}Ax$, where $A = A^{\mathrm{T}}$, then $\operatorname{disc}(Q) = \det(2A)\square$.

From the classification of nondegenerate quadratic forms over finite fields of odd order, we know that there are two isometry classes of such orthogonal spaces for any given dimension. We may use the discriminant to distinguish between these isometry classes. For even algebraic dimension, a nondegenerate quadratic form Q on F^{2n} gives rise to an $O^+(2n, q)$ space if and only if $\operatorname{disc}(Q) = (-1)^n\square$.

The following results can be found in Bader et al. [4], the treatment of which we follow closely here, with the final corollary originally being due to Bader et al. [3].

Lemma. *Let $x, y, z \in V = F^5$ be linearly independent singular vectors with respect to the nondegenerate quadratic form Q with polar form f defined on V, and denote by $\perp$ the polarity defined by Q. Then $\langle x, y, z\rangle^\perp$ is an external line with respect to the quadric defined by Q if and only if*

$$\frac{-2f(x,y)f(x,z)f(y,z)\square}{\mathrm{disc}(Q)} = \boxtimes \in F^*/\square.$$

Proof. Since x, y, z are pairwise linearly independent singular vectors, $W = \langle x, y, z\rangle$ is a nondegenerate subspace of V. Let $W^\perp = \langle u, v\rangle$. Then $S = \{x, y, z, u, v\}$ is a basis for V. Let B be the matrix of f with respect to S, and let B_W be the matrix of f restricted to W. Then

$$\det(B) = \det(B_W)\det(B_{W^\perp}).$$

So

$$\frac{\square}{\mathrm{disc}(Q\mid_{W^\perp})} = \frac{\mathrm{disc}(Q\mid_W)}{\mathrm{disc}(Q)}.$$

Now $\det(B_W) = 2f(x,y)f(x,z)f(y,z)$ since f is symmetric. So

$$\frac{\square}{\mathrm{disc}(Q\mid_{W^\perp})} = \frac{2f(x,y)f(x,z)f(y,z)\square}{\mathrm{disc}(Q)}.$$

Now consider the subspace $W^\perp$. This has algebraic dimension 2, i.e., it is a line in $PV = PG(4,q)$. Hence $W^\perp$ is an external line (an $O^-(2,q)$ space) if and only if $-\mathrm{disc}(Q\mid_{W^\perp}) = \boxtimes$. The result follows.

Corollary. *Let q be odd. Let $\mathcal{P}$ be a set of at least 3 points of $Q(4,q)$ and let f be the bilinear form corresponding to the quadratic form Q underlying $Q(4,q)$. Then $\mathcal{P}$ is a partial BLT-set if and only if for all $\langle x\rangle, \langle y\rangle, \langle z\rangle \in \mathcal{P}$ we have:*

$$\frac{-2f(x,y)f(x,z)f(y,z)\square}{\mathrm{disc}(Q)} = \boxtimes \in F^*/\square.$$

Lemma. *Let q be odd. Let $\mathcal{P} = \{\langle x\rangle, \langle y\rangle, \langle z\rangle, \langle w\rangle\}$ be a set of 4 points of $Q(4,q)$ such that $\{\langle x\rangle, \langle y\rangle, \langle z\rangle\}, \{\langle x\rangle, \langle y\rangle, \langle w\rangle\}$ and $\{\langle x\rangle, \langle z\rangle, \langle w\rangle\}$ are partial BLT-sets. Then $\mathcal{P}$ is a partial BLT-set.*

Proof. It suffices to check that $\{\langle y\rangle, \langle z\rangle, \langle w\rangle\}$ is a partial BLT-set. If the defining quadratic form is Q, we know that

$$-2f(x,y)f(x,z)f(y,z)\square/\mathrm{disc}(Q), \qquad -2f(x,y)f(x,w)f(y,w)\square/\mathrm{disc}(Q),$$
$$-2f(x,z)f(x,w)f(z,w)\square/\mathrm{disc}(Q)$$

are all equal to $\boxtimes \in F^*/\square$, hence their product

$$\left(\frac{-2f(x,y)f(x,z)f(x,w)\square}{\mathrm{disc}(Q)}\right)^2 \frac{-2f(y,z)f(y,w)f(z,w)\square}{\mathrm{disc}(Q)}$$

also equals $\boxtimes \in F^*/\square$, and the result follows.

Corollary. *Let q be odd. Let $\mathcal{P}$ be a set of at least 3 points of $Q(4, q)$. Suppose there exists $\langle x \rangle \in \mathcal{P}$ such that $\{\langle x \rangle, \langle y \rangle, \langle z \rangle\}$ is a partial BLT-set for all $\langle y \rangle, \langle z \rangle \in \mathcal{P} \backslash \{\langle x \rangle\}$. Then $\mathcal{P}$ is a partial BLT-set.*

These results can be used to form a straightforward check for whether a set of points of $Q(4, q)$ is a partial BLT-set, and we use this in the next section to search for BLT-sets using a computer. The significance of these results is that we may test a set of points for being a partial BLT-set by performing a simple field calculation on all triples in the set containing a chosen element of the set, instead of having to test collinearity of all points of $Q(4, q)$ with each point of the set.

3. Search techniques

As a result of recent upgrades, the computer algebra package MAGMA [8] now has built in to it the projective general orthogonal groups of five dimensions over fields of order $q \leq 97$ as permutation groups on the points of the associated parabolic quadric $Q(4, q)$. This has provided the possibility of extending the searches of Penttila and Royle [30] past $q = 25$. Furthermore, using the results in the previous section we can form a cheap and easily implemented check for whether a set of points of $Q(4, q)$ is a partial BLT-set.

In this section, we will give a brief outline of the techniques used to search for BLT-sets using MAGMA, illustrated with some fragments of code as examples of the MAGMA syntax. Note, however, that the code fragments presented are by no means the most efficient, but merely for instructive purposes.

When we call the projective general orthogonal group $PGO(5, q)$ of five dimensions over the Galois field of order $q = p^h$, MAGMA indexes the points of the parabolic quadric $Q(4, q)$ and computes $PGO(5, q)$ as a permutation group acting on this set of indices. We label the field, group and indexed set `I` as follows:

```
> q:=p^h;
> F:=GaloisField(q);
> G,I:=ProjectiveGeneralOrthogonalGroup(5,F);
```

We remark that `I` is a sequence of q^3+q^2+q+1 points of $PG(4, q)$, each represented by a (homogeneous) 5-tuple from $GF(q)$, normalised so that the leftmost nonzero coordinate is a 1.

The quadratic form used by MAGMA is

$$Q(x) = x_1 x_5 + x_2 x_4 + \tfrac{1}{4} x_3^2$$

(up to a scalar multiple) as can be verified by the following code

```
> Q:=func<x|F!(x[1]*x[5]+x[2]*x[4]+x[3]^2/4)>;
> for i:=1 to #I do
>  if Q(I[i]) ne 0 then print i;end if;
> end for;
```

and observing no output occurring. (This does not seem to be documented in the manual for MAGMA.)

The full group of the quadric $Q(4, q)$ is the projective semilinear orthogonal group $P\Gamma O(5, q)$ of five dimensions over the Galois field of order q, which for prime order fields is equivalent to $PGO(5, q)$. However, in general, we must work with $P\Gamma O(5, q)$, and can construct it by forming the subgroup of the symmetric group on the set of indices generated by $PGO(5, q)$ and a field automorphism, namely, the Frobenius automorphism.

```
> V:=VectorSpace(F,5);
> a:=[];
> for i:=1 to #I do x:=I[i];
>  Append(~a,Index(I,V![x[1]^p,x[2]^p,x[3]^p,x[4]^p,x[5]^p]));
> end for;
> Gam:=sub<Sym(#I)|G,a>;
```

We need to be able to check easily and cheaply whether a set of points of $Q(4, q)$ is a partial BLT-set. It should be clear that the results of the previous section combine to give such a check, which lends itself well to being programmed on a computer. This can be done as follows:

```
> Q:=func<x|x[1]*x[5]+x[2]*x[4]+x[3]^2/4>;
> f:=func<x,y|Q(x+y)-Q(x)-Q(y)>;

> d:=((q-1) div 2);
> blt:=function(X);
>  if #X ge 3 then x:=Random(X);
>   for y in X diff {x} do flag:=true;
>    for z in X diff {x,y} do
>     flag:=(-4*f(I[x],I[y])*f(I[x],I[z])*f(I[y],I[z]))^d eq F!(-1);
>     if not flag then break;end if;
>    end for;
>    if not flag then break;end if;
>   end for;
>   return flag;
>  else return true;
>  end if;
> end function;
```

Our basic pattern of searching is for each prime p dividing $|P\Gamma O(5, q)|$ to compute a Sylow p-subgroup of $P\Gamma O(5, q)$

```
> P:=Sylow(Gam,p);
```

loop over the elements of P of order p

```
> list:={};
> for g in P do
>  if Order(g) eq p then
>   H:=sub<P|g>; FP:=Fix(H);
>   if not done(H,list) then
>     O:=Orbits(H);
```

```
>      cand:={i:i in {1..#O}|blt(O[i])};
>      c:={i:i in cand|#O[i] eq p};
>      edges:={{i,j}:i in c,j in c|i lt j and blt(O[i] join O[j])};
>      Gamma:=Graph<c|edges>;
>      d:=((q+1-#FP) div p)+1;e:=(q+1) div p;
>      for k:=d to e do
>       Cl:=AllCliques(Gamma,k);
>       for j:=1 to #Cl do Y:={};
>        for k in {x:x in c|VertexSet(Gamma)!x in Cl[j]} do
>         Y:=Y join O[k];
>        end for;
>        for X in Subsets(FP,(q+1)-k*p) do
>         if blt(Y join X) then
>          PrintFile(‘‘bltsets’’,Y join X);list:=list join {H};
>         end if;
>        end for;
>       end for;
>      end for;
>    end if;
>   end if;
> end for;
```

and provided that the subgroup H it generates has not already been dealt with up to conjugacy, calculate and write all H-invariant BLT-sets to a file. Here the function for testing whether a subgroup has already been dealt with can be implemented as follows

```
> done:=function(H,list);
>  flag:=false;
>  for X in list do flag:=IsConjugate(Gam,H,X);
>   if flag then break;end if;
>  end for;
>  return flag;
> end function;
```

In the interest of efficiency, not all of these computations need be performed. For instance, conjugacy testing need only be performed if (cheap) invariants indicate the subgroups may be conjugate, and normalisers may be used to not have to calculate all H-invariant BLT-sets, reducing, in the process, the size of the cliques that need to be computed.

Finally, we must have some way of determining whether a BLT-set that is discovered is in fact new. For most of the searching done the order of the stabiliser of the BLT-set in $P\Gamma O(5, q)$ was used.

```
> GB:=Stabiliser(Gam,B);
```

This proved to be a reasonably effective test in most cases, though it should be noted that it was not always sufficient, and stricter testing was sometimes needed, for example, comparing the order and number of fixed points of elements of the stabilisers. Indeed, this is how the new example for $q = 37$ with group order 72 was found to be different from K_2/JP, and how the two examples with group order 4 were distinguished. This highlights the fact that our searches have been far from exhaustive, and that possibly some new examples were overlooked during searching because they had the same group orders as known examples. Other more complex invariants were also used, including an adaptation of the conic invariant of Williams [34]. However, these simple invariants are enough to distinguish the examples presented here for $q \leq 59$.

4. The examples

We refer the reader to the prequel paper of Penttila and Royle [30] for a list of all but the two most recent of known families of BLT-sets, a description of which can be found in the papers of Penttila [28] and Law and Penttila [20].

The results of the searches are summarised in the following table, along with the known examples not yet belonging to an infinite family. The group order listed is that of the stabiliser of the BLT-set in $P\Gamma O(5, q)$, and the orbit structure is listed as a multiset, so, e.g., $\{2, 4, 8^3\}$ indicates that there is one orbit of length 2, one of length 4 and 3 of length 8.

We abbreviate the attributions as follows:

DCH—De Clerck–Herssens (1992) [10]

DCP—De Clerck–Penttila (1997) [11]

PR—Penttila–Royle (1998) [30]

PW—Penttila–Williams (2000) [31]

BLP—Bader–Lunardon–Pinneri (1999) [2].

For $q = 27$ we find 1 new example, which has been generalised to an infinite family in characteristic 3 (see [20]). For $q = 29$ we find 5 new examples, for $q = 31$ 5 new examples, for $q = 37$ 3 new examples, for $q = 41$ 3 new examples, for $q = 43$ 1 new example, for $q = 47$ 2 new examples, for $q = 49$ 2 new examples, for $q = 53$ 2 new examples, for $q = 59$ 3 new examples, and for $q = 125$ 1 new example.

In the subsequent tables we present the new BLT-sets discovered for $27 \leq q \leq 59$ as sets of $q + 1$ points of the nonsingular quadric $Q(4, q)$ defined by the quadratic equation

$$Q(x) = x_1x_5 + x_2x_4 + x_3^2.$$

Each example has been presented containing the points $\langle(0, 0, 0, 0, 1)\rangle$ and $\langle(1, 0, 0, 0, 0)\rangle$. We identify each BLT-set by the order of its stabiliser in $P\Gamma O(5, q)$. The orbits of the stabiliser on the BLT-set are distinguished by vertical bars, reading across the table from left to right. The example X_4^{cyc} for $q = 31$ has a cyclic stabiliser, whilst X_4^{noncyc} has stabiliser isomorphic to the noncyclic group of order 4. The example X_4^2 for $q = 37$ has 2 fixed points on the BLT-set, whilst X_4^0 has 2 fixed points not on the BLT-set. The new example for $q = 125$ is then presented using interpolated polynomials.

q	Attribution	Group order	Orbit structure
17	DCH	144	$\{12, 6\}$
	PR	24	$\{12, 6\}$
19	PR	20	$\{20\}$
	PR	16	$\{8^2, 2^2\}$
23	DCH	72	$\{18, 6\}$
	PR	1 152	$\{24\}$
	PR	24	$\{24\}$
	PR	16	$\{8^2, 4^2\}$
	PR	6	$\{6^3, 3^2\}$
25	PR	16	$\{16, 8, 2\}$
	PR	8	$\{8^3, 2\}$
27	New	6	$\{1^2, 2, 6^4\}$
29	New	720	$\{30\}$
	New	48	$\{24, 6\}$
	New	8	$\{8^3, 4, 2\}$
	New	6	$\{6^4, 3^2\}$
	New	3	$\{3^{10}\}$
31	New	96	$\{24, 6, 2\}$
	New	10	$\{10^2, 5^2, 2\}$
	New	8	$\{8^3, 4^2\}$
	New	4	$\{4^8\}$
	New	4	$\{4^7, 2^2\}$
37	New	72	$\{36, 2\}$
	New	4	$\{4^9, 2\}$
	New	4	$\{4^9, 1^2\}$
41	New	60	$\{30, 12\}$
	New	24	$\{24, 12, 6\}$
	New	8	$\{8^5, 2\}$
43	New	4	$\{4^{11}\}$
47	DCP	2 304	$\{48\}$
	New	24	$\{24^2\}$
	New	3	$\{3^{16}\}$
49	New	40	$\{40, 10\}$
	New	20	$\{20^2, 5^2\}$
53	New	24	$\{24^2, 6\}$
	New	12	$\{12^4, 6\}$
59	New	120	$\{60\}$
	New	24	$\{24^2, 12\}$
	New	5	$\{5^{12}\}$
125	New	72	$\{72, 24^2, 6\}$
243	PW/BLP	26 730	$\{243, 1\}$

X_6	$(0, 0, 0, 0, 1)$	$\mid$	$(1, 0, 0, 0, 0)$	$(1, w, w^{19}, 2, w^{20})$	$\mid$
$q = 27$	$(1, w^{14}, w^6, 1, w^{20})$	$\mid$	$(1, w^{20}, w^{18}, w^{17}, w^6)$	$(1, w^{25}, w^{24}, w^{25}, w^4)$	
$w^3 = w - 1$	$(1, w^3, w^6, w^{24}, w^{24})$		$(1, w^{16}, w^5, w^{23}, w^{24})$	$(1, w^{15}, w^7, w^8, w^4)$	
	$(1, w^5, w^{21}, w^{20}, w^6)$	$\mid$	$(1, w^{24}, w^5, w^{14}, w^{18})$	$(1, w^7, w^2, w^{16}, w^{14})$	
	$(1, 2, w^{12}, w^{11}, w^{24})$		$(1, w^{23}, w^6, w^7, w^6)$	$(1, w^{17}, w^3, w^{21}, w^4)$	
	$(1, w^{11}, w^{16}, w^{22}, w^2)$	$\mid$	$(1, w^8, w^{24}, w^{15}, w^{18})$	$(1, w^{21}, 0, w^{12}, w^{20})$	
	$(1, w^{18}, w^{14}, w^4, w^{20})$		$(1, 1, w^{22}, w^{19}, w^{14})$	$(1, w^2, w^7, w^{10}, w^{20})$	
	$(1, w^6, w^9, w^5, w^2)$	$\mid$	$(1, w^4, w^8, w^6, w^8)$	$(1, w^{19}, w^{23}, w^2, w^{16})$	
	$(1, w^{10}, w^5, w^9, 1)$		$(1, w^9, w^{22}, w, w^{12})$	$(1, w^{12}, w^{14}, w^{18}, w^{10})$	
	$(1, w^{22}, w^5, w^3, w^{22})$				

X_{720}	(1, 26, 16, 3, 14)	(1, 0, 0, 0, 0)	(1, 20, 11, 4, 2)
$q = 29$	(1, 6, 27, 27, 8)	(1, 16, 20, 2, 3)	(1, 19, 15, 9, 10)
	(1, 10, 19, 28, 26)	(0, 0, 0, 0, 1)	(1, 9, 28, 24, 15)
	(1, 13, 9, 7, 2)	(1, 14, 11, 15, 17)	(1, 27, 24, 23, 21)
	(1, 23, 22, 16, 18)	(1, 7, 17, 6, 17)	(1, 15, 10, 25, 18)
	(1, 5, 20, 26, 21)	(1, 24, 26, 11, 17)	(1, 22, 0, 17, 3)
	(1, 3, 8, 5, 8)	(1, 8, 25, 21, 19)	(1, 2, 24, 8, 17)
	(1, 1, 0, 12, 17)	(1, 25, 27, 22, 26)	(1, 18, 8, 18, 18)
	(1, 21, 0, 20, 15)	(1, 12, 24, 14, 10)	(1, 17, 23, 19, 18)
	(1, 11, 0, 13, 2)	(1, 28, 17, 1, 2)	(1, 4, 4, 10, 2)

X_{48}	(1, 26, 24, 25, 21)	(1, 15, 3, 24, 8)	(1, 5, 14, 3, 21)	
$q = 29$	(1, 18, 23, 20, 10)	(1, 3, 28, 13, 18)	(1, 14, 26, 5, 8)	$\mid$
	(0, 0, 0, 0, 1)	(1, 0, 0, 0, 0)	(1, 2, 23, 17, 17)	
	(1, 12, 17, 18, 17)	(1, 20, 19, 7, 21)	(1, 21, 12, 27, 14)	
	(1, 7, 28, 1, 21)	(1, 24, 8, 28, 18)	(1, 19, 19, 15, 21)	
	(1, 11, 25, 21, 14)	(1, 25, 11, 2, 3)	(1, 13, 7, 8, 21)	
	(1, 17, 18, 26, 17)	(1, 28, 19, 23, 10)	(1, 6, 18, 6, 17)	
	(1, 23, 11, 11, 3)	(1, 4, 12, 4, 14)	(1, 8, 21, 12, 14)	
	(1, 9, 11, 10, 21)	(1, 27, 9, 22, 21)	(1, 10, 18, 9, 21)	
	(1, 22, 2, 16, 21)	(1, 16, 19, 19, 2)	(1, 1, 19, 14, 2)	

X_8	(1, 18, 7, 11, 14)		(1, 15, 20, 28, 21)	$\mid$	(1, 13, 20, 8, 18)	
$q = 29$	(1, 24, 23, 24, 26)		(1, 22, 26, 1, 27)		(1, 7, 6, 13, 18)	$\mid$
	(1, 0, 0, 0, 0)		(1, 16, 11, 26, 14)		(1, 27, 18, 12, 19)	
	(1, 14, 23, 20, 3)		(1, 6, 5, 22, 17)		(1, 11, 10, 6, 8)	
	(1, 26, 3, 2, 26)		(1, 28, 22, 23, 3)	$\mid$	(1, 25, 24, 25, 17)	
	(1, 1, 22, 27, 11)		(1, 2, 28, 5, 18)		(1, 21, 13, 4, 8)	
	(1, 5, 24, 14, 21)		(1, 10, 24, 16, 18)		(0, 0, 0, 0, 1)	
	(1, 8, 25, 3, 18)	$\mid$	(1, 23, 15, 15, 10)		(1, 4, 10, 21, 19)	
	(1, 3, 3, 10, 19)		(1, 12, 16, 7, 8)		(1, 9, 20, 18, 18)	
	(1, 17, 2, 17, 26)		(1, 20, 27, 9, 19)		(1, 19, 17, 19, 17)	

<table>
<tr><td>X_6</td><td>(1, 0, 0, 0, 0)</td><td>(1, 9, 21, 11, 11)</td><td>(1, 8, 16, 17, 14)</td><td>|</td></tr>
<tr><td>$q = 29$</td><td>(1, 2, 20, 28, 8)</td><td>(1, 1, 24, 1, 3)</td><td>(1, 10, 7, 26, 10)</td><td>|</td></tr>
<tr><td></td><td>(1, 3, 1, 25, 11)</td><td>(1, 20, 1, 21, 14)</td><td>(1, 15, 17, 3, 14)</td><td></td></tr>
<tr><td></td><td>(1, 14, 0, 23, 26)</td><td>(1, 21, 3, 19, 27)</td><td>(0, 0, 0, 0, 1)</td><td>|</td></tr>
<tr><td></td><td>(1, 11, 28, 5, 2)</td><td>(1, 24, 1, 24, 3)</td><td>(1, 17, 4, 15, 19)</td><td></td></tr>
<tr><td></td><td>(1, 27, 11, 4, 3)</td><td>(1, 4, 6, 10, 11)</td><td>(1, 25, 26, 22, 21)</td><td>|</td></tr>
<tr><td></td><td>(1, 19, 18, 8, 17)</td><td>(1, 26, 21, 7, 15)</td><td>(1, 16, 7, 9, 10)</td><td></td></tr>
<tr><td></td><td>(1, 5, 3, 16, 27)</td><td>(1, 22, 17, 14, 12)</td><td>(1, 23, 27, 12, 10)</td><td>|</td></tr>
<tr><td></td><td>(1, 6, 16, 13, 14)</td><td>(1, 12, 15, 27, 2)</td><td>(1, 18, 5, 6, 12)</td><td></td></tr>
<tr><td></td><td>(1, 7, 26, 18, 10)</td><td>(1, 13, 3, 20, 21)</td><td>(1, 28, 4, 2, 15)</td><td></td></tr>
</table>

<table>
<tr><td>X_3</td><td>(1, 0, 0, 0, 0)</td><td>(1, 23, 28, 3, 17)</td><td>(1, 22, 20, 12, 3)</td><td>|</td></tr>
<tr><td>$q = 29$</td><td>(1, 16, 25, 7, 17)</td><td>(1, 20, 27, 9, 19)</td><td>(0, 0, 0, 0, 1)</td><td>|</td></tr>
<tr><td></td><td>(1, 2, 16, 18, 27)</td><td>(1, 19, 4, 14, 8)</td><td>(1, 8, 0, 25, 3)</td><td>|</td></tr>
<tr><td></td><td>(1, 25, 9, 16, 12)</td><td>(1, 14, 20, 11, 26)</td><td>(1, 9, 28, 26, 26)</td><td>|</td></tr>
<tr><td></td><td>(1, 6, 9, 15, 3)</td><td>(1, 12, 6, 13, 11)</td><td>(1, 27, 5, 4, 12)</td><td>|</td></tr>
<tr><td></td><td>(1, 21, 22, 1, 17)</td><td>(1, 17, 16, 24, 3)</td><td>(1, 1, 25, 2, 11)</td><td>|</td></tr>
<tr><td></td><td>(1, 13, 26, 8, 3)</td><td>(1, 18, 13, 27, 12)</td><td>(1, 11, 1, 20, 11)</td><td>|</td></tr>
<tr><td></td><td>(1, 26, 6, 19, 21)</td><td>(1, 7, 9, 17, 3)</td><td>(1, 4, 16, 5, 14)</td><td>|</td></tr>
<tr><td></td><td>(1, 15, 5, 28, 19)</td><td>(1, 5, 5, 21, 15)</td><td>(1, 28, 7, 10, 19)</td><td>|</td></tr>
<tr><td></td><td>(1, 24, 11, 22, 18)</td><td>(1, 3, 9, 6, 17)</td><td>(1, 10, 9, 23, 8)</td><td></td></tr>
</table>

<table>
<tr><td>X_{96}</td><td>(1, 0, 0, 0, 0)</td><td>(0, 0, 0, 0, 1)</td><td>|</td><td>(1, 30, 30, 3, 2)</td></tr>
<tr><td>$q = 31$</td><td>(1, 1, 1, 28, 2)</td><td>(1, 6, 6, 13, 10)</td><td></td><td>(1, 5, 5, 16, 19)</td></tr>
<tr><td></td><td>(1, 25, 25, 18, 10)</td><td>(1, 26, 26, 15, 19)</td><td>|</td><td>(1, 12, 3, 1, 10)</td></tr>
<tr><td></td><td>(1, 21, 13, 25, 19)</td><td>(1, 22, 10, 27, 19)</td><td></td><td>(1, 4, 24, 14, 19)</td></tr>
<tr><td></td><td>(1, 3, 15, 22, 19)</td><td>(1, 16, 24, 19, 19)</td><td></td><td>(1, 28, 11, 10, 2)</td></tr>
<tr><td></td><td>(1, 23, 2, 24, 2)</td><td>(1, 2, 13, 30, 19)</td><td></td><td>(1, 11, 28, 20, 19)</td></tr>
<tr><td></td><td>(1, 19, 15, 6, 2)</td><td>(1, 8, 16, 21, 10)</td><td></td><td>(1, 29, 15, 5, 2)</td></tr>
<tr><td></td><td>(1, 27, 18, 4, 2)</td><td>(1, 15, 13, 17, 10)</td><td></td><td>(1, 18, 27, 2, 10)</td></tr>
<tr><td></td><td>(1, 7, 11, 9, 2)</td><td>(1, 24, 16, 7, 10)</td><td></td><td>(1, 14, 28, 29, 19)</td></tr>
<tr><td></td><td>(1, 17, 19, 11, 10)</td><td>(1, 9, 18, 12, 2)</td><td></td><td>(1, 10, 3, 26, 10)</td></tr>
<tr><td></td><td>(1, 20, 27, 8, 10)</td><td>(1, 13, 3, 23, 2)</td><td></td><td></td></tr>
</table>

<table>
<tr><td>X_{10}</td><td>(1, 0, 0, 0, 0)</td><td></td><td>(0, 0, 0, 0, 1)</td><td>|</td><td>(1, 30, 3, 18, 9)</td><td></td></tr>
<tr><td>$q = 31$</td><td>(1, 14, 14, 20, 20)</td><td></td><td>(1, 28, 17, 17, 10)</td><td></td><td>(1, 12, 10, 16, 18)</td><td></td></tr>
<tr><td></td><td>(1, 9, 18, 22, 5)</td><td>|</td><td>(1, 15, 8, 14, 5)</td><td></td><td>(1, 2, 1, 6, 18)</td><td></td></tr>
<tr><td></td><td>(1, 11, 5, 11, 9)</td><td></td><td>(1, 24, 18, 30, 10)</td><td></td><td>(1, 10, 30, 1, 20)</td><td>|</td></tr>
<tr><td></td><td>(1, 4, 18, 7, 20)</td><td></td><td>(1, 13, 4, 10, 9)</td><td></td><td>(1, 23, 16, 21, 5)</td><td></td></tr>
<tr><td></td><td>(1, 8, 18, 19, 20)</td><td></td><td>(1, 18, 24, 29, 18)</td><td></td><td>(1, 3, 2, 3, 18)</td><td></td></tr>
<tr><td></td><td>(1, 25, 2, 28, 9)</td><td></td><td>(1, 19, 2, 27, 10)</td><td></td><td>(1, 26, 6, 2, 5)</td><td></td></tr>
<tr><td></td><td>(1, 16, 1, 9, 10)</td><td>|</td><td>(1, 5, 20, 25, 2)</td><td></td><td>(1, 7, 28, 4, 25)</td><td></td></tr>
<tr><td></td><td>(1, 21, 9, 24, 4)</td><td></td><td>(1, 6, 13, 13, 1)</td><td></td><td>(1, 20, 14, 5, 14)</td><td></td></tr>
<tr><td></td><td>(1, 17, 25, 15, 19)</td><td></td><td>(1, 1, 10, 8, 16)</td><td></td><td>(1, 29, 10, 23, 8)</td><td></td></tr>
<tr><td></td><td>(1, 22, 14, 26, 7)</td><td></td><td>(1, 27, 12, 12, 28)</td><td></td><td></td><td></td></tr>
</table>

X_8	(1, 0, 0, 0, 0)		(1, 29, 27, 17, 18)		(1, 5, 22, 20, 5)	
$q = 31$	(1, 26, 8, 6, 28)	$\mid$	(1, 8, 20, 2, 18)		(1, 27, 15, 10, 1)	
	(1, 21, 0, 4, 9)		(1, 9, 22, 15, 1)	$\mid$	(1, 22, 7, 30, 4)	
	(1, 28, 14, 5, 5)		(1, 4, 24, 7, 16)		(1, 10, 3, 19, 18)	
	(0, 0, 0, 0, 1)		(1, 17, 23, 14, 8)		(1, 30, 26, 8, 14)	
	(1, 11, 12, 11, 14)	$\mid$	(1, 18, 6, 27, 5)		(1, 24, 14, 13, 19)	
	(1, 6, 24, 29, 25)		(1, 2, 23, 25, 10)		(1, 13, 3, 23, 2)	
	(1, 20, 28, 1, 2)		(1, 12, 1, 12, 10)		(1, 1, 26, 21, 16)	$\mid$
	(1, 3, 17, 26, 5)		(1, 25, 5, 16, 9)		(1, 23, 6, 9, 5)	
	(1, 14, 7, 18, 9)		(1, 19, 12, 3, 16)		(1, 16, 21, 28, 10)	
	(1, 15, 14, 22, 1)		(1, 7, 4, 24, 2)			

X_4^{cyc}	(1, 16, 1, 13, 8)		(1, 29, 12, 19, 18)		(1, 0, 0, 0, 0)	
$q = 31$	(1, 14, 4, 7, 10)	$\mid$	(1, 28, 15, 11, 25)		(1, 27, 27, 1, 19)	
	(1, 19, 11, 18, 2)		(1, 20, 4, 27, 2)	$\mid$	(1, 5, 1, 5, 5)	
	(1, 12, 10, 4, 7)		(1, 25, 12, 6, 16)		(1, 1, 26, 2, 4)	$\mid$
	(1, 6, 16, 3, 5)		(1, 24, 5, 9, 7)		(1, 8, 4, 20, 10)	
	(1, 4, 14, 24, 18)	$\mid$	(0, 0, 0, 0, 1)		(1, 9, 15, 29, 10)	
	(1, 3, 25, 23, 19)		(1, 7, 11, 8, 9)	$\mid$	(1, 18, 23, 14, 25)	
	(1, 17, 18, 15, 10)		(1, 22, 18, 21, 20)		(1, 11, 25, 16, 5)	$\mid$
	(1, 30, 10, 26, 19)		(1, 26, 1, 30, 25)		(1, 21, 26, 25, 8)	
	(1, 2, 13, 17, 14)	$\mid$	(1, 13, 4, 12, 14)		(1, 15, 1, 22, 10)	
	(1, 23, 0, 10, 18)		(1, 10, 20, 28, 2)			

X_4^{noncyc}	(1, 14, 29, 2, 2)		(1, 28, 26, 18, 9)	$\mid$	(1, 7, 21, 19, 28)	
$q = 31$	(1, 5, 13, 3, 28)	$\mid$	(1, 15, 27, 12, 2)		(1, 30, 24, 29, 9)	
	(1, 23, 5, 30, 9)		(1, 0, 0, 0, 0)	$\mid$	(0, 0, 0, 0, 1)	
	(1, 18, 26, 6, 2)		(1, 4, 1, 28, 4)		(1, 21, 12, 10, 2)	$\mid$
	(1, 9, 22, 7, 2)		(1, 10, 28, 11, 4)		(1, 27, 23, 1, 19)	
	(1, 20, 15, 9, 4)	$\mid$	(1, 22, 25, 15, 2)		(1, 26, 1, 16, 10)	
	(1, 25, 21, 25, 1)		(1, 11, 13, 5, 19)	$\mid$	(1, 2, 0, 21, 20)	
	(1, 1, 15, 27, 2)		(1, 3, 14, 26, 28)		(1, 16, 3, 22, 10)	$\mid$
	(1, 8, 8, 13, 4)		(1, 29, 1, 24, 9)		(1, 6, 15, 17, 20)	
	(1, 13, 8, 4, 25)	$\mid$	(1, 24, 0, 8, 25)		(1, 12, 25, 23, 25)	
	(1, 19, 8, 14, 28)		(1, 17, 5, 20, 18)			

X_{72}	(0, 0, 0, 0, 1)	(1, 0, 0, 0, 0)	$\mid$	(1, 20, 1, 33, 5)
$q = 37$	(1, 6, 23, 19, 23)	(1, 32, 11, 35, 17)		(1, 33, 12, 14, 23)
	(1, 25, 8, 12, 6)	(1, 16, 6, 1, 22)		(1, 23, 8, 5, 6)
	(1, 8, 6, 13, 8)	(1, 22, 19, 20, 13)		(1, 21, 23, 16, 23)
	(1, 28, 6, 9, 8)	(1, 2, 26, 7, 13)		(1, 5, 9, 36, 35)
	(1, 12, 23, 10, 17)	(1, 14, 16, 25, 23)		(1, 9, 8, 26, 35)
	(1, 1, 11, 8, 19)	(1, 19, 12, 11, 17)		(1, 18, 27, 22, 22)
	(1, 24, 36, 17, 35)	(1, 26, 27, 23, 5)		(1, 27, 25, 24, 22)
	(1, 36, 21, 32, 35)	(1, 7, 16, 31, 8)		(1, 35, 5, 15, 5)
	(1, 10, 36, 6, 13)	(1, 30, 4, 3, 5)		(1, 4, 3, 30, 19)
	(1, 17, 13, 2, 19)	(1, 11, 28, 18, 17)		(1, 13, 16, 27, 22)
	(1, 29, 12, 28, 6)	(1, 15, 10, 34, 19)		(1, 3, 30, 4, 13)
	(1, 31, 9, 21, 8)	(1, 34, 9, 29, 6)		

X_4^0	(1, 9, 16, 8, 5)		(1, 13, 20, 26, 2)	$\vert$	(1, 0, 0, 0, 0)	
$q = 37$	(1, 4, 12, 14, 22)		(1, 19, 23, 5, 5)		(1, 24, 17, 19, 32)	$\vert$
	(0, 0, 0, 0, 1)		(1, 18, 18, 10, 14)		(1, 2, 4, 36, 23)	
	(1, 7, 5, 24, 29)	$\vert$	(1, 36, 20, 22, 29)		(1, 14, 10, 13, 14)	
	(1, 28, 22, 7, 23)		(1, 32, 28, 20, 19)	$\vert$	(1, 30, 27, 23, 24)	
	(1, 3, 30, 28, 15)		(1, 1, 20, 31, 13)		(1, 11, 36, 33, 6)	$\vert$
	(1, 29, 22, 30, 15)		(1, 20, 10, 4, 5)		(1, 8, 14, 32, 29)	
	(1, 25, 15, 9, 31)	$\vert$	(1, 26, 13, 18, 29)		(1, 23, 19, 6, 19)	
	(1, 35, 11, 15, 20)		(1, 22, 11, 25, 32)	$\vert$	(1, 10, 16, 2, 20)	
	(1, 17, 18, 27, 31)		(1, 15, 29, 16, 29)		(1, 31, 18, 1, 15)	$\vert$
	(1, 12, 34, 35, 15)		(1, 33, 23, 17, 20)		(1, 27, 23, 21, 14)	
	(1, 34, 22, 3, 6)	$\vert$	(1, 21, 11, 11, 18)		(1, 5, 33, 12, 35)	
	(1, 6, 22, 34, 15)		(1, 16, 31, 29, 18)			

X_4^2	(1, 3, 7, 25, 24)	$\vert$	(1, 35, 30, 32, 15)	$\vert$	(1, 0, 0, 0, 0)	
$q = 37$	(1, 4, 10, 30, 2)		(1, 14, 8, 17, 31)		(1, 30, 18, 2, 23)	$\vert$
	(0, 0, 0, 0, 1)		(1, 19, 0, 10, 32)		(1, 8, 18, 12, 24)	
	(1, 34, 2, 20, 19)	$\vert$	(1, 28, 34, 15, 15)		(1, 2, 21, 23, 31)	
	(1, 13, 29, 36, 23)		(1, 23, 13, 8, 17)	$\vert$	(1, 25, 12, 35, 17)	
	(1, 1, 14, 4, 22)		(1, 17, 2, 19, 6)		(1, 12, 17, 1, 32)	$\vert$
	(1, 27, 23, 3, 19)		(1, 20, 2, 21, 20)		(1, 33, 30, 22, 2)	
	(1, 11, 16, 31, 32)	$\vert$	(1, 29, 6, 13, 31)		(1, 16, 17, 7, 6)	
	(1, 6, 25, 14, 31)		(1, 32, 27, 29, 8)	$\vert$	(1, 15, 8, 9, 23)	
	(1, 5, 17, 24, 35)		(1, 18, 13, 33, 14)		(1, 7, 36, 26, 2)	$\vert$
	(1, 21, 34, 28, 32)		(1, 10, 22, 16, 22)		(1, 26, 34, 27, 29)	
	(1, 31, 13, 34, 35)	$\vert$	(1, 9, 13, 18, 2)		(1, 22, 28, 5, 31)	
	(1, 24, 3, 11, 23)		(1, 36, 27, 6, 17)			

X_{60}	(1, 27, 36, 26, 11)	(1, 0, 0, 0, 0)	(0, 0, 0, 0, 1)	
$q = 41$	(1, 22, 11, 6, 34)	(1, 11, 40, 30, 38)	(1, 24, 20, 33, 38)	
	(1, 38, 11, 3, 11)	(1, 28, 1, 1, 12)	(1, 35, 25, 31, 12)	
	(1, 12, 27, 36, 28)	(1, 15, 31, 27, 28)	(1, 34, 3, 12, 34)	$\vert$
	(1, 32, 24, 37, 3)	(1, 26, 1, 7, 22)	(1, 30, 23, 2, 26)	
	(1, 4, 2, 34, 24)	(1, 29, 29, 10, 17)	(1, 8, 32, 25, 6)	
	(1, 16, 16, 22, 7)	(1, 20, 1, 28, 13)	(1, 36, 19, 29, 30)	
	(1, 18, 12, 20, 29)	(1, 37, 10, 18, 13)	(1, 14, 7, 35, 35)	
	(1, 33, 40, 14, 29)	(1, 9, 21, 40, 19)	(1, 40, 28, 32, 27)	
	(1, 23, 26, 16, 22)	(1, 10, 20, 13, 3)	(1, 17, 16, 39, 24)	
	(1, 13, 26, 38, 19)	(1, 3, 13, 21, 14)	(1, 5, 4, 8, 26)	
	(1, 21, 39, 5, 14)	(1, 2, 39, 15, 7)	(1, 1, 24, 9, 30)	
	(1, 19, 9, 23, 15)	(1, 31, 14, 17, 15)	(1, 25, 12, 24, 35)	
	(1, 7, 32, 11, 6)	(1, 6, 8, 19, 27)	(1, 39, 27, 4, 17)	

X_{24}	(1, 25, 4, 20, 17)	(1, 16, 12, 25, 30)	(1, 22, 5, 30, 12)	
$q = 41$	(1, 15, 0, 8, 3)	(1, 14, 3, 39, 19)	(1, 26, 23, 2, 34)	$\vert$
	(1, 3, 14, 40, 12)	(1, 10, 2, 5, 28)	(1, 5, 26, 23, 29)	
	(1, 37, 9, 34, 14)	(1, 39, 40, 6, 11)	(1, 13, 30, 33, 24)	
	(1, 11, 22, 18, 15)	(1, 38, 34, 7, 13)	(1, 28, 24, 29, 6)	
	(1, 23, 35, 17, 24)	(1, 32, 29, 19, 27)	(1, 33, 36, 4, 7)	$\vert$
	(1, 21, 38, 36, 14)	(1, 0, 0, 0, 0)	(1, 7, 12, 13, 11)	
	(1, 20, 26, 28, 35)	(1, 27, 36, 3, 17)	(1, 24, 14, 16, 35)	
	(1, 40, 24, 9, 7)	(1, 30, 17, 31, 11)	(1, 36, 6, 37, 26)	
	(1, 4, 28, 21, 34)	(1, 35, 23, 15, 12)	(1, 1, 7, 14, 19)	
	(1, 18, 2, 35, 22)	(1, 17, 30, 26, 11)	(1, 31, 21, 27, 34)	
	(1, 34, 39, 22, 27)	(1, 2, 13, 12, 12)	(1, 9, 19, 24, 38)	
	(1, 12, 8, 1, 6)	(1, 6, 25, 11, 6)	(0, 0, 0, 0, 1)	
	(1, 29, 37, 38, 30)	(1, 19, 40, 32, 6)	(1, 8, 7, 10, 35)	

X_8	(1, 28, 16, 28, 26)		(1, 5, 1, 25, 38)	$\vert$	(1, 8, 12, 30, 26)	
$q = 41$	(1, 0, 0, 0, 0)		(1, 2, 27, 10, 30)		(1, 10, 32, 18, 26)	
	(1, 36, 10, 32, 19)		(1, 22, 8, 21, 7)		(1, 9, 2, 9, 38)	
	(1, 35, 10, 26, 15)	$\vert$	(1, 6, 12, 33, 27)		(0, 0, 0, 0, 1)	
	(1, 37, 29, 6, 3)		(1, 24, 10, 34, 27)		(1, 13, 24, 40, 11)	
	(1, 34, 15, 39, 7)		(1, 17, 22, 17, 6)		(1, 18, 13, 5, 28)	$\vert$
	(1, 40, 31, 31, 13)		(1, 12, 14, 3, 14)		(1, 38, 40, 22, 24)	
	(1, 29, 6, 13, 38)		(1, 31, 11, 27, 26)		(1, 23, 12, 35, 35)	
	(1, 26, 15, 36, 28)		(1, 30, 17, 29, 30)	$\vert$	(1, 39, 11, 14, 30)	
	(1, 33, 35, 12, 19)		(1, 11, 24, 2, 17)		(1, 21, 33, 24, 6)	
	(1, 27, 12, 20, 13)		(1, 19, 16, 11, 27)		(1, 25, 9, 16, 11)	
	(1, 1, 3, 15, 17)	$\vert$	(1, 16, 39, 4, 14)		(1, 7, 36, 19, 6)	
	(1, 3, 11, 7, 22)		(1, 4, 19, 8, 17)		(1, 15, 14, 1, 35)	
	(1, 32, 21, 38, 24)		(1, 20, 15, 37, 19)		(1, 14, 10, 23, 29)	

X_4	(1, 13, 23, 21, 15)		(1, 0, 0, 0, 0)		(1, 27, 39, 11, 31)	
$q = 43$	(1, 1, 7, 40, 40)	$\vert$	(1, 39, 7, 4, 10)		(1, 2, 6, 29, 35)	
	(0, 0, 0, 0, 1)		(1, 16, 8, 25, 9)	$\vert$	(1, 32, 16, 41, 23)	
	(1, 10, 6, 1, 40)		(1, 18, 37, 14, 13)		(1, 12, 27, 18, 1)	$\vert$
	(1, 3, 6, 19, 36)		(1, 20, 13, 36, 14)		(1, 26, 31, 12, 17)	
	(1, 11, 19, 37, 6)	$\vert$	(1, 40, 0, 26, 35)		(1, 42, 36, 27, 21)	
	(1, 25, 18, 2, 13)		(1, 7, 39, 23, 38)	$\vert$	(1, 34, 28, 22, 16)	
	(1, 19, 9, 7, 1)		(1, 33, 2, 24, 21)		(1, 14, 19, 10, 15)	$\vert$
	(1, 6, 21, 39, 13)		(1, 31, 3, 8, 1)		(1, 17, 35, 13, 16)	
	(1, 41, 6, 38, 40)	$\vert$	(1, 21, 27, 30, 17)		(1, 29, 14, 31, 23)	
	(1, 5, 9, 42, 10)		(1, 24, 6, 9, 6)	$\vert$	(1, 37, 35, 17, 38)	
	(1, 36, 34, 15, 24)		(1, 35, 10, 33, 35)		(1, 38, 14, 34, 17)	$\vert$
	(1, 4, 23, 16, 9)		(1, 8, 33, 20, 41)		(1, 15, 24, 3, 24)	
	(1, 30, 29, 5, 41)	$\vert$	(1, 28, 40, 28, 24)		(1, 22, 2, 6, 36)	
	(1, 23, 25, 32, 15)		(1, 9, 31, 35, 14)			

X_{2304} = DCP	(1, 12, 6, 19, 18)	(1, 0, 0, 0, 0)	(1, 23, 22, 36, 4)
$q = 47$	(1, 25, 40, 13, 2)	(1, 13, 29, 5, 34)	(1, 44, 44, 30, 34)
	(1, 15, 43, 11, 7)	(1, 8, 41, 3, 34)	(1, 37, 7, 16, 17)
	(1, 16, 32, 22, 34)	(1, 41, 25, 34, 2)	(1, 27, 17, 17, 4)
	(1, 6, 11, 1, 14)	(1, 5, 35, 2, 34)	(1, 1, 23, 33, 2)
	(1, 31, 24, 31, 14)	(1, 10, 12, 27, 9)	(1, 38, 29, 45, 34)
	(1, 33, 12, 35, 17)	(1, 20, 33, 15, 21)	(1, 46, 38, 8, 21)
	(1, 42, 26, 6, 12)	(1, 21, 4, 12, 14)	(1, 36, 1, 9, 4)
	(1, 32, 6, 28, 8)	(1, 40, 43, 37, 8)	(1, 26, 11, 10, 42)
	(1, 11, 5, 38, 27)	(1, 34, 18, 42, 34)	(1, 7, 18, 4, 24)
	(0, 0, 0, 0, 1)	(1, 29, 5, 21, 24)	(1, 39, 25, 40, 24)
	(1, 3, 17, 25, 12)	(1, 17, 23, 29, 12)	(1, 2, 24, 14, 7)
	(1, 28, 23, 24, 21)	(1, 45, 33, 41, 27)	(1, 24, 28, 43, 17)
	(1, 18, 3, 46, 9)	(1, 19, 4, 18, 18)	(1, 22, 0, 7, 34)
	(1, 30, 38, 26, 32)	(1, 43, 6, 23, 9)	(1, 35, 36, 39, 18)
	(1, 9, 38, 32, 7)	(1, 14, 38, 44, 8)	(1, 4, 38, 20, 27)

X_{24}	(1, 42, 19, 42, 37)	(1, 0, 0, 0, 0)	(1, 11, 4, 24, 2)
$q = 47$	(1, 8, 30, 36, 34)	(1, 19, 42, 1, 3)	(1, 22, 36, 19, 25)
	(1, 28, 19, 7, 7)	(1, 16, 38, 12, 9)	(1, 34, 23, 11, 37)
	(1, 33, 25, 17, 36)	(1, 31, 23, 25, 12)	(1, 25, 3, 16, 14)
	(1, 23, 25, 30, 1)	(1, 13, 30, 3, 1)	(1, 35, 16, 40, 36)
	(1, 5, 4, 2, 21)	(1, 46, 35, 9, 6)	(1, 38, 3, 28, 8)
	(1, 43, 22, 5, 6)	(1, 29, 28, 35, 34)	(1, 9, 4, 33, 16)
	(1, 2, 24, 13, 9)	(1, 17, 14, 38, 4)	(1, 12, 23, 22, 6)
	(1, 44, 21, 15, 27)	(1, 39, 33, 18, 42)	(1, 18, 41, 44, 18)
	(1, 20, 38, 45, 6)	(1, 7, 24, 34, 32)	(1, 36, 23, 10, 4)
	(1, 24, 30, 27, 3)	(0, 0, 0, 0, 1)	(1, 45, 32, 32, 27)
	(1, 30, 43, 20, 42)	(1, 4, 18, 26, 42)	(1, 21, 32, 39, 37)
	(1, 37, 12, 43, 4)	(1, 26, 46, 23, 12)	(1, 14, 7, 31, 34)
	(1, 27, 11, 4, 6)	(1, 15, 17, 8, 14)	(1, 32, 17, 46, 25)
	(1, 41, 43, 41, 42)	(1, 6, 34, 21, 34)	(1, 1, 46, 29, 17)
	(1, 10, 11, 6, 7)	(1, 40, 25, 14, 37)	(1, 3, 34, 37, 2)

X_3	(1, 0, 0, 0, 0)	(1, 36, 44, 32, 14)	(1, 12, 12, 28, 37)
$q = 47$	(1, 30, 43, 2, 18)	(1, 32, 37, 10, 3)	(0, 0, 0, 0, 1)
	(1, 24, 16, 40, 6)	(1, 43, 43, 18, 9)	(1, 18, 32, 44, 17)
	(1, 3, 38, 42, 28)	(1, 9, 35, 22, 34)	(1, 4, 23, 20, 2)
	(1, 17, 13, 37, 1)	(1, 44, 3, 35, 2)	(1, 5, 45, 25, 12)
	(1, 40, 37, 1, 1)	(1, 7, 43, 43, 12)	(1, 21, 35, 4, 7)
	(1, 39, 28, 16, 2)	(1, 1, 22, 6, 27)	(1, 26, 28, 8, 42)
	(1, 42, 43, 21, 42)	(1, 16, 4, 38, 34)	(1, 46, 43, 17, 1)
	(1, 45, 21, 26, 34)	(1, 33, 7, 36, 32)	(1, 13, 21, 13, 1)
	(1, 14, 43, 29, 1)	(1, 35, 33, 19, 32)	(1, 37, 29, 5, 8)
	(1, 25, 13, 23, 8)	(1, 11, 39, 11, 3)	(1, 19, 26, 46, 1)
	(1, 31, 3, 33, 2)	(1, 10, 8, 39, 16)	(1, 38, 25, 15, 27)
	(1, 27, 44, 27, 14)	(1, 15, 10, 34, 1)	(1, 22, 32, 30, 8)
	(1, 2, 14, 41, 4)	(1, 34, 17, 9, 16)	(1, 29, 12, 24, 6)
	(1, 6, 31, 3, 8)	(1, 8, 14, 14, 21)	(1, 41, 28, 45, 3)
	(1, 20, 9, 31, 4)	(1, 28, 36, 7, 12)	(1, 23, 40, 12, 4)

X_{40}	$(1, 0, 0, 0, 0)$	$(1, w^{44}, w^{27}, 5, w^{33})$	$(1, w^{6}, w^{27}, w^{3}, w^{27})$
$q = 49$	$(1, w^{21}, w^{11}, w^{7}, w^{45})$	$(1, w^{35}, w^{34}, 1, w^{33})$	$(1, w^{26}, w^{33}, w^{17}, w)$
$w^2 = w - 3$	$(1, 5, 1, w^{10}, w^{35})$	$(1, w^{17}, 4, w, w^{3})$	$(1, 4, 0, w^{33}, w^{41})$
	$(1, w^{25}, 2, w^{25}, w^{29})$	$(1, w^{27}, w^{26}, w^{45}, w^{23})$	$(1, w^{20}, w^{43}, 2, w^{23})$
	$(1, w^{14}, w^{25}, w^{43}, w^{13})$	$(1, 3, w^{37}, w^{22}, w^{19})$	$(1, w^{41}, w^{7}, 6, w^{35})$
	$(1, w^{30}, w^{34}, w^{11}, w^{23})$	$(1, w^{9}, 3, w^{14}, w^{27})$	$(1, w^{15}, w^{41}, w^{5}, w^{25})$
	$(1, w^{7}, w^{21}, w^{39}, w^{35})$	$(0, 0, 0, 0, 1)$	$(1, w^{23}, 6, w^{21}, w^{37})$
	$(1, w, 6, w^{20}, w^{3})$	$(1, w^{12}, w^{41}, w^{26}, w^{27})$	$(1, w^{18}, w^{22}, 4, w^{19})$
	$(1, w^{19}, w^{27}, w^{44}, w)$	$(1, w^{13}, 6, w^{37}, w^{35})$	$(1, w^{10}, w^{19}, w^{9}, w^{39})$
	$(1, w^{36}, 5, w^{46}, w^{19})$	$(1, w^{43}, w^{14}, w^{15}, w^{7})$	$(1, w^{11}, w^{17}, w^{27}, w^{27})$
	$(1, w^{33}, w^{5}, w^{34}, w^{5})$	$(1, w^{2}, w^{11}, w^{38}, w^{19})$	$(1, w^{29}, w^{2}, w^{18}, w^{13})$
	$(1, w^{37}, w^{29}, w^{35}, w^{15})$	$(1, 2, w^{38}, w^{41}, w^{29})$	$(1, w^{47}, w^{7}, w^{13}, w^{47})$
	$(1, w^{31}, w^{10}, w^{31}, w^{37})$	$(1, w^{28}, w^{17}, w^{4}, w^{19})$	$(1, w^{3}, w^{29}, w^{36}, w^{11})$
	$(1, w^{34}, w^{25}, w^{2}, w^{41})$	$(1, w^{22}, w^{44}, w^{47}, w^{41})$	$(1, w^{4}, w^{7}, w^{12}, w)$
	$(1, w^{39}, w^{15}, w^{19}, w^{29})$	$(1, w^{45}, w^{39}, w^{6}, w^{33})$	$(1, w^{46}, w^{20}, w^{28}, w^{31})$
	$(1, w^{38}, 0, w^{29}, w^{43})$	$(1, 6, 4, w^{42}, w^{3})$	$(1, w^{5}, w^{35}, w^{23}, w^{45})$
	$(1, w^{42}, w^{19}, w^{30}, w^{29})$	$(1, 1, w^{25}, 3, w^{25})$	

X_{20}	$(1, 0, 0, 0, 0)$	$(1, w^{21}, w^{2}, w^{27}, w^{41})$	$(1, w^{41}, 1, w^{20}, w^{15})$
$q = 49$	$(1, w, w^{12}, w^{37}, w^{29})$	$(1, w^{4}, 2, w^{43}, w^{45})$	$(1, w^{14}, w^{5}, w^{2}, w^{33})$
$w^2 = w - 3$	$(1, w^{28}, w^{46}, w^{19}, w^{17})$	$(1, w^{39}, w^{25}, 5, w^{3})$	$(1, w^{38}, w^{34}, 1, w^{17})$
	$(1, 1, w^{39}, w^{13}, w^{7})$	$(1, w^{7}, w^{22}, w^{46}, w^{39})$	$(1, w^{20}, w^{35}, w^{30}, w^{21})$
	$(1, w^{9}, w^{18}, w^{9}, w^{15})$	$(1, 4, w^{31}, 6, w^{31})$	$(1, w^{10}, w, w^{6}, w^{7})$
	$(1, w^{11}, w^{7}, w^{21}, w^{11})$	$(1, w^{29}, w^{45}, w^{28}, w^{7})$	$(1, w^{17}, w^{13}, w^{38}, w^{27})$
	$(1, w^{37}, 3, w^{45}, w^{13})$	$(1, 5, w^{17}, w, w^{45})$	$(1, w^{6}, w^{13}, w^{35}, w^{39})$
	$(1, w^{18}, w^{6}, w^{12}, w^{9})$	$(1, w^{2}, 6, w^{23}, w^{31})$	$(0, 0, 0, 0, 1)$
	$(1, w^{22}, w^{37}, w^{47}, w^{35})$	$(1, w^{44}, w^{33}, w^{31}, w^{13})$	$(1, w^{3}, w^{41}, 3, w^{17})$
	$(1, 3, w^{29}, w^{5}, w^{31})$	$(1, w^{23}, w^{22}, w^{34}, w^{1}1)$	$(1, w^{19}, w^{34}, w^{10}, w^{15})$
	$(1, w^{5}, w^{35}, 4, w^{35})$	$(1, w^{43}, w^{33}, w^{29}, w^{41})$	$(1, w^{46}, w^{33}, w^{26}, w^{41})$
	$(1, w^{42}, w^{20}, w^{42}, w^{29})$	$(1, w^{26}, w^{25}, w^{22}, w^{35})$	$(1, w^{13}, w^{35}, w^{7}, w^{7})$
	$(1, w^{47}, w^{3}, w^{36}, w^{7})$	$(1, w^{45}, w^{4}, w^{15}, w)$	$(1, w^{31}, w^{45}, w^{41}, w^{21})$
	$(1, w^{25}, w^{38}, w^{18}, w^{41})$	$(1, w^{30}, w^{31}, w^{4}, w^{33})$	$(1, w^{27}, w^{30}, w^{14}, w^{13})$
	$(1, w^{33}, 3, w^{33}, w^{3})$	$(1, 2, w^{25}, w^{17}, w^{27})$	$(1, w^{15}, w^{17}, w^{44}, w^{17})$
	$(1, w^{36}, w^{33}, w^{39}, w^{13})$	$(1, 6, w^{19}, w^{25}, w^{45})$	$(1, w^{35}, w^{17}, w^{3}, w^{27})$
	$(1, w^{12}, w^{45}, w^{11}, w^{43})$	$(1, w^{34}, w^{15}, 2, w)$	

X_{24}	(1, 4, 23, 6, 30)	(1, 20, 37, 26, 19)	(1, 47, 33, 37, 34)
$q = 53$	(1, 2, 40, 47, 2)	(1, 29, 10, 21, 33)	(1, 11, 32, 14, 41)
	(1, 0, 0, 0, 0)	(1, 24, 7, 35, 12)	(1, 26, 34, 16, 18)
	(1, 17, 51, 20, 27)	(1, 8, 11, 9, 19)	(1, 45, 6, 15, 31)
	(1, 41, 6, 12, 2)	(1, 25, 45, 52, 14)	(1, 14, 2, 19, 48)
	(1, 40, 25, 23, 45)	(1, 12, 10, 2, 35)	(1, 22, 14, 44, 2)
	(1, 51, 32, 28, 39)	(1, 23, 41, 43, 33)	(1, 10, 22, 27, 41)
	(1, 43, 23, 42, 50)	(1, 16, 21, 40, 32)	(1, 48, 24, 4, 27)
	(1, 38, 18, 31, 35)	(1, 28, 41, 33, 45)	(1, 30, 44, 49, 39)
	(1, 19, 31, 48, 35)	(1, 52, 17, 36, 12)	(1, 6, 1, 30, 31)
	(1, 18, 9, 32, 32)	(1, 37, 32, 3, 31)	(1, 36, 20, 39, 51)
	(1, 13, 43, 24, 12)	(1, 1, 1, 1, 51)	(1, 42, 25, 34, 14)
	(1, 31, 42, 22, 45)	(1, 3, 32, 13, 50)	(1, 44, 46, 51, 39)
	(1, 27, 14, 38, 50)	(1, 49, 3, 7, 19)	(1, 15, 19, 17, 20)
	(1, 50, 43, 45, 35)	(1, 46, 14, 25, 32)	(1, 39, 20, 50, 35)
	(1, 33, 19, 11, 18)	(1, 35, 31, 46, 26)	(1, 9, 49, 41, 39)
	(0, 0, 0, 0, 1)	(1, 34, 1, 10, 30)	(1, 5, 12, 18, 31)
	(1, 21, 30, 8, 45)	(1, 32, 31, 29, 19)	(1, 7, 35, 5, 12)

X_{12}	(1, 49, 15, 31, 5)	(0, 0, 0, 0, 1)	(1, 12, 24, 48, 14)
$q = 53$	(1, 21, 37, 15, 12)	(1, 20, 39, 37, 18)	(1, 52, 42, 46, 31)
	(1, 0, 0, 0, 0)	(1, 41, 26, 30, 2)	(1, 2, 26, 20, 26)
	(1, 38, 47, 33, 35)	(1, 45, 23, 39, 48)	(1, 16, 33, 49, 35)
	(1, 46, 21, 32, 48)	(1, 43, 14, 7, 33)	(1, 3, 0, 47, 18)
	(1, 28, 15, 35, 14)	(1, 8, 14, 1, 8)	(1, 44, 37, 26, 31)
	(1, 9, 30, 51, 19)	(1, 7, 3, 8, 41)	(1, 5, 27, 38, 35)
	(1, 51, 12, 43, 48)	(1, 27, 42, 34, 21)	(1, 26, 41, 27, 2)
	(1, 48, 15, 29, 26)	(1, 30, 51, 24, 18)	(1, 18, 31, 23, 3)
	(1, 47, 22, 13, 18)	(1, 32, 29, 36, 21)	(1, 42, 39, 52, 5)
	(1, 35, 46, 4, 23)	(1, 24, 42, 25, 21)	(1, 36, 37, 17, 33)
	(1, 14, 30, 2, 26)	(1, 15, 35, 3, 2)	(1, 22, 11, 6, 12)
	(1, 39, 41, 9, 35)	(1, 34, 5, 28, 30)	(1, 10, 33, 10, 30)
	(1, 29, 37, 5, 23)	(1, 25, 17, 18, 3)	(1, 1, 4, 19, 18)
	(1, 11, 38, 12, 14)	(1, 17, 15, 16, 33)	(1, 19, 11, 44, 50)
	(1, 23, 32, 45, 8)	(1, 40, 51, 42, 12)	(1, 33, 43, 22, 22)
	(1, 37, 35, 40, 51)	(1, 31, 3, 11, 21)	(1, 50, 45, 14, 31)
	(1, 13, 34, 21, 2)	(1, 6, 40, 50, 8)	(1, 4, 44, 41, 20)

X_{120}	(1, 0, 0, 0, 0)	(1, 20, 23, 24, 53)	(1, 31, 45, 58, 12)
$q = 59$	(1, 5, 41, 43, 51)	(1, 55, 28, 36, 9)	(1, 32, 47, 32, 12)
	(1, 34, 19, 51, 29)	(1, 46, 58, 22, 49)	(1, 41, 13, 3, 3)
	(1, 18, 54, 50, 19)	(1, 28, 46, 9, 51)	(1, 33, 10, 26, 45)
	(1, 54, 19, 47, 51)	(1, 26, 2, 56, 15)	(1, 4, 53, 2, 15)
	(1, 49, 20, 35, 9)	(1, 44, 7, 34, 48)	(1, 47, 41, 30, 36)
	(1, 17, 50, 29, 16)	(1, 56, 50, 54, 22)	(0, 0, 0, 0, 1)
	(1, 11, 55, 49, 35)	(1, 35, 18, 23, 51)	(1, 21, 1, 44, 19)
	(1, 23, 37, 6, 27)	(1, 2, 4, 4, 35)	(1, 30, 7, 48, 45)
	(1, 51, 5, 33, 3)	(1, 48, 49, 16, 17)	(1, 36, 32, 15, 29)
	(1, 14, 19, 18, 36)	(1, 57, 10, 38, 35)	(1, 13, 0, 53, 19)
	(1, 58, 53, 55, 19)	(1, 16, 32, 27, 19)	(1, 25, 36, 1, 36)
	(1, 27, 1, 20, 49)	(1, 10, 4, 31, 28)	(1, 42, 38, 45, 29)
	(1, 19, 38, 25, 28)	(1, 7, 12, 11, 15)	(1, 8, 35, 12, 36)
	(1, 50, 5, 19, 28)	(1, 43, 16, 14, 27)	(1, 1, 30, 46, 57)
	(1, 6, 31, 8, 53)	(1, 40, 34, 52, 9)	(1, 38, 38, 57, 48)
	(1, 29, 12, 17, 12)	(1, 9, 24, 39, 17)	(1, 39, 13, 40, 41)
	(1, 24, 18, 37, 27)	(1, 37, 7, 41, 27)	(1, 53, 21, 28, 22)
	(1, 52, 40, 42, 51)	(1, 22, 58, 21, 9)	(1, 12, 23, 7, 36)
	(1, 3, 58, 13, 19)	(1, 15, 58, 10, 26)	(1, 45, 54, 5, 45)

X_{24}	(1, 36, 40, 12, 53)	(1, 54, 38, 46, 46)	(1, 5, 25, 4, 45)
$q = 59$	(1, 58, 45, 9, 19)	(1, 41, 13, 21, 26)	(1, 18, 6, 34, 28)
	(1, 13, 7, 42, 17)	(1, 55, 7, 58, 28)	(1, 21, 44, 35, 20)
	(1, 12, 3, 51, 20)	(1, 8, 48, 38, 5)	(1, 3, 22, 30, 25)
	(1, 25, 38, 50, 41)	(1, 30, 21, 32, 36)	(1, 34, 46, 56, 45)
	(1, 19, 1, 31, 45)	(0, 0, 0, 0, 1)	(1, 1, 25, 53, 12)
	(1, 24, 28, 2, 51)	(1, 51, 16, 14, 48)	(1, 44, 4, 52, 9)
	(1, 11, 1, 5, 48)	(1, 40, 35, 36, 9)	(1, 0, 0, 0, 0)
	(1, 33, 54, 11, 29)	(1, 53, 43, 49, 53)	(1, 43, 52, 20, 57)
	(1, 28, 35, 18, 1)	(1, 27, 50, 57, 19)	(1, 26, 12, 41, 19)
	(1, 45, 17, 26, 41)	(1, 35, 35, 48, 5)	(1, 14, 23, 28, 51)
	(1, 39, 31, 43, 15)	(1, 10, 5, 7, 27)	(1, 49, 34, 3, 36)
	(1, 17, 39, 45, 20)	(1, 4, 42, 47, 20)	(1, 2, 36, 40, 9)
	(1, 32, 1, 23, 16)	(1, 38, 13, 22, 51)	(1, 7, 16, 44, 41)
	(1, 16, 58, 1, 28)	(1, 20, 11, 16, 48)	(1, 56, 45, 39, 9)
	(1, 23, 21, 19, 28)	(1, 9, 17, 33, 29)	(1, 42, 16, 17, 48)
	(1, 31, 38, 27, 41)	(1, 15, 48, 29, 51)	(1, 46, 18, 13, 29)
	(1, 37, 58, 24, 41)	(1, 47, 24, 10, 35)	(1, 48, 37, 25, 36)
	(1, 57, 23, 15, 1)	(1, 6, 30, 55, 35)	(1, 22, 1, 54, 36)
	(1, 29, 22, 8, 1)	(1, 52, 57, 6, 41)	(1, 50, 18, 37, 16)

X_5	(1, 0, 0, 0, 0)		(1, 24, 54, 52, 25)		(1, 55, 0, 21, 25)	
$q = 59$	(1, 23, 52, 12, 29)		(1, 3, 36, 35, 15)	|	(0, 0, 0, 0, 1)	
	(1, 53, 5, 54, 4)		(1, 19, 28, 27, 1)		(1, 41, 15, 13, 9)	
	(1, 11, 13, 18, 46)	|	(1, 10, 30, 10, 3)		(1, 56, 17, 20, 7)	
	(1, 6, 30, 31, 35)		(1, 15, 36, 41, 36)		(1, 43, 21, 44, 27)	|
	(1, 58, 11, 30, 27)		(1, 8, 35, 16, 4)		(1, 45, 16, 6, 5)	
	(1, 44, 35, 39, 9)		(1, 12, 46, 25, 3)	|	(1, 35, 53, 37, 26)	
	(1, 52, 40, 47, 27)		(1, 30, 31, 40, 22)		(1, 51, 15, 58, 3)	
	(1, 34, 15, 34, 35)	|	(1, 13, 15, 38, 48)		(1, 21, 4, 49, 17)	
	(1, 7, 14, 45, 20)		(1, 50, 42, 24, 45)		(1, 16, 22, 46, 19)	|
	(1, 40, 19, 11, 25)		(1, 22, 11, 42, 17)		(1, 29, 15, 2, 12)	
	(1, 5, 33, 29, 5)		(1, 57, 39, 7, 27)	|	(1, 14, 19, 5, 41)	
	(1, 49, 39, 15, 45)		(1, 38, 41, 23, 41)		(1, 54, 24, 26, 26)	
	(1, 25, 3, 57, 41)	|	(1, 31, 0, 53, 9)		(1, 27, 43, 50, 46)	
	(1, 17, 19, 48, 3)		(1, 26, 42, 55, 51)		(1, 28, 41, 19, 29)	|
	(1, 1, 30, 32, 12)		(1, 48, 19, 14, 29)		(1, 39, 24, 3, 15)	
	(1, 46, 55, 43, 12)		(1, 18, 50, 28, 5)	|	(1, 37, 49, 33, 36)	
	(1, 32, 31, 17, 29)		(1, 42, 1, 8, 17)		(1, 9, 14, 22, 19)	
	(1, 47, 45, 56, 4)	|	(1, 4, 12, 1, 29)		(1, 36, 57, 4, 29)	
	(1, 33, 50, 9, 35)		(1, 2, 17, 51, 22)		(1, 20, 20, 36, 1)	

The new example for $q = 125$:

$$X_{72} = \{\langle(1, t, f(t), -g(t), tg(t) - f(t)^2)\rangle : t \text{ in } GF(q)\} \cup \{\langle(0, 0, 0, 0, 1)\rangle\}$$

where f and g are as follows:

$$\begin{aligned} f &= x^{59} + 2x^{57} + 4x^{55} + 2x^{53} + x^{51} + 3x^{49} + 4x^{45} + 4x^{43} + 4x^{39} + 2x^{37} \\ &\quad + 4x^{35} + 3x^{33} + 4x^{31} + 4x^{29} + 3x^{25} + 4x^{23} + x^{19} + 4x^{17} + 4x^{15} \\ &\quad + 4x^{13} + 3x^{11} + 2x^{9} + x^{7} + 4x^{5} + 4x \\ g &= x^{117} + 3x^{115} + x^{113} + x^{107} + 3x^{105} + 3x^{103} + 3x^{101} + 3x^{95} + 2x^{91} + 4x^{87} \\ &\quad + 2x^{85} + 3x^{83} + 2x^{77} + 2x^{75} + 4x^{73} + 4x^{71} + x^{67} + 2x^{65} + 2x^{61} \\ &\quad + 4x^{59} + x^{57} + x^{55} + 4x^{53} + 4x^{49} + 3x^{47} + 2x^{45} + 4x^{43} + 2x^{41} + 4x^{39} \\ &\quad + x^{33} + 2x^{31} + 4x^{29} + 3x^{25} + 4x^{23} + x^{21} + x^{19} + 4x^{17} + x^{15} + 3x^{13} \\ &\quad + 4x^{11} + 2x^{7} + 4x^{3} + 4x. \end{aligned}$$

Noting that the points of the BLT-set other than $(\infty) = \langle(0, 0, 0, 0, 1)\rangle$ are indexed by the elements of $GF(125)$ (with respect to the second coordinate), we describe the orbits on X_{72} as a partition of $GF(125) \cup \{\infty\}$, where $w^3 = 2w + 2$:

$\{0, 1, 2, 3, 4, (\infty)\}$,
$\{w^{6}, w^{19}, w^{21}, w^{26}, w^{29}, w^{30}, w^{33}, w^{34}, w^{41}, w^{43}, w^{44}, w^{46}, w^{68}, w^{81}, w^{83}, w^{88},$
$w^{91}, w^{92}, w^{95}, w^{96}, w^{103}, w^{105}, w^{106}, w^{108}\}$,
$\{w^{7}, w^{9}, w^{17}, w^{23}, w^{35}, w^{37}, w^{39}, w^{45}, w^{51}, w^{53}, w^{57}, w^{61}, w^{69}, w^{71}, w^{79}, w^{85},$
$w^{97}, w^{99}, w^{101}, w^{107}, w^{113}, w^{115}, w^{119}, w^{123}\}$,
$\{w, w^{2}, w^{3}, w^{4}, w^{5}, w^{8}, w^{10}, w^{11}, w^{12}, w^{13}, w^{14}, w^{15}, w^{16}, w^{18}, w^{20}, w^{22}, w^{24},$
$w^{25}, w^{27}, w^{28}, w^{32}, w^{36}, w^{38}, w^{40}, w^{42}, w^{47}, w^{48}, w^{49}, w^{50}, w^{52}, w^{54}, w^{55},$
$w^{56}, w^{58}, w^{59}, w^{60}, w^{63}, w^{64}, w^{65}, w^{66}, w^{67}, w^{70}, w^{72}, w^{73}, w^{74}, w^{75}, w^{76},$

$w^{77}, w^{78}, w^{80}, w^{82}, w^{84}, w^{86}, w^{87}, w^{89}, w^{90}, w^{94}, w^{98}, w^{100}, w^{102}, w^{104}, w^{109}, w^{110}, w^{111}, w^{112}, w^{114}, w^{116}, w^{117}, w^{118}, w^{120}, w^{121}, w^{122}\}$.

5. Final remarks

We conclude with some remarks about the data. The results of [24] imply that the stabiliser in $P\Gamma O(5, q)$ of the Ganley semifield BLT-set over $GF(q)$ has order $2qh$, where $q = 3^h$ and $h \geq 3$. While this is true for $h > 3$, for $h = 3$, the Ganley BLT-set has a group of order $648 = 8qh$. This was first noted by us via our software. See [19] for a complete list of group orders of the known infinite families (in particular, orders for small values of q).

The example X_{2304} over the field of order 47 was constructed in 1997 by De Clerck and Penttila [11] by different methods, but not published. That this example and PR_{1152} over the field of order 23 are sporadic is suggested by the results of [1].

That the example of [2] has a group of order 26 730 follows from the calculation of the group of the corresponding ovoid first described in [31], and the fact that by [14] the group of the BLT-set has a unique fixed point on the BLT-set. An alternative proof, using the fundamental theorem of q-clan geometry appears in [19].

The example X_{72} over the field of order 125 was constructed in MAGMA using different techniques to those described earlier. To determine the group of this example, an adaptation of the conic invariant of Williams [34] was used. From this, we have a unique conic C meeting X_{72} in 6 points, these being the points over $GF(5)$, and all others meeting X_{72} in at most 4 points. Hence the stabiliser H of X_{72} in $P\Gamma O(5, 125)$ is contained in the stabiliser G of $X_{72} \cap C$ in $P\Gamma O(5, 125)$. Since $|G| = 4320$, we can loop over all elements of G and see if they stabilise X_{72} in order to determine H. We find that $|H| = 72$ and $H \cong S_4 \times C_3$, with orbit structure $\{72, 24^2, 6\}$. (The 4 orbits are actually distinguished by how many conics meeting X_{72} in 4 points a point lies on.) See [19] for further detail.

We finish with a description of the example X_{720} over the field of order 29, which suggests that it is sporadic.

Let V be $GF(29)^6$ and Q be the quadratic form on V given by

$$Q(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2.$$

Then $Q(1, 1, 1, 1, 1, 1)$ is not 0, so $W = \{x | x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 0\}$ is a nondegenerate subspace. Let $\mathcal{P}$ be the set of all images of the point $(1, 1, 1, 1, 7, 18)$ under permutations of the coordinates. Then $\mathcal{P}$ is a set of 30 points of W. $\mathcal{P}$ is a BLT-set of W (considered as $O(5, 29)$) whose group is S_6.

Theorem. *Let $V = GF(29)^6$ and $Q : V \to GF(29)$ be given by $Q(x_1, x_2, x_3, x_4, x_5, x_6) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2$. Let $e = (1, 1, 1, 1, 1, 1)$ and $W = e^{\perp}$, so that W is a nondegenerate subspace of V. Let S_6 act on V by permuting the coordinates. Let $\mathcal{P}$ be the orbit of $(1, 1, 1, 1, 7, 18)$ under S_6. Then $\mathcal{P}$ is a BLT-set of W, and the full stabiliser of $\mathcal{P}$ in $PGO(5, 29)$ is S_6.*

Proof. The polar form of Q is $f(x, y) = 2x_1y_1 + 2x_2y_2 + 2x_3y_3 + 2x_4y_4 + 2x_5y_5 + 2x_6y_6$. S_6 fixes W and $f(e, (1, 1, 1, 1, 7, 18)) = 0 \pmod{29}$, so $\mathcal{P}$ is a subset of W. A calculation shows that the discriminant of the restriction of Q to W is a nonsquare, so we must show that $f(x, y) f(x, z) f(y, z)$ is a nonsquare, for all distinct $x, y, z \in \mathcal{P}$. It is sufficient to show that $f(x, y)$ is a nonsquare, for distinct $x, y \in \mathcal{P}$. There are 5 cases:

$$f(x, y) = 2(18^2 + 7.1 + 1.7 + 1^2 + 1^2 + 1^2) = 15 \pmod{29};$$
$$f(x, y) = 2(18.7 + 7.18 + 1^2 + 1^2 + 1^2 + 1^2) = 19 \pmod{29};$$
$$f(x, y) = 2(18.7 + 7.1 + 1.18 + 1^2 + 1^2 + 1^2) = 18 \pmod{29};$$
$$f(x, y) = 2(18.1 + 7.7 + 1.18 + 1^2 + 1^2 + 1^2) = 2 \pmod{29};$$
$$f(x, y) = 2(18.1 + 7.1 + 1.18 + 1.7 + 1^2 + 1^2) = 17 \pmod{29}.$$

These are all nonsquares modulo 29, so $\mathcal{P}$ is a BLT-set. $\mathcal{P}$ admits S_6, which is a maximal subgroup of $PGO(5, 29)$, so the full stabiliser of $\mathcal{P}$ in $PGO(5, 29)$ is S_6.

The previously known BLT-sets in $Q(4, 29)$ were the classical BLT-set, with a group of order 1461 600, the Fisher-Thas/Walker/Kantor BLT-set, with a group of order 24 360, the Fisher BLT-set, with a group of order 1800, and the Penttila BLT-set, with a group of order 60. Hence $\mathcal{P}$ is a new BLT-set.

Since S_6 acts transitively on $\mathcal{P}$, $\mathcal{P}$ gives rise to just one flock of the quadratic cone of $PG(3, 29)$ and just one translation plane of order 29^2. It also gives a GQ of order $(29^2, 29)$ via the Knarr construction.

References

[1] L. Bader, N. Durante, M. Law, G. Lunardon, T. Penttila, Symmetries of BLT-sets, Des. Codes Cryptogr. (accepted).

[2] L. Bader, G. Lunardon, I. Pinneri, A new semifield flock, J. Combin. Theory Ser. A 86 (1) (1999) 49–62.

[3] L. Bader, G. Lunardon, J.A. Thas, Derivation of flocks of quadratic cones, Forum Math. 2 (2) (1990) 163–174.

[4] L. Bader, C.M. O'Keefe, T. Penttila, Some remarks on flocks, Australas J. Combin. (in press).

[5] R.D. Baker, J.M. Dover, G.L. Ebert, K.L. Wantz, Hyperbolic fibrations of PG(3, q), European J. Combin. 19 (1998) 1–16.

[6] R.D. Baker, G.L. Ebert, T. Penttila, Hyperbolic fibrations and q-clans, (submitted to J. Geom.).

[7] F. Buekenhout (Ed.), Handbook of Incidence Geometry, Buildings and Foundations, North-Holland, Amsterdam, Netherlands, 1995.

[8] J. Cannon, C. Playoust, An Introduction to MAGMA, University of Sydney Press, Sydney, Australia, 1993.

[9] F. De Clerck, H. Gevaert, J.A. Thas, Flocks of a quadratic cone in PG(3, q), $q \leq 8$, Geom. Dedicata 26 (1988) 215–230.

[10] F. De Clerck, C. Herssens, Flocks of the quadratic cone in PG(3, q), for q small, The CAGe Reports 8 (1992).

[11] F. De Clerck, T. Penttila, personal communication, 1997.

[12] Y. Hiramine, M. Matsumoto, T. Oyama, On some extension of 1-spread sets, Osaka J. Math. 24 (1) (1987) 123–137.

[13] N.L. Johnson, Sequences of derivable translation planes, Osaka J. Math. 25 (1988) 519–530.

[14] N.L. Johnson, G. Lunardon, F.W. Wilke, Semifield skeletons of conical flocks, J. Geom. 40 (1–2) (1991) 105–112.

[15] N.L. Johnson, S.E. Payne, Flocks of Laguerre planes and associated geometries, in: Mostly Finite Geometries (Iowa City, IA, 1996), Dekker, New York, USA, 1997, pp. 51–122.

[16] W.M. Kantor, Generalized quadrangles associated with $G_2(q)$, J. Combin. Theory Ser. A 29 (2) (1980) 212–219.

[17] W.M. Kantor, Generalized quadrangles, flocks, and BLT sets, J. Combin. Theory Ser. A 58 (1) (1991) 153–157.
[18] N. Knarr, A geometric construction of generalized quadrangles from polar spaces of rank three, Results Math. 21 (3–4) (1992) 332–344.
[19] M. Law, Flocks, generalised quadrangles and translation planes from BLT-sets, Ph.D. Thesis, The University of Western Australia, 2003.
[20] M. Law, T. Penttila, Some flocks in characteristic 3, J. Combin. Theory Ser. A 94 (2) (2001) 387–392.
[21] M. Law, T. Penttila, Classification of flocks of the quadratic cone over fields of order at most 29, Adv. Geom. (accepted).
[22] C.M. O'Keefe, T. Penttila, A new hyperoval in $PG(2, 32)$, J. Geom. 44 (1–2) (1992) 117–139.
[23] S.E. Payne, Generalized quadrangles as group coset geometries, Congr. Numer. 29 (1980) 717–734.
[24] S.E. Payne, Collineations of the generalized quadrangles associated with q-clans, Combinatorics' 90 (Gaeta, 1990) Ann. Discrete Math., vol. 52. North-Holland, Amsterdam, Netherlands, 1992, pp. 449–461.
[25] S.E. Payne, The fundamental theorem of q-clan geometry, Des. Codes Cryptogr. 8 (1–2) (1996) 181–202.
[26] S.E. Payne, L.A. Rogers, Local group actions on generalized quadrangles, Simon Stevin 64 (3–4) (1990) 249–284.
[27] S.E. Payne, J.A. Thas, Finite generalized quadrangles, in: Research Notes in Mathematics, vol. 110, Pitman, Boston, USA, 1984.
[28] T. Penttila, Regular cyclic BLT-sets, Rend. Circ. Mat. Palermo (2) Suppl. (53) (1998) 167–172.
[29] T. Penttila, G.F. Royle, On hyperovals in small projective planes, J. Geom. 54 (1–2) (1995) 91–104.
[30] T. Penttila, G.F. Royle, BLT-sets over small fields, Australas J. Combin. 17 (1998) 295–307.
[31] T. Penttila, B. Williams, Ovoids of parabolic spaces, Geom. Dedicata 82 (1–3) (2000) 1–19.
[32] J.A. Thas, Generalized quadrangles and flocks of cones, European J. Combin. 8 (4) (1987) 441–452.
[33] M. Walker, A class of translation planes, Geom. Dedicata 5 (2) (1976) 135–146.
[34] B. Williams, Ovoids of parabolic and hyperbolic spaces, Ph.D. Thesis, The University of Western Australia, 1998.