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The coherence of Łukasiewicz assessments is NP-complete

Simone Bova [a], Tommaso Flaminio [b,*]

[a] Department of Mathematics, Vanderbilt University, 1326 Stevenson Center, 37240-001 Nashville, TN, USA
[b] Department of Mathematics and Computer Science, University of Siena, Pian dei Mantellini 44, I-53100 Siena, Italy

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ABSTRACT

The problem of deciding whether a rational assessment of formulas of infinite-valued Łukasiewicz logic is coherent has been shown to be decidable by Mundici [1] and in PSPACE by Flaminio and Montagna [10]. We settle its computational complexity proving an NP-completeness result. We then obtain NP-completeness results for the satisfiability problem of certain many-valued probabilistic logics introduced by Flaminio and Montagna in [9].

1. Introduction

De Finetti's foundation of probability theory relies on the *coherence* of betting odds as follows [3–5][1]: Let $\phi_1,\dots,\phi_k$ be classical events and let $\mathbf{a}:\{\phi_1,\dots,\phi_k\}\to[0,1]$ be an assessment of $\phi_1,\dots,\phi_k$. Then $\mathbf{a}$ is said to be coherent if and only if there is no system of reversible bets on the events leading to a win independently of the truth of $\phi_1,\dots,\phi_k$. Precisely, the assessment $\mathbf{a}$ is coherent if and only if, for every $\mathbf{b}:\{\phi_1,\dots,\phi_k\}\to\mathbb{R}$, there exists a Boolean valuation $\mathbf{v}:\{\phi_1,\dots,\phi_k\}\to\{0,1\}$ such that

$$\sum_{i=1}^{k}\mathbf{b}(\phi_i)(\mathbf{a}(\phi_i)-\mathbf{v}(\phi_i))\geqslant 0. \tag{1}$$

The celebrated de Finetti's theorem states that an assessment $\mathbf{a}$ is coherent if and only if $\mathbf{a}$ coincides with the restriction to $\{\phi_1,\dots,\phi_k\}$ of a probability measure P from the free Boolean algebra generated by the ϕ_i's to $[0,1]$. In this case, we say that P *extends* $\mathbf{a}$, or that $\mathbf{a}$ *extends to* the probability measure P. The problem of checking whether or not a *rational* assessment $\mathbf{a}:\{\phi_1,\dots,\phi_k\}\to\mathbb{Q}\cap[0,1]$ is coherent is NP-complete [23].

A natural generalization of de Finetti's coherence criterion is obtained allowing an infinite-valued interpretation of events $\phi_1,\dots,\phi_k$, instead of their classical two-valued interpretation. A first attempt in this direction has been made by Paris [24], who firstly extended de Finetti's theorem to deal with a generalization of the classical Boolean semantics of the events, namely the semantics of $(n+1)$-valued Łukasiewicz logic [2]: an assessment $\mathbf{a}:\{\phi_1,\dots,\phi_k\}\to[0,1]$ is coherent if and only if $\mathbf{a}$ extends to a *state*[2] on the finite $(n+1)$-valued MV-algebra over $\{0,1/n,\dots,1\}$ freely generated by the ϕ_i's, if and only if for every $\mathbf{b}:\{\phi_1,\dots,\phi_k\}\to\mathbb{R}$, there exists a valuation $\mathbf{v}:\{\phi_1,\dots,\phi_k\}\to\{0,1/n,\dots,1\}$ satisfying (1). As a straightforward consequence of [13, Theorem 1] and [7, Theorem 4.4.1], deciding the coherence of $\mathbf{a}$ above is an NP-complete problem. In light

* Corresponding author.
E-mail addresses: simone.bova@vanderbilt.edu (S. Bova), flaminio@unisi.it (T. Flaminio).

[1] We refer the reader to Section 2.1 for the exact definitions of the relevant notions.

[2] *Compare Definition 2.2.*

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of Paris work, in [22] Mundici approaches the infinite-valued semantics for the events, showing that the coherence of an assessment $\mathbf{a} : \{\phi_1, \ldots, \phi_k\} \to [0,1]$ with respect to $[0,1]$-valued Łukasiewicz valuations is characterized by the existence of a state on the free MV-algebra generated by the ϕ's, extending $\mathbf{a}$. In recent work [15], Mundici and Kühr further extend this result to every $[0,1]$-valued algebraizable logic with continuous connectives.

In [22], Mundici shows that the coherence of rational Łukasiewicz assessments is decidable, and, as regards to the computational complexity of the problem, Flaminio and Montagna show that the problem is in PSPACE [10]. In this paper we settle the computational complexity issue, showing that the problem is NP-complete (Section 2). In light of this, in Section 3 we obtain NP-completeness results for the satisfiability problem of several classes of formulas of probabilistic logics introduced in [9,10], settling a problem raised by [13,10].

2. Łukasiewicz assessments complexity

In this section, we introduce the required background (Section 2.1), and we prove that the coherence of rational Łukasiewicz assessments is NP-complete (Section 2.2). Throughout, $[n] = \{1, 2, \ldots, n\}$.

2.1. Background

Let $\varsigma = (\odot, \oplus, \neg, \bot, \top)$ be a signature of type $(2,2,1,0,0)$. The set T of *formulas* over ς is the smallest set containing a countable set $X = \{X_1, X_2, \ldots\}$ of variables, $\bot, \top$, or strings of the form $(\neg\varphi), (\varphi \odot \psi), (\varphi \oplus \psi)$, with $\varphi, \psi \in T$. We let T_n denote the subset of T containing formulas over variables $X_1, \ldots, X_n$. Further binary operation symbols are defined as follows over the signature $\varsigma : \varphi \to \psi$ is $\neg\varphi \oplus \psi, \varphi \leftrightarrow \psi$ is $(\varphi \to \psi) \odot (\psi \to \varphi), \varphi \ominus \psi$ is $\neg(\varphi \to \psi), \varphi \vee \psi$ is $(\varphi \to \psi) \to \psi$, and $\varphi \wedge \psi$ is $\neg(\neg\varphi \vee \neg\psi)$. The algebra

$$[0,1]_{\mathrm{MV}} = ([0,1], \odot^{[0,1]}, \oplus^{[0,1]}, \neg^{[0,1]}, \bot^{[0,1]}, \top^{[0,1]}),$$

where $x \odot^{[0,1]} y = \max(0, x + y - 1), x \oplus^{[0,1]} y = \min(1, x + y), \neg^{[0,1]} x = 1 - x, \bot^{[0,1]} = 0$, and $\top^{[0,1]} = 1$, is called the *standard MV-algebra*. The variety of MV-algebras is generated, as a quasivariety, by $[0,1]_{\mathrm{MV}}$ [1].

Let $T_{\mathrm{MV}} = (T, \odot^T, \oplus^T, \neg^T, \bot^T, \top^T)$ be the MV-algebra over T defined by putting $\bot^T = \bot, \top^T = \top$, and for every $\varphi, \psi \in T, \varphi \odot^T \psi = (\varphi \odot \psi), \varphi \oplus^T \psi = (\varphi \oplus \psi), \neg^T \varphi = (\neg\varphi)$. We say that a formula $\phi \in T$ is *satisfiable* (respectively, *positive* satisfiable) in Łukasiewicz logic[3] if there exists a homomorphism (or a *valuation*) $\mathbf{v}$ from T_{MV} to $[0,1]_{\mathrm{MV}}$ such that $\mathbf{v}(\phi) = 1$ (respectively, $\mathbf{v}(\phi) > 0$). We let T_{MV_n} denote the subalgebra of T_{MV} generated by T_n.

Let $n \geqslant 1$. For every $\phi \in T_n$, we let $\phi^{[0,1]}$ denote the function from $[0,1]^n$ to [0,1] inductively defined as follows. For every $\mathbf{x} = (x_1, \ldots, x_n) \in [0,1]^n : \bot^{[0,1]}(\mathbf{x}) = 0, \top^{[0,1]}(\mathbf{x}) = 1, X_i^{[0,1]}(\mathbf{x}) = x_i, (\neg\varphi)^{[0,1]}(\mathbf{x}) = \neg^{[0,1]} \varphi^{[0,1]}(\mathbf{x}), (\varphi \odot \psi)^{[0,1]}(\mathbf{x}) = \varphi^{[0,1]}(\mathbf{x}) \odot^{[0,1]} \psi^{[0,1]}(\mathbf{x})$, $(\varphi \oplus \psi)^{[0,1]}(\mathbf{x}) = \varphi^{[0,1]}(\mathbf{x}) \oplus^{[0,1]} \psi^{[0,1]}(\mathbf{x})$. Letting

$$F_n = \{\phi^{[0,1]} | \phi \in T_n\},$$

the free n-generated MV-algebra is the algebra

$$F_{MV_n} = (F_n, \odot^F, \oplus^F, \neg^F, \bot^F, \top^F),$$

where $\bot^F = \bot^{[0,1]}, \top^F = \top^{[0,1]}$, and the operations are the operations of $[0,1]_{\mathrm{MV}}$ defined pointwise; for instance, if $f \in F_n$, then $\neg^F f$ is the function such that for every $\mathbf{x} = (x_1, \ldots, x_n) \in [0,1]^n, (\neg^F f)(\mathbf{x}) = \neg^{[0,1]}(f(\mathbf{x})) = 1 - (f(\mathbf{x}))$.

The free n-generated MV-algebra has a nice functional representation in terms of n-ary *McNaughton functions*. An n-ary McNaughton function is a continuous function $f : [0,1]^n \to [0,1]$ defined as follows: there exist linear polynomials $p_1, \ldots, p_k$ with integer coefficients such that for every $\mathbf{x} \in [0,1]^n$, there exists $j \in [k]$ such that $f(\mathbf{x}) = p_j(\mathbf{x})$. From now on, we call the polynomials $p_1, \ldots, p_k$ the *linear pieces* of f. A routine induction on $\phi \in T_n$ shows that the function $\phi^{[0,1]} : [0,1]^n \to [0,1]$ is a McNaughton function. Conversely, for every n-ary McNaughton function f, there exists a formula $\phi \in T_n$ such that $\phi^{[0,1]} = f$ [16]. Indeed, in [20] Mundici describes an explicit nontrivial construction of ϕ, which we now sketch.

Let f be a given n-ary McNaughton function, and let $p_1, \ldots, p_k$ be its linear pieces. For every permutation π of $[k]$, let:

$$P_\pi = \{\mathbf{x} \in [0,1]^n : p_{\pi(1)}(\mathbf{x}) \leqslant p_{\pi(2)}(\mathbf{x}) \leqslant \cdots \leqslant p_{\pi(k)}(\mathbf{x})\}, \quad (2)$$

$$C = \{P_\pi : P_\pi \text{ is } n\text{-dimensional}\}. \quad (3)$$

We observe that C is a finite set of n-dimensional polyhedra with rational vertices V, that is, every $P_\pi \in C$ is the convex hull of a finite set of rational points in $[0,1]^n$. Along the lines of [2, Proposition 3.3.1], C can be manufactured into a *unimodular partition* of $[0,1]^n$ that *linearizes* f, that is, a finite set S of n-dimensional *unimodular simplexes* over the rational vertices V,

[3] See [2] for an axiomatization of Łukasiewicz logic.

enjoying the following three properties[4]: (i) the union of all simplexes in S is equal to $[0,1]^n$; (ii) any two simplexes in S intersect in a common face; (iii) for each simplex $T \in S$, there exists $j \in [k]$ such that the restriction of f to T coincides with p_j. We also say that f is *linear* over S.

Let $\mathbf{q}$ be a vertex of a simplex in the unimodular partition S. The *Schauder hat* at $\mathbf{q}$ is the McNaughton function $h_{\mathbf{q}}$ linearized by S such that $h_{\mathbf{q}}(\mathbf{q}) = 1/\mathrm{den}(\mathbf{q})$ and $h_{\mathbf{q}}(\mathbf{r}) = 0$ for every vertex $\mathbf{r}$ distinct from $\mathbf{q}$ in S. The *normalized* Schauder hat at $\mathbf{q}$ is the function $k_{\mathbf{q}} = \mathrm{den}(\mathbf{q}) \cdot h_{\mathbf{q}}$. Note that every McNaughton function that is linear over S is a linear combination of the family of Schauder hats corresponding to S, where each hat $h_{\mathbf{q}}$ has a uniquely determined integer coefficient between 0 and $\mathrm{den}(\mathbf{q})$. Thus in particular, as the given function f is linear over S,

$$f = \sum_{\mathbf{q} \in V} a_{\mathbf{q}} \cdot h_{\mathbf{q}}, \tag{4}$$

for uniquely determined integers $0 \leqslant a_{\mathbf{q}} \leqslant \mathrm{den}(\mathbf{q})$.

The core of the construction (appealing to unimodularity) yields for every vertex $\mathbf{q}$ in S a formula $\chi_{\mathbf{q}} \in T_n$ such that

$$\chi_{\mathbf{q}}^{[0,1]} = h_{\mathbf{q}};$$

the desired formula in T_n is then attained by putting

$$\phi = \bigoplus_{\mathbf{q} \in V} a_{\mathbf{q}} \chi_{\mathbf{q}},$$

where $a_{\mathbf{q}} \chi_{\mathbf{q}} = \chi_{\mathbf{q}} \oplus \cdots \oplus \chi_{\mathbf{q}}$ ($a_{\mathbf{q}}$ times).

The functional representation of the free n-generated MV-algebra, sketched above, plays a central rôle in the technical development of the rest of the paper. In particular, the following two facts will be used in the proof of the main lemma.

Proposition 2.1. *Let S be a unimodular partition of $[0,1]^n$ over vertices $\mathbf{q}_1, \ldots, \mathbf{q}_m$. Then:*

(i) Let $h_{\mathbf{q}_i}$ and $h_{\mathbf{q}_j}$ be (normalized) Schauder hats at vertices $\mathbf{q}_i$ and $\mathbf{q}_j$, respectively for $i \neq j \in [m]$. Then,

$$h_{\mathbf{q}_i} \odot^{[0,1]} h_{\mathbf{q}_j} = \perp^{[0,1]}.$$

(ii) Let $k_{\mathbf{q}_1}, \ldots, k_{\mathbf{q}_m}$ be the normalized Schauder hats at the vertices of S. Then,

$$\bigoplus_{i=1}^{m} k_{\mathbf{q}_i} = \top^{[0,1]}.$$

Proof. (i) Simply notice that for each vertex $\mathbf{q}_i$, all (normalized) Schauder hats over S, with the exception of $h_{\mathbf{q}_i}$, vanish. Hence, for each $\mathbf{x} \in [0,1]^n$, it holds that $h_{\mathbf{q}_i} \odot^{[0,1]} h_{\mathbf{q}_j}(\mathbf{x}) = 0 = \perp^{[0,1]}(\mathbf{x})$. In order to prove (*ii*) recall that, for any $\mathbf{q}_i$, $k_{\mathbf{q}_i}(\mathbf{q}_i) = 1$, whence the claim follows from the linearity of each normalized hat over every simplex of S. □

As already mentioned in the introduction, the notion of *state* is key for the investigation of coherent assessments of formulas of several [0,1]-valued logics and Łukasiewicz logic in particular. Normalized and additive maps on MV-algebras have been introduced by Kôpka and Chovanec in [14], and then by Mundici under the name of *MV-algebraic states* (or simply *states*) in [21].

Definition 2.2. Let $A = (A, \odot, \oplus, \neg, \perp, \top)$ be an MV-algebra. A *state* of A is a map $\mathbf{s} : A \to [0,1]$ satisfying *normality*, that is,

$$\mathbf{s}(\top) = 1,$$

and *additivity*, that is, for all $x, y \in A$,

$$\mathbf{s}(x \oplus y) = \mathbf{s}(x) + \mathbf{s}(y)$$

whenever $x \odot y = \perp$.

Proposition 2.3 [11]. *For any $m \in \mathbb{N}$, and any Schauder hat $h_{\mathbf{q}} \in F_{MV_m}$, if $\mathbf{s} : F_{MV_m} \to [0,1]$ is a state, then* $\mathbf{s}(\mathrm{den}(\mathbf{q}) \cdot h_{\mathbf{q}}) = \mathrm{den}(\mathbf{q}) \cdot \mathbf{s}(h_{\mathbf{q}})$.

Proof. Immediate from Proposition 2.1, and Definition 2.2. □

As already mentioned in the introduction, in this paper we study the complexity of the problem of deciding whether or not a Łukasiewicz assessment is coherent. This purpose allows for restricting to rational, decidable assessments.

[4] An n-dimensional *simplex* is the convex hull of $n+1$ affinely independent vertices. The empty set $\emptyset$ is a (-1)-dimensional simplex. A k-dimensional *face* of the n-simplex T over vertices $\mathbf{q}_1, \ldots, \mathbf{q}_{n+1}$ is the k-simplex spanned by $k+1$ vertices of T. Let T be an n-dimensional simplex over rational vertices. Let $\mathbf{q} = (a_1/d, \ldots, a_n/d)$ be a vertex of T, for uniquely determined relatively prime integers $a_1, \ldots, a_n, d$ with $d \geqslant 1$. Call $(a_1, \ldots, a_n, d)$ the *homogeneous coordinates* of $\mathbf{q}$, and call $\mathrm{den}(\mathbf{q}) = d$ the *denominator* of $\mathbf{q}$. Then, T is *unimodular* if the absolute value of the determinant of the integer square matrix having the homogeneous coordinates of the ith vertex as its ith row, for all $i \in [n+1]$, is equal to 1. A k-dimensional simplex ($k \leqslant n$) is unimodular if it is a face of some unimodular n-dimensional simplex.

Definition 2.4. Let $\phi_1, \ldots, \phi_k$ be formulas in T_m. A *(Łukasiewicz) assessment* is a map

$$\mathbf{a} : \{\phi_1, \ldots, \phi_k\} \to [0,1]. \tag{5}$$

An assessment $\mathbf{a}$ is *rational* if $\mathbf{a}(\phi_i) \in \mathbb{Q}$ for every $i \in [k]$, and is *coherent* if for every $\mathbf{b} : \{\phi_1, \ldots, \phi_k\} \to \mathbb{R}$, there exists a homomorphism $\mathbf{v}$ from T_{MV_m} to $[0,1]_{\mathrm{MV}}$ such that

$$\sum_{i \in [k]} \mathbf{b}(\phi_i)(\mathbf{a}(\phi_i) - \mathbf{v}(\phi_i)) \geqslant 0. \tag{6}$$

The main technical lemma of the present work, Lemma 2.7, is a sharpening of the following result of Kühr and Mundici.

Theorem 2.5 (Kühr–Mundici). *Let* $\mathbf{a}$ *be an assessment on* $\phi_1, \ldots, \phi_k \in T_m$. *Then, the following are equivalent:*

(i) $\mathbf{a}$ is coherent.
(ii) There exists a state $\mathbf{s}$ over F_{MV_m} such that, for all $i \in [k]$,

$$\mathbf{s}(\phi_i^{[0,1]}) = \mathbf{a}(\phi_i).$$

(iii) There exist $l \leqslant k+1$ homomorphisms $\mathbf{v}_1, \ldots, \mathbf{v}_l$ from T_{MV_m} to $[0,1]_{\mathrm{MV}}$ such that $(\mathbf{a}(\phi_i))_{i\in[k]}$ is a convex combination of $(\mathbf{v}_1(\phi_i))_{i\in[k]}, \ldots, (\mathbf{v}_l(\phi_i))_{i\in[k]}$.

Proof. (i) $\iff$ (ii) is [22, Theorem 2.1]. (i) $\iff$ (iii) is [15, Theorem 3.2]. $\square$

It is worth noticing that the above theorem is stated in a more general way in [15]. Indeed, the authors characterize coherent assessments of formulas of any $[0,1]$-valued logic whose connectives are continuous functions with respect to the usual topology of the real unit interval [0,1], including the logic RŁ [11], Rational Pavelka logic [12], and the logic PMV+ [17,18].

We now prepare some terminology and notation in view of the main lemma.

We assume a reasonably compact binary encoding of $\phi \in T$, such that the number size(ϕ) of bits in the encoding of ϕ is bounded above by a polynomial $e_1 : \mathbb{N} \to \mathbb{N}$ of the number $c(\phi)$ of symbols $\odot, \to$ occurring in ϕ, that is,

$$\mathrm{size}(\phi) \leqslant e_1(c(\phi_i)).$$

We similarly assume that the length in bits of the encoding of a finite set of formulas $\{\phi_1, \ldots, \phi_k\} \subseteq T$, in symbols size($\{\phi_1, \ldots, \phi_k\}$) satisfies

$$\mathrm{size}(\{\phi_1, \ldots, \phi_k\}) \leqslant e_2(\mathrm{size}(\phi_1) + \cdots + \mathrm{size}(\phi_k)),$$

for some polynomial $e_2 : \mathbb{N} \to \mathbb{N}$. Also, letting $\mathbf{a} : \{\phi_1, \ldots, \phi_k\} \to [0,1]$ be a rational assessment such that $\mathbf{a}(\phi_i) = n_i/d_i$ with n_i and d_i relatively prime integers for all i in $[k]$, we assume a binary encoding of $\mathbf{a}$ such that the number of bits in the encoding of $\mathbf{a}$, in symbols, size($\mathbf{a}$), satisfies

$$\mathrm{size}(\mathbf{a}) \leqslant e_3(\mathrm{size}(\{\phi_1, \ldots, \phi_k\}) + k \cdot \log_2 \max\{d_1, \ldots, d_k\}),$$

for some polynomial $e_3 : \mathbb{N} \to \mathbb{N}$.

Proposition 2.6. *Let* $\phi_1, \ldots, \phi_k \in T_m$ *for some* $k \geqslant 1$. *Then, there exist a unary polynomial* $q : \mathbb{N} \to \mathbb{N}$, *and a unimodular partition* S *of* $[0,1]^m$ *linearizing* $\phi_1, \ldots, \phi_k$, *such that each rational vertex* $\mathbf{x}$ *of* S *satisfies*

$$\log_2 \mathrm{den}(\mathbf{x}) \leqslant q(\mathrm{size}(\{\phi_1, \ldots, \phi_k\})).$$

Proof. The statement is an application of [2, Proposition 9.3.3].

For all $i \in [k]$, let $f_i = \phi_i^{[0,1]}$ be the m-ary McNaughton function corresponding to formula $\phi_i \in T_m$. Let $p_1, \ldots, p_l$ be the list of *all* linear pieces of the functions $f_1, \ldots, f_k$, together with $x_1, \ldots, x_m, 0, 1$, and define P_π as in (2) and C as in (3) based on these pieces. Let V be the vertices of C, and let S be a unimodular partition manufactured from C without adding new vertices, as explained in Section 2.1. We show that S satisfies the statement.

First, since C includes all the linear pieces of all the functions $f_1, \ldots, f_k$, and S is a subdivision of C, it follows that S linearizes each of the functions $f_1, \ldots, f_k$.

Second, recall that by the definition of McNaughton function, each piece p_i has the form

$$p_i(x_1, \ldots, x_m) = c_{i,1}x_1 + \cdots + c_{i,m}x_m + d_m,$$

with $c_{i,1}, \ldots, c_{i,m}, d_m \in \mathbb{Z}$. Thus, by inspection of (2), each $\mathbf{x} \in V$ is the rational solution of a system of m linear equations in m unknowns, each equation having either form $p_h(x_1, \ldots, x_m) = p_i(x_1, \ldots, x_m), p_h(x_1, \ldots, x_m) = 0, p_h(x_1, \ldots, x_m) = 1$ for $h, i \in [l]$, or $x_i = 0, x_i = 1$ for $i \in [m]$. Suppose that p_i is a linear piece of f_j. A routine induction on ϕ_j shows that

$$|c_{i,1}|, \ldots, |c_{i,m}| \leqslant \mathrm{size}(\phi_j).$$

Hence, the largest coefficient (in absolute value) of any linear piece amongst $p_1,\dots,p_l$ is bounded above by

$$\max\{\mathrm{size}(\phi_j) | j\in[k]\}\leqslant \mathrm{size}(\{\phi_1,\dots,\phi_k\}),$$

so that the nth equation in the linear system having $\mathbf{x}$ as solution has the form

$$a_{n,1}x_1+\cdots+a_{n,m}x_m=b_n,$$

with

$$|a_{n,1}|,\dots,|a_{n,m}|\leqslant 2\cdot \mathrm{size}(\{\phi_1,\dots,\phi_k\}). \tag{7}$$

As

$$\mathbf{x}=\begin{pmatrix} a_{1,1} & \dots & a_{1,m}\\ \vdots & \ddots & \vdots\\ a_{m,1} & \dots & a_{m,m}\end{pmatrix}^{-1}\begin{pmatrix} b_1\\ \vdots\\ b_m\end{pmatrix}=A^{-1}\mathbf{b},$$

it follows that $\mathrm{den}(\mathbf{x})\leqslant|\det(A)|$ by elementary linear algebra [25]. In light of (7), an application of Hadamard's inequality now yields the desired bound,

$$\begin{aligned}|\det(A)| &\leqslant \prod_{i\in[m]}(a_{i,1}^2+\cdots+a_{i,m}^2)^{1/2}\\ &\leqslant \prod_{i\in[m]}|a_{i,1}|+\cdots+|a_{i,m}|\\ &\leqslant \prod_{i\in[m]} m\cdot 2\cdot \mathrm{size}(\{\phi_1,\dots,\phi_k\})\\ &\leqslant 2^{2m\log_2 m\cdot \mathrm{size}(\{\phi_1,\dots,\phi_k\})}\\ &\leqslant 2^{q(\mathrm{size}(\{\phi_1,\dots,\phi_k\}))},\end{aligned}$$

by putting

$$q(n)=n^2$$

and noticing that $m\leqslant \mathrm{size}(\{\phi_1,\dots,\phi_k\})$, because the size of a set of formulas over m distinct variables is greater than or equal to m. $\square$

Lemma 2.7. *Let $\phi_1,\dots,\phi_k\in T_m$, and let $\mathbf{a}:\{\phi_1,\dots,\phi_k\}\to[0,1]\cap\mathbb{Q}$ be a Łukasiewicz assessment. The following are equivalent:*

(i) $\mathbf{a}$ is coherent.
(ii) There exist a unary polynomial $p:\mathbb{N}\to\mathbb{N}$, and $l\leqslant k+1$ homomorphisms $\mathbf{v}_1,\dots,\mathbf{v}_l$ from T_{MV_m} to $[0,1]_{\mathrm{MV}}$ satisfying the following. For all $i\in[l]$, $\mathbf{v}_i$ ranges over $\{0,1/d_i,\dots,(d_i-1)/d_i,1\}$, where

$$\log_2 d_i\leqslant p(\mathrm{size}(\mathbf{a})),$$

and $(\mathbf{a}(\phi_i))_{i\in[k]}$ is a convex combination of $(\mathbf{v}_1(\phi_i))_{i\in[k]},\dots,(\mathbf{v}_l(\phi_i))_{i\in[k]}$.

Proof. (i $\Rightarrow$ ii) Let $\mathbf{s}$ be a state over F_{MV_m} satisfying Theorem 2.5, and let S be a unimodular partition satisfying Proposition 2.6. Let $\mathbf{q}_1,\dots,\mathbf{q}_n$ be the rational vertices of S, and let $d_1=\mathrm{den}(\mathbf{q}_1),\dots,d_n=\mathrm{den}(\mathbf{q}_n)$. Let $\chi_i^{[0,1]}=k_{\mathbf{q}_i}$ be the normalized Schauder hat at vertex $\mathbf{q}_i$, $i\in[n]$. Let $x_1,\dots,x_n\in\mathbb{R}$ be defined by putting, for all $i\in[n]$,

$$x_i=\mathbf{s}(\chi_i^{[0,1]}).$$

Then,

$$\begin{aligned}x_1+\cdots+x_n &= \mathbf{s}(\chi_1^{[0,1]})+\cdots+\mathbf{s}(\chi_n^{[0,1]})\\ &= \mathbf{s}((\chi_1\oplus\cdots\oplus\chi_n)^{[0,1]}) \quad \text{by Proposition 2.1(1)} \quad \text{and additivity of } \mathbf{s}\\ &= \mathbf{s}(\top^{[0,1]}) \quad \text{by Proposition 2.1(2)}\\ &= 1.\end{aligned}$$

Let $\vartheta_i^{[0,1]}=\chi_i^{[0,1]}/d_i=h_{\mathbf{q}_i}$ be the Schauder hat at vertex $\mathbf{q}_i$, $i\in[n]$. For all $i\in[k]$, $\phi_i^{[0,1]}$ is a McNaughton function linearized by S, and for $j\in[n]$, $\vartheta_j^{[0,1]}$ is the Schauder hat at vertex $\mathbf{q}_j$ of S. Thus by (4) there is a unique choice of integers $0\leqslant a_{i,j}\leqslant \mathrm{den}(\mathbf{q}_j)\leqslant 1$ such that $\phi_i^{[0,1]}$ satisfies the following equation:

$$\phi_i^{[0,1]}=\sum_{j\in[n]}a_{i,j}\cdot\vartheta_j^{[0,1]}.$$

For all $i \in [n]$, let $\mathbf{w}_i$ be the homomorphism from T_{MV_m} to $[0,1]_{\mathrm{MV}}$ defined by putting, for every $\varphi \in T_m$,

$$\mathbf{w}_i(\varphi) = \varphi^{[0,1]}(\mathbf{q}_i).$$

Note that $\mathbf{w}_i$ ranges over $\{0, 1/d_i, \ldots, (d_i-1)/d_i, 1\}$, and by Proposition 2.6,

$$\log_2 d_i = \log_2 \mathrm{den}(\mathbf{q}_i) \leqslant q(\mathrm{size}(\{\phi_1, \ldots, \phi_k\})) \leqslant p(\mathrm{size}(\mathbf{a})),$$

letting

$$p(n) = q(n) = n^2,$$

as $\mathrm{size}(\{\phi_1, \ldots, \phi_k\}) \leqslant \mathrm{size}(\mathbf{a})$.

Now compute, for every $j \in [k]$,

$$\begin{aligned}
\sum_{i\in[n]} x_i \cdot \mathbf{w}_i(\phi_j) &= \sum_{i\in[n]} x_i \cdot \phi_j^{[0,1]}(\mathbf{q}_i)\\
&= \sum_{i\in[n]} \mathbf{s}\left(\chi_i^{[0,1]}\right) \cdot a_{j,i}/d_i\\
&= \sum_{i\in[n]} \mathbf{s}\left(d_i \cdot \vartheta_i^{[0,1]}\right) \cdot a_{j,i}/d_i\\
&= \sum_{i\in[n]} \mathbf{s}\left(a_{j,i} \cdot \vartheta_i^{[0,1]}\right) \quad \text{by Proposition 2.3}\\
&= \sum_{i\in[n]} \mathbf{s}\left(a_{j,i}/d_i \cdot \chi_i^{[0,1]}\right)\\
&= \mathbf{s}\left(\bigoplus_{i\in[n]} a_{j,i}/d_i \cdot \chi_i^{[0,1]}\right) \quad \text{by Proposition 2.1(1)} \quad \text{and additivity of } \mathbf{s}\\
&= \mathbf{s}(\phi_j^{[0,1]}).
\end{aligned}$$

As $x_1 + \cdots + x_n = 1$, the point $(\mathbf{a}(\phi_i))_{i\in[k]}$ is a convex combination of points $(\mathbf{w}_1(\phi_i))_{i\in[k]}, \ldots, (\mathbf{w}_n(\phi_i))_{i\in[k]}$. By Carathéodory's theorem [6, Theorem 2.3], there exists a choice of $l \leqslant k+1$ homomorphisms $\mathbf{v}_1, \ldots, \mathbf{v}_l$ amongst $\mathbf{w}_1, \ldots, \mathbf{w}_n$ such that $(\mathbf{a}(\phi_i))_{i\in[k]}$ is a convex combination of

$$(\mathbf{v}_1(\phi_i))_{i\in[k]}, \ldots, (\mathbf{v}_l(\phi_i))_{i\in[k]},$$

and we are done.

(ii ⇒ i) A routine verification shows that the map $\mathbf{s}$ from F_{MV_m} to $[0,1]$ defined by putting

$$\mathbf{s}(\varphi^{[0,1]}) = \sum_{i\in[n]} x_i \cdot \mathbf{v}_i(\varphi)$$

for every $\varphi \in T_m$, is a state; hence, $\mathbf{a}$ is coherent by Theorem 2.5. We refer the reader to [22] for details.[5] □

2.2. Complexity

Let $\langle \mathbf{a} \rangle$ denote the binary encoding of a rational Łukasiewicz assessment $\mathbf{a}$. The problem of deciding coherence of rational Łukasiewicz assessments is defined as follows:

$$\text{LUK-COH} = \{\langle \mathbf{a} \rangle | \mathbf{a} \text{ is a coherent rational } \textit{Łukasiewicz assessment}\}.$$

In the next two paragraphs we prove that LUK-COH is in NP (Lemma 2.9), and is NP-hard (Lemma 2.10), thus

Theorem 2.8. LUK-COH *is NP-complete.*

Upper bound. It is known that the feasibility problem of linear systems is decidable in polynomial time in the size of the binary encoding of the linear system [25]. Therefore, Lemma 2.7 directly furnishes a nondeterministic polynomial time algorithm for the coherence problem, as follows.

Lemma 2.9. LUK-COH *is in NP.*

Proof. Let $\mathbf{a} : \{\phi_1, \ldots, \phi_k\} \to [0,1] \cap \mathbb{Q}$ be a Łukasiewicz assessment, where $\phi_1, \ldots, \phi_k$ are over variables $X_1, \ldots, X_m$. Following Lemma 2.7, the algorithm guesses a natural number $l \leqslant k+1$ and, for all $i \in [l]$, the algorithm guesses the denominator d_i, the restriction of homomorphism v_i to variables $X_1, \ldots, X_m$, and eventually checks the feasibility of the following linear system:

[5] We do no appeal to this direction later.

$$
\begin{aligned}
&x_1 + \cdots + x_{l-1} + x_l = 1\\
&\mathbf{v}_1(\phi_1)x_1 + \cdots + \mathbf{v}_{l-1}(\phi_1)x_{l-1} + \mathbf{v}_l(\phi_1)x_l = \mathbf{a}(\phi_1)\\
&\qquad\qquad\vdots\\
&\mathbf{v}_1(\phi_k)x_1 + \cdots + \mathbf{v}_{l-1}(\phi_k)x_{l-1} + \mathbf{v}_l(\phi_k)x_l = \mathbf{a}(\phi_k)
\end{aligned}
$$

By Lemma 2.7, for all $i \in [l]$, the denominator d_i has a polynomial-space encoding. Hence, the restriction of $\mathbf{v}_i$ to $X_1, \ldots, X_m$, as well as the coefficients $\mathbf{v}_i(\phi_1), \ldots, \mathbf{v}_i(\phi_k)$, are in $\{0, 1/d_i, \ldots, (d_i - 1)/d_i, 1\}$. So, the size of the system is polynomial in size($\mathbf{a}$), and the algorithm terminates in time polynomial in size($\mathbf{a}$). Noticing that the linear system is feasible if and only if $\mathbf{a}$ is a convex combination of $\mathbf{v}_1, \ldots, \mathbf{v}_l$ if and only if $\mathbf{a}$ is coherent, we are done. □

Lower bound. Let $\langle\phi\rangle$ denote the binary encoding of the formula $\phi \in T$. In [19], it is proved that the problem

$$\mathsf{LUK\text{-}SAT} = \{\langle\phi\rangle \mid \phi \text{ is satisfiable in } \textit{Łukasiewicz logic}\}$$

is NP-complete.

Lemma 2.10. LUK-COH *is NP-hard.*

Proof. We describe a logarithmic-space reduction from the NP-hard problem LUK-SAT to LUK-COH.

Let $\phi \in T_m$. Let $\mathbf{a}$ be the assessment sending formulas $X_1 \oplus \neg X_1, \ldots, X_m \oplus \neg X_m$, and ϕ to 1, that is,

$$\mathbf{a}(X_1 \oplus \neg X_1) = \cdots = \mathbf{a}(X_m \oplus \neg X_m) = \mathbf{a}(\phi) = 1.$$

The construction of the assessment $\mathbf{a}$ is feasible in space logarithmic in size(ϕ). We show that $\mathbf{a}$ is coherent if and only if ϕ is satisfiable in Łukasiewicz logic.

($\Rightarrow$) Suppose that $\mathbf{a}$ is coherent. Let $b_i = -1$ for all $i \in [m+1]$, and let $\mathbf{v}$ be a homomorphism from T_{MV} to $[0,1]_{\mathrm{MV}}$ such that (6) holds that is,

$$\mathbf{a}(\phi) - \mathbf{v}(\phi) \leqslant \sum_{i \in [m]} (\mathbf{v}(X_i \oplus \neg X_i) - \mathbf{a}(X_i \oplus \neg X_i)).$$

As $\mathbf{v}(X_i \oplus \neg X_i) = 1 = \mathbf{a}(X_i \oplus \neg X_i)$ for every $i \in [m]$, the right hand side vanishes so that

$$1 = \mathbf{a}(\phi) \leqslant \mathbf{v}(\phi) \leqslant 1.$$

($\Leftarrow$) Let $\mathbf{v}$ be a homomorphism from T_{MV} to $[0,1]_{\mathrm{MV}}$ such that $\mathbf{v}(\phi) = 1$. Let $\mathbf{s}$ be a map from F_{MV_m} to [0,1] defined by putting, for every $\varphi^{[0,1]} \in F_m$,

$$\mathbf{s}(\varphi^{[0,1]}) = \mathbf{v}(\varphi).$$

We claim that $\mathbf{s}$ is a state over the free m-generated MV-algebra: For normality, $\mathbf{s}(\top^{[0,1]}) = \mathbf{v}(\top) = 1$. For additivity, if $(\varphi \odot \psi)^{[0,1]} = \bot^{[0,1]}$, then

$$0 = \mathbf{v}(\bot) = \mathbf{s}(\bot^{[0,1]}) = \mathbf{s}((\varphi \odot \psi)^{[0,1]}) = \mathbf{v}(\varphi \odot \psi) = \max(0, \mathbf{v}(\varphi) + \mathbf{v}(\psi) - 1),$$

hence $\mathbf{v}(\varphi) + \mathbf{v}(\psi) \leqslant 1$. Thus,

$$\mathbf{s}((\varphi \oplus \psi)^{[0,1]}) = \mathbf{v}(\varphi \oplus \psi) = \min(1, \mathbf{v}(\varphi) + \mathbf{v}(\psi)) = \mathbf{v}(\varphi) + \mathbf{v}(\psi) = \mathbf{s}(\varphi^{[0,1]}) + \mathbf{s}(\psi^{[0,1]}).$$

As

$$\mathbf{s}(\phi^{[0,1]}) = \mathbf{v}(\phi) = 1 = \mathbf{a}(\phi),$$

and for every $i \in [m]$,

$$\mathbf{s}((X_i \oplus \neg X_i)^{[0,1]}) = \mathbf{v}(X_i \oplus \neg X_i) = 1 = \mathbf{a}(X_i \oplus \neg X_i),$$

$\mathbf{a}$ is coherent by Theorem 2.5. □

3. Probabilistic *SFP*-formulas complexity

In [9] the authors introduce the logic *SFP*(Ł, Ł) as the natural algebraizable extension of the probabilistic logic *FP*(Ł,Ł) introduced in [8] and studied by Hájek in [13].

The language of *SFP*(Ł, Ł) is obtained by extending that of Łukasiewicz logic by the unary modality Pr, and defining formulas in the usual way. The axioms of *SFP*(Ł, Ł) are those of Łukasiewicz logic [12], together with the following schemata for the modality Pr:

($P1$) $\Pr(\bot) \leftrightarrow \bot$
($P2$) $\Pr(\Pr(\varphi) \oplus \Pr(\psi)) \leftrightarrow \Pr(\varphi) \oplus \Pr(\psi)$

$(P3)$ $\Pr(\varphi \to \psi) \to (\Pr(\varphi) \to \Pr(\psi))$
$(P4)$ $\Pr(\varphi \oplus \psi) \leftrightarrow [\Pr(\varphi) \oplus \Pr(\psi \ominus (\varphi \odot \psi))]$

Rules are modus ponens: from φ and $\varphi \to \psi$, derive ψ, and necessitation: from φ, derive $\Pr(\varphi)$.

We let SFP denote the set of formulas of *SFP*(Ł, Ł). In the sequel we assume, without loss of generality, that formulas in SFP do not contain occurrences of subformulas of the form $\Pr(\Pr(\phi))$. Indeed, $\Pr(\Pr(\phi))$ is logically equivalent to $\Pr(\phi)$ by axiom $(P2)$, and in any formula in SFP we can substitute subformulas by a logically equivalent formulas [10]. Therefore, in logarithmic space, it is possible to replace in a formula of SFP any occurrence of the subformula $\Pr(\Pr(\phi))$ with the formula $\Pr(\phi)$, until no nested applications of Pr occur.

The equivalent algebraic semantics of *SFP*(Ł, Ł) is the variety of SMV-algebras [9]. These are pairs (A, σ), where A is an MV-algebra, and σ is a unary operation on A satisfying the following equations:

$(\sigma 1)$ $\sigma(0) = 0$.
$(\sigma 2)$ $\sigma(\neg x) = \neg\sigma(x)$.
$(\sigma 3)$ $\sigma(\sigma(x) \oplus \sigma(y)) = \sigma(x) \oplus \sigma(y)$.
$(\sigma 4)$ $\sigma(x \oplus y) = \sigma(x) \oplus \sigma(y \ominus (x \odot y))$.

In any SMV-algebra (A, σ), the image of A under σ, in symbols, $\sigma(A)$, is the domain of an MV-subalgebra of A. An SMV-algebra (A, σ) is said to be *σ-simple* provided that A is semisimple, and $\sigma(A)$ is simple.[6] Hence, up to isomorphism, any σ-simple SMV-algebra (A, σ) is made of an MV-algebra A of real-valued continuous functions, such that $\sigma(A)$ is an MV-subalgebra of $[0,1]_{\mathrm{MV}}$ [10].

A valuation of *SFP*(Ł, Ł) into an SMV-algebra A is a map $\mathbf{v}$ sending the variables X to the domain A. Any valuation $\mathbf{v}$ extends uniquely to a map from formulas in SFP by the following inductive stipulations: $\mathbf{v}$ commutes with Łukasiewicz connectives, and for every $\psi \in \mathrm{SFP}, \mathbf{v}(\Pr(\psi)) = \sigma(\mathbf{v}(\psi))$.

Formulas in SFP can be interpreted by a different class of structures, as follows [12,10].

Definition 3.1 (Probabilistic Kripke models). A *probabilistic Kripke model* (*Kripke model* henceforth) is a triple $K = (W, e, \mu)$ where:

$(K1)$ W is a finite or countable set of nodes, called *worlds*.
$(K2)$ e is a map from pairs in $W \times X$ to [0,1]. For any fixed world $w \in W, e(w, \cdot)$ extends uniquely to a valuation of formulas in T in [0,1], in the usual way.
$(K3)$ $\mu : W \to [0,1]$ is a function such that $\sum_{w \in W} \mu(w) = 1$.

Let $\phi \in \mathrm{SFP}$, let $K = (W, e, \mu)$ be a Kripke model, and let $w \in W$. Then the truth value of ϕ in K at the world w, in symbols $\|\phi\|_{K,w}$, is inductively defined as follows:

(i) if ϕ is a variable X_i, then $\|X_i\|_{K,w} = e(w, X_i)$;
(ii) if $\phi = \Pr(\psi), \|\Pr(\psi)\|_{K,w} = \sum_{w' \in W} \mu(w') \cdot \|\psi\|_{K,w'}$;
(iii) $\|\cdot\|_{K,w}$ commutes with Łukasiewicz connectives.

Remark 1. If ϕ is such that all of its Łukasiewicz subformulas occur under the scope of the modality Pr, then its truth value does not depend on the chosen world w, so that, in this case, we simply write $\|\phi\|_K$.

Definition 3.2. Let ϕ be in SFP. Then:

(i) ϕ is *SMV-satisfiable*, or $\phi \in$ SMV-SAT, if there is an SMV-algebra (A, σ) and a valuation $\mathbf{v}$ into A such that $\mathbf{v}(\phi) = 1$. ϕ is *standard satisfiable*, or $\phi \in$ STD-SAT, if there is a σ-simple SMV-algebra (A, σ) and a valuation $\mathbf{v}$ into A such that $\mathbf{v}(\phi) = 1$. ϕ is *Kripke-satisfiable*, or $\phi \in$ KR-SAT, if there is a Kripke model K such that, for each world w, $\|\phi\|_{K,w} = 1$.
(ii) ϕ is *locally positive Kripke satisfiable*, or $\phi \in$ L-KR-POS-SAT, if there is a Kripke model K and a world w of K, such that $\|\phi\|_{K,w} > 0$.
(iii) ϕ is *locally standard satisfiable*, or $\phi \in$ L-STD-SAT, if there is a σ-simple SMV-algebra (A, σ), a valuation $\mathbf{v}$ into A, and a homomorphism $h : A \to [0,1]_{\mathrm{MV}}$ such that $h(\mathbf{v}(A)) = 1$. ϕ is *locally Kripke satisfiable*, or $\phi \in$ L-KR-SAT, if there is a Kripke model K and a world w of K such that $\|\phi\|_{K,w} = 1$.
(iv) ϕ is a *standard-tautology*, or $\phi \in$ STD-TAUT, if for every σ-simple SMV-algebra (A, σ) and every valuation $\mathbf{v}$ into $(A, \sigma), \mathbf{v}(\phi) = 1.\phi$ is a *Kripke-tautology*, or $\phi \in$ KR-TAUT, if $\|\phi\|_{K,w} = 1$ for every Kripke model K and every world w of K.

[6] We refer the reader to [2] for a complete discussion about simple and semi-simple MV-algebras. For what it concerns the understanding of the rest of this section, it is sufficient to recall that, up to isomorphisms, any simple MV-algebra is an MV-subalgebra of $[0,1]_{\mathrm{MV}}$ [2, Theorem 3.5.1], while a semi-simple MV-algebra is an algebra of [0,1]-valued continuos functions on a compact Hausdorff space X [2, Corollary 3.6.8].

Theorem 3.3. *From* [10]*:*

(i) $\phi \in$ KR-TAUT if and only if $\neg\phi \notin$ L-KR-POS-SAT.
(ii) KR-TAUT = STD-TAUT.
(ii) KR-SAT = STD-SAT = SMV-SAT.
(iv) L-KR-SAT = L-STD-SAT.

Lemma 3.4. *Let* $\phi \in$ SFP, *and let* $\Pr(\gamma_1), \ldots, \Pr(\gamma_r)$ *be the subformulas of* ϕ *of the from* $\Pr(\cdot)$. *Then:*

(*i*) For every Kripke model $K = (W, e, \mu)$ and every $w \in W$, there exists a Kripke model $K' = (W', e', \mu')$, with $|W'| = r + 2$, and a world $w' \in W'$, such that $\|\phi\|_{K,w} = \|\phi\|_{K',w'}$. Moreover $\mu'(v) > 0$ for at most $r + 1$ worlds in W'.
(*ii*) $\phi \in$ KR-SAT if and only if there exists a Kripke model $K' = (W', e', \mu')$ with at most $r + 2$ worlds, such that, for all $w' \in W', \|\phi\|_{K',w'} = 1$. Moreover $\mu'(v) > 0$ for at most $r + 1$ worlds in W'.
(*iii*) $\phi \in$ KR-SAT iff $\Pr(\phi) \in$ L-KR-SAT iff $\Pr(\phi) \in$ KR-SAT.

Proof. By [10, Corollary 3.10 and Theorem 3.6]. □

3.1. Complexity

In this section we characterize the computational complexity of the sets KR-SAT, KR-TAUT, and L-KR-SAT, obtaining as a corollary the desired complexity bounds for SMV-SAT, STD-TAUT, and L-STD-SAT.

A *MIP-problem* is a tuple (A, b, c, d, k) where (A, b, c, d) is a *linear programming problem*, and $k \leqslant n$ represents the additional request that $x_k, \ldots, x_n$ must be in $\{0, 1\}$. MIP-problems are in NP [25]. A polynomial reduction to MIP-problems shows that LUK-SAT, as well as the problem

LUK-POS-SAT = $\{\langle\phi\rangle | \phi$ is positive satisfiable in Łukasiewicz logic$\}$

are in NP [12]. Formally,

Lemma 3.5. *From* [12]:

(i) For every $\phi \in T$, there exists a MIP-problem $\mathscr{SAT}(\phi)$, polynomial-time computable in size(ϕ), such that $\phi \in$ LUK-SAT if and only if $\mathscr{SAT}(\phi)$ has solution.
(ii) For every $\phi \in T$, there exists a MIP-problem $\mathscr{SAT}$-$\mathscr{POS}(\phi)$, polynomial-time computable in size(ϕ), such that $\phi \in$ LUK-POS-SAT if and only if $\mathscr{SAT}$-$\mathscr{POS}(\phi)$ has solution.

In particular, a solution to $\mathscr{SAT}(\phi)$ gives a valuation of Łukasiewicz formulas $\mathbf{v}$ into $[0,1]$ such that $\mathbf{v}(\phi) = 1$, and a solution to $\mathscr{SAT}-\mathscr{POS}(\phi)$ gives a valuation of Łukasiewicz formulas $\mathbf{v}$ into $[0,1]$ such that $\mathbf{v}(\phi) > 0$.

Lemma 3.6. *The sets* KR-SAT, L-KR-SAT, *and* L-KR-POS-SAT *are in NP.*

Proof. Let $\phi(X_1, \ldots, X_t, \Pr(\psi_1), \ldots, \Pr(\psi_k))$ be a formula in SFP. Let $\{Z_1, \ldots, Z_k\}$ be a set of fresh variables, and let $\phi^* \in T$ be obtained by substituting $\Pr(\psi_j)$ with Z_j, for every $j \in [k]$. By Lemma 3.5, let $\mathscr{SAT}(\phi^*)$ be the MIP-problem having a solution if and only if ϕ^* is satisfiable, and let $z_1, \ldots, z_k$ be the variables of $\mathscr{SAT}(\phi^*)$ corresponding to the fresh variables $Z_1, \ldots, Z_k$. We now guess, in polynomial time, a partial solution to the MIP-problem $\mathscr{SAT}(\phi^*)$. The guessed partial solution, in particular, covers all variables constrained to have an integer solution, thus turning the original MIP into a linear programing problem.

First, the algorithm guesses the solution to the variables in $\mathscr{SAT}(\phi^*)$, distinct from $z_1, \ldots, z_k$, constrained to be solved over $\{0, 1\}$. Second, the algorithm deals with the z_j's. These variables cope with probabilistic formulas, hence we must ensure them to be evaluated by a coherent Łukasiewicz assessment $\mathbf{a} : \{\psi_1, \ldots, \psi_k\} \to [0, 1]$. By Lemma 2.7, the coherence of $\mathbf{a}$ is witnessed by the existence of an $l \leqslant k + 1$, non-negative real numbers $b_1, \ldots, b_l$, natural numbers $a_1, \ldots, a_l \leqslant 2^{\text{size}(\{\psi_1, \ldots, \psi_k\})}$, and homomorphisms $\mathbf{v}_1, \ldots, \mathbf{v}_l : T_{MV_m} \to [0, 1]_{\text{MV}}$ (where m is the number of distinct variables occurring in $\psi_1, \ldots, \psi_k$), such that, for each $j \in [l]$, and each $\gamma \in T_{MV_m}$, we have that $\mathbf{v}_j(\gamma) \in \{0, 1/a_j, \ldots, (a_j - 1)/a_j, 1\}$,

$$\sum_{j=1}^{l} b_j = 1, \quad \text{and} \quad \sum_{j=1}^{l} b_j \cdot \mathbf{v}_j(\psi_i) = z_i (\text{for all } i \in [k]). \tag{8}$$

Accordingly, the algorithm guesses the natural numbers $a_1, \ldots, a_l$, and the restriction of homomorphisms $\mathbf{v}_1, \ldots, \mathbf{v}_l$ to the m variables occurring in $\psi_1, \ldots, \psi_k$. Each $\mathbf{v}_j$ is a vector in $\{0, 1/a_j, \ldots, (a_j - 1)/a_j, 1\}^m$. Finally, in polynomial time, it computes the values $\mathbf{v}_1(\psi_1), \ldots, \mathbf{v}_l(\psi_k)$, and extends $\mathscr{SAT}(\phi^*)$ by the $k + 1$ equations (in the real unknowns $b_1, \ldots, b_l$) from (8) to obtain a linear problem, $\mathscr{P}$-$\mathscr{SAT}(\phi)$.

$\mathscr{P}$-$\mathscr{SAT}(\phi)$ is solvable in polynomial time [25], and moreover, as we now show, $\phi \in$ L-KR-SAT if and only if $\mathscr{P}$-$\mathscr{SAT}(\phi)$ has a positive solution.

Claim 1. *$\phi \in$ L-KR-SAT if and only if $\mathscr{P}$-$\mathscr{SAT}(\phi)$ has a positive solution.*

($\Rightarrow$) Assume that $\phi \in$ L-KR-SAT, whence, there exists a Kripke model $K = (W, e, \mu)$, and a node $w \in W$, such that $\|\phi\|_{K,w} = 1$. From Lemma 3.4(i) this is equivalent to the existence of a Kripke model $K' = (W', e'\mu')$ such that $|W'| = k+2, \mu'(v) > 0$ for $l \leqslant k+1$ nodes v in W' (call them $v_1, \ldots, v_l$), and $\|\phi\|_{K',w'} = 1$ for a given $w' \in W'$.

For all $j \in [l]$, call $\alpha_j = \mu'(v_j)$. Then $(\alpha_1, \ldots, \alpha_l)$ solves (8). Moreover, for $w' \in W'$ satisfying $\|\phi\|_{K',w'} = 1$, the array

$$(e(w', X_1), \ldots, e(w', X_t), \|\Pr(\psi_1)\|_{K',w'}, \ldots, \|\Pr(\psi_k)\|_{K',w'}),$$

is a solution for $\mathscr{SAT}(\phi^*)$. Then $\mathscr{P}$-$\mathscr{SAT}(\phi)$ has a positive solution.

($\Leftarrow$). Conversely assume that $\mathscr{P}$-$\mathscr{SAT}(\phi)$ has a positive solution. This means that both there is a map $\mathbf{v}_0 : \{X_1, \ldots, X_t, Z_1, \ldots, Z_k\} \to [0,1]$ such that $\mathbf{v}_0(\phi^*) = 1$, and there are $l \leqslant k+1$ real numbers $b_1, \ldots, b_l \in [0,1]$, and homomorphisms $\mathbf{v}_1, \ldots, \mathbf{v}_l$ such that (8) holds.

Therefore let $K = (W, e, \mu)$ be so defined:

- $W = \{\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_l\}$, and for each propositional variable X_j, and for all $i = 0, \ldots, l$, $e(\mathbf{v}_i, X_j) = \mathbf{v}_i(X_j)$.
- $\mu : W \to [0,1]$ is $\mu(\mathbf{v}_i) = b_i$ for all $i \in [l]$, and $\mu(\mathbf{v}_0) = 0$. Then it is clear that $\sum_{j=0}^{l} \mu(\mathbf{v}_j) = 1$.

Then K is a Kripke model. Moreover $\|\phi\|_{K,\mathbf{v}_0} = 1$, and finally $\phi \in$ L-KR-SAT, and Claim 1 is settled and L-KR-SAT is in NP.

In order to show that KR-SAT $\in$ NP, just apply the above construction to the formula $\Pr(\phi)$. Using the same argument one can easily show that there exists a MIP-problem $\mathscr{P}$-$\mathscr{SAT}(\Pr(\phi))$ that has a positive solution if and only if $\Pr(\phi) \in$ L-KR-SAT if and only if (from Lemma 3.4(iii)) $\phi \in$ KR-SAT. Since $\Pr(\phi)$ is obviously computed from ϕ in polynomial time, it shows KR-SAT $\in$ NP.

Finally, to show L-KR-POS-SAT $\in$ NP, just run through the proof of L-KR-SAT $\in$ NP, replacing $\mathscr{SAT}(\phi^*)$ by $\mathscr{SAT}$-$\mathscr{POS}(\phi^*)$, and appealing to Lemma 3.5(ii). $\square$

Lemma 3.7. *The sets* KR-SAT, L-KR-SAT, *and* L-KR-POS-SAT *are NP-hard.*

Proof. Let $\phi(X_1, \ldots, X_t)$ be a formula, and consider $\phi(\Pr(Z_1), \ldots, \Pr(Z_t))$ where the $\{Z_1, \ldots, Z_t\}$ is a set of t many fresh variables. Clearly, if the latter formula belongs to L-KR-SAT (L-KR-POS-SAT, respectively), then the former belongs to LUK-SAT (LUK-POS-SAT, respectively). Conversely, let $\mathbf{v}$ be a valuation such that $\mathbf{v}(\phi(X_1, \ldots, X_t)) = 1$ ($\mathbf{v}(\phi(X_1, \ldots, X_t)) > 0$, respectively). Let us assume that $\mathbf{v}(X_1) \leqslant \cdots \leqslant \mathbf{v}(X_t)$. Then define $K = (W, e, \mu)$ as follows[7]:

(i) $W = \{w_1, \ldots, w_{t+1}\}$, and for all $i \in [t+1]$ and $j \in [t], e(w_j, Z_i) = 1$ if $j \leqslant i$, and 0 otherwise.
(ii) $\mu : W \to [0,1]$ defined as follows:

$$\mu(w_j) = \begin{cases} \mathbf{v}(X_1) & \text{if } j = 1, \\ \mathbf{v}(X_j) - \mathbf{v}(X_{j-1}) & \text{if } 1 < j \leqslant t, \\ 1 - \mathbf{v}(X_t) & \text{if } j = t+1. \end{cases}$$

Then $\sum_{j=1}^{t+1} \mu(w_j) = 1$, and, for every $w_j \in W$, and every $i \in [t]$,

$$\|\Pr(Z_i)\|_{K,w_j} = \sum_{j=1}^{t+1} e(w_j, Z_i) \cdot \mu(w_j) = \sum_{j \leqslant i} \mu(w_j) = \mathbf{v}(X_i).$$

Then $\|\phi(\Pr(Z_1), \ldots, \Pr(Z_t))\|_{K,w_j} = \mathbf{v}(\phi(X_1, \ldots, X_t))$. In particular this shows that $\phi(\Pr(Z_1), \ldots, \Pr(Z_t)) \in$ L-KR-SAT ($\phi(\Pr(Z_1), \ldots, \Pr(Z_t)) \in$ L-KR-POS-SAT, respectively).

In order to prove that KR-SAT is NP-hard, a direct inspection of the above construction, and Remark 1, show that a formula of the form $\phi(\Pr(Z_1), \ldots, \Pr(Z_t))$ is in L-KR-SAT if and only if it belongs to KR-SAT, and this settles our claim. $\square$

Theorem 3.8. *The sets* KR-SAT = STD-SAT = SMV-SAT, L-KR-SAT = L-STD-SAT, *and* L-KR-POS-SAT *are NP-complete. The sets* KR-TAUT *and* STD-TAUT *are coNP-complete.*

Proof. By Lemmas 3.6, 3.7, and Theorem 3.3. $\square$

[7] The following construction is due to Hájek, see [13, Theorem 1].

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