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G. R. Cirmi, S. D'Asero, and S. Leonardi*

Morrey estimates for a class of elliptic equations with drift term

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Abstract: We consider the following boundary value problem

$$\begin{cases} -\mathrm{div}[M(x)\nabla u - E(x)u] = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, with $N > 2$, $M : \Omega \to \mathbb{R}^{N^2}$ is a symmetric matrix, $E(x)$ and $f(x)$ are respectively a vector field and function both belonging to suitable Morrey spaces and we study the corresponding regularity of u and Du.

Keywords: Morrey regularity, higher differentiability, drift term

MSC: 35J25, 35B65

1 Introduction

This paper is devoted to the study of the regularity of a weak solution u of the following homogeneous Dirichlet problem

$$\begin{cases} -\mathrm{div}[M(x)\nabla u - E(x)u] = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, with $N > 2$, $M : \Omega \to \mathbb{R}^{N^2}$ is a matrix with measurable bounded entries $M_{ij}(x)$ satisfying the standard ellipticity condition, $E(x)$ and $f(x)$ are respectively a vector field and function both belonging to suitable Morrey spaces to be specified later on.

The study of the above problem goes back to the papers [53, 54] by G. Stampacchia and it presents a difficulty due to the noncoercivity of the differential operator $u \to -\mathrm{div}[M(x)\nabla u - E(x)u]$. In the quoted papers the existence and uniqueness of a weak solution $u \in W_0^{1,2}(\Omega)$ have been proved assuming that

$$|E| \in L^N(\Omega), \text{ with } \|E\|_{L^N(\Omega)} \text{ sufficiently small,} \tag{1.2}$$

and

$$f \in L^{\frac{2N}{N+2}}(\Omega). \tag{1.3}$$

Moreover, the following regularity results hold:

(i) if $f \in L^m(\Omega)$, $\frac{2N}{N+2} \le m < \frac{N}{2}$, then $u \in L^{m^{**}}(\Omega)$, $m^{**} = \frac{Nm}{N-2m}$;

(ii) if $f \in L^m(\Omega)$, $m > \frac{N}{2}$ and $|E| \in L^r(\Omega)$, with $r > N$, then $u \in L^\infty(\Omega)$.

G. R. Cirmi, Dipartimento di Matematica e Informatica, Viale A. Doria 6 – 95125 Catania, Italy, E-mail: cirmi@dmi.unict.it
S. D'Asero, Dipartimento di Matematica e Informatica, Viale A. Doria 6 – 95125 Catania, Italy, E-mail: dasero@dmi.unict.it
***Corresponding Author: S. Leonardi,** Dipartimento di Matematica e Informatica, Viale A. Doria 6 – 95125 Catania, Italy, E-mail: leonardi@dmi.unict.it

Later, using a nonlinear approach and exploiting techniques issued from those of G. Stampacchia, L. Boccardo in [3] retrieved the previous results without the smallness condition in (1.2).

The aforementioned results are stated in a global fashion and they have been obtained when the data belong to usual Lebesgue spaces. On the other hand, G. Stampacchia in [54] also studies local properties of the solution of the problem (1.1). In particular, under the hypotheses stated in (ii), he proves that a $W^{1,2}$–solution of the problem (1.1) is locally bounded and locally Hölder continuous ([1]).

So that, naturally it raises the question of studying the problem in the setting of Morrey spaces, in order to try to obtain results similar to those obtained by S. Campanato in [6], for example. As far as we know the only available results in this framework are contained in the paper [21], where Hölder continuity and a Morrey estimate of a weak solution u have been proved. The technique used is essentially based on the representation formula of the solution but it doesn't allow to obtain any Morrey estimate of the gradient of the solution.

In this paper we suppose that the right–hand side f and the vector field E belong to suitable Morrey spaces and we recover the gradient estimate of a solution u in the corresponding Morrey space, so to retrieve the regularity theory at the "gradient level" as in Campanato's work.

Moreover, we also weaken the assumptions on the data when dealing with the existence of bounded weak solutions.

Namely, under the following minimal assumptions

$$f \in L^{1,\lambda}(\Omega) \quad \text{and} \quad |E| \in L^{2,\lambda}(\Omega) \quad \text{with } N-2<\lambda<N,$$

we will prove that there exists a weak solution $u \in W_0^{1,2}(\Omega)$ such that $u \in L^\infty(\Omega)$ and $|Du| \in L^{2,\lambda_0}(\Omega)$, for some $\lambda_0 \in]N-2,\lambda]$. The latter information brings back the Hölder continuity of u (see Theorem 3.2 below). Moreover, if f has an higher summability exponent and a lower Morrey exponent, that is

$$f \in L^{\frac{2N}{N+2},\frac{\lambda N}{N+2}}(\Omega) \qquad \text{with } 0<\lambda<N-2,$$

and

$$|E| \in L^{2,\mu}(\Omega) \quad \text{with } N-2<\mu<N,$$

then any weak solution u of problem (1.1) satisfies $|Du| \in L^{2,\lambda}(\Omega)$ (see Theorem 3.7 below).

At last, Morrey estimate obtained for $|Du|$ allows us to extend to the problem (1.1) the Calderon–Zygmund theory introduced in the paper [50] by G. Mingione (see also [2]).

In the framework of regularity theory of weak solutions the reader can also refer to the following papers [7–10, 16, 22, 23, 27–30, 32–36, 43, 45, 51, 52].

2 Main notations, functions spaces and auxiliary lemmas

In this section, for reader's convenience, we recall some useful properties of functions spaces and some lemmas that we are going to exploit.

In the sequel, Ω is a bounded open subset of $\mathbb{R}^N$, $N>2$, with a sufficiently smooth boundary $\partial\Omega$.

Definition 2.1 (Morrey space). *Let $p \geq 1$ and $0 \leq \lambda < N$. $L^{p,\lambda}(\Omega)$ is the space of all functions $u \in L^p(\Omega)$ such that*

$$\sup_{x_0\in\Omega,\, 0<R\leq d_\Omega} R^{-\lambda} \int_{\Omega\cap B(x_0,R)} |u|^p\,dx < +\infty$$

where $B(x_0,R)$ is the ball centered at x_0 with radius R and d_Ω is the diameter of Ω.

Remark 2.2. *We list below some well-known features of Morrey spaces which will be tacitly used throughout the paper.*

1 Further local properties of solutions depending on local properties of the data are object of study in the forthcoming paper [11].

- $L^{p,\lambda}(\Omega) \nsubseteq L^{p+\varepsilon}(\Omega), \forall \varepsilon > 0$;
- *if* $p \geq q$ *and* $\frac{N-\lambda}{p} \leq \frac{N-\mu}{q}$ *then* $L^{p,\lambda}(\Omega) \hookrightarrow L^{q,\mu}(\Omega)$;
- *if* $\lambda > N - p$, *with* $p > 1$, *then* $L^{p,\lambda}(\Omega) \subset W^{-1,p'}(\Omega)$;
- *if* $u \in W_0^{1,p}(\Omega)$ *and* $|Du| \in L^{p,\lambda}(\Omega)$, *with* $\lambda > N - p$, *then* $u \in C^{0,\mu}(\Omega)$ *with* $\mu = 1 - \frac{N-\lambda}{p}$.

Definition 2.3 (Fractional Sobolev space). *Let* $t \in]0, 1]$ *and* $p \geq 1$. $W^{t,p}(\Omega)$ *is the space of all functions* $u \in L^p(\Omega)$ *such that*

$$\|u\|_{W^{t,p}(\Omega)} = \|u\|_p + [u]_{t,p,\Omega} < +\infty$$

where

$$[u]_{t,p,\Omega} = \begin{cases} \left(\int\limits_\Omega \int\limits_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+tp}} \, \mathrm{d}x \, \mathrm{d}y \right)^{\frac{1}{p}} & \text{if } t < 1 \\ \\ \|\mathrm{D}u\|_{L^p(\Omega)} & \text{if } t = 1. \end{cases}$$

We will then need the following technical lemma.

Lemma 2.4 ([6]). *Let* $\varphi(t)$ *be a non–negative and nondecreasing function defined in* $[0, \delta]$ *such that for every couple* ρ, R $(0 < \rho < R \leq \delta)$ *the following inequality*

$$\varphi(\rho) \leq A \left(\frac{\rho}{R}\right)^\alpha \varphi(R) + B\, R^\beta$$

holds with A, B, α *and* β *positive constants,* $\alpha - \beta > 0$.
Then there exists a positive constant C *depending on* A *and* $\alpha - \beta$ *such that for every couple* ρ, R $(0 < \rho < R \leq \delta)$, *the following inequality*

$$\varphi(\rho) \leq A \left(\frac{\rho}{R}\right)^\beta \varphi(R) + C\, B\, \rho^\beta$$

holds.

Next, we state the following Sobolev like embedding Lemma due to D. Adams.

Lemma 2.5 ([1]). *Let* m *be a positive Radon measure supported in* Ω, $1 < p < N$, $q > p$ *and* $\sigma = q\left(\frac{N}{p} - 1\right)$. *Assume that there exists* $M > 0$ *such that*

$$m(B(x_0, r)) \leq M r^\sigma \quad \text{for all } x_0 \in \mathbb{R}^N \text{ and } r > 0.$$

Then, there exists a positive constant H *depending on* p, N, q, M *such that*

$$\left(\int\limits_\Omega |u|^q \, \mathrm{d}\, m \right)^{\frac{1}{q}} \leq H ||u||_{W_0^{1,p}(\Omega)}$$

for every $u \in W_0^{1,p}(\Omega)$.

Next lemma concerns the product of two functions belonging to Morrey spaces.

Lemma 2.6 ([21]). *Let* $N - 2 < \mu < N$, $E \in L^{2,\mu}(\Omega)$ *and* $u \in L^{2,\nu+2}(\Omega)$ *such that* $|\mathrm{D}u| \in L^{2,\nu}(\Omega)$ *for some* $\nu \in [0, N-2[$. *Then*

$$E\, u \in L^{2,\mu+\nu-N+2}(\Omega)$$

and moreover

$$||E\, u||_{L^{2,\mu+\nu-N+2}(\Omega)} \leq C\, ||E||_{L^{2,\mu}(\Omega)} \left(||\mathrm{D}u||_{L^{2,\nu}(\Omega)} + ||u||_{L^{2,2+\nu}(\Omega)} \right)$$

for some $C > 0$ *independent of* u *and* E.

Finally, the last result we state is a Sobolev-Morrey embedding Lemma.

LEMMA 2.7 (CFR. [14], [15]). *Assume that $\partial\Omega \in C^1$. Let $u \in W_0^{1,2}(\Omega)$ such that $\mathrm{D}u \in L^{2,\nu}(\Omega)$ with $\nu \in]0, N-2[$. Then*

$$u \in L^{2_\nu,\nu}(\Omega) \qquad \text{where } \frac{1}{2_\nu} = \frac{1}{2} - \frac{1}{N-\nu},$$

and moreover there exists a positive constant C depending on N, ν such that

$$||u||_{L^{2_\nu,\nu}(\Omega)} \le C\,||\mathrm{D}u||_{L^2(\Omega)}.$$

3 Statement of the main results

Let $M : \Omega \to \mathbb{R}^{N^2}$ be a matrix with measurable entries M_{ij} such that

$$\alpha|\xi|^2 \le M(x)\xi \cdot \xi, \quad |M(x)| \le \beta, \quad \text{for a.e. } x \in \Omega, \quad \forall\, \xi \in \mathbb{R}^N \tag{3.4}$$

We will initially consider the problem (1.1) under the following assumptions on tha data

$$f \in L^1(\Omega), \quad |E| \in L^2(\Omega).$$

DEFINITION 3.1. *By a weak solution of the problem (1.1) we mean a function u such that*

$$\begin{cases} u \in W_0^{1,2}(\Omega), \quad |Eu| \in L^2(\Omega) \\ \int\limits_\Omega M(x)\mathrm{D}u\mathrm{D}\varphi\,\mathrm{d}x = \int\limits_\Omega E\,u\mathrm{D}\varphi\,\mathrm{d}x + \int\limits_\Omega f\varphi\,\mathrm{d}x \end{cases} \tag{3.5}$$

for all $\varphi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.

Our first result concerns the existence and uniqueness of a bounded solution of the problem (1.1) with data E and f belonging to Morrey spaces having a large Morrey exponent but a lower summability exponent.

THEOREM 3.2. *Assume that (3.4) holds and*

$$f \in L^{1,\lambda}(\Omega), \tag{3.6}$$

$$|E| \in L^{2,\lambda}(\Omega), \tag{3.7}$$

with $\lambda \in]N-2, N[$.

Then there exists a unique weak solution u of the problem (1.1) such that

$$u \in L^\infty(\Omega)$$

and

$$||u||_{L^\infty(\Omega)} \le c_1$$

for some positive constant c_1 depending on $\alpha, \beta, N, \Omega, ||E||_{L^{2,\lambda}(\Omega)}, ||f||_{L^{1,\lambda}(\Omega)}$.

Moreover, there exists $\lambda_0 \in]N-2, \lambda]$ such that

$$|Du| \in L^{2,\lambda_0}(\Omega)$$

and

$$u \in C^{0,1-\frac{N-\lambda_0}{2}}(\Omega)$$

with corresponding norms estimates

$$\|Du\|_{L^{2,\lambda_0}(\Omega)} \le c_2, \tag{3.8}$$

$$[u]_{C^{0,1-\frac{N-\lambda_0}{2}}(\Omega)} \le c_3, \tag{3.9}$$

where c_2, c_3 are two positive constants depending on c_1 and $\|u\|_{W_0^{1,2}(\Omega)}$.

Remark 3.3. *It is worth noticing that, with respect to the results in [3, 20, 21], the boundedness of the solution u and its Hölder's continuity have been obtained under weaker assumptions on data (see Remark 2.2).*

Namely, in [3] Theorem 5.6 or in [20] Theorem 4.1 and pag. 203, the boundedness of u is obtained under the hypotheses

$$|E| \in L^m(\Omega), \text{ with } m > N, \quad f \in L^r(\Omega), \text{ with } r > N/2;$$

while in [21] Theorem 5.2, Hölder continuity of u is achieved under the hypotheses

$$|E| \in L^{2,\lambda}(\Omega), \quad f \in L^{\frac{2N}{N+2},\lambda}(\Omega), \quad \text{with } \lambda \in]N-2, N[.$$

Remark 3.4. *We stress that, for some constants $c_4, c_5 > 0$, one has*

$$|Eu| \le c_4|E|^{\frac{N}{2}} + c_5|u|^{\frac{2^*}{2}}$$

so that our lower order term lies in the framework of "controlli limite" as described in Campanato's book [6] pages 122 and 125 (see, in particular, Osservazione 4.I).

For this problem, also in the nonlinear setting, it is proven an L^p–estimate for $|Du|$ under the stronger assumption $|E|^{N/2} \in L^p(\Omega)$, $p > 2$ (see Theorem 4.III, pag. 125 in [6]).

Remark 3.5. *We observe that if $|E| = 0$ then our result is consistent with the classical Morrey–Campanato regularity result proved for Du in Theorem 8.V, pag. 92 of [6], where it assumed that*

$$f \in L^{\frac{2N}{N+2},\frac{N\lambda}{N+2}}(\Omega), \qquad 0 < \lambda < N.$$

Indeed, by embedding properties of Morrey spaces (see Remark 2.2) one has

$$L^{\frac{2N}{N+2},\frac{N}{N+2}\lambda}(\Omega) \hookrightarrow L^{1,\frac{N-2+\lambda}{2}}(\Omega)$$

where, if $\lambda > N-2$,

$$\frac{N-2+\lambda}{2} > N-2.$$

Consequently our result improves the previous one, at least in the case when $\lambda > N-2$ as stated in the following

Corollary 3.6. *Assume* (3.4), *$f \in L^{\frac{2N}{N+2},\frac{N\lambda}{N+2}}(\Omega)$, $\lambda \in]N-2, N[$ and $|E| \in L^{2,\frac{N-2+\lambda}{2}}(\Omega)$.*

Then there exists a unique weak solution u of the problem (1.1) *satisfying (3.8) and (3.9).*

If we assume

$$f \in L^{\frac{2N}{N+2},\frac{N}{N+2}\lambda}(\Omega), \qquad 0 < \lambda < N-2 \tag{3.10}$$

then

$$f \in L^{1,\frac{N-2+\lambda}{2}}(\Omega) \text{ with } \frac{N-2+\lambda}{2} < N-2$$

and we cannot expect any bounded solution u of the problem (1.1). However, a regularity result similar to Campanato's one (see [6] pag. 91) can be proved for weak solutions of problem (1.1).

We point out that, in this case, since the datum f has higher integrability (i.e. equal to the duality exponent $\frac{2N}{N+2}$) by a *weak solution* of problem (1.1) we mean a function u such that

$$\begin{cases} u \in W_0^{1,2}(\Omega), \quad |Eu| \in L^2(\Omega) \\ \int\limits_\Omega M(x)\mathrm{D}u\mathrm{D}\varphi \,\mathrm{d}\,x = \int\limits_\Omega E\,u\mathrm{D}\varphi \,\mathrm{d}\,x + \int\limits_\Omega f\varphi \,\mathrm{d}\,x \quad \forall \varphi \in W_0^{1,2}(\Omega). \end{cases} \tag{3.11}$$

Theorem 3.7. *Assume that hypotheses* (3.4), (3.10) *and*

$$|E| \in L^{2,\mu}(\Omega) \qquad N-2 < \mu < N \tag{3.12}$$

hold and let u be a weak solution of the problem (1.1). *Then,*
i) $\mathrm{D}u \in L^{2,\lambda}(\Omega)$
ii) $u \in L^{2_\lambda,\lambda}(\Omega)$ *where* $\frac{1}{2_\lambda} = \frac{1}{2} - \frac{1}{N-\lambda}$
with corresponding norms estimates.

Remark 3.8. *We point out that the existence and uniqueness of a weak solution of the problem* (1.1) *is ensured by the additional assumption* $|E| \in L^N(\Omega)$ *(see [20], [3] and [4]).*

We recall that, being $N \geq 3$*, in general it is not true that* $L^{2,\mu}(\Omega) \hookrightarrow L^N(\Omega)$*.*

Remark 3.9. *The above theorem improves Theorem 5.1 from [21] where it is proved that* $u \in L^{p,\lambda}(\Omega)$ *for any* $p \in [1, 2_\lambda[$*. Moreover our result provides as well information on the gradient of the solution.*

Remark 3.10. *Observe that if* $|E| = 0$ *then we retrive the result of Theorem 8.V, pag. 92 of [6].*

Finally we state a Theorem on the fractional differentiability of Du. For the sake of brevity we will focus only on the case of lower Morrey exponent. The same calculations can be repeated also in the previous case $\lambda > N-2$ (see [50], Theorem 1.10 for the case $|E| = 0$).

Theorem 3.11. *Assume that hypotheses* (3.4), (3.10), (3.12) *and*

$$M_{ij} \in C^{0,\eta}(\Omega), \qquad 0 < \eta \leq 1$$

hold. Let $u \in W_0^1(\Omega)$ *be a weak solution u of* (1.1) *. Then*

$$\mathrm{D}u \in \mathrm{W}^{t,2}_{\mathrm{loc}}(\Omega) \tag{3.13}$$

for every $t \in [0, \eta\delta[$ *and for every* $\delta \in [0, \min\{1, \frac{\lambda}{2}\}[$.

Moreover, for every couple of open subset $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ *there exists a constant* $c_6 = c_6(\alpha, \beta, N, \Omega, \|E\|_{L^{2,\mu}(\Omega)}, \|f\|_{L^{\frac{2N}{N+2}, \frac{N\lambda}{N+2}}(\Omega)})$*, independent of u, such that*

$$[\mathrm{D}u]^2_{W^{t,2}(\Omega')} \leq c_6 \left[\int\limits_{\Omega''} |\mathrm{D}u|^2 \, dx + \|\mathrm{D}u\|^2_{L^{2,\lambda}(\Omega'')} \right]. \tag{3.14}$$

REMARK 3.12. *It is worthwhile to observe that, in the case when M_{ij}, $|E| \in C^{0,1}(\bar{\Omega})$ and $f \in L^2(\Omega) \subset L^{\frac{2N}{N+2},\frac{2N}{N+2}}(\Omega)$, the weak solution $u \in W^{2,2}_{loc}(\Omega)$ (see e.g. [24] pag. 183).*

A result similar to the aforementioned one is also present in [6], Theorem 1.1 pag. 167, where it is proved that if $M_{ij} \in C^1(\bar{\Omega})$, $|E| = 0$ and $f \in L^{\frac{2N}{N+2}}(\Omega)$ then $u \in W^{2,2}(\Omega)$.

Previously, in [5] Theorem 10.1 pag. 346, it was proved that if $M_{ij} \in C^1(\bar{\Omega})$, $|E| = 0$ and $f \in L^{2,\lambda}(\Omega)$, with $0 \le \lambda < N$ (2), then $D^2u \in L^{2,\lambda}(\Omega)$.

Finally we notice that, at least formally, if $\lambda = 0$ then no fractional differentiability property of Du seems to be achievable.

Further details can also be found in [11–13, 17–19, 25, 26, 31, 37–42, 44, 46–49].

4 Proofs of Theorems 3.2, 3.7 and 3.11.

Proof of Theorem 3.2.

The proof will be performed in several steps.

Step 1 (Global boundedness)

For every $n \in \mathbb{N}$ and for a.e. $x \in \Omega$, let us introduce the bounded functions

$$f_n(x) = \frac{f(x)}{1 + \frac{1}{n}|f(x)|} \qquad \text{and} \qquad E_n(x) = \frac{E(x)}{1 + \frac{1}{n}|E(x)|}.$$

Note that

$$f_n \to f \quad \text{strongly in } L^1(\Omega), \tag{4.15}$$

with

$$||f_n||_{L^{1,\lambda}(\Omega)} \le ||f||_{L^{1,\lambda}(\Omega)},$$

and

$$E_n \to E \quad \text{strongly in } L^2(\Omega), \tag{4.16}$$

with

$$||E_n||_{L^{2,\lambda}(\Omega)} \le ||E||_{L^{2,\lambda}(\Omega)}.$$

We consider the following approximating problems

$$\begin{cases} u_n \in W^{1,2}_0(\Omega) \\ -\operatorname{div}\big(M(x)\mathrm{D}u_n\big) = -\operatorname{div}\Big(E_n(x)\dfrac{u_n}{1 + \frac{1}{n}|u_n|}\Big) + f_n(x) \quad \text{in } \Omega. \end{cases} \tag{4.17}$$

Thanks to Schauder fixed point Theorem, for every fixed $n \in \mathbb{N}$, there exists a weak solution u_n of the problem (4.17) i.e.

$$\begin{cases} u_n \in W^{1,2}_0(\Omega), \\ \displaystyle\int_\Omega M(x)\mathrm{D}u_n\mathrm{D}\varphi\,\mathrm{d}x = \int_\Omega E_n\frac{u_n}{1 + \frac{1}{n}|u_n|}\mathrm{D}\varphi\,\mathrm{d}x + \int_\Omega f_n\varphi\,\mathrm{d}x, \end{cases} \tag{4.18}$$

for every $\varphi \in W^{1,2}_0(\Omega)$.

Moreover, due to the boundedness of the functions $E_n(x)\dfrac{u_n}{1 + \frac{1}{n}|u_n|}$ and f_n, every u_n is bounded (see [53], [54]).

Next, we prove that the sequence $\{u_n\}$ is uniformly bounded in $L^\infty(\Omega)$.

2 Recall that $L^{2,\lambda}(\Omega) \hookrightarrow L^{\frac{2N}{N+2},\frac{N(2+\lambda)}{N+2}}(\Omega) \hookrightarrow L^{\frac{2N}{N+2},\frac{N\lambda}{N+2}}(\Omega)$ (if $\lambda > 0$).

Let (see [3])

$$\psi(s)=\begin{cases}0 & \text{if } |s|\le k\\ \frac{s}{1+s}-\frac{k}{1+k} & \text{if } s>k,\\ \frac{s}{1-s}+\frac{k}{1+k} & \text{if } s<-k\end{cases}$$

and take $\varphi=\psi(u_n)$ as test function in the weak formulation (4.18). Using Young's inequality and taking into account that $|\psi(u_n)|\le 1$, we have

$$\begin{aligned}\int_\Omega M(x)|\mathrm{D}u_n|^2\psi'(u_n)\,\mathrm{d}x &\le \int_\Omega |E_n|\frac{|u_n|}{1+\frac{1}{n}|u_n|}|\mathrm{D}u_n|\psi'(u_n)\,\mathrm{d}x+\int_\Omega |f_n||\psi(u_n)|\,\mathrm{d}x\\ &\le \int_{|u_n|>k}|E|\frac{|\mathrm{D}u_n|}{1+|u_n|}\,\mathrm{d}x+\int_{|u_n|>k}|f|\,\mathrm{d}x\\ &\le \frac{\alpha}{2}\int_{|u_n|>k}\frac{|\mathrm{D}u_n|^2}{\big(1+|u_n|\big)^2}\,\mathrm{d}x+\frac{1}{2\alpha}\int_{|u_n|>k}|E|^2\,\mathrm{d}x+\int_{|u_n|>k}|f|\,\mathrm{d}x\end{aligned} \tag{4.19}$$

Exploiting (3.4) from (4.19) we obtain

$$\frac{\alpha}{2}\int_{|u_n|>k}\frac{|\mathrm{D}u_n|^2}{(1+|u_n|)^2}\,\mathrm{d}x\le\frac{1}{2\alpha}\int_{|u_n|>k}|E|^2\,\mathrm{d}x+\int_{|u_n|>k}|f|\,\mathrm{d}x,$$

which implies (for $k=\mathrm{e}^h-1$)

$$\frac{\alpha}{2}\int_{\log(1+|u_n|)>h}|\mathrm{D}\log(1+|u_n|)|^2\,\mathrm{d}x\le\int_{\log(1+|u_n|)>h}\left[\frac{1}{2\alpha}|E|^2+|f|\right]\mathrm{d}x. \tag{4.20}$$

For any measurable set $A\subseteq\Omega$, the function

$$m(A):=\int_A(1+|E|^2+|f|)\,\mathrm{d}x,$$

is a positive Radon measure and, moreover,

$$m(B_r)\le C\,r^\lambda\big(||f||_{L^{1,\lambda}(\Omega)}+||E||_{L^{2,\lambda}}\big) \tag{4.21}$$

for every $r>0$.

Let now $v_n(x)=\log(1+|u_n(x)|)$ and set

$$G_k(v_n)=v_n-\max\{-k,\min\{v_n,k\}\}\in W_0^{1,2}(\Omega).$$

Due to Lemma 2.5, applied with $\sigma=\lambda$ and $q=\lambda\frac{2}{N-2}$ (note that $q>2$ since $\lambda>N-2$), and by virtue of (4.20), (4.21) we deduce the inequality

$$\left(\int_\Omega|G_k(v_n)|^q\,\mathrm{d}m\right)^{\frac{2}{q}}\le c_7\int_\Omega|\mathrm{D}G_k(v_n)|^2\,\mathrm{d}x=c_7\int_{A(k)}|\mathrm{D}\log(1+|u_n|)|^2\,\mathrm{d}x\le c_8 m(A(k))$$

where $A(k)=\{x\in\Omega\,:\,v_n(x)>k\}$, which in turn implies

$$(h-k)^q\cdot m(A(h))=\int_{A(h)}|h-k|^q\,\mathrm{d}m\quad\le\int_\Omega|G_k(v_n)|^q\,\mathrm{d}m\le c_9\big[m(A(k))\big]^{\frac{q}{2}} \tag{4.22}$$

for every $h>k$.

Thanks to the inequality (4.22) and a well–known Stampacchia's Lemma we obtain that

$$\log(1+|u_n(x)|) \le L \quad \text{a.e. in } \Omega,$$

with L independent of n, and so

$$||u_n||_{L^\infty(\Omega)} \le M_1, \qquad \forall\, n \in \mathbb{N} \tag{4.23}$$

where the constant M_1 is independent of n.

Now, choosing $\varphi = u_n$ in (4.18) and taking into account (3.4), we deduce

$$\alpha \int_\Omega |\mathrm{D}u_n|^2 \,\mathrm{d}x \le \int_\Omega |E_n| \frac{|u_n|^2}{1+\frac{1}{n}|u_n|} \,\mathrm{d}x + \int_\Omega |f_n|\,|u_n| \,\mathrm{d}x \quad \le C(M_1) \int_\Omega \big[|E| + |f|\big] \,\mathrm{d}x. \tag{4.24}$$

Therefore, for every $n \in \mathbb{N}$

$$||u_n||_{W_0^{1,2}(\Omega)} \le M_2, \tag{4.25}$$

where M_2 is a positive constant independent of $n \in \mathbb{N}$.

Step 2 (Local Morrey regularity)

In a fixed ball $B_R \subset\subset \Omega$ we write

$$u_n(x) = v_n(x) + w_n(x),$$

where $w_n \in W_0^{1,2}(B_R)$ is the unique usual weak solution of the problem

$$\begin{cases} -\operatorname{div}(M(x)\mathrm{D}w_n) = -\operatorname{div}\left(E_n(x)\, \dfrac{u_n}{1+\frac{1}{n}|u_n|}\right) + f_n(x) & \text{in } B_R \\ w_n = 0 & \text{on } \partial B_R, \end{cases} \tag{4.26}$$

and $v_n \in W^{1,2}(B_R)$ is the usual weak solution of the problem

$$\begin{cases} \operatorname{div}(M(x)\mathrm{D}v_n) = 0 & \text{in } B_R \\ v_n = u_n & \text{on } \partial B_R. \end{cases} \tag{4.27}$$

The existence and uniqueness of the solution w_n of the problem (4.26) is ensured by the Lax–Milgram Theorem.

Moreover the classical Maximum Principle and the boundedness of u_n provide us

$$||v_n||_{L^\infty(B_R)} \le ||u_n||_{L^\infty(\Omega)}$$

and consequently

$$||w_n||_{L^\infty(B_R)} \le ||u_n||_{L^\infty(B_R)} + ||v_n||_{L^\infty(B_R)} \le 2||u_n||_{L^\infty(\Omega)}.$$

Now we choose w_n as test function in the weak formulation of the problem (4.26) and we use Young's inequality, (3.4) and the boundedness of u_n and w_n we obtain

$$\begin{aligned} \alpha \int_{B_R} |\mathrm{D}w_n|^2 \,\mathrm{d}x &\le \int_{B_R} |E|\,|u_n|\,|\mathrm{D}w_n| \,\mathrm{d}x + \int_{B_R} |f|\,|w_n| \,\mathrm{d}x \\ &\le \frac{\alpha}{2} \int_{B_R} |\mathrm{D}w_n|^2 \,\mathrm{d}x + \frac{1}{2\alpha}||u_n||^2_{L^\infty(\Omega)} \int_{B_R} |E|^2 \,\mathrm{d}x + ||w_n||_{L^\infty(B_R)} \int_{B_R} |f| \,\mathrm{d}x. \end{aligned} \tag{4.28}$$

Thus, the above inequality and hypotheses (3.6) and (3.7) imply

$$\int_{B_R} |\mathrm{D}w_n|^2 \,\mathrm{d}x \le c\big(\alpha, M_1, ||E||_{L^{2,\lambda}(\Omega)}, ||f||_{L^{1,\lambda}(\Omega)}\big) R^\lambda. \tag{4.29}$$

On the other hand, it is well known that v_n satisfies the so-called Saint Venaint's principle (see [6] pag. 91, Theorem 8.IV), that is, there exist two constants $c_{10} = c_{10}(\alpha, \beta, N) > 0$ and $y = y\left(\frac{\alpha}{\beta}, N\right) \in]0, 1[$ such that

$$\int_{B_\rho} |Dv_n|^2 \, dx \le c_{10} \left(\frac{\rho}{R}\right)^{N-2+2y} \int_{B_R} |Dv_n|^2 \, dx, \quad \forall\, 0 < \rho \le R. \tag{4.30}$$

From (4.29) and (4.30) we deduce

$$\begin{aligned}\int_{B_\rho} |Du_n|^2 \, dx &\le c_{11} \left(\frac{\rho}{R}\right)^{N-2+2y} \int_{B_R} |Dv_n|^2 \, dx + c_{12} \int_{B_R} |Dw_n|^2 \, dx \\ &\le c_{13} \left(\frac{\rho}{R}\right)^{N-2+2y} \int_{B_R} |Du_n|^2 \, dx + c_{14} \int_{B_R} |Dw_n|^2 \, dx \\ &\le c_{15} \left(\frac{\rho}{R}\right)^{N-2+2y} \int_{B_R} |Du_n|^2 \, dx + c_{16} R^\lambda. \end{aligned} \tag{4.31}$$

Taking into account (4.31) and using Lemma 2.4, we obtain

$$\int_{B_\rho} |Du_n|^2 \, dx \le c_{15} \left(M_1, M_2, ||E||_{L^{2,\lambda}(\Omega)}, ||f||_{L^{1,\lambda}(\Omega)}\right) \rho^{\lambda_0} \tag{4.32}$$

where $\lambda_0 = \lambda$ if $\lambda \in]N-2, N-2+2y[$, otherwise λ_0 is arbitrarily chosen in $]N-2, N-2+2y[$.

Consequently,

$$Du_n \in L^{2,\lambda_0}_{\mathrm{loc}}(\Omega) \quad \text{and} \quad u_n \in C^{0,\tau}_{\mathrm{loc}}(\Omega) \quad \text{with } \tau = 1 - \frac{N-\lambda_0}{2}.$$

Now the linearity of the equation (4.17), the convergences (4.15) and (4.16), allow us to apply a standard limiting procedure as in [3] to the problems (4.18). Indeed, thanks to the estimates (4.23) and (4.25) up to a subsequence, still denoted by $\{u_n\}$, $\{u_n\}$ converges weakly in $W^{1,2}_0(\Omega)$ and a.e. in Ω to a function $u \in W^{1,2}_0(\Omega) \cap L^\infty(\Omega)$ such that

$$\int_\Omega M(x) Du D\varphi \, dx = \int_\Omega E\, u D\varphi \, dx + \int_\Omega f\varphi \, dx, \quad \forall \varphi \in W^{1,2}_0(\Omega) \cap L^\infty(\Omega).$$

Moreover, by the lower semicontinuity of norms, taking the limit as $n \to +\infty$ in the estimate (4.32), Du satisfies

$$\int_{B_\rho} |Du|^2 \, dx \le c_{15} \left(M_1, M_2, ||E||_{L^{2,\lambda}(\Omega)}, ||f||_{L^{1,\lambda}(\Omega)}\right) \rho^{\lambda_0} \tag{4.33}$$

which implies

$$Du \in L^{2,\lambda_0}_{\mathrm{loc}}(\Omega) \quad \text{and} \quad u \in C^{0,\tau}_{\mathrm{loc}}(\Omega) \quad \text{with } \tau = 1 - \frac{N-\lambda_0}{2}.$$

Finally, we remark that the uniqueness of u is a consequence of Theorem 6.1 of [3], where only the assumption $|E| \in L^2(\Omega)$ was used.

Step 3 (Boundary Morrey regularity)

Now we prove the regularity of $|Du|$ up to the boundary of Ω. For this purpose, following the idea of G. M. Troianiello [55] we will deduce boundary and then global regularity from the previous interior result through an extension technique and successive standard "flattening and covering" arguments.

This technique is illustrated in Lemma 2.18 and in Theorem 2.19 of the cited book. We will reproduce here the main steps for reader's convenience.

We denote a vector of $\mathbb{R}^N$ by $x = (x_1, \cdots, x_{N-1}, x_N) \equiv (x', x_N)$.

If $y = (y', 0)$ we define

$$B^+_\rho(y) = \{x \in B(y, \rho) : x_N > 0\},$$

$$\Gamma_\rho(y) = \{x \in B(y, \rho) : x_N = 0\}.$$

Fixed $R_1 > 0$, let M' be a simmetrix matrix with bounded coefficients M'_{ij}, E' be a vector field and f' be a function defined in $\Omega = B^+_{R_1}(y)$. We begin by investigating a solution of the problem

$$\begin{cases} u' \in W^{1,2}(B^+_{R_1}(y)) \\ u'_{|\Gamma_{R_1}(y)} = 0 \\ \displaystyle\int_{B^+_{R_1}(y)} M'(x)\mathrm{D}u'\mathrm{D}\varphi \,\mathrm{d}\,x = \int_{B^+_{R_1}} E'(x)\,u'\mathrm{D}\varphi\,\mathrm{d}\,x + \int_{B^+_{R_1}} f'\,\varphi\,\mathrm{d}\,x \\ \qquad\qquad\qquad\qquad \text{for all } \varphi \in W^{1,2}_0(B^+_{R_1}(y)) \cap L^\infty(B^+_{R_1}(y)). \end{cases} \tag{4.34}$$

We state the following

Lemma 4.1. *Assume that M', E' and f' satisfy respectively* (3.4), (3.6) *and* (3.7) *with $\Omega = B^+_{R_1}$. Let u be a solution of problem* (4.34).

Then, for every $R \in]0, R_1[$, we have

$$|\mathrm{D}u'| \in L^{2,\lambda_0}(B^+_R)(^3)$$

and there exists a positive constant c_{16} depending only on $\alpha, \beta, N, R_1, \|u'\|_{L^\infty(\Omega)}, \|E\|_{L^{2,\lambda}(\Omega)}, \|f\|_{L^{1,\lambda}(\Omega)}$ such that

$$||\mathrm{D}u'||_{L^{2,\lambda_0}(B^+_R)} \le c_{16}. \tag{4.35}$$

Proof. We extend the functions $M'_{ij}(x)$, $E'_i(x)$, f' and u' a.e. to $B_{R_1}(y)$ by setting

$$\overline{M_{iN}}(x', x_N) = \begin{cases} M'_{iN}(x', x_N) & \text{if } x_N > 0 \\ -M'_{iN}(x', -x_N) & \text{if } x_N < 0 \end{cases}$$

$$\overline{M_{Ni}}(x', x_N) = \begin{cases} M'_{Ni}(x', x_N) & \text{if } x_N > 0 \\ -M'_{Ni}(x', -x_N) & \text{if } x_N < 0 \end{cases}$$

$$\text{for } i = 1, \dots, N-1,$$

$$\overline{M_{ij}}(x', x_N) = \begin{cases} M'_{ij}(x', x_N) & \text{if } x_N > 0 \\ M'_{ij}(x', -x_N) & \text{if } x_N < 0 \end{cases}$$

$$\text{for all the remaining values of } i, j,$$

$$\overline{E_N}(x', x_N) = \begin{cases} E'_N(x', x_N) & \text{if } x_N > 0 \\ -E'_N(x', -x_N) & \text{if } x_N < 0 \end{cases}$$

$$\overline{E_i}(x', x_N) = \begin{cases} E'_i(x', x_N) & \text{if } x_N > 0 \\ E'_i(x', -x_N) & \text{if } x_N < 0 \end{cases}$$

$$\text{for } i = 1, \dots, N-1,$$

$$\overline{f}(x', x_N) = \begin{cases} f'(x', x_N) & \text{if } x_N > 0 \\ -f'(x', -x_N) & \text{if } x_N < 0 \end{cases}$$

and finally

$$\overline{u}(x', x_N) = \begin{cases} u'(x', x_N) & \text{if } x_N > 0 \\ -u'(x', -x_N) & \text{if } x_N < 0 \end{cases}$$

Note that functions $\overline{M_{ij}}$, $\overline{E}$ and $\overline{f}$ satisfy the assumption (3.4), (3.7) and (3.6) in $B_{R_1}(y)$, respectively and $\overline{u} \in W^{1,2}_0(B_{R_1}(y))$.

3 λ_0 is the number introduced in Step 2.

Given a function $v(x)$, with $x = (x', x_N) \in B_{R_1}(y)$, we set

$$\tilde{v}(x', x_N) \equiv v(x', -x_N)$$

Fixed a function $v \in C_0^1(B_{R_1}(y))$, we note that $v - \tilde{v} \in C_0^1(B^+_{R_1}(y))$. Therefore, simple calculations show that

$$\int_{B_{R_1}(y)} \overline{M}(x)\mathrm{D}\overline{u}\mathrm{D}v\,\mathrm{d}x - \int_{B_{R_1}(y)} \overline{E}(x)\,\overline{u}\mathrm{D}v\,\mathrm{d}x = \int_{B^+_{R_1}(y)} M'(x)\mathrm{D}u'\mathrm{D}(v-\tilde{v})\,\mathrm{d}x - \int_{B^+_{R_1}(y)} E'(x)\,u'\mathrm{D}(v-\tilde{v})\,\mathrm{d}x$$
$$= \int_{B^+_{R_1}(y)} f'(v-\tilde{v})\,\mathrm{d}x = \int_{B_{R_1}(y)} \overline{f}\,v\,\mathrm{d}x \tag{4.36}$$

and by density argument function $\overline{u}$ is solution of the problem (1.1) in $\Omega = B_{R_1}(y)$ with M', E' and f' replaced by $\overline{M}$, $\overline{E}$ and $\overline{f}$. Therefore $\overline{u}$ verifies (4.33) and the Lemma follows by changing back the coordinates. □

Step 4 (Global Morrey regularity)

Now we can prove the global Morrey regularity.

Since $\partial\Omega \in C^1$, for each $\bar{y} \in \partial\Omega$ there is a ball $B_{R_0}(\bar{y})$ and a $C^1(B_{R_0}(\bar{y}))$–diffeomorphism $\Lambda : \overline{B_{R_0}(\bar{y})} \to \overline{B_{\alpha_2 R_0}(0)}$, which straighten $\partial\Omega \cap B_{R_0}(\bar{y})$ and such that

1. $\Lambda(\bar{y}) = 0$
2. $B^+_{\alpha_1 R_0}(0) \subset \Lambda(B_{R_0}(\bar{y}) \cap \Omega) \subset B^+_{\alpha_2 R_0}(0)$ for some $0 < \alpha_1 \leq \alpha_2$.

Put $R_1 = \alpha_1 R_0$. If $z \in B^+_{R_1}(0) \equiv B^+_{R_1}$, we set

$$\begin{aligned}
M'_{ij}(z) &= M_{hk}(\Lambda^{-1}(z))\frac{\partial\Lambda_i}{\partial y_h}(\Lambda^{-1}(z))\frac{\partial\Lambda_j}{\partial y_k}(\Lambda^{-1}(z))J(z)\\
E'_i(z) &= E_h(\Lambda^{-1}(z))\frac{\partial\Lambda_i}{\partial y_h}(\Lambda^{-1}(z))J(z)\\
f'(z) &= f(\Lambda^{-1}(z))J(z)\\
u'(z) &= u(\Lambda^{-1}(z))J(z)
\end{aligned} \tag{4.37}$$

where $z = \Lambda(y)$, $y = \Lambda^{-1}(z)$ and $J(z)$ denotes the absolute value of the Jacobian determinant of Λ^{-1} at z.

Let us observe that M'_{ij} belong to $L^\infty(B^+_{R_1})$, E'_i belong to $L^N(B^+_{R_1}) \cap L^{2,\mu}(B^+_{R_1})$.

Moreover from the definition (4.37), it follows that

$$M'_{ij}\xi_i\xi_j = M_{hk}\frac{\partial\Lambda_i}{\partial y_h}\frac{\partial\Lambda_j}{\partial y_k}\xi_i\xi_j J \geq \alpha\sum_{h=1}^N\left(\sum_{i=1}^N\frac{\partial\Lambda_i}{\partial y_h}\xi_i\right)^2\min_{\overline{B^+_{R_1}}} J \geq \alpha'|\xi|^2$$

for all $\xi \in \mathbb{R}^N$.

Thus, a change of variables in (1.1) yields

$$\begin{cases}
u' \in W^{1,2}(B^+_{R_1})\\
u'_{|\Gamma_{R_1}(0)} = 0\\
\displaystyle\int_{B^+_{R_1}} M'(z)\mathrm{D}u'\mathrm{D}\varphi\,\mathrm{d}z = \int_{B^+_{R_1}} E'(z)\,u'\mathrm{D}\varphi\,\mathrm{d}z + \int_{B^+_{R_1}} f'\,\varphi\,\mathrm{d}z\\
\qquad\qquad \text{for all } \varphi \in W_0^{1,2}(B^+_{R_1}) \cap L^\infty(B^+_{R_1}).
\end{cases} \tag{4.38}$$

To (4.38) we apply Lemma 4.1 and thus we get the membership of $\mathrm{D}u'$ in $L^{2,\lambda}(B^+_R)$, $0 < R < R_1$, with norm estimate (4.35).

We extend $\mathrm{D}u'$ a.e. to B_R setting $\mathrm{D}u'(x', x_N) = \mathrm{D}u'(x', -x_N)$ if $x_N < 0$. Thus $\mathrm{D}u' \in L^{2,\lambda}(B_R)$.

As a consequence, since for some $r > 0$ $B_r(\bar{y}) \cap \Omega \subset \Lambda^{-1}(B_R^+)$, the vector–function $\mathrm{D}u'(\Lambda(y))$, $y \in B_r(\bar{y})$ belongs to $L^{2,\lambda}(B_r(\bar{y}))$ that is, by the chain rule, $\mathrm{D}u$ belongs to $L^{2,\lambda}(\Omega \cap B_r(\bar{y}))$; moreover, by virtue of (4.37) and (4.35),

$$||\mathrm{D}u||^2_{L^{2,\lambda}(B_r(\bar{y})\cap\Omega)} \le c. \tag{4.39}$$

Because $\partial\Omega$ is compact, there is a finite number of balls such as $B_r(\bar{y})$, say $B^1, B^2,\dots, B^m$, which cover $\partial\Omega$. Moreover, there exists an open set $\Omega \setminus \cup_{i=1}^m B_i \subset H_0 \subset\subset \Omega$ such that $H_0, B^1, B^2, \dots, B^m$ cover $\overline{\Omega}$.

If $\{g_i\}_{i=0,1,\dots,m}$ is a partition of the unity relative to the above covering then it turns out

$$||\mathrm{D}u||_{L^{2,\lambda}(\Omega)} \le c_{18}\left[||\mathrm{D}u||_{L^{2,\lambda}(H_0))} + \sum_{i=1}^m ||\mathrm{D}u||_{L^{2,\lambda}(B_r^m(\bar{y})\cap\Omega))}.\right]. \tag{4.40}$$

Then (3.8) follows from (4.40) by joining together (4.39) and the interior estimate (4.32) of Step 2.

□

Proof of Theorem 3.7.

Let $u \in W_0^{1,2}(\Omega)$ be a weak solution of the problem (1.1) in the weak sense (3.11). As we have done in the proof of the previous theorem we fix $B_R \subset\subset \Omega$ and we set

set

$$u(x) = w(x) + v(x), \qquad x \in B_R,$$

where $w \in W_0^{1,2}(B_R)$ and $v \in W^{1,2}(B_R)$ are respectively solutions of the following boundary problems

$$\begin{cases} -\operatorname{div}(M(x)\mathrm{D}w) = -\operatorname{div}(E(x)\,u) + f & \text{in } B_R \\ w = 0 & \text{on } \partial B_R, \end{cases} \tag{4.41}$$

and

$$\begin{cases} \operatorname{div}(M(x)\mathrm{D}v) = 0 & \text{in } B_R \\ v = u & \text{on } \partial B_R. \end{cases}. \tag{4.42}$$

Choosing w as test function in the weak formulation of the problem (4.41) and using hypothesis (3.4), by standard calculations, we obtain

$$\alpha\int_{B_R}|\mathrm{D}w|^2\,\mathrm{d}x \le \frac{1}{2\alpha}\int_{B_R}|Eu|^2\,\mathrm{d}x + \frac{\alpha}{2}\int_{B_R}|\mathrm{D}w|^2\,\mathrm{d}x + C(\sigma)\Bigg(\int_{B_R}|f|^{\frac{2N}{N+2}}\,\mathrm{d}x\Bigg)^{\frac{N+2}{N}} + \sigma\mathcal{S}\int_{B_R}|\mathrm{D}w|^2\,\mathrm{d}x, \quad \forall\varepsilon,\sigma > 0. \tag{4.43}$$

where S is the Sobolev's constant and σ is a positive constant chosen sufficiently small in order to get

$$\int_{B_R}|\mathrm{D}w|^2\,\mathrm{d}x \le c_{19}\int_{B_R}|Eu|^2\,\mathrm{d}x + c_{20}\Bigg(\int_{B_R}|f|^{\frac{2N}{N+2}}\,\mathrm{d}x\Bigg)^{\frac{N+2}{N}}. \tag{4.44}$$

As in (4.30) the solution v of the problem (4.27) satisfies the Saint Venaint's principle; therefore, by (4.44), we deduce for every $0 < \rho \le R$

$$\begin{aligned} \int_{B_\rho}|\mathrm{D}u|^2\,\mathrm{d}x &= c_{21}\int_{B_\rho}|\mathrm{D}v|^2\,\mathrm{d}x + c_{22}\int_{B_\rho}|\mathrm{D}w|^2\,\mathrm{d}x \\ &\le c_{23}\left(\frac{\rho}{R}\right)^{N-2+2y}\int_{B_R}|\mathrm{D}v|^2\,\mathrm{d}x + c_{24}\int_{B_R}|Eu|^2\,\mathrm{d}x + c_{25}\Bigg(\int_{B_R}|f|^{\frac{2N}{N+2}}\,\mathrm{d}x\Bigg)^{\frac{N+2}{N}} \\ &\le c_{26}\left(\frac{\rho}{R}\right)^{N-2+2y}\int_{B_R}|\mathrm{D}u|^2\,\mathrm{d}x + c_{27}\int_{B_R}|Eu|^2\,\mathrm{d}x + c_{28}\Bigg(\int_{B_R}|f|^{\frac{2N}{N+2}}\,\mathrm{d}x\Bigg)^{\frac{N+2}{N}}. \end{aligned} \tag{4.45}$$

We point out that $u \in L^{2^*}(\Omega) \subset L^{2,2}(\Omega)$. Consequently, by virtue of Lemma 2.6

$$Eu \in L^{2,\mu_0}(\Omega) \qquad \text{with} \quad \mu_0 = \mu - N + 2$$

and from (4.45) we obtain

$$\begin{aligned}\int_{B_\rho} |\mathrm{D}u|^2 \,\mathrm{d}\,x &\le c_{26}\left(\frac{\rho}{R}\right)^{N-2+2y} \int_{B_R} |Du|^2 \,\mathrm{d}\,x + c_{27}||E\,u||^2_{L^{2,\mu_0}(\Omega)} R^{\mu_0} + c_{28}||f||^2_{L^{\frac{2N}{N+2},\frac{N\lambda}{N+2}}(\Omega)} R^\lambda \\ &\le c_{26}\left(\frac{\rho}{R}\right)^{N-2+2y} \int_{B_R} |Du|^2 \,\mathrm{d}\,x + c_{29}\left(||E\,u||_{L^{2,\mu_0}(\Omega)}, ||f||_{L^{\frac{2N}{N+2},\frac{N\lambda}{N+2}}(\Omega)}\right) R^{\mu_1}\end{aligned} \tag{4.46}$$

where

$$\mu_1 = \min\left\{\mu_0, \lambda\right\} < N - 2. \tag{4.47}$$

Now, iterating the above inequality (see Campanato (cfr. [6]), we establish

$$|\mathrm{D}u| \in L^{2,\mu_1}_{\mathrm{loc}}(\Omega).$$

and

$$||\mathrm{D}u||^2_{L^{2,\mu_1}_{\mathrm{loc}}(\Omega)} \le c_{30}\left[||\mathrm{D}u||^2_{L^2(\Omega)} + ||Eu||^2_{L^{2,\mu_0}(\Omega)} + ||f||^2_{L^{\frac{2N}{N+2},\frac{N\lambda}{N+2}}(\Omega)}\right] \tag{4.48}$$

with c_{30} positive constant independent of u, E and f.

Next, using the extension technique and the successive standard “flattening and covering" arguments as we have done in Steps 3 and 4 of proof of Theorem 3.2, we can prove the regularity of $\mathrm{D}u$ up to the boundary of Ω, with the norm estimate

$$||\mathrm{D}u||^2_{L^{2,\mu_1}(\Omega)} \le c_{31}\left[||\mathrm{D}u||^2_{L^2(\Omega)} + ||Eu||^2_{L^{2,\mu_0}(\Omega)} + ||f||^2_{L^{\frac{2N}{N+2},\frac{2\lambda}{N+2}}(\Omega)}\right]. \tag{4.49}$$

Now, we compare μ_0 with λ. If $\mu_0 \ge \lambda$ then $\mu_1 = \lambda$ and the thesis follows.

Otherwise $\mu_1 = \mu_0$ and we can apply Lemma 2.7 to the function $u \in W^{1,2}_0(\Omega)$; since $\mathrm{D}u \in L^{2,\mu_1}(\Omega)$ we obtain

$$u \in L^{2_{\mu_1},\mu_1}(\Omega) \quad \text{where } \frac{1}{2_{\mu_1}} = \frac{1}{2} - \frac{1}{N-\mu_1}.$$

But $L^{2_{\mu_1},\mu_1}(\Omega)$ is embedded into $L^{2,2+\mu_1}(\Omega)$ therefore a new application of Lemma 2.6 gives us

$$E\,u \in L^{2,\mu_0+\mu_1}(\Omega)$$

with the norm estimate

$$||Eu||_{L^{2,\mu_0+\mu_1}(\Omega)} \le c_{32}||E||_{L^{2,\mu}(\Omega)}\left(||\mathrm{D}u||_{L^{2,\mu_1}(\Omega)} + ||u||_{L^{2,2+\mu_1}(\Omega)}\right) \tag{4.50}$$

for some $c_{32} > 0$ independent of u and E

Using in (4.45) the improved norm estimate (4.50) of $|Eu|$, we have

$$\begin{aligned}\int_{B_\rho} |\mathrm{D}u|^2 \,\mathrm{d}\,x &\le c_{26}\left(\frac{\rho}{R}\right)^{N-2+2y} \int_{B_R} |Du|^2 \,\mathrm{d}\,x + c_{33}||E\,u||^2_{L^{2,\mu_0+\mu_1}(\Omega)} R^{\mu_0+\mu_1} + c_{34}||f||^2_{L^{\frac{2N}{N+2},\frac{N\lambda}{N+2}}(\Omega)} R^\lambda \\ &\le c_{26}\left(\frac{\rho}{R}\right)^{N-2+2y} \int_{B_R} |Du|^2 \,\mathrm{d}\,x + c_{35}\left(||E\,u||^2_{L^{2,\mu_0+\mu_1}(\Omega)} + ||f||^2_{L^{\frac{2N}{N+2},\frac{N\lambda}{N+2}}(\Omega)}\right) R^{\mu_2}\end{aligned} \tag{4.51}$$

where

$$\mu_2 = \min\left\{\mu_0 + \mu_1, \lambda\right\} < N - 2.$$

As usually Lemma 2.4 provides us

$$\mathrm{D}u \in L^{2,\mu_2}_{\mathrm{loc}}(\Omega)$$

and as in the previous step we deduce the regularity of Du up to the boundary of Ω, that is

$$\mathrm{D}u \in L^{2,\mu_2}(\Omega)$$

and the corresponding estimate

$$||\mathrm{D}u||^2_{L^{2,\mu_2}(\Omega)} \le c_{36}\left[||\mathrm{D}u||^2_{L^2(\Omega)} + ||Eu||^2_{L^{2,\mu_0+\mu_1}(\Omega)} + ||f||^2_{L^{\frac{2N}{N+2},\frac{N\lambda}{N+2}}(\Omega)}\right] \tag{4.52}$$

where $c_{36} > 0$ is a constant independent of u, E and f.

Iterating the previous procedure and setting for every $k = 1, 2, \ldots$

$$\mu_k = \min\left\{\mu_0 + \mu_{k-1}, \lambda\right\} < N - 2, \tag{4.53}$$

it follows

i) $E\,u \in L^{2,\mu_0+\mu_k}(\Omega)$, with the norm estimate

$$||Eu||_{L^{2,\mu_0+\mu_k}(\Omega)} \le C_k\,||E||^k_{L^{2,\mu}(\Omega)}\left(||\mathrm{D}u||_{L^{2,\mu_k}(\Omega)} + ||u||_{L^{2,2+\mu_k}(\Omega)}\right),$$

for some constant $C_k > 0$ independent of u and E;

ii)

$$\int_{B_\rho} |\mathrm{D}u|^2\,\mathrm{d}\,x \le c_{26}\left(\frac{\rho}{R}\right)^{N-2+2y}\int_{B_R} |Du|^2\,\mathrm{d}\,x + c_{37}\left(||E\,u||^2_{L^{2,\mu_0+\mu_k}(\Omega)} + ||f||^2_{L^{\frac{2N}{N+2},\frac{N\lambda}{N+2}}(\Omega)}\right)R^{\mu_{k+1}}$$

iii) $\mathrm{D}u \in L^{2,\mu_{k+1}}(\Omega)$, with norm estimate

$$||\mathrm{D}u||^2_{L^{2,\mu_{k+1}}(\Omega)} \le c_{38}\left[||\mathrm{D}u||^2_{L^2(\Omega)} + ||Eu||^2_{L^{2,\mu_0+\mu_k}(\Omega)} + ||f||^2_{L^{\frac{2N}{N+2},\frac{2\lambda}{N+2}}(\Omega)}\right]$$

for some constant $c_{38} > 0$ independent of u, E and f.

After a finite number of steps we will have $\mu_0 + \mu_k > \lambda$, that is $\mu_{k+1} = \lambda$, and so $\mathrm{D}u \in L^{2,\lambda}(\Omega)$.

Finally, a further application of Lemma 2.7 provides us

$$u \in L^{2_\lambda,\lambda}(\Omega)$$

with $\frac{1}{2_\lambda} = \frac{1}{2} - \frac{1}{N-\lambda}$. □

Proof of Theorem 3.11.

We will proceed as in the proof of Theorem 4 of [16].

Let $B \subset\subset \Omega$ be a ball of radius R and let $\hat{B}$ be the enlarged ball of radius $32R$. We shall denote by $Q_{inn}(B)$ and $Q_{out}(B)$ the largest and the smallest cubes, concentric to B and with sides parallel to the coordinate axes, contained in B and containing B respectively. If we put

$$Q_{inn} = Q_{inn}(B),\ Q_{out} = Q_{out}(B)$$

and

$$\hat{Q}_{inn} = Q_{inn}(\hat{B}),\ \hat{Q}_{out} = Q_{out}(\hat{B}),$$

we have the following inclusions

$$Q_{inn} \subset B \subset\subset 4B \subset\subset \hat{Q}_{inn} \subset \hat{B} \subset \hat{Q}_{out}\ ^{4}. \tag{4.54}$$

Let Ω' and Ω'' be a couple of open subset such that $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ and $x_0 \in \Omega'$. For any $\tau \in]0, 1[$ (that will be chosen later) we fix $h \in \mathbb{R}$ with $0 < |h| << \min\left\{1, d(\Omega', \partial\Omega'')\right\}$ such that, denoted with $B = B(x_0, |h|^\tau)$ the ball centered in x_0 and with radius $|h|^\tau$, the outer cube of B, $\hat{Q}_{out}$ is included in Ω''.

4 With KB we denote the ball with radius KR, $K \in \mathbb{N}$.

Let $v \in W^{1,2}(\hat{B})$ be the unique weak solution to the problem

$$\begin{cases} \operatorname{div}\big(M(x)\mathrm{D}v\big) = 0 & \text{in } \hat{B} \\ v = u & \text{on } \partial\hat{B}, \end{cases} \tag{4.55}$$

and let $v_0 \in W^{1,2}(8B)$ be the unique weak solution to the problem

$$\begin{cases} \operatorname{div}\big(M(x_0)\mathrm{D}v_0\big) = 0 & \text{in } 8B \\ v_0 = v & \text{on } \partial 8B. \end{cases} \tag{4.56}$$

Then we have

$$\int_B |\tau_{ih}(\mathrm{D}u)|^2 \,\mathrm{d}x \le c_{39} \left[\int_B |\tau_{ih}(\mathrm{D}v_0)|^2 \,\mathrm{d}x + \int_{\hat{B}} |\mathrm{D}u - \mathrm{D}v|^2 \,\mathrm{d}x + \int_{8B} |\mathrm{D}v - \mathrm{D}v_0|^2 \,\mathrm{d}x \right] \tag{4.57}$$

The first term on the right–hand side of (4.57) can be estimated as

$$\int_B |\tau_{ih}(Dv_0)|^2 \,dx \le c_{40}|h|^{2(1-\tau)} \int_{8B} |Dv_0 - z_0|^2 \,dx, \tag{4.58}$$

for all $z_0 \in \mathbb{R}^n$ (see [50]).

The third term on the right–hand side of (4.57) can be estimated as

$$\int_{8B} |Dv - Dv_0|^2 \,dx \le c_{41}|h|^{2\eta\tau} \int_{\hat{B}} |Du|^2 \,dx \tag{4.59}$$

(see [50]).

Finally, we estimate

$$\int_{\hat{B}} |\mathrm{D}u - \mathrm{D}v|^2 \,\mathrm{d}x.$$

Let us observe that the function $w = v - u \in \mathrm{W}_0^{1,2}(\hat{B})$ is the weak solution to the equation

$$\operatorname{div}[M(x)\mathrm{D}(w + u)] = 0 \quad \text{in } \hat{B}, \tag{4.60}$$

whence, by assumption (3.4), we deduce

$$\begin{aligned} \int_{\hat{B}} |\mathrm{D}u - \mathrm{D}v|^2 \,\mathrm{d}x &= \int_{\hat{B}} |\mathrm{D}w|^2 \,\mathrm{d}x \le \frac{1}{\alpha} \int_{\hat{B}} |M(x)||\mathrm{D}u||\mathrm{D}w| \,\mathrm{d}x \le \frac{\beta}{\alpha} \int_{\hat{B}} |\mathrm{D}u||\mathrm{D}w| \,\mathrm{d}x \\ &\le c_{42} \left(\varepsilon \int_{\hat{B}} |\mathrm{D}(u - v)|^2 \,\mathrm{d}x + C(\varepsilon) \int_{\hat{B}} |\mathrm{D}u|^2 \,\mathrm{d}x \right) \end{aligned} \tag{4.61}$$

with ε, $C(\varepsilon)$ positive constants independent of the radius of $\hat{B}$.

In turn, for a sufficiently small ε, inequality (4.61) yields

$$\int_{\hat{B}} |\mathrm{D}u - \mathrm{D}v|^2 \,\mathrm{d}x \le c_{43} \int_{\hat{B}} |\mathrm{D}u|^2 \,\mathrm{d}x. \tag{4.62}$$

From this point on, we gather together inequalities (4.57), (4.58), (4.59) and (4.62) and we can argue as in the proof of Theorem 4 in [16], exploiting the method introduced in [50].

□

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