Duality in spaces of finite linear combinations of atoms

Fulvio Ricci and Joan Verdera

Abstract

In this note we describe the dual and the completion of the space of finite linear combinations of (p, ∞)-atoms, $0 < p \leq 1$. As an application, we show an extension result for operators uniformly bounded on (p, ∞)-atoms, $0 < p < 1$, whose analogue for $p = 1$ is known to be false. Let $0 < p < 1$ and let T be a linear operator defined on the space of finite linear combinations of (p, ∞)-atoms, $0 < p < 1$, which takes values in a Banach space B. If T is uniformly bounded on (p, ∞)-atoms, then T extends to a bounded operator from $H^p(\mathbb{R}^n)$ into B.

1 Introduction

For each $0 < p \leq 1$ consider the space F^p of finite linear combinations of (p, ∞)-atoms, endowed with its natural norm (or quasi-norm for $p < 1$)

$$\|f\|_{F^p} = \inf \left\{ \left(\sum_j{}' |\lambda_j|^p \right)^{\frac{1}{p}} : f = \sum_j{}' \lambda_j a_j \,, \ a_j \text{ a } (p, \infty)\text{-atom}\,, \ \lambda_j \in \mathbb{C} \right\}, \quad (1)$$

where $\sum'$ denotes a finite sum. Recall that a is a (p, ∞)-atom if a is a measurable function supported on a ball B, satisfying the cancellation condition

$$\int a(x) x^\alpha \, dx = 0, \quad |\alpha| \leq n(\frac{1}{p} - 1),$$

and the size condition

$$|a| \leq \frac{1}{|B|^{\frac{1}{p}}}.$$

The space F^p is clearly contained in $H^p = H^p(\mathbb{R}^n)$, the standard real Hardy space on $\mathbb{R}^n$. The elements of H^p are the distributions that admit an atomic decomposition, $f = \sum_{j=1}^\infty \lambda_j a_j$, converging in the sense of distributions, for some (p, ∞)-atoms a_j and scalars λ_j with $\sum_{j=1}^\infty |\lambda_j|^p < \infty$ (for $p = 1$, $H^1 \subset L^1$ and atomic sums converge in the L^1-norm). In [MTW] Meyer,Taibleson and Weiss observed that the

1

F^p-norm is not comparable to the H^p-norm on F^p. Recently, it was shown in [B] that the Meyer-Taibleson-Weiss result leads to the following conclusion in the case $p = 1$: there exists a bounded linear functional on F^1 which does not extend to a bounded linear functional on H^1. In other words, there is a linear operator which is uniformly bounded on $(1, \infty)$-atoms but does not extend to a bounded linear operator on H^1.

In this paper we describe the structure of the completion $\widetilde{F^p}$ of F^p, $0 < p \leq 1$, and of its dual space. We show in particular that, when $p < 1$, F^p and H^p have the same dual, and therefore no example like the one in [B] can be exhibited for $p < 1$. An immediate consequence of this is that if $0 < p < 1$ and the linear operator

$$T : F^p \to B, \tag{2}$$

maps F^p into a Banach space B satisfying the inequality

$$\|T(a)\|_B \leq C,$$

for some positive constant C and all (p, ∞)-atoms, then T extends to a bounded linear operator from H^p into B. The argument proceeds by duality as follows. Take any u in the dual B^* of B. Since $u \circ T \in (F^p)^* = (H^p)^*$,

$$|u(T(f))| \leq C \|u\| \|f\|_{H^p},$$

and so, by the dual expression of the norm in a Banach space,

$$\|T(f)\|_B \leq C \|f\|_{H^p}.$$

We prove the following facts about $\widetilde{F^p}$, $0 < p \leq 1$.

(i) The closed subspace $\widetilde{F^{p,c}}$ of $\widetilde{F^p}$ spanned by the continuous (p, ∞)-atoms is isomorphic to H^p as a Banach space, and $\widetilde{F^p}$ splits as the direct sum of $\widetilde{F^{p,c}}$ and a *non-trivial* complementary closed subspace N^p.

(ii) Every element ξ of $\widetilde{F^p}$ admits an atomic decomposition

$$\xi = \sum_{j=1}^{\infty} \lambda_j a_j,$$

for (p, ∞)-atoms a_j and scalars λ_j with $\sum_{j=1}^{\infty} |\lambda_j|^p < \infty$. Moreover, the $\widetilde{F^p}$-norm of ξ is equivalent to its atomic norm

$$\inf \left\{ \left(\sum_{j=1}^{\infty} |\lambda_j|^p \right)^{\frac{1}{p}} : \sum_{j=1}^{\infty} \lambda_j a_j = \xi \text{ in } \widetilde{F^p} \right\}.$$

(iii) If an atomic sum $\sum_{j=1}^{\infty} \lambda_j a_j$, with λ_j and a_j as above, converges to 0 in $\widetilde{F^p}$, it also converges to 0 in H^p, but not viceversa. In fact, N^p consists of those elements of $\widetilde{F^p}$ that are represented by atomic sums converging to 0 in H^p.

In other words, H^p and $\widetilde{F^p}$ are both quotients of the space of "formal series" of (p, ∞)-atoms with ℓ^p coefficients, but the equivalence relation defining $\widetilde{F^p}$ is finer than that defining H^p.

So, the reason why $(F^1)^*$ is strictly larger than $(H^1)^*$ is that it is the direct sum of $(\widetilde{F^{1,c}})^* = (H^1)^*$ and $(N^1)^*$. Notice that $(N^1)^*$ is non-trivial, as the dual of the non-trivial Banach space N^1. On the other hand, it turns out that $(N^p)^*$ is trivial for $p < 1$.

To describe our results we need to introduce some notation and recall some basic classical facts in the theory of Banach algebras (see Section 3 for details).

Denote by $L_0^\infty(\mathbb{R}^n)$ the space of bounded measurable functions on $\mathbb{R}^n$ vanishing at infinity. Then $L_0^\infty(\mathbb{R}^n)$ is a commutative C^*-algebra without unit, and its maximal ideal space is a locally compact, non-compact space, which we call $\widehat{\mathbb{R}^n}$ (cf. [F]).

By the Gelfand-Naimark theorem, the Gelfand transform $f \to \hat{f}$ establishes an isometric isomorphism between $L_0^\infty(\mathbb{R}^n)$ and the algebra $C_0(\widehat{\mathbb{R}^n})$ of all continuous functions on $\widehat{\mathbb{R}^n}$ vanishing at ∞. On the other hand, $C_0(\mathbb{R}^n)$ is a closed subalgebra of $L_0^\infty(\mathbb{R}^n)$, and its maximal ideal space is $\mathbb{R}^n$. This embedding induces a continuous projection π from $\widehat{\mathbb{R}^n}$ onto $\mathbb{R}^n$. Clearly, if $f \in C_0(\mathbb{R}^n)$, then $\hat{f} = f \circ \pi$.

In a similar way, given any ball B in $\mathbb{R}^n$, the maximal ideal space of $L^\infty(B)$ is a compact space $\widehat{B}$, endowed with a projection π_B onto $\bar{B}$ induced by the inclusion of $C(\bar{B})$ in $L^\infty(B)$. Moreover, $L^\infty(B) \cong C(\widehat{B})$, again by the Gelfand-Naimark theorem.

The restriction map $f \mapsto f_{|B}$ from $L_0^\infty(\mathbb{R}^n)$ to $L^\infty(B)$ induces a natural embedding $\iota_B : \widehat{B} \to \widehat{\mathbb{R}^n}$, which is compatible with the projections π and π_B, in the sense that

$$\pi_B = \pi \circ \iota_B .$$

Similar embeddings $\iota_{B,B'} : \widehat{B'} \to \widehat{B}$ exist for pairs of balls B, B' with $B' \subset B$, with the same compatibility with respect to the corresponding projections.

Denote by m the Lebesgue measure on $\mathbb{R}^n$. The continuous linear functional $f \mapsto \int f \, dm$ on $L^\infty(B)$ is represented by a positive Borel measure $\widehat{m}_B$ on $\widehat{B}$, that is,

$$\int f \, dm = \int \hat{f} \, d\widehat{m}_B, \quad f \in L^\infty(B) . \tag{3}$$

If B is contained in a second ball B', then the restriction of $\widehat{m}_{B'}$ to $\widehat{B}$ is precisely $\widehat{m}_B$ and thus we can define a positive Borel measure $\widehat{m}$ globally on $\widehat{\mathbb{R}^n}$ by requiring that its restriction to $\widehat{B}$ be $\widehat{m}_B$ for each ball B.

We can now state our main result.

Theorem.

(A) *Let ℓ be a bounded linear functional on F^1. Then there exist a function $b \in BMO(\mathbb{R}^n)$ and a Radon measure μ on $\widehat{\mathbb{R}^n}$, singular with respect to $\widehat{m}$, satisfying*

$$|\mu|(\widehat{B}) \le C\, m(B), \quad \text{for each ball} \quad B, \tag{4}$$

such that

$$\ell(f) = \int f\, b\, dm + \int \hat{f}\, d\mu, \quad f \in F^1. \tag{5}$$

Conversely, if b and μ are as above, then the identity (5) defines a bounded linear functional on F^1 and

$$\|\ell\|_{(F^1)^*} \cong \|b\|_{BMO} + \sup_B \frac{|\mu|(\widehat{B})}{m(B)}\ .$$

(B) *Each bounded linear functional on F^p, $0 < p < 1$, extends uniquely to a bounded linear functional on $H^p(\mathbb{R}^n)$. Thus $(F^p)^* = H^p(\mathbb{R}^n)^*$, $0 < p < 1$.*

It is clear that relation (5) determines the function b and the measure μ uniquely. Therefore $(F^1)^*$ differs from $(H^1)^* = BMO$ by the presence of the complementary subspace S of singular measures satisfying (4). We will show that S is non-trivial; in fact, the Meyer, Taibleson and Weiss argument may be interpreted as the construction of a non-zero measure in S. The decomposition of $(F^1)^*$ as $BMO \oplus S$ is the dual counterpart of the decomposition of $\widetilde{F^1}$ as $\widetilde{F^{1,c}} \oplus N^1$, although S and BMO do not coincide with the annihilators of $\widetilde{F^{1,c}}$ and N^1 respectively.

The nature of the elements of N^p, including $p = 1$, is somehow mysterious. It is not clear at all to us if they can be represented by concrete analytic objects.

Section 2 contains the discussion of the completion of F^p and a constructive argument which proves the non-triviality of N^p. In Section 3 we prove the Theorem. We also give an example of a non-zero singular measure satisfying (4).

We remark here that a variation of the main argument in the proof of the Theorem provides an alternative proof of some results in [MSV] and [YZ]) on the equivalence of the finite and infinite atomic norms of $(1, q)$-atoms, $q < \infty$, and on extension of bounded operators defined on finite linear combinations of (p, q)-atoms with $1 < q < \infty$.

2   The completion of F^p

Let $F^{p,c}$ stand for the subspace of H^p consisting of finite linear combinations of continuous (p, ∞) atoms. A surprising recent result in [MSV] states that the H^p and the F^p norms are equivalent on $F^{p,c}$, $0 < p \le 1$. Indeed, the result is proved in [MSV] only for $p = 1$, but, as suggested in Remark 3.2 there, the same argument extends to the case $0 < p < 1$.

More precisely, we can quote Lemma 3.1 and Remark 3.2 in [MSV] as follows.

Lemma 1. *The following norms are equivalent on $F^{p,c}$:*

(a) *the H^p-norm;*

(b) *the F^p-norm (1);*

(c) *the $F^{p,c}$-norm*

$$\|f\|_{F^{p,c}} = \inf\{\sum_j{}' |\lambda_j| : f = \sum_j{}' \lambda_j a_j \, , \ a_j \text{ a continuous } (p,\infty)\text{-atom} \, , \ \lambda_j \in \mathbb{C}\} \, .$$

Since $F^{p,c}$ is dense in H^p, the natural inclusion of $F^{p,c}$ in F^p extends uniquely to a continuous linear operator T from H^p to $\widetilde{F^p}$. By Lemma 1, T maps H^p isomorphically onto the closure $\widetilde{F^{p,c}}$ of $F^{p,c}$ in $\widetilde{F^p}$. Notice that, again by Lemma 1, $\widetilde{F^{p,c}}$ is the completion of $F^{p,c}$ endowed either with the norm $\|\cdot\|_{F^{p,c}}$ or with the norm inherited from F^p.

On the other hand, the inclusion of F^p (endowed with its natural norm) into H^p is continuous, and it extends to a continuous linear operator U from $\widetilde{F^p}$ to H^p. We then have the diagram

$$H^p \xrightarrow{\ T\ } \widetilde{F^p} \xrightarrow{\ U\ } H^p \, ,$$

with $U \circ T$ being the identity map. In particular U is surjective. Set $P = T \circ U$, so that P is a projection, that is, $P^2 = P$. The kernel of P is the kernel of U, which we denote by N^p, and the kernel of $I - P$ is $T(H^p) = \widetilde{F^{p,c}}$. Hence we get the topological direct sum decomposition

$$\widetilde{F^p} = \widetilde{F^{p,c}} \oplus N^p \, .$$

Notice that N^p is non-trivial, since otherwise the H^p and the F^p norms would be comparable on F^p.

To better understand the space $\widetilde{F^p}$ we prove now the following.

Proposition. *Given any sequence of (p,∞) atoms a_j and any ℓ^p-sequence of scalars λ_j, the series $\sum_{j=1}^{\infty} \lambda_j \, a_j$ converges in $\widetilde{F^p}$ to an element ξ such that $\|\xi\|_{\widetilde{F^p}}^p \leq \sum_{j=1}^{\infty} |\lambda_j|^p$. Conversely, each $\xi \in \widetilde{F^p}$ can be written as*

$$\xi = \sum_{j=1}^{\infty} \lambda_j \, a_j \, , \tag{6}$$

where each a_j is a (p,∞) atom and the sum is convergent in $\widetilde{F^p}$. Moreover,

$$\|\xi\|_{\widetilde{F^p}}^p = \inf\left\{ \sum_{j=1}^{\infty} |\lambda_j|^p \right\} \, , \tag{7}$$

where the infimum is taken over all decompositions (6) of ξ.

Proof. Let ξ be an element of $\widetilde{F^p}$. To prove (6), express ξ as the limit in $\widetilde{F^p}$ of a sequence S_k of elements of F^p. Given $\epsilon > 0$, we may assume that $\|S_1\|_{F^p}^p < (1+\epsilon)\|\xi\|_{\widetilde{F^p}}^p$ and that $\|S_k - S_{k+1}\|_{F^p}^p < \epsilon^k \|\xi\|_{\widetilde{F^p}}^p$. Thus

$$\xi = \lim_{k\to\infty} S_1 + (S_2 - S_1) + \cdots + (S_k - S_{k-1}) \,.$$

Set

$$S_1 = \sum_{j=1}^{N_1} \lambda_j \, a_j \,,$$

where the above expression of has been chosen so that

$$\sum_{j=1}^{N_1} |\lambda_j|^p < (1+\epsilon)\|\xi\|_{\widetilde{F^p}}^p \,.$$

Similarly, set

$$S_\ell - S_{\ell-1} = \sum_{j=N_{\ell-1}+1}^{N_\ell} \lambda_j \, a_j \,, \quad \ell \geq 2 \,,$$

with

$$\sum_{j=N_{\ell-1}+1}^{N_\ell} |\lambda_j|^p < \epsilon^\ell \|\xi\|_{\widetilde{F^p}}^p \,.$$

Then $\sum_{j=1}^\infty |\lambda_j|^p < (1-\epsilon)^{-1}\|\xi\|_{\widetilde{F^p}}^p$ and the partial sums $\xi_m = \sum_{j=1}^m \lambda_j a_j$ form a Cauchy sequence in F^p. This shows that (6) holds.

Notice also that, for each $\xi \in \widetilde{F^p}$, the inequality $\|\xi\|_{\widetilde{F^p}}^p \leq \inf\{\sum_{j=1}^\infty |\lambda_j|^p\}$, where the infimum is taken over all possible expressions (6), is due to the fact that $\|\cdot\|_{\widetilde{F^p}}^p$ satisfies the triangle inequality. $\qquad\square$

The atomic decomposition of elements of $\widetilde{F^p}$ given above provides an explicit description of the operator U.

Corollary. *Let $\xi \in \widetilde{F^p}$ be represented by the sum (6). Then $U(\xi)$ is the sum of the same series in H^p.*

We end this section by providing a constructive proof of the non triviality of N^p. Let us first describe the Meyer, Taibleson and Weiss construction as presented in [B]. Let B denote the open ball centered at the origin with radius 1. Take a sequence of open disjoint balls B_j, $j \geq 1$, such that $\cup_j B_j$ is dense in B. Notice that we may also choose the B_j so that the Lebesgue measure of their union $\sum_{j\geq 1} |B_j|$ is as small as we wish. As shown in [B], for each j there exists a (non-continuous) (p,∞) atom a_j supported on B_j with the property that $|a_j| \geq c|B_j|^{-\frac{1}{p}}$, where c is a small positive constant depending only on n. Thus, setting

$$f = \sum_{j\geq 1} |B_j|^{\frac{1}{p}} \, a_j \,, \tag{8}$$

6

we get $|f| \geq c$ on $\cup B_j$. From that is not difficult to conclude (see [B]) that

$$\|f\|_{F^p} \geq c\,|B|^{\frac{1}{p}}\,.$$

On the other hand, we clearly have $\|f\|_{H^p}^p \leq \sum_{j\geq 1} |B_j|$, so that the ratio between H^p-norm and F^p-norm can be made as small as we wish.

We can now construct a sequence $\{f_m\}$ in F^p satisfying

$$\|f_m\|_{F^p}^p \geq c^p\,|B|\,,$$

$$\|f_m - f_{m+1}\|_{F^p}^p \leq 2^p\,\frac{|B|}{2^m}\,,$$

$$\|f_m\|_{H^p}^p \leq \frac{|B|}{2^m}\,.$$

The first two conditions imply that $\{f_m\}$ has a non-zero limit $\xi \in \widetilde{F^p}$, whereas the third implies that $U f_m = f_m$ tends to 0 in H^p. Hence $\xi \in N^p$.

The functions f_m have the form (8), precisely

$$f_m = \sum_{j\geq 1} |B_j^m|^{\frac{1}{p}}\, a_j^m\,, \tag{9}$$

where, for each m, $\{B_j^m\}_j$ is a disjoint family of balls contained in B with dense union and small total measure, and each a_j^m is a (p,∞)-atom with $|a_j^m| \geq c\,|B_j^m|^{-\frac{1}{p}}$.

The first function f_1 can be any function as in (8) with, say, $\sum_{j\geq 1} |B_j^1| < |B|/2$. We then construct inductively f_{m+1} from f_m as follows.

Take N so large that $\sum_{j>N} |B_j^m| < (1/4) \sum_{j\geq 1} |B_j^m|$. Inside each $B_j^m, 1 \leq j \leq N$, we take open disjoint balls $B'_{j\,l}, l \geq 1$, such that $\cup_{l\geq 1} B'_{j\,l}$ is dense in B_j^m and $\sum_{l\geq 1} |B'_{j\,l}| < |B_j^m|/4$.

Then

$$\sum_{j=1}^{N} \sum_{l\geq 1} |B'_{j\,l}| + \sum_{j>N} |B_j^m| \leq \frac{1}{2} \sum_{j\geq 1} |B_j^m|\,. \tag{10}$$

Let $a'_{j\,l}$ be a (p,∞) atom supported on $B'_{j\,l}$ with $|a'_{j\,l}| \geq c\,|B'_{j\,l}|^{-\frac{1}{p}}$. Set

$$f_{m+1} = \sum_{j=1}^{N} \sum_{l\geq 1} |B'_{j\,l}|^{\frac{1}{p}}\, a'_{j\,l} + \sum_{j>N} |B_j^m|^{\frac{1}{p}}\, a_j^m\,.$$

Since $|f_{m+1}| \geq c$ on an open dense subset of B, $\|f_{m+1}\|_{F^p}^p \geq c^p\,|B|$. Moreover,

$$f_m - f_{m+1} = \sum_{j=1}^{N} \left(|B_j^m|^{\frac{1}{p}}\, a_j^m - \sum_{l\geq 1} |B'_{j\,l}|^{\frac{1}{p}}\, a'_{j\,l} \right)\,.$$

For each j, the function

$$|B_j^m|^{\frac{1}{p}}\, a_j^m - \sum_{l \geq 1} |B_{jl}'|^{\frac{1}{p}}\, a_{jl}'$$

is supported on B_j^m and its absolute value is not greater than 2. Hence

$$\|f_m - f_{m+1}\|_{F^p}^p \leq \sum_{j=1}^{N} 2^p\, |B_j^m|\,.$$

We relabel now the balls in such a way that $\{B_j^{m+1}\}_{j \geq 1} = \{B_j^m\}_{j > N} \cup \{B_{jl}'\}_{j \leq N,\, l \geq 1}$, and rename the atoms in f_{m+1} as a_j^{m+1} accordingly. Then, inductively from (10),

$$\sum_{j \geq 1} |B_j^m| \leq 2^{-m} |B|$$

for every m, and the required estimates can be easily verified.

3   Proof of the Theorem

We start by proving, for the reader's sake, a few statements made (explicitly or not) in the last part of the introduction concerning the Gelfand spectrum $\widehat{\mathbb{R}^n}$ and its projection π on $\mathbb{R}^n$.

The first statement we want to prove is that π is in fact well defined. Given ϕ in $\widehat{\mathbb{R}^n}$, i.e., a nontrivial multiplicative functional on $L_0^\infty(\mathbb{R}^n)$, it is clear that its restriction to $C_0(\mathbb{R}^n)$ is also multiplicative. We must show that this restriction is evaluation at some point $x = \pi(\phi)$ of $\mathbb{R}^n$, or, equivalently, that it is not identically zero.

Since $L_0^\infty(\mathbb{R}^n)$ is a C^*-algebra, it is symmetric, so that $\phi(\bar{f}) = \overline{\phi(f)}$ for every f. Therefore, $f \geq 0$ implies that $\phi(f) \geq 0$, so that ϕ is monotonic on real-valued functions. If ϕ vanishes identically on $C_0(\mathbb{R}^n)$, it also vanishes on characteristic functions of compact sets. By linearity and continuity, this would be a contradiction.

The second statement is that the mapping π is surjective. We know that to each $\phi \in \widehat{\mathbb{R}^n}$ we can associate a point $\pi(\phi)$ in $\mathbb{R}^n$. Given $y \in \mathbb{R}^n$, we can define a translate $\tau_y \phi \in \widehat{\mathbb{R}^n}$ by

$$\tau_y \phi(f) = \phi\big(f(\cdot + y)\big)\,. \tag{11}$$

It is quite clear that $\pi(\tau_y \phi) = \pi(\phi) + y$. Since $\widehat{\mathbb{R}^n}$ is nonempty, π is surjective.

The last statement which remained unproved in the introduction is that $\widehat{\mathbb{R}^n}$ is the union of the $\widehat{B}$ over all balls B. This is a direct consequence of (ii) in the following lemma.

Lemma 2. *Let B be an open ball in $\mathbb{R}^n$. Then*

(i)
$$\widehat{B} = \{\phi \in \widehat{\mathbb{R}^n} : \phi(\chi_B) = 1\} = \operatorname{supp} \widehat{\chi_B} \ , \tag{12}$$

where $\widehat{f}$ stands for the Gelfand transform of $f \in L_0^\infty(\mathbb{R}^n)$.

(ii)
$$\pi^{-1}(B) \subset \widehat{B} \subset \pi^{-1}(\overline{B}) \ .$$

Proof. To prove (12) notice that $\phi(\chi_B)$ is either 0 or 1 by the multiplicative property.

If $\phi(\chi_B) = 1$, then $\phi(f) = \phi(f\chi_B)$, $f \in L_0^\infty(\mathbb{R}^n)$, which means that ϕ factors through a character of $L^\infty(B)$. Thus $\phi \in \widehat{B}$. The argument can be reversed, so (12) is proved.

Assume now that for some $\phi \in \widehat{\mathbb{R}^n}$ we have $\pi(\phi) \in B$. Let f be a continuous function on $\mathbb{R}^n$, with $f(\pi(\phi)) = 1$ and compact support contained in B. Then $f\chi_B = f$ and so
$$1 = \phi(f) = \phi(f)\phi(\chi_B) = \phi(\chi_B).$$
Then $\phi \in \widehat{B}$ because of (12).

If $\pi(\phi)$ is not in $\overline{B}$, then there is a continuous function f on $\mathbb{R}^n$, with $f(\pi(\phi)) = 1$ and compact support in $\mathbb{R}^n \setminus \overline{B}$. Thus $f\chi_B = 0$ and so $\phi(\chi_B) = 0$, that is, ϕ is not in $\widehat{B}$. $\qquad\square$

We turn now to the proof of the Theorem. We begin by discussing the converse statement in part (A) of the Theorem. Obviously, given $b \in BMO$, the linear functional $f \mapsto \int fb\,dm$ is bounded on $\widetilde{F^1}$ with a norm controlled from above by the BMO-norm of b. On the other hand, restriction of the functional to $\widetilde{F^{1,c}}$ gives a control from below by the same BMO-norm.

We first remark that (12) clearly implies that, given $f \in L_0^\infty(\mathbb{R}^n)$, the support of f is contained in B if and only if the support of $\widehat{f}$ is contained in $\widehat{B}$.

Let μ be a Radon measure on $\widehat{\mathbb{R}^n}$ satisfying (4). For each $(1,\infty)$-atom a supported on a ball B one has
$$\left| \int \widehat{a}\,d\mu \right| \leq \|a\|_\infty\,|\mu|(\widehat{B}) \leq \frac{|\mu|(\widehat{B})}{m(B)} < C \ .$$

Hence μ determines a bounded linear functional on $\widetilde{F^1}$.

Assume now that ℓ is a bounded linear functional on F^1. Fix a ball B and let $L_0^\infty(B)$ stand for the set of functions in $L^\infty(B)$ with zero integral. Given $f \in L_0^\infty(B)$,
$$\frac{1}{m(B)}\frac{f}{\|f\|_\infty}$$

is a $(1,\infty)$-atom. Thus
$$|\ell(f)| \leq \|\ell\|\,\|f\|_\infty\,m(B), \quad f \in L_0^\infty(B). \tag{13}$$

9

The restriction of ℓ to $L_0^\infty(B)$ extends to a bounded linear functional on $L^\infty(B) = C(\widehat{B})$. Thus there exists a measure ν_B on $\widehat{B}$ such that

$$\ell(f) = \int \widehat{f}\, d\nu_B, \quad f \in L_0^\infty(B). \tag{14}$$

If $f \in L^\infty(B)$, then clearly $\widehat{f}_{\widehat{B}} = f_B$, where g_E stands for the mean of the function g on the set E with respect to the underlying measure ($\widehat{m}$ or m in the case at hand). Then

$$\begin{aligned}
\ell(f - f_B) &= \int_{\widehat{B}} (\widehat{f} - \widehat{f}_{\widehat{B}})\, d\nu_B \\
&= \int_{\widehat{B}} (\widehat{f} - \widehat{f}_{\widehat{B}}) \left(d\nu_B - \nu_B(\widehat{B}) \frac{d\widehat{m}}{\widehat{m}(\widehat{B})} \right) \\
&= \int_{\widehat{B}} \widehat{f} \left(d\nu_B - \frac{\nu_B(\widehat{B})}{\widehat{m}(\widehat{B})} d\widehat{m} \right),
\end{aligned} \tag{15}$$

for each $f \in L^\infty(B)$. Therefore, if ν_B represents ℓ on $L_0^\infty(B)$, that is, if (14) holds, then $d\nu_B - \nu_B(\widehat{B})\frac{\widehat{m}}{\widehat{m}(\widehat{B})}$ is uniquely determined.

Let B_N stand for the open ball with center at the origin and radius N, $N = 1, 2, ...$ Take any measure ν_1 on $\widehat{B_1}$ that represents ℓ on $L_0^\infty(B_1)$. Each other such measure differs from ν_1 by a constant multiple of $\chi_{\widehat{B_1}}\widehat{m}$. By the preceding remark applied to B_N there exists a unique measure ν_N on $\widehat{B_N}$ which represents ℓ on $L_0^\infty(B_N)$ and $\nu_N(B_1) = \nu_1(B_1)$. Clearly ν_N restricted to $\widehat{B_{N-1}}$ is precisely ν_{N-1}. Therefore we can define a measure ν on $\widehat{\mathbb{R}^n}$ by requiring that ν restricted to $\widehat{B_N}$ be ν_N.

Given any ball B take N such that $B \subset B_N$. Since the restriction of ν to $\widehat{B_N}$ represents ℓ on $L_0^\infty(B_N)$, which contains $L_0^\infty(B)$, the restriction of ν to $\widehat{B}$ represents ℓ on $L_0^\infty(B)$ as well. By (15)

$$\left| \int_{\widehat{B}} \widehat{f} \left(d\nu - \nu(\widehat{B}) \frac{d\widehat{m}}{\widehat{m}(\widehat{B})} \right) \right| \le 2 \|\ell\| \|f\|_\infty m(B), \quad f \in L^\infty(B),$$

or

$$\left\| \nu - \nu(\widehat{B}) \frac{\widehat{m}}{\widehat{m}(\widehat{B})} \right\|_{\widehat{B}} \le 2 \|\ell\| m(B). \tag{16}$$

Let us now consider the Radon-Nikodym decomposition of ν

$$\nu = g\, \widehat{m} + \mu,$$

where $g \in L_{loc}^1(\widehat{m})$ and μ is singular with respect to $\widehat{m}$. By (16)

$$|\mu|(\widehat{B}) \le 2 \|\ell\| m(B). \tag{17}$$

and

$$\int_{\widehat{B}} \left| g - g_{\widehat{B}} - \frac{\mu(\widehat{B})}{\widehat{m}(\widehat{B})} \right| d\widehat{m} \leq 2 \left\| \ell \right\| m(B). \tag{18}$$

We are left with the task of finding the BMO-function b.

Combining (17) and (18) we readily get

$$\int_{\widehat{B}} |g - g_{\widehat{B}}| \, d\widehat{m} \leq 4 \left\| \ell \right\| m(B). \tag{19}$$

We need a Lemma.

Lemma 3. *For each function $g \in L^1_{\text{loc}}(\widehat{m})$ there exists a unique function $f \in L^1_{\text{loc}}(m)$ with the property that for each ball B,*

$$\int g \, \widehat{\varphi} \, d\widehat{m} = \int f \, \varphi \, dm, \quad \varphi \in L^\infty(B).$$

Such f satisfies

$$\int_{\widehat{B}} |g - g_{\widehat{B}}| \, d\widehat{m} = \int_B |f - f_B| \, dm,$$

for each ball B.

Once the lemma is proved we complete the proof of part (A) of the Theorem by just calling b the function f associated with g in Lemma 2. Inequality (19) tells us that $b \in BMO(\mathbb{R}^n)$ and that its $BMO(\mathbb{R}^n)$ norm is not greater than $4 \left\| \ell \right\|$.

Proof of Lemma 3. We will show that for each ball B the Gelfand transform, which is an isometry between $L^\infty(B)$ and $C(\widehat{B})$, extends to an isometry between $L^1(B, m)$ and $L^1(\widehat{B}, \widehat{m})$. This immediately provides a further extension of the Gelfand transform to a topological isomorphism between $L^1_{\text{loc}}(m)$ and $L^1_{\text{loc}}(\widehat{m})$.

We begin by showing that, for each ball B in $\mathbb{R}^n$ and every $f \geq 0$ in $L^\infty(B)$,

$$\int_{\widehat{B}} \widehat{f} \, d\widehat{m} = \int_B f \, dm, \quad . \tag{20}$$

This follows from

$$\int_B f \, dm = \sup_\varphi \int_B f \, \varphi \, dm$$

$$= \sup_\varphi \int_{\widehat{B}} \widehat{f} \, \widehat{\varphi} \, d\widehat{m}$$

$$= \int_{\widehat{B}} \widehat{f} \, d\widehat{m},$$

where the supremum is taken on the closed unit ball of $L^\infty(B)$.

By linearity, (20) provides an extension of the Gelfand transform to a topological isomorphisms $f \to \widehat{f}$ of $L^1_{\text{loc}}(m)$ onto $L^1_{\text{loc}}(\widehat{m})$. Given $g \in L^1_{\text{loc}}(\widehat{m})$ take $f \in L^1_{\text{loc}}(m)$ with $g = \widehat{f}$. The first identity in the statement of Lemma 2 follows by approximating $f \in L^1(B, m)$ by functions in $L^\infty(B)$ and the second follows from (20). $\qquad\square$

Before proving part (B) of the Theorem we give an explicit example, modeled on the Meyer-Taibleson-Weiss argument, of a non-zero measure which is singular with respect to $\widehat{m}$ and satisfies (4).

Take an open set U of $\mathbb{R}^n$, $U \subset B_0 = \{x : |x| \le 1\}$, such that U is dense in B_0 and $m(U) < m(B_0)$. Then the compact set $E = B_0 \setminus U$ has positive Lebesgue measure. Set $V = \pi^{-1}(U)$, so that $V \subset \widehat{B_0}$ by Lemma 1. Then $U \subset \pi(\overline{V})$ and so $\pi(\overline{V}) = B_0$, because U is dense in B_0. Hence $\pi(\partial V) = E$. Now, the boundary of each open set in $\widehat{B_0}$ has zero $\widehat{m}$ measure ([R, p. 286]). Therefore $\widehat{m}(\partial V) = 0$ but $m(\pi(\partial V)) = m(E) > 0$. Identify $C(E)$ to the subspace S of continuous functions on ∂V of the form $f \circ \pi$, $f \in C(E)$. The bounded linear functional on S defined by $f \to \int f \, dm$ extends by Hahn-Banach to a bounded linear functional on $C(\partial V)$ with the same norm. Thus there exists a positive measure μ on ∂V such that

$$\int (f \circ \pi) \, d\mu = \int f \, dm, \quad f \in C(E).$$

If B is an open ball, then by Lemma 2

$$\mu(\widehat{B}) \le \mu(\pi^{-1}(\overline{B})) = m(\overline{B} \cap E) \le m(B),$$

and condition (4) is satisfied.

Proof of (B) of the Theorem. The argument is analogous to the proof of part (A), except for minor technical details. If $0 < p < 1$, then, as we will see, the singular measure μ vanishes and so we will conclude that $(F^p)^* = H^p(\mathbb{R}^n)^*$.

Let ℓ be a bounded linear functional on F^p, $0 < p < 1$. Let d be the integer part of $n(\frac{1}{p} - 1)$. Given a ball B let $L^\infty_d(B)$ stand for the set of functions $f \in L^\infty(B)$ such that

$$\int f(x) \, x^\alpha \, dx = 0, \quad |\alpha| \le d.$$

For each $f \in L^\infty_d(B)$,

$$\frac{1}{m(B)^{\frac{1}{p}}} \frac{f}{\|f\|_\infty}$$

is a (p, ∞) atom and so

$$|\ell(f)| \le \|\ell\| \, \|f\|_\infty \, m(B)^{\frac{1}{p}}, \quad f \in L^\infty_d(B). \tag{21}$$

For each $f \in L^\infty(B)$ let $P_B(f)$ be (the restriction to B of) the unique polynomial of degree not greater than d such that

$$\int f(x) \, x^\alpha \, dx = \int_B P_B(f)(x) \, x^\alpha \, dx, \quad |\alpha| \le d.$$

Since $P_B(f)$ is the orthogonal projection (in $L^2(B)$) of f into the subspace of polynomials of degree not greater than d,

$$\|P_B(f)\|_2 \leq \|f\|_2 \leq \|f\|_\infty,$$

where the L^2 norms are taken with respect to the normalized Lebesgue measure on B. We want now to compare the norms $\| \cdot \|_2$ and $\| \cdot \|_\infty$ on the space $P_d(B)$ of restrictions to B of polynomials of degree not greater than d. After appropriate translation and dilation we may assume that B has center 0 and radius 1. Since $P_d(B)$ is finite dimensional, there is a constant $C(d, n)$, depending only on d and n, such that

$$\|P\|_\infty \leq C(d, n)\, \|P\|_2, \quad P \in P_d(B),$$

and so

$$\|P_B(f)\|_\infty \leq C(d, n)\, \|f\|_\infty, \quad f \in L^\infty(B).$$

Therefore by (21)

$$|\ell(f - P_B(f))| \leq (1 + C(d, n))\, \|\ell\|\, \|f\|_\infty\, m(B)^{\frac{1}{p}}, \quad f \in L^\infty(B). \tag{22}$$

By (21) there is a measure ν_B on $\widehat{B}$ such that

$$\ell(f) = \int \widehat{f}\, d\nu_B, \quad f \in L^\infty_d(B). \tag{23}$$

Given a measure λ on $\widehat{B}$ there is a unique polynomial $P_B(\lambda) \in P_d(B)$ such that

$$\int_{\widehat{B}} \left(\pi(\phi) \right)^\alpha d\lambda(\phi) = \int_B P_B(\lambda)(x)\, x^\alpha\, dx, \quad |\alpha| \leq d\,.$$

Hence, for every polynomial Q of degree $\leq d$,

$$\int_{\widehat{B}} \widehat{Q}\, d\lambda = \int_B P_B(\lambda)\, Q\, dm = \int_{\widehat{B}} \widehat{P_B(\lambda)}\, \widehat{Q}\, d\widehat{m}\,.$$

Therefore, by (23),

$$\ell(f - P_B(f)) = \int_{\widehat{B}} (\widehat{f} - \widehat{P_B(f)})\, d\nu_B$$

$$= \int_{\widehat{B}} (\widehat{f} - \widehat{P_B(f)}) \left(d\nu_B - \widehat{P_B(\nu_B)}\, d\widehat{m} \right) \tag{24}$$

$$= \int_{\widehat{B}} \widehat{f} \left(d\nu_B - \widehat{P_B(\nu_B)}\, d\widehat{m} \right),$$

for each $f \in L^\infty(B)$. Hence the measure $\nu_B - \widehat{P_B(\nu_B)}\, \widehat{m}$ is determined by ℓ.

As before, with B_N denoting the ball of radius N centered at the origin, we fix a measure ν_1 on $\widehat{B_1}$ that represents ℓ on $L_d^\infty(B_1)$ and then take the unique measure ν_N on $\widehat{B_N}$ which represents ℓ on $L_d^\infty(B_N)$ and such that $P_{B_1}(\nu_N) = P_{B_1}(\nu_1)$. Then ν_N restricted to B_{N-1} is ν_{N-1} and so we can define a measure ν on $\widehat{\mathbb{R}^n}$ by requiring that ν restricted to B_N be ν_N.

Given any ball B, take N such that $B \subset B_N$. Then the restriction of ν to $L_d^\infty(B)$ is ℓ and so, by (22) and (24),

$$\left| \int_{\widehat{B}} \widehat{f} \, (d\nu - \widehat{P_B(\nu)} \, d\widehat{m}) \right| \le C \, \|f\|_\infty \, m(B)^{\frac{1}{p}}, \quad f \in L^\infty(B).$$

Hence

$$\left| \nu - \widehat{P_B(\nu)} \widehat{m} \right|(\widehat{B}) \le C \, m(B)^{\frac{1}{p}}. \tag{25}$$

Consider now the Radon-Nikodym decomposition of ν,

$$\nu = g \, \widehat{m} + \mu,$$

with μ singular with respect to $\widehat{m}$. We get, by (25) and Lemma 2,

$$|\mu|(\pi^{-1}(B)) \le |\mu|(\widehat{B}) \le C \, m(B)^{\frac{1}{p}},$$

for each open ball B. Since $0 < p < 1$, we readily conclude that $\mu = 0$. Indeed, let r be the radius of B. Covering B by $A_n k^n$ balls of radius r/k, we see that the constant C in the right-hand side of the above inequality can be replaced by $C A_n k^{n(1-\frac{1}{p})}$. Letting k tend to ∞, we get the conclusion.

Take now $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ with $g = \widehat{f}$. Then

$$\int_B |f - P_B(f)| \, dm \le C \, m(B)^{\frac{1}{p}},$$

which is precisely the condition that guarantees that f determines a bounded linear functional on $H^p(\mathbb{R}^n)$ ([TW]). Thus ℓ is a bounded linear functional on $H^p(\mathbb{R}^n)$ and the proof is complete. $\qquad\square$

References

[B] M. Bownik, *Boundedness of operators on Hardy spaces via atomic decompositions,* Proc. Amer. Math. Soc. 133(2005), 3535-3542.

[F]   G. Folland, *A Course in Abstract Harmonic Analysis,* Studies in Advanced Mathematics, CRC Press, 1995.

[MSV] S. Meda, P. Sjogren and M. Vallarino, *On the $H^1 - L^1$ boundedness of operators,* Proc. Amer. Math. Soc. to appear.

[MTW] Y. Meyer, M. H. Taibleson and G. Weiss *Some functional analytic properties of the spaces B_q generated by blocks,* Indiana U. Math. J. 34 (3)(1985), 493-515.

[R]   W. Rudin, *Functional Analysis,* International Series in Pure and Applied Mathematics, McGraw-Hill, Inc., New York, 1991, Second edition.

[TW]   M. H. Taibleson and G. Weiss, *The molecular characterization of certain Hardy spaces,* Asterisque 77, Societe Mathematique de France, (1980).

[YZ]   D. Yang and Y. Zhou, *A boundedness criterion via atoms for linear operators in Hardy spaces,* to appear in Constructive Approximation.

Fulvio Ricci
Scuola Normale Superiore di Pisa
Piazza dei Cavalieri 7
56126 Pisa, Italia
E-mail: `fricci@sns.it`

Joan Verdera
Departament de Matemàtiques
Universitat Autònoma de Barcelona
08193 Bellaterra, Barcelona, Catalonia
E-mail: `jvm@mat.uab.cat`