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Finite-dimensional non-associative algebras and codimension growth ☆

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ABSTRACT

Let A be a (non-necessarily associative) finite-dimensional algebra over a field of characteristic zero. A quantitative estimate of the polynomial identities satisfied by A is achieved through the study of the asymptotics of the sequence of codimensions of A. It is well known that for such an algebra this sequence is exponentially bounded.

Here we capture the exponential rate of growth of the sequence of codimensions for several classes of algebras including simple algebras with a special non-degenerate form, finite-dimensional Jordan or alternative algebras and many more. In all cases such rate of growth is integer and is explicitly related to the dimension of a subalgebra of A. One of the main tools of independent interest is the construction in the free non-associative algebra of multialternating polynomials satisfying special properties.

1. Introduction

The main purpose of this paper is to study the exponential rate of growth of the sequence of codimensions of a non-necessarely associative finite-dimensional algebra over a field of characteristic zero.

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Let $F\{X\}$ be the free non-associative algebra over a field F on a countable set X and let A be an F-algebra. A polynomial of $F\{X\}$ vanishing under every evaluation in A is called a polynomial identity of A and let $Id(A)$ denote the T-ideal of polynomial identities satisfied by A. If P_n is the space of multilinear polynomials in the indeterminates $x_1, \dots, x_n$, we denote by $c_n(A) = \dim \frac{P_n}{P_n \cap Id(A)}$, $n = 1, 2, \dots,$ the sequence of codimensions of A.

In general such sequence has overexponential growth and several methods have been developed in the years [2,8,15] in order to study its properties. So far the most significant results have been obtained when $c_n(A)$ is exponentially bounded, and in this setting a celebrated theorem of Regev [17] states that any associative algebra satisfying a non-trivial polynomial identity (PI-algebra) has sequence of codimensions exponentially bounded.

The class of non-associative algebras sharing such property is quite wide and includes the object of our study, that is, finite-dimensional algebras [1]. In case $c_n(A)$ is exponentially bounded, one can construct the bounded sequence $\sqrt[n]{c_n(A)}$, $n = 1, 2, \dots,$ and ask if $\exp(A) = \lim_{n\to\infty} \sqrt[n]{c_n(A)}$ exists.

In [7] it was proved that for any associative PI-algebra A the PI-exponent $\exp(A)$ exists and is an integer. In case of finite-dimensional Lie algebras, in [19] it was shown that the PI-exponent also exists and is an integer. The same conclusion was achieved in [9,10] for the special simple Jordan algebras.

These results about the integrality of the PI-exponent are quite surprising since in [5] the authors constructed, for any real number $\alpha > 1$, a non-associative algebra whose exponential rate of growth of the codimensions equals α.

In general in [3] it was proved that if A is a finite-dimensional algebra, $\dim A = d$, then either $c_n(A)$ is polynomially bounded or $c_n(A) > \frac{1}{n^2} 2^{\frac{n}{3d^2}}$ for n large enough. Moreover, given any real numbers $1 < \alpha < \beta < 2$ there exists a finite-dimensional algebra B such that $\alpha < \exp(B) < \beta$. Despite these results, the exponential rate of growth of the codimensions cannot be less than 2 for a wide class of algebras. In fact, if A is either an associative algebra [13] or a Lie algebra [14] or a finite-dimensional special Jordan algebra [9], then the asymptotic inequality $c_n(A) < 2^n$ implies $c_n(A) \leqslant f(n)$ for some polynomial f. The same phenomenon appears in case of two and three-dimensional non-associative algebras, but does not hold any more for five-dimensional algebras [4,6].

The purpose of this paper is twofold. First we prove that for a wide class of simple algebras A which includes noncommutative Jordan algebras, $\exp(A)$ exists and equals $\dim A$. On the other hand we determine finite-dimensional algebras, including Jordan and alternative algebras, for which $\exp(A)$ exists and is a non-negative integer.

Throughout F will be a field of characteristic zero. We shall often use other symbols like y, z, x_i^j for extra new indeterminates in $F\{X\}$. We shall use the left-normed notation on monomials i.e., $x_1 \cdots x_n = ((x_1 x_2) \cdots) x_n$.

2. Multialternating polynomials

Let A be an algebra over a field F and let $\mathrm{End}(A)$ be the algebra of endomorphisms of A. For $a \in A$ we denote by R_a and L_a the right and left multiplication by a, respectively. Then we define $\mathrm{M}(A)$ to be the subalgebra of $\mathrm{End}(A)$ generated by the right and left multiplications by elements of A. $\mathrm{M}(A)$ is the multiplication algebra of A.

We remark that if A is a finite-dimensional central simple algebra over F then $\mathrm{M}(A) = \mathrm{End}_F A$ [16]. In fact, considered as a module over the multiplication algebra $\mathrm{M}(A)$, A is an irreducible faithful module. Therefore, by the density theorem, $\mathrm{M}(A)$ is a dense subring of the ring of endomorphism of the module A considered as a vector space over its centralizer, which is F. In other words, $\mathrm{M}(A)$ is dense in $\mathrm{End}_F A$. Since A is finite-dimensional over F, $\mathrm{M}(A) = \mathrm{End}_F A$.

Given a finite-dimensional simple algebra A, $\dim A = d$, next we prove the existence of a multilinear polynomial $f = f(x_1, \dots, x_d, y_1, \dots, y_k)$ which is not an identity of A and is alternating on $x_1, \dots, x_d$.

Lemma 1. *Let A be a finite-dimensional simple algebra over an algebraically closed field of characteristic zero, $\dim A = d$. Then there exists a multilinear polynomial $f = f(x_1, \dots, x_d, y_1, \dots, y_m)$ such that f is alternating on $x_1, \dots, x_d$ and f is not an identity of A.*

Proof. Let $t \geqslant 1$ be the largest number of alternating indeterminates in a multilinear polynomial which is not an identity of A and suppose that $t < d$. Let $h = h(x_1, \dots, x_t, y_1, \dots, y_m)$ be such a polynomial. Hence h is not an identity of A and is alternating on $x_1, \dots, x_t$. Since h is multilinear, there exists a basis $e_1, \dots, e_d$ of A and indices $1 \leqslant i_1, \dots, i_m \leqslant d$ such that

$$b = h(e_1, \dots, e_t, e_{i_1}, \dots, e_{i_m}) \neq 0 \tag{1}$$

in A.

Now consider $\mathrm{M}(A)$, the multiplication algebra of A. Since A is simple, as we remarked above, $\mathrm{M}(A) = \mathrm{End}(A) \simeq M_d(F)$.

It is well known that for $M_d(F)$ there exists a central polynomial

$$C(x_1, \dots, x_{d^2}, y_1, \dots, y_{d^2})$$

alternating on $x_1, \dots, x_{d^2}$ and on $y_1, \dots, y_{d^2}$ which is not an identity of $M_d(F)$ (see [8, Theorem 5.7.4]). In particular for any two bases $\{\bar{x}_1, \dots, \bar{x}_{d^2}\}$ and $\{\bar{y}_1, \dots, \bar{y}_{d^2}\}$ of $M_d(F)$ we have

$$C(\bar{x}_1, \dots, \bar{x}_{d^2}, \bar{y}_1, \dots, \bar{y}_{d^2}) = \lambda E \tag{2}$$

where E is the unit matrix of $M_d(F)$ and $\lambda \in F$ is a non-zero scalar. Moreover if b is the element of A defined in (1), we may assume that $\bar{x}_1 = R_b$ or $\bar{x}_1 = L_b$. All other elements in $\{\bar{x}_2, \dots, \bar{x}_{d^2}, \bar{y}_1, \dots, \bar{y}_{d^2}\}$ will be products (of one or more factors) of left and right multiplications by $e_1, \dots, e_d$.

Let us say that $\bar{x}_1 = L_b$. Now, from (2) it follows that

$$C(\bar{x}_1, \dots, \bar{x}_{d^2}, \bar{y}_1, \dots, \bar{y}_{d^2})(e_{t+1}) = \lambda e_{t+1}. \tag{3}$$

On the other hand

$$C\big(\bar{x}'_1, \bar{x}_2, \dots, \bar{x}_{d^2}, \bar{y}_1, \dots, \bar{y}_{d^2}\big)(e_j) = \mu e_j \tag{4}$$

for any $1 \leqslant j \leqslant t$ and for any $\bar{x}'_1 = L_{b'}$, $b' \in A$ and for some $\mu = \mu(b', j) \in F$.

Since all $\bar{x}_i, \bar{y}_j$ are products of left and right multiplications by $e_1, \dots, e_d$, the left-hand side of (3) can be viewed as an evaluation φ in A of some non-associative polynomial

$$w = w\big(h(x_1, \dots, x_t, y_1, \dots, y_m), z_1, \dots z_n, x_0\big)$$

such that $\varphi(h) = b$ and $\varphi(x_0) = e_{t+1}$. Now we alternate the polynomial w on $x_1, \dots, x_t$ and x_0 and we get

$$\tilde{w} = \sum_{\sigma \in S_{t+1}} (-1)^\sigma w_\sigma,$$

where S_{t+1} is the symmetric group on $\{0, 1, \dots, t\}$ and

$$w_\sigma = w\big(h(x_{\sigma(1)}, \dots, x_{\sigma(t)}), y_1, \dots, y_m, z_1, \dots, z_n, x_{\sigma(0)}\big).$$

Consider the same evaluation φ. Clearly, if $\sigma(0)=0$ then $\varphi((-1)^\sigma w_\sigma)=\varphi(w)=D$ where D is the left-hand side of (3) since h is alternating on $x_1,\dots,x_t$. If $\sigma(0)=j>0$, then $\varphi(w_\sigma)=D'$, the left-hand side of (4), and $\varphi((-1)^\sigma w_\sigma)=\mu e_j$ with $j<t+1$. Hence, for suitable $\mu_1,\dots,\mu_t\in F$, we have $\varphi(\tilde{w})=t!\lambda e_{t+1}+\sum_{i=1}^t \mu_i e_i$, and $\varphi(\tilde{w})\neq 0$ since $\lambda\neq 0$. It follows that $\tilde{w}$ is a multilinear polynomial alternating on $t+1$ indeterminates and is not an identity of A. This contradiction completes the proof of our lemma. □

The following technical lemma will be of use.

Lemma 2. *Let $f=f(x_1,\dots,x_m,y_1,\dots,y_k)$ be a polynomial multilinear and alternating on $x_1,\dots,x_m$. Then, for any $\Psi\in \mathrm{M}(F\{X\})$, the polynomial*

$$g=\sum_{i=1}^m f\bigl(x_1,\dots,x_{i-1},\Psi(x_i),x_{i+1},\dots,x_m,y_1,\dots,y_k\bigr)$$

is also alternating on $x_1,\dots,x_m$.

Proof. Clearly it is enough to check that g vanishes when we identify any two variables $x_\alpha=x_\beta$ with $1\leqslant\alpha<\beta\leqslant m$. Suppose for instance that $\alpha=1$ and $\beta=2$. The polynomial

$$\sum_{i=3}^m f\bigl(x_1,\dots,\Psi(x_i),\dots,x_m,y_1,\dots,y_k\bigr)$$

is alternating on x_1 and x_2, hence

$$\begin{aligned} g(x_1,x_1,x_3,\dots) &= f\bigl(\Psi(x_1),x_1,\dots\bigr)+f\bigl(x_1,\Psi(x_1),\dots\bigr)\\ &= f\bigl(\Psi(x_1),x_1,\dots\bigr)-f\bigl(\Psi(x_1),x_1,\dots\bigr)=0, \end{aligned}$$

since $f(x,x,x_3,\dots,x_m)=0$. □

In order to simplify the notation, we shall often write $f=f(x_1,\dots,x_m,y_1,\dots,y_n)=f(x_1,\dots,x_m,Y)$ where $Y=\{y_1,\dots,y_n\}$.

Throughout we let $\alpha(x,y)\in \mathrm{M}(F\{X\})$ be a fixed linear combination of elements of the type $T_uT'_v, T_{uv}$, where $T,T'\in\{R,L\}$ and $\{u,v\}=\{x,y\}$. Moreover, in case A is a finite-dimensional algebra, we denote by $\langle x,y\rangle=\mathrm{tr}(\alpha(x,y))$ the bilinear form determined by α, where tr is the usual trace. The following lemma generalizes [10, Lemma 3].

Lemma 3. *Let A be a simple algebra, $\dim A=d$. Let $Y=Y_0\cup Y_1\cup\cdots\cup Y_r\subseteq X$ be a disjoint union with $r\geqslant 0$. Let $f=f(x_1,\dots,x_d,Y)$ be a polynomial multilinear and alternating on each Y_i, $1\leqslant i\leqslant r$, and on $x_1,\dots,x_d$. Then, for any $k\geqslant 1$ and for any $v_1,z_1,\dots,v_k,z_k\in X$, there exists a multilinear polynomial*

$$g=g(x_1,\dots,x_d,v_1,z_1,\dots,v_k,z_k,Y)$$

such that, for any evaluation $\varphi: X\to A$, $\varphi(x_i)=\bar{x}_i$, $1\leqslant i\leqslant d$, $\varphi(v_j)=\bar{v}_j$, $\varphi(z_j)=\bar{z}_j$, $1\leqslant j\leqslant k$, $\varphi(y)=\bar{y}$, for $y\in Y$, we have

$$\begin{aligned} \varphi(g) &= g(\bar{x}_1,\dots,\dots,\bar{x}_d,\bar{v}_1,\bar{z}_1,\dots,\bar{v}_k,\bar{z}_k,\bar{Y})\\ &= \langle\bar{v}_1,\bar{z}_1\rangle\cdots\langle\bar{v}_k,\bar{z}_k\rangle f(\bar{x}_1,\dots,\bar{x}_d,\bar{Y}). \end{aligned}$$

Moreover g is alternating on each set Y_i, $1\leqslant i\leqslant r$, and on $x_1,\dots,x_d$.

Proof. The proof is by induction of k. Suppose first that $k=1$ and define

$$g = g(x_1, \ldots, \ldots, x_d, v, z, Y) = \sum_{i=1}^{d} f\bigl(x_1, \ldots, \alpha(v,z)(x_i), \ldots, x_d, Y\bigr).$$

Then g is alternating on each set Y_i, $1 \leqslant i \leqslant r$ and, by Lemma 2, is also alternating on $x_1, \ldots, x_d$. Consider an evaluation $\varphi : X \to A$ such that $\varphi(x_i) = \bar{x}_i$, $1 \leqslant i \leqslant d$, $\varphi(v) = \bar{v}$, $\varphi(z) = \bar{z}$, $\varphi(y) = \bar{y}$, for $y \in Y$. Suppose first that the elements $\bar{x}_1, \ldots, \bar{x}_d$ are linearly dependent over F. Then, since g is alternating on $x_1, \ldots, x_d$, it follows that $\varphi(g) = 0$ and we are done.

Therefore we may assume that $\bar{x}_1, \ldots, \bar{x}_d$ are linearly independent over F and, so, since $\dim A = d$, they form a basis of A. Hence, for all $i = 1, \ldots, d$, we write

$$\alpha(\bar{v}, \bar{z})(\bar{x}_i) = \alpha_{ii}\bar{x}_i + \sum_{j \neq i} \alpha_{ij}\bar{x}_j,$$

for some scalars $\alpha_{ij} \in F$. Since f is alternating on $x_1, \ldots, x_d$,

$$f\bigl(\bar{x}_1, \ldots, \alpha(\bar{v}, \bar{z})(\bar{x}_i), \ldots, \bar{x}_d, \bar{Y}\bigr) = \alpha_{ii} f(\bar{x}_1, \ldots, \bar{x}_i, \ldots, \bar{x}_d, \bar{Y}).$$

Therefore

$$g(\bar{x}_1, \ldots, \bar{x}_d, \bar{v}, \bar{z}, \bar{Y}) = (\alpha_{11} + \cdots + \alpha_{dd}) f(\bar{x}_1, \ldots, \bar{x}_d, \bar{Y}),$$

and, since $\alpha_{11} + \cdots + \alpha_{dd} = \mathrm{tr}(\alpha(\bar{v}, \bar{z})) = \langle \bar{v}, \bar{z} \rangle$, the lemma is proved in case $k = 1$.

Now let $k > 1$ and let $g = g(x_1, \ldots, \ldots, x, v_1, z_1, \ldots, v_{k-1}, z_{k-1}, Y)$ be a multilinear polynomial satisfying the conclusion of the lemma. Then we write $g = g(x_1, \ldots, \ldots, x_d, Y')$ where $Y' = Y_0' \cup Y_1 \cup \cdots \cup Y_r$ and $Y_0' = Y_0 \cup \{v_1, z_1, \ldots, v_{k-1}, z_{k-1}\}$. If we now apply to g the same arguments as in the case $k = 1$, we obtain a polynomial satisfying the conclusion of the lemma. □

Theorem 1. *Let A be a finite-dimensional simple algebra,* $\dim A = d$. *Suppose that the form* $\langle x, y \rangle = \mathrm{tr}(\alpha(x, y))$ *is non-degenerate on A. Then, for any $k \geqslant 0$ there exists a multilinear polynomial*

$$g_k = g_k\bigl(x_1^{(1)}, \ldots, x_d^{(1)}, \ldots, x_1^{(2k+1)}, \ldots, x_d^{(2k+1)}, y_1, \ldots, y_N\bigr)$$

satisfying the following conditions:

1) *g_k is alternating on each set* $\{x_1^{(i)}, \ldots, x_d^{(i)}\}$, $1 \leqslant i \leqslant 2k+1$;
2) *g_k is not an identity of A*;
3) *the integer N does not depend on k.*

Proof. Let $f = f(x_1, \ldots, x_d, y_1, \ldots, y_m)$ be the multilinear polynomial constructed in Lemma 1. Hence f is alternating on $x_1, \ldots, x_d$ and does not vanish on A.

Suppose first that $k = 1$ and write $Y = \{y_1, \ldots, y_m\}$. By Lemma 3 there exists a multilinear polynomial

$$g = g\bigl(x_1, \ldots, x_d, v_1^{(1)}, z_1^{(1)}, \ldots, v_d^{(1)}, z_d^{(1)}, Y\bigr)$$

such that under any evaluation $\bar{\ }$ we have

$$g\big(\bar{x}_1,\ldots,\bar{x}_d,\bar{v}_1^{(1)},\bar{z}_1^{(1)},\ldots,\bar{v}_d^{(1)},\bar{z}_d^{(1)},\bar{Y}\big) = \big\langle\bar{v}_1^{(1)},\bar{z}_1^{(1)}\big\rangle\cdots\big\langle\bar{v}_d^{(1)},\bar{z}_d^{(1)}\big\rangle f(\bar{x}_1,\ldots,\bar{x}_d,\bar{Y}).$$

Now, for any $\sigma,\tau\in S_d$, define the polynomial

$$g_{\sigma,\tau}=g_{\sigma,\tau}\big(x_1,\ldots,x_d,v_1^{(1)},z_1^{(1)},\ldots,v_d^{(1)},z_d^{(1)},Y\big) = g\big(x_1,\ldots,x_d,v_{\sigma(1)}^{(1)},z_{\tau(1)}^{(1)},\ldots,v_{\sigma(d)}^{(1)},z_{\tau(d)}^{(1)},Y\big).$$

Then set

$$g_1\big(x_1,\ldots,x_d,v_1^{(1)},z_1^{(1)},\ldots,v_d^{(1)},z_d^{(1)},Y\big)=\frac{1}{d!}\sum_{\sigma,\tau\in S_d}(\operatorname{sgn}\sigma)(\operatorname{sgn}\tau)g_{\sigma,\tau}.$$

The polynomial g_1 is alternating on each of the sets $\{x_1,\ldots x_d\}$, $\{v_1^{(1)},\ldots,v_d^{(1)}\}$ and $\{z_1^{(1)},\ldots,z_d^{(1)}\}$. Next we show that for any evaluation φ,

$$\varphi(g_1)=\det\bar{\Delta}_1\varphi(f),$$

where

$$\bar{\Delta}_1=\begin{pmatrix}\langle\bar{v}_1^{(1)},\bar{z}_1^{(1)}\rangle & \cdots & \langle\bar{v}_1^{(1)},\bar{z}_d^{(1)}\rangle\\ \vdots & & \vdots\\ \langle\bar{v}_d^{(1)},\bar{z}_1^{(1)}\rangle & \cdots & \langle\bar{v}_d^{(1)},\bar{z}_d^{(1)}\rangle\end{pmatrix}.$$

Now, by Lemma 3, for any evaluation $\varphi: X\to A$ we have

$$\varphi(g_1)=\gamma\varphi(f),$$

where

$$\gamma=\frac{1}{d!}\sum_{\sigma,\tau\in S_d}(\operatorname{sgn}\sigma)(\operatorname{sgn}\tau)\big\langle\bar{v}_{\sigma(1)}^{(1)},\bar{z}_{\tau(1)}^{(1)}\big\rangle\cdots\big\langle\bar{v}_{\sigma(d)}^{(1)},\bar{z}_{\tau(d)}^{(1)}\big\rangle.$$

We fix $\sigma\in S_m$ and compute the sum

$$\gamma_\sigma=\sum_{\tau\in S_d}(\operatorname{sgn}\tau)\big\langle\bar{v}_{\sigma(1)}^{(1)},\bar{z}_{\tau(1)}^{(1)}\big\rangle\cdots\big\langle\bar{v}_{\sigma(d)}^{(1)},\bar{z}_{\tau(d)}^{(1)}\big\rangle.$$

Write simply $\bar{v}_{\sigma(i)}^{(1)}=a_i$, $\bar{z}_i^{(1)}=b_i$, $i=1,\ldots,d$. Then

$$\gamma_\sigma=\sum_{\tau\in S_d}(\operatorname{sgn}\tau)\langle a_1,b_{\tau(1)}\rangle\cdots\langle a_d,b_{\tau(d)}\rangle=\det\begin{pmatrix}\langle a_1,b_1\rangle & \cdots & \langle a_1,b_d\rangle\\ \vdots & & \vdots\\ \langle a_d,b_1\rangle & \cdots & \langle a_d,b_d\rangle\end{pmatrix} = (\operatorname{sgn}\sigma)\det\begin{pmatrix}\langle a_{\sigma^{-1}(1)},b_1\rangle & \cdots & \langle a_{\sigma^{-1}(1)},b_d\rangle\\ \vdots & & \vdots\\ \langle a_{\sigma^{-1}(d)},b_1\rangle & \cdots & \langle a_{\sigma^{-1}(d)},b_d\rangle\end{pmatrix}=(\operatorname{sgn}\sigma)\det\bar{\Delta}_1.$$

Hence

$$\gamma = \frac{1}{d!} \sum_{\sigma \in S_d} (\operatorname{sgn} \sigma) \gamma_\sigma = \det \bar{\Delta}_1$$

and $\varphi(g_1) = \det \bar{\Delta}_1 \varphi(f)$. Thus, since $\langle -,- \rangle$ is a non-degenerate form, g_1 does not vanish in A. This completes the proof in case $k = 1$.

If $k > 1$, by the inductive hypothesis there exists a multilinear polynomial

$$g_{k-1}\big(x_1, \dots, x_d, v_1^{(1)}, z_1^{(1)}, \dots, v_d^{(1)}, z_d^{(1)}, \dots, v_1^{(k-1)}, z_1^{(k-1)}, \dots, v_d^{(k-1)}, z_d^{(k-1)}, Y\big)$$

satisfying the conclusion of the theorem. Then we write

$$g_{k-1} = g_{k-1}\big(x_1, \dots, \dots, x_d, Y'\big)$$

where $Y' = Y \cup \{v_1^{(1)}, z_1^{(1)}, \dots, v_d^{(1)}, z_d^{(1)}, \dots, v_1^{(k-1)}, z_1^{(k-1)}, \dots, v_d^{(k-1)}, z_d^{(k-1)}\}$ and we apply to g_{k-1} Lemma 3 and the previous arguments. In this way we can construct the polynomial g_k and, for any evaluation φ, we have

$$\varphi(g_k) = \det \bar{\Delta}_k \varphi(g_{k-1}) = \det \bar{\Delta}_1 \cdots \det \bar{\Delta}_k \varphi(f).$$

This completes the proof of the theorem. □

3. Simple algebras and growth of the identities

In this section we restrict our attention to the multilinear identities of a finite-dimensional simple algebra A. Let $Id(A) = \{f \in F\{X\} \mid f \equiv 0 \text{ in } A\}$ be the T-ideal of polynomial identities of A and, for any $n \geqslant 1$, define $P_n \subseteq F\{X\}$ to be the space of multilinear polynomials in the variables $x_1, \dots, x_n$. Then $c_n(A) = \dim \frac{P_n}{P_n \cap Id(A)}$ is the nth codimension of A and our aim is to study the sequence $c_n(A)$, $n = 1, 2, \dots$. Since A is finite-dimensional such sequence is exponentially bounded (see [1]), and here we want to capture its exponential rate of growth by proving that $\exp(A) = \lim_{n\to\infty} \sqrt[n]{c_n(A)}$ exists and equals $\dim A = d$, for some classes of simple algebras.

It is well known that the symmetric group S_n acts on P_n: if $\sigma \in S_n$ and $f(x_1, \dots, x_n) \in P_n$, then $\sigma f(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$ (see [8, Chapter 2]). Then it is easily seen that $\frac{P_n}{P_n \cap Id(A)}$ becomes an S_n-module and we consider its decomposition into irreducible submodules. We refer the reader to [12] for a description of the representation theory of S_n.

Here we recall how to construct an irreducible S_n-module. Let $\lambda \vdash n$ be a partition of n. Given a Young tableau T_λ of shape $\lambda \vdash n$, let R_{T_λ} and C_{T_λ} denote the subgroups of S_n stabilizing the rows and the columns of T_λ, respectively. Then set $\bar{R}_{T_\lambda} = \sum_{\sigma \in R_{T_\lambda}} \sigma$ and $\bar{C}_{T_\lambda} = \sum_{\tau \in C_{T_\lambda}} (\operatorname{sgn} \tau)\tau$. It follows that the element $e_{T_\lambda} = \bar{R}_{T_\lambda} \bar{C}_{T_\lambda}$ is an essential idempotent of the group algebra FS_n, generating an irreducible S_n-module corresponding to λ.

In the next theorem we prove the existence of the exponent $\exp(A)$ for some finite-dimensional algebras.

Theorem 2. *Let A be a finite-dimensional simple algebra over an algebraically closed field F of characteristic zero and suppose that for some α, the form $\langle x, y \rangle = \operatorname{tr}(\alpha(x, y))$ is non-degenerate on A. Then, for all $n \geqslant 1$, there exist constants $C > 0$, t such that*

$$Cn^t d^n \leqslant c_n(A) \leqslant d^{n+1}.$$

Hence the exponent $\exp(A) = \lim_{n\to\infty} \sqrt[n]{c_n(A)}$ *exists and* $\exp(A) = \dim A = d$.

Proof. By Theorem 1, for all $k \geqslant 1$ there exists a multilinear polynomial

$$g_k = g_k\big(x_1^{(1)}, \dots, x_d^{(1)}, \dots, x_1^{(2k+1)}, \dots, x_d^{(2k+1)}, y_1, \dots, y_N\big)$$

such that g_k is alternating on each set of indeterminates $\{x_1^{(i)}, \dots, x_d^{(i)}\}$, $1 \leqslant i \leqslant 2k+1$, and g_k is not a polynomial identity of A. Rename the variables and write

$$g_k = h(x_1, \dots, x_{d(2k+1)}, Y),$$

where $Y = \{y_1, \dots, y_N\}$.

Fix k and let $n = d(2k+1)$. Let P_{n+N} be the space of multilinear polynomials in $x_1, \dots, x_n$, $y_1, \dots, y_N$. If we let S_n act on $x_1, \dots, x_n$, P_{n+N} is an S_n-module and let FS_nh be the S_n-submodule generated by h. Since $h \notin Id(A)$, there exists a partition $\lambda = (\lambda_1, \dots, \lambda_m) \vdash n$ and a Young tableau T_λ such that $FS_ne_{T_\lambda}h \nsubseteq Id(A)$. Our next goal is to show that $\lambda = ((2k+1)^d)$ is a rectangle of width $2k+1$ and height d.

If $\lambda_1 \geqslant 2k+2$, then $e_{T_\lambda}h$ is a polynomial symmetric on at least $2k+2$ variables among $x_1, \dots, x_n$. But for any $\sigma \in \bar{R}_{T_\lambda}$ these variables in $\sigma\bar{C}_{T_\lambda}h$ are divided into $2k+1$ disjoint alternating subsets. It follows that $\sigma\bar{C}_{T_\lambda}h$ is alternating and symmetric on at least two variables and, so, $e_{T_\lambda}h = 0$ in the zero polynomial, a contradiction. Thus $\lambda_1 \leqslant 2k+1$.

Suppose now that $m \geqslant d+1$. Since the first column of T_λ is of height at least $d+1$, the polynomial $\bar{C}_{T_\lambda}h$ is alternating on at least $d+1$ variables among $x_1, \dots, x_n$. Since $\dim A = d$ we get that for any σ, $\sigma\bar{C}_{T_\lambda}h \equiv 0$ on A and, so, also $e_{T_\lambda}h = \bar{R}_{T_\lambda}\bar{C}_{T_\lambda}h \equiv 0$ on A, a contradiction.

We have proved that $FS_ne_{T_\lambda}h \nsubseteq Id(A)$, for some Young tableau T_λ of shape $\lambda = ((2k+1)^d)$.

Let now $n \geqslant d+N$ be an arbitrary integer, and write $n = d(2k+1)+N+r$, for some $k \geqslant 0$ and $0 \leqslant r < 2d$. Let $g_k = h(x_1, \dots, x_{d(2k+1)}, Y)$ be the above polynomial. If $r = 0$, set $h' = h$. If $r > 0$, let a be a non-zero value of h and consider all multilinear monomials $m(a, a_1, \dots, a_r)$, where $a_1, \dots, a_r \in A$. Since there exists at least one of them, say m, which is not zero on A, we define

$$h' = m(h, x_{d(2k+1)+1}, \dots, x_{d(2k+1)+r}).$$

Then $h' \in P_n$ and, if T_λ is the Young tableau of shape $\lambda = ((2k+1)^d)$ given above such that $e_{T_\lambda}h \notin Id(A)$, we also have that $e_{T_\lambda}h' \notin Id(A)$.

Decompose $FS_n = \bigoplus_{\mu\vdash n} I_\mu$ into the sum of minimal two-sided ideals I_μ and let $d_\mu = \sqrt{\dim I_\mu}$ be the dimension of an irreducible S_n-module corresponding to μ. By the branching theorem of $S_{d(2k+1)}$ (see [12, Theorem 2.4.3]) we have that

$$FS_ne_{T_\lambda}h' \subseteq \bigoplus_{\substack{\mu\supseteq\lambda \\ \mu\vdash n}} I_\mu h',$$

and, since $e_{T_\lambda}h' \notin Id(A)$, there exists a partition $\mu \vdash n$, $\mu \supseteq \lambda$, and a tableau T_μ such that $FS_ne_{T_\mu}h' \nsubseteq Id(A)$. This says that $c_n(A) \geqslant \dim FS_ne_{T_\mu}$. For any $\lambda \vdash n$ let us write $\dim FS_ne_{T_\lambda} = d_\lambda$. Then $c_n(A) \geqslant d_\mu$. But again by the branching rule, $d_\mu \geqslant d_{((2k+1)^d)}$. Since $n - d(2k+1) \leqslant N + 2d$ and asymptotically $d_{((2k+1)^d)} \simeq C_0m^sd^m$, where $m = d(2k+1)$, for some constants C_0, s (see [8, Lemma 6.2.5]), we obtain that

$$c_n(A) \geqslant Cn^td^n,$$

for some constants $C > 0$, t. We have found a lower bound for $c_n(A)$.

For the upper bound, recall that by [10, Proposition 2] for any finite-dimensional algebra A, $\dim A = d$, the nth codimension $c_n(A)$ is bounded by d^{n+1}. Hence we obtain $Cn^td^n \leqslant c_n(A) \leqslant d^{n+1}$. It follows that $\exp(A) = \lim_{n\to\infty} \sqrt[n]{c_n(A)} = d$ and we are done. □

Next we apply the above theorem to Jordan algebras.

Corollary 1. *If A is a finite-dimensional simple unitary noncommutative Jordan algebra over an algebraically closed field of characteristic zero, then* $\exp(A)$ *exists and equals* $\dim A$.

Proof. If A is a finite-dimensional semisimple unitary noncommutative Jordan algebra, then by [18, pp. 141–142], $\alpha(x, y) = \mathrm{tr}(R_{xy+yx} + L_{xy+yx})$ is a non-degenerate bilinear form, and the conclusion follows from Theorem 2. □

We remark that in particular the above theorem is true for commutative Jordan algebras, for quasi-associative algebras, for flexible quadratic algebras which include octonions (see [18]).

4. An upper bound for the PI-exponent

Throughout this section we shall assume that A is a finite-dimensional algebra over a field F of characteristic zero with a Wedderburn–Malcev type decomposition. That is, there exist simple unitary subalgebras $C_1, \dots, C_m$ of A such that

$$A = C_1 \oplus \cdots \oplus C_m + R, \tag{5}$$

where $R = \mathrm{Rad}\, A$ is the radical of A. We shall also assume that R is a strongly nilpotent ideal i.e., there exists an integer $T \geqslant 1$ such that any product of elements of A containing at least T elements of R must be zero.

We fix a basis $B = B_0 \cup B_1$ of A such that B_0 is a basis of R and B_1 is the union of bases of $C_1, \dots, C_m$, respectively. In what follows any product of elements of B will be called a monomial of A. Next we define the height of a monomial as follows.

Let $M = M(a_1, \dots, a_k, b_1, \dots, b_n)$ be a non-zero monomial of A where $a_1, \dots, a_k \in B_1$ and $b_1, \dots, b_n \in B_0$. Then the height of M is

$$ht(M) = \dim(C_{i_1} + \cdots + C_{i_k})$$

where $a_1 \in C_{i_1}, \dots, a_k \in C_{i_k}$. Since A is a finite-dimensional algebra we can define the integer

$$d = d(A) = \max\bigl\{ht(M) \bigm| 0 \neq M \in A\bigr\}. \tag{6}$$

We shall prove that under suitable hypotheses, the PI-exponent of A equals the integer d defined in (6). We start with the following

Lemma 4. *Let d be the integer defined in* (6)*. Then there exists an integer T such that any multilinear polynomial*

$$f = f\bigl(x_1^1, \dots, x_{d+1}^1, \dots, x_1^T, \dots, x_{d+1}^T, y_1, y_2, \dots\bigr)$$

alternating on each set $\{x_1^i, \dots, x_{d+1}^i\}$, $1 \leqslant i \leqslant T$, is an identity of A.

Proof. Let T be the smallest integer such that any product of elements of A containing at least T elements of R is equal to zero.

Denote by Alt_i the operator of alternation on the set $\{x_1^i, \dots, x_{d+1}^i\}$, $i = 1, \dots, T$. We claim that

$$Alt_1 \cdots Alt_T(m) \equiv 0 \tag{7}$$

for any multilinear monomial $m = m(x_1^1, \dots, x_{d+1}^1, \dots, x_1^T, \dots, x_{d+1}^T, y_1, y_2, \dots)$.

In fact, since m is multilinear, it is enough to check that

$$Alt_1 \cdots Alt_T\big(m\big(b_1, \ldots, b_{T(d+1)}, b'_1, b'_2, \ldots\big)\big) = 0, \tag{8}$$

for any $b_i, b'_i \in B$. First suppose that $b_1, \ldots, b_{d+1} \in B_1$ and let $b_1 \in C_{i_1}, \ldots, b_{d+1} \in C_{i_{d+1}}$. If $\dim(C_{i_1} + \cdots + C_{i_{d+1}}) > d$ then $m(b_1, \ldots, b_{T(d+1)}, b'_1, b'_2, \ldots) = 0$ by the definition of d. In case $\dim(C_{i_1} + \cdots + C_{i_{d+1}}) \leqslant d$, then $b_1, \ldots, b_{d+1}$ are linearly dependent over F and since the polynomial in (7) is alternating in the corresponding variables, we get that (8) still holds.

Note that the discussion of the previous paragraph applies to the sets

$$\{b_{i(d+1)+1}, \ldots, b_{(i+1)(d+1)}\}, \quad 1 \leqslant i \leqslant T-1. \tag{9}$$

Hence we may assume that every set in (9) contains at least one $b_j \in B_0$. But in this case (8) holds since each monomial on the left-hand side of (8) contains at least T elements of R and by hypothesis such a product is zero.

Since any polynomial f alternating on T disjoint sets of size $d+1$ is a linear combination of polynomials of the type $Alt_1 \cdots Alt_T(m)$ the proof is complete. □

Lemma 5. *Let A be a finite-dimensional algebra with a Wedderburn–Malcev decomposition and strongly nilpotent radical. If $d = d(A)$ is the integer defined in* (6)*, then there exist constants C, k such that*

$$c_n(A) \leqslant Cn^k d^n,$$

for all $n \geqslant 1$.

Proof. As we mentioned at the beginning of Section 3, $P_n/(P_n \cap Id(A))$ is a left S_n-module and we let $\chi_n(A)$ be its character, called the nth cocharacter of A. Clearly $\deg \chi_n(A) = c_n(A)$. By complete reducibility we decompose $\chi_n(A)$ into irreducible S_n-characters

$$\chi_n(A) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda, \tag{10}$$

where χ_λ is the irreducible S_n-character associated to the partition λ and $m_\lambda \geqslant 0$ is the corresponding multiplicity. Clearly it is enough to prove the lemma for $n \geqslant T(d+1)$.

Recall that, given a partition $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$ and a Young tableau T_λ, then $e_{T_\lambda} = \bar{R}_{T_\lambda} \bar{C}_{T_\lambda}$ is a minimal essential idempotent of FS_n. Moreover, for any multilinear polynomial $f(x_1, \ldots, x_n)$, the polynomial $\bar{C}_{T_\lambda} f(x_1, \ldots, x_n)$ is alternating on at least k indeterminates. It follows that $m_\lambda = 0$ in (10) as soon as $k > D = \dim A$. This says that the cocharacter $\chi_n(A)$ lies in a strip of height D.

Consider the rectangular partition $\mu = (T, T, \ldots, T) = (T^{d+1}) \vdash T(d+1)$ where T is the integer determined in Lemma 4. Then by the above property and Lemma 4, $m_\lambda = 0$ in (10) for any $\lambda \geqslant \mu$, i.e., $\lambda_i \geqslant \mu_i$, for all i.

Hence if $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$ is such that $m_\lambda \neq 0$ we must have $\lambda_{d+1} \leqslant T$. Let us define $n' = \lambda_1 + \cdots + \lambda_d$ and $\lambda^\circ = (\lambda_1, \ldots, \lambda_d) \vdash n'$. Then, recalling that $k \leqslant D$, we must have $n - n' \leqslant (T-1)(D-d)$. But then, by [8, Lemma 6.2.4], we have that

$$d_\lambda = \deg \chi_\lambda \leqslant n^{(T-1)(D-d)} d_{\lambda^\circ},$$

and since by [8, Lemma 6.2.5],

$$d_{\lambda^\circ} \leqslant C'\big(n'\big)^r d^{n'},$$

for some constants C', r, we get the following conclusion: if $m_\lambda \neq 0$ in (10) we must have

$$d_\lambda \leqslant C'' n^{r'} d^n$$

for some constants C'', r'.

On the other hand by [3, Theorem 1],

$$\sum_{\lambda \vdash n} m_\lambda \leqslant D(n+1)^{D^2+D}.$$

Therefore, by computing degrees in (10), from the above two upper bounds we get the desired conclusion of the lemma. □

5. A lower bound for the PI-exponent

As in the previous section, here we shall assume that A is a finite-dimensional algebra with a Wedderburn–Malcev decomposition and strongly nilpotent radical. We shall also assume that in the decomposition $A = C_1 \oplus \cdots \oplus C_m + R$, all simple algebras are unitary and have a non-degenerate bilinear form $\langle x, y\rangle = \mathrm{tr}(\alpha(x, y))$, as in Theorem 1 and F is algebraically closed. Recall that B_0 is a basis of R and B_1 is the union of bases of $C_1, \ldots, C_m$, respectively.

The following remark is obvious.

Lemma 6. *Let $d = d(A)$ be as in (6). Then there exist $t \geqslant 0$ and a monomial $M = M(x_1, \ldots, x_{k+t+l})$ such that*

$$M(a_1, \ldots, a_{k+l}, b_1, \ldots, b_t) \neq 0$$

for some $a_1, \ldots, a_{k+l} \in B_1$, $b_1, \ldots, b_t \in B_0$, $l \geqslant 0$, where $a_1, \ldots, a_k$ belong to distinct simple components $C_{i_1}, \ldots, C_{i_k}$, respectively, and $\dim(C_{i_1} \oplus \cdots \oplus C_{i_k}) = d$.

If $a_1, \ldots, a_n \in A$ are elements of an algebra A, we denote by $p(a_1, \ldots, a_n)$ the set of all products $a_{i_1} \cdots a_{i_n}$ where $i_1, \ldots, i_n$ is a permutation of $1, \ldots, n$, with all possible arrangements of brackets.

Lemma 7. *Let A be a finite-dimensional simple algebra with 1, $\dim A = t$, and let $a \in A$ be non-zero. Then*

$$A = \mathrm{span}\bigl\{p(a, a_1, \ldots, a_{t-1}) \bigm| a_1, \ldots, a_{t-1} \in A\bigr\}. \tag{11}$$

Proof. For $j \geqslant 1$ let

$$A_j = \mathrm{span}\bigl\{p(a, a_1, \ldots, a_i) \bigm| a_1, \ldots, a_i \in A,\ 1 \leqslant i \leqslant j\bigr\}.$$

Then

$$\mathrm{span}\{a\} = A_0 \subseteq A_1 \subseteq \cdots. \tag{12}$$

Since A is finite-dimensional, the chain (12) stabilizes, say $A_j = A_{j+1}$. Then A_j is an ideal containing a and, by the simplicity of A, $A_j = A$. If j is minimal such that $A = A_j$ then

$$\dim A_0 < \dim A_1 < \cdots < \dim A_j$$

and $j \leqslant t-1$. In particular, $A_{t-1} = A_j = A$. Finally, since A is an algebra with 1, A_{t-1} coincides with the right-hand side of (11). □

In the next lemma we shall construct multialternating polynomials of arbitrarily large degree for finite-dimensional algebras satisfying our hypotheses. We shall do so by "gluing" the polynomials constructed in Theorem 1.

Recall that $A/R = C_1 \oplus \cdots \oplus C_m$ is a sum of simple algebras where R is the radical of A and $B = B_0 \cup B_1$ is a basis of A such that $B_1 \subseteq C_1 \oplus \cdots \oplus C_m$, $B_0 \subseteq R$.

Lemma 8. *If $d = d(A)$ is defined as in (6) then, for any $k \geqslant 1$, there exists a multilinear polynomial*

$$f = f\left(x_1^1, \ldots, x_d^1, \ldots, x_1^{2k}, \ldots, x_d^{2k}, y_1, \ldots, y_N\right)$$

alternating on each set $\{x_1^i, \ldots, x_d^i\}$, $1 \leqslant i \leqslant 2k$, and f is not an identity of A. Moreover N does not depend on k.

Proof. Let $M(a_1, \ldots, a_{r+l}, b_1, \ldots, b_t) \neq 0$ be a monomial with $a_1 \in C_1, \ldots, a_r \in C_r$ and $\dim(C_1 \oplus \cdots \oplus C_r) = d$, as in Lemma 6. We rename the elements of M and we write

$$M(a_1, \ldots, a_r, b_1, \ldots, b_t)$$

where $b_1, \ldots, b_t \in B_0 \cup B_1$. Denote $p_0 = 0$, $p_i = p_{i-1} + d_i$, $i = 1, \ldots, r-1$ where $d_i = \dim C_i$. By Theorem 1, for any C_i there exists a multialternating polynomial

$$h_i = h_i\left(x^{(1)}_{p_{i-1}+1}, \ldots, x^{(1)}_{p_{i-1}+d_i}, \ldots, x^{(2k)}_{p_{i-1}+1}, \ldots, x^{(2k)}_{p_{i-1}+d_i}, y_1^i, \ldots, y_N^i\right)$$

which is not an identity of C_i. Let φ_i be an evaluation of h_i in C_i such that $\varphi_i(h_i) \neq 0$.

Then, according to Lemma 7, we can write a_i as a monomial $a_i = w_i(\varphi_i(h_i), e_1^i, \ldots, e_{d_i-1}^i)$, for suitable $e_1^i, \ldots, e_{d_i-1}^i \in C_i$. Let

$$g_i = w_i\left(h_i, z_1^i, \ldots, z_{d_i-1}^i\right) \tag{13}$$

be a polynomial of the free algebra such that

$$\bar{g}_i = w_i\left(\varphi_i(h_i), e_1^i, \ldots, e_{d_i-1}^i\right) = a_i.$$

Then define

$$f = Alt_1 \cdots Alt_{2k} M\left(g_1 z_d^1, \ldots, g_r z_d^r, u_1, \ldots, u_t\right)$$

where $z_d^1, \ldots, z_d^r, u_1, \ldots, u_t$ are further distinct variables and Alt_j denotes alternation on the set $\{x_1^{(j)}, \ldots, x_d^{(j)}\}$. We will prove that f is not an identity of A for any $k \geqslant 1$.

For every i, $1 \leqslant i \leqslant r$, consider the valuation above $\bar{g}_i = w_i(\varphi_i(h_i), e_1^i, \ldots, e_{d_i-1}^i) = a_i$, $u_1 = b_1, \ldots, u_t = b_t$, and set $\bar{z}_d^i = e_0^i$, where e_0^i is the unit of C_i. Then

$$W = M\left(\bar{g}_1 e_0^1, \ldots, \bar{g}_r e_0^r, b_1, \ldots, b_t\right) = M(a_1, \ldots, a_r, b_1, \ldots, b_t) \neq 0.$$

Now we recall that each h_i is alternating on $\{x^{(j)}_{p_{i-1}+1}, \ldots, x^{(j)}_{p_{i-1}+d_i}\}$, that any product $C_i C_j$ is zero as soon as $i \neq j$ and that each $\bar{z}_d^i$ equals e_0^i, the unit of C_i. These facts imply that under the above evaluation, the polynomial f evaluates into

$$\varphi(f) = (d_1!)^{2k} \cdots (d_r!)^{2k} \cdot M(a_1, \dots, a_r, b_1, \dots, b_t) \neq 0.$$

Thus f is the desired polynomial and the proof is complete. □

We remark that if we replace any of the polynomials g_i in (13) with $g_i y_{N+1}$ then we obtain another multilinear non-identical polynomial f depending on $2kd$ alternating variables and on $y_1, \dots, y_{N+1}$. A repeated application of this argument gives the following.

Remark 1. Under the hypotheses of Lemma 8, for any $N' \geqslant N$ there exists a multilinear polynomial f' satisfying the same properties as f, depending on d $2k$-element alternating sets and on $y_1, \dots, y_{N'}$.

We are now able to find a lower bound for $c_n(A)$. In fact we have the following.

Lemma 9. *Let A be a finite-dimensional algebra over an algebraically closed field F of characteristic zero and let $d = d(A)$ be defined as in* (6)*. Suppose that A has a Wedderburn–Malcev decomposition $A = C_1 \oplus \cdots \oplus C_m + R$ with strongly nilpotent radical R and each simple algebra C_i has a non-degenerate bilinear form $\langle x, y\rangle = \mathrm{tr}(\alpha(x, y))$ as in Theorem* 1*. If $d = d(A)$ is defined as in* (6)*, there exist constants $C > 0$, q such that*

$$c_n(A) \geqslant Cn^q d^n$$

for all $n \geqslant 1$.

Proof. Let $n \geqslant 2d + N$ be an integer where N is as in Lemma 8 and write $n = 2kd + N'$, for some $k \geqslant 1$ where $N' < N + 2d$. Fix a polynomial f as in Remark 1 and regard P_n as the space of multilinear polynomials in the variables appearing in f. We consider P_n as an S_{2kd}-module by letting S_{2kd} act on the $2kd$ variables of the sets $\{x^i_1, \dots, x^i_d\}$, $1 \leqslant i \leqslant 2k$, on which the polynomial f is alternating.

Since $f \notin P_n \cap Id(A)$, there exists a partition $\lambda = (\lambda_1, \dots, \lambda_p) \vdash 2kd$ and a tableau T_λ such that the polynomial $e_{T_\lambda} f$ does not lie in $P_n \cap Id(A)$.

We now apply to $e_{T_\lambda} f$ the same argument as in the proof of Theorem 2: if $\lambda_1 \geqslant 2k + 1$ then $e_{T_\lambda} f = 0$ since f contains $2k$ alternating sets of variables and $e_{T_\lambda} f$ is symmetric on a set of order λ_1. On the other hand $e_{T_\lambda} f \in Id(A)$ as soon as $p > D = \dim A$. Thus $\lambda_1 \leqslant 2k$ and $p \leqslant D$.

Let T be the smallest integer such that any product of elements of A containing at least T elements of R is equal to zero. If $\lambda_{d+1} \geqslant T$, then $e_{T_\lambda} f$ is a linear combination of polynomials each alternating on at least T disjoint sets of variables. By Lemma 4 this implies that $e_{T_\lambda} f \in Id(A)$, a contradiction. Hence $\lambda_{d+1} < T$.

It follows that $2k \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_{d-1}$ and $\lambda_{d+1} + \cdots + \lambda_p < DT$ and from this we get

$$\begin{aligned} \lambda_d &= 2kd - (\lambda_1 + \cdots + \lambda_{d-1}) - (\lambda_{d+1} + \cdots + \lambda_p) \\ &> 2kd - 2k(d-1) - DT = 2k - DT. \end{aligned}$$

If we set $2k - DT = k_0$, then $\lambda_d > k_0$ says that λ contains a rectangular subdiagram $\mu = (k_0^d) \vdash n_0 = k_0 d$. In particular,

$$d_\lambda = \deg \chi_\lambda \geqslant d_\mu \simeq a n_0^b d^{n_0}$$

(see [8, Lemma 6.2.5]). Since $n - n_0 = N' + DTd$ is constant, one can find $C > 0$ and k such that

$$c_n(A) \geqslant d_\lambda \geqslant Cn^q d^n. \qquad \square$$

Next we want to apply the above results to Jordan algebras. In order to do so we need to prove the following.

Lemma 10. *The radical* Rad J *of a finite-dimensional Jordan algebra* J *is strongly nilpotent.*

Proof. Let $\mathrm{M}(J)$ be the multiplication algebra of J and let $R : J \to \mathrm{M}(J)$, $a \mapsto R_a$, be the canonical mapping of J into $\mathrm{M}(J)$. It is well known (see [11]) that for any $a \in \mathrm{Rad}\, J$ the image R_a lies in $\mathrm{Rad}\,\mathrm{M}(J)$. Let n be the degree of nilpotency of $\mathrm{Rad}\,\mathrm{M}(J)$. Then in $\mathrm{M}(J)$ every associative word containing at least n elements of $\mathrm{Rad}\,\mathrm{M}(J)$, in particular of $R(\mathrm{Rad}\, J)$, equals 0.

We claim that Rad J is strongly nilpotent of degree $N = 2n+1$. In fact, let w be a non-associative word in J containing at least $2n+1$ elements of Rad J. Considering J as an $\mathrm{M}(J)$-module, we can write $w = W \cdot a$, where $a \in J$, $W \in \mathrm{M}(J)$ and W contains at least $2n$ elements from Rad J. But then the equality in $\mathrm{M}(J)$

$$R_{a_1\circ(a_3\circ a_2)} = -R_{a_1}R_{a_2}R_{a_3} - R_{a_3}R_{a_2}R_{a_1} + R_{a_1}R_{a_2\circ a_3} + R_{a_2}R_{a_1\circ a_3} + R_{a_3}R_{a_1\circ a_2},$$
$$a_1, a_2, a_3 \in J,$$

shows that $W \in (\mathrm{Rad}\,\mathrm{M}(J))^n = 0$, and this proves the claim. □

We remark that the statement of Lemma 10 is evidently true for alternative algebras. In fact, in an alternative algebra A, for every ideal I of A and for every natural number n the power I^n is again an ideal of A. Therefore, every nilpotent ideal of A is strongly nilpotent.

At the light of the above, from Lemmas 5 and 9 we now immediately get the following.

Theorem 3. *Let A be a finite-dimensional alternative or Jordan algebra over a field of characteristic zero. Then* $\exp(A)$ *exists and is a non-negative integer. Moreover if A is nilpotent, then* $\exp(A) = 0$*. If A is not nilpotent, then either* $\exp(A) \geqslant 2$ *or* $\exp(A) = 1$*, and $c_n(A)$ is polynomially bounded.*

Proof. As we have already mentioned, an extension of the base field does not change the codimensions of an algebra. Hence we may assume F to be algebraically closed. The conclusion of the theorem is obvious for a nilpotent algebra. Hence assume A non-nilpotent and let $d = d(A)$ be as in (6). Then $\exp(A) = d$ as follows from Lemmas 5 and 9.

Notice that if $d = 1$ then the polynomial upper bound also follows from Lemma 5. □

We remark that the absence of intermediate codimension growth (i.e., faster than any polynomial and slower than any exponential $\alpha^n, \alpha > 1$) was proved in [3] for any finite-dimensional algebra, but for any real numbers $1 < \beta < \alpha$, there exists a finite-dimensional algebra B with $\beta < \exp(B) < \alpha$.

References

[1] Yu.A. Bahturin, V. Drensky, Identities of bilinear mappings and graded polynomial identities of matrices, Linear Algebra Appl. 369 (2003) 95–112.
[2] A. Berele, A. Regev, Asymptotic behaviour of codimensions of P.I. algebras satisfying Capelli identities, Trans. Amer. Math. Soc. 360 (2008) 5155–5172.
[3] A. Giambruno, S. Mishchenko, M. Zaicev, Algebras with intermediate growth of the codimensions, Adv. in Appl. Math. 37 (2006) 360–377.
[4] A. Giambruno, S. Mishchenko, M. Zaicev, Codimension growth of two-dimensional non-associative algebras, Proc. Amer. Math. Soc. 135 (2007) 3405–3415.
[5] A. Giambruno, S. Mishchenko, M. Zaicev, Codimensions of algebras and growth functions, Adv. Math. 217 (2008) 1027–1052.
[6] A. Giambruno, S. Mishchenko, M. Zaicev, Polynomial identities of algebras of small dimension, Comm. Algebra 37 (2009) 1934–1948.
[7] A. Giambruno, M. Zaicev, Exponential codimension growth of P.I. algebras: an exact estimate, Adv. Math. 142 (1999) 221–243.

[8] A. Giambruno, M. Zaicev, Polynomial Identities and Asymptotic Methods, Math. Surveys Monogr., vol. 122, American Mathematical Society, Providence, RI, 2005.
[9] A. Giambruno, M. Zaicev, Multialternating Jordan polynomials and codimension growth of matrix algebras, Linear Algebra Appl. 422 (2007) 372–379.
[10] A. Giambruno, M. Zaicev, Codimension growth of special simple Jordan algebras, Trans. Amer. Math. Soc. 362 (2010) 3107–3123.
[11] N. Jacobson, Structure and Representations of Jordan Algebras, Amer. Math. Soc. Colloq. Publ., vol. XXXIX, American Mathematical Society, Providence, RI, 1968.
[12] G. James, A. Kerber, The Representation Theory of the Symmetric Group, Encyclopedia Math. Appl., vol. 16, Addison–Wesley, London, 1981.
[13] A.R. Kemer, The Spechtian nature of T-ideals whose codimensions have power growth, Sibirsk. Mat. Zh. 19 (1978) 54–69 (in Russian); translation in: Siberian Math. J. 19 (1978) 37–48.
[14] S. Mishchenko, Lower bounds on the dimensions of irreducible representations of symmetric groups and of the exponents of the exponential of varieties of Lie algebras, Mat. Sb. 187 (1996) 83–94 (in Russian); translation in: Sb. Math. 187 (1996) 81–92.
[15] V. Petrogradsky, On types of superexponential growth of identities in Lie PI-algebras, Fundam. Prikl. Mat. 1 (1995) 989–1007 (in Russian).
[16] S.V. Polikarpov, I.P. Shestakov, Nonassociative affine algebras, Algebra Logic 29 (1990) 709–723.
[17] A. Regev, Existence of identities in $A \otimes B$, Israel J. Math. 11 (1972) 131–152.
[18] R.D. Schafer, An Introduction to Nonassociative Algebras, Academic Press, New York, 1966.
[19] M. Zaicev, Integrality of exponents of growth of identities of finite-dimensional Lie algebras, Izv. Ross. Akad. Nauk Ser. Mat. 66 (2002) 23–48 (in Russian); English translation: Izv. Math. 66 (2002) 463–487.