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Reduced order observer design for nonlinear systems

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Abstract

This work is a geometric study of reduced order observer design for nonlinear systems. Our reduced order observer design is applicable for Lyapunov stable nonlinear systems with a linear output equation and is a generalization of Luenberger's reduced order observer design for linear systems. We establish the error convergence for the reduced order estimator for nonlinear systems using the center manifold theory for flows. We illustrate our reduced order observer construction for nonlinear systems with a physical example, namely a nonlinear pendulum without friction.
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1. Introduction

The nonlinear observer design problem was introduced by Thau [1]. Over the past three decades, many significant works have been carried out on the construction of observers for nonlinear systems in the control systems literature [2–17]. This work is an extension of our recent work [14–17] on the full order observer design for nonlinear control systems.

The reduced order observer design for nonlinear systems presented in this work is a generalization of the construction of reduced order observers for linear systems devised by Luenberger [18].

To explain the concept of reduced order observers, consider the nonlinear system modelled by the equations

$$\begin{aligned}
\dot{x} &= f(x) \\
y &= Cx
\end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the *state* and $y \in \mathbb{R}^p$ is the *output* of the nonlinear system (1). For all practical situations, $p \leq n$. Suppose that C has full rank, i.e. rank$(C) = p$. Then we can make a linear change of coordinates

$$\xi = \begin{bmatrix} \xi_m \\ \xi_u \end{bmatrix} = \Lambda x = \begin{bmatrix} C \\ Q \end{bmatrix} x \tag{2}$$

where Q is chosen so that Λ is an invertible matrix. Note that $\xi_m \in \mathbb{R}^p$ and $\xi_u \in \mathbb{R}^{n-p}$. Such a choice of Q is made possible by the assumption that C has full rank.

Under the coordinates transformation (2), the plant (1) takes the form

$$\begin{bmatrix} \dot{\xi}_m \\ \dot{\xi}_u \end{bmatrix} = \begin{bmatrix} F_1(\xi_m, \xi_u) \\ F_2(\xi_m, \xi_u) \end{bmatrix}$$
$$y = \xi_m$$
(3)

where

$$F(\xi) = f(\Lambda^{-1}\xi).$$

The motivation for the reduced order state estimator or observer stems from the fact that in the plant model (3), the state ξ_m is directly available for measurement and hence it suffices to build an observer that estimates only the unmeasured state ξ_u. The order of such an observer will correspond to the dimension of the unmeasured state, namely $n - p \leq n$. This type of observer is called a *reduced order observer* [18] and it has many important applications in design problems.

In this work, we present a reduced order exponential observer designed for a Lyapunov stable plant of the form (3). We establish that the associated estimation error decays to zero exponentially using the center manifold theory for flows [19].

This work is organized as follows. In Section 2, we give the problem statement for reduced order observer design. In Section 3, we present our main results, namely reduced order exponential order design for Lyapunov stable nonlinear systems. In Section 4, we illustrate our main results with a physical example, namely a nonlinear pendulum without friction.

2. Problem statement

In this work, we consider nonlinear plants of the form

$$\begin{bmatrix} \dot{x}_m \\ \dot{x}_u \end{bmatrix} = \begin{bmatrix} F_1(x_m, x_u) \\ F_2(x_m, x_u) \end{bmatrix}$$
$$y = x_m$$
(4)

where $x_m \in \mathbb{R}^p$ is the *measured state*, $x_u \in \mathbb{R}^{n-p}$ the *unmeasured state* and $y \in \mathbb{R}^p$ the *output* of the plant (4). We assume that the state vector

$$x = \begin{bmatrix} x_m \\ x_u \end{bmatrix}$$

is defined in a neighborhood X of the origin of $\mathbb{R}^n$ and $F : X \to \mathbb{R}^n$ is a $\mathcal{C}^1$ vector field vanishing at the origin.

We can define the *reduced order exponential observers* for the plant (4) as follows.

Definition 1. Consider a $\mathcal{C}^1$ dynamical system defined by

$$\dot{z}_u = G(z_u, y)$$
(5)

where $z_u \in \mathbb{R}^{n-p}$ and $G : \mathbb{R}^{n-p} \times \mathbb{R}^p \to \mathbb{R}^{n-p}$ is a locally $\mathcal{C}^1$ mapping with $G(0, 0) = 0$. Then the system (5) is called a **reduced order exponential observer** for the plant (4) if the following conditions are satisfied:

(O1) If $z_u(0) = x_u(0)$, then $z_u(t) = x_u(t)$ for all $t \geq 0$. (Basically, this requirement states that if the initial estimation error is zero, then the estimation error stays zero for all future time.)

(O2) For any given ϵ-ball, $B_\epsilon(0)$, of the origin of $\mathbb{R}^{n-p}$, there exists a δ-ball, $B_\delta(0)$, of the origin of $\mathbb{R}^{n-p}$ such that

$$z_u(0) - x_u(0) \in B_\delta(0) \Longrightarrow z_u(t) - x_u(t) \in B_\epsilon(0) \qquad \text{for all } t \geq 0$$

and, moreover,

$$\|z_u(t) - x_u(t)\| \leq M \exp(-\alpha t) \|z_u(0) - x_u(0)\| \qquad \text{for all } t \geq 0$$

for some positive constants M and α. (Basically, this requirement states that if the initial estimation error is sufficiently small, then the future estimation error can be made to stay in any arbitrarily assigned neighborhood of the origin, and in addition, the estimation error decays to zero exponentially with time.) $\quad\square$

In this work, we consider the problem of finding reduced order exponential observers of the form (5) for Lyapunov stable nonlinear plants of the form (4).

3. Main results

Linearizing the plant (4) at $x = 0$, we obtain the following:

$$
\begin{bmatrix} \dot{x}_m \\ \dot{x}_u \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_m \\ x_u \end{bmatrix} + \begin{bmatrix} \phi(x_m, x_u) \\ \psi(x_m, x_u) \end{bmatrix}
$$
$$
y = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} x_m \\ x_u \end{bmatrix}
$$

(6)

where ϕ, ψ are $\mathcal{C}^1$ functions vanishing at the origin together with all their first partial derivatives.

First, we prove a basic lemma.

Lemma 1. *The system linearization pair of the plant* (6) *is detectable if, and only if, the pair* (A_{12}, A_{22}) *is detectable.*

Proof. By the PBH test for detectability [20, p. 286], the system linearization pair of the plant (6) is detectable if, and only if,

$$
\text{rank} \begin{bmatrix} C \\ \lambda I - A \end{bmatrix} = n \qquad \text{for all } \lambda \text{ with } \text{Re}(\lambda) \le 0
$$

i.e.

$$
\text{rank} \begin{bmatrix} I & 0 \\ \lambda I - A_{11} & -A_{12} \\ -A_{21} & \lambda I - A_{22} \end{bmatrix} = n \qquad \text{for all } \lambda \text{ with } \text{Re}(\lambda) \le 0
$$

i.e.

$$
\text{rank} \begin{bmatrix} -A_{12} \\ \lambda I - A_{22} \end{bmatrix} = n - p \qquad \text{for all } \lambda \text{ with } \text{Re}(\lambda) \le 0
$$

i.e. the pair (A_{12}, A_{22}) is detectable. This completes the proof. $\quad\square$

Our reduced order exponential observer design for the nonlinear system (4) is based on the following basic assumptions:

(H1) The equilibrium $x = 0$ of the system (4) is Lyapunov stable.
(H2) The pair (A_{12}, A_{22}) is detectable.

We can justify the above assumptions as follows. As pointed out in [15], the stability assumption of the plant dynamics is because of the conceptual problem, namely what does the existence of a local exponential observer mean in terms of the nonlinear dynamics of the system to be observed? For example, it must mean that the trajectories do not have finite escape time, but what does local existence mean for unbounded trajectories? In view of this crucial factor, we have focused our efforts on treating the local existence of reduced order exponential observers for those nonlinear systems which are Lyapunov stable. This justifies the assumption **(H1)**. By Lemma 1, the assumption **(H2)** is equivalent to the assumption that the linearization pair of the system (6) is detectable, which is quite standard in nonlinear observer design. In fact, detectability is a necessary condition for the existence of local exponential observers (see [7,14–17]).

Like in the full order observer case for local exponential observers presented in [15], we may consider the following generalization of the Luenberger observer:

$$
\dot{z_u} = G(z_u, y) \triangleq F_2(y, z_u) + L[\text{correction term}]
$$

where L is the observer gain matrix and the correction term $= y - x_m = 0$. (Note that x_m is the measured state.) Thus, as in the reduced order observer design for linear systems [18], we can construct a suitable correction term using the derivative of y, and the state equation for the derivative of x_m. If the measurement y is subject to noise, then the derivative operation amplifies the noise, and so this approach may not look very effective. However, in the actual implementation of the reduced order nonlinear estimator, we can avoid the appearance of $\dot{y}$ in its equation through a suitable change of coordinates as in [18].

Note that

$$\dot{y} = \dot{x}_m = F_1(x_m, x_u) = A_{11}x_m + A_{12}x_u + \phi(x_m, x_u).$$

Hence, it follows that

$$\dot{y} - A_{11}y = A_{12}x_u + \phi(x_m, x_u). \tag{7}$$

Note that in Eq. (7), the L.H.S. consists of terms that are available for measurement. Hence, we may view (7) as the new output equation, and accordingly, a correction term can be constructed.

Hence, we consider the following candidate observer:

$$\dot{z}_u = F_2(y, z_u) + L[\dot{y} - F_1(y, z_u)] \tag{8}$$

or equivalently,

$$\dot{z}_u = A_{21}y + A_{22}z_u + \psi(y, z_u) + L[\dot{y} - A_{11}y - A_{12}z_u - \phi(y, z_u)] \tag{9}$$

where $z_u \in \mathbb{R}^{n-p}$ and the observer gain matrix L is chosen such that $A_{22} - LA_{12}$ is Hurwitz. (Such a matrix L is guaranteed to exist in view of assumption **(H2)**.)

Since $y = x_m$, from the plant dynamics in (6), it follows that

$$\dot{x}_u = A_{21}y + A_{22}x_u + \psi(y, x_u). \tag{10}$$

The estimation error e is defined by

$$e \triangleq z_u - x_u.$$

From (9), (10) and (7), it follows that

$$\dot{e} = (A_{22} - LA_{12})e + \psi(y, e + x_u) - \psi(y, x_u) - L[\phi(y, x_u + e) - \phi(y, x_u)]. \tag{11}$$

Note that by construction, the linearization matrix $A_{22} - LA_{12}$ in the error dynamics (11) is Hurwitz, and also that $e = 0$ is an invariant manifold for the composite system consisting of the plant dynamics (6) and (11). Note also that by assumption **(H1)**, $x = 0$ is a Lyapunov stable equilibrium of the plant dynamics in (6). Hence, by an argument using center manifold theory for flows [19] similar to the proof of Theorem 3 in [17], it can be established that the candidate observer defined by (8) is an exponential observer that estimates the unmeasured state x_u of the plant dynamics in (4).

Next, we note that the implementation of the reduced order estimator given in the formula (8) may pose a problem as it involves the derivative of the measurement vector y. It is known that differentiation amplifies noise, so if y is noisy, the use of $\dot{y}$ is unacceptable. To get around this difficulty, we define the new estimator state to be

$$\zeta_u = z_u - Ly.$$

Then it can be easily shown that

$$\dot{\zeta}_u = F_2(y, Ly + \zeta_u) - LF_1(y, Ly + \zeta_u).$$

We may summarize the above results in the following main theorem.

Theorem 1. *Consider the plant* (4) *that satisfies the assumptions* **(H1)** *and* **(H2)**. *Linearizing the plant equations in* (4) *at the origin, we obtain the equivalent form for the plant given by* (6). *Let L be any matrix (observer gain) such*

that $A_{12} - LA_{22}$ is Hurwitz. Then a reduced order state estimator (or observer) for the plant (4) is given by the estimator dynamics (8) having the estimator state z_u which can be implemented through the following equations:

$$\dot{\zeta}_u = F_2(y, \zeta_u + Ly) - LF_1(y, \zeta_u + Ly)$$
$$z_u = \zeta_u + Ly. \tag{12}$$

If the pair (A_{12}, A_{22}) is observable, then we can construct a reduced order estimator of the form (12) with the steady-state error dictated by error poles, which are the eigenvalues of the error matrix $A_{22} - LA_{12}$, and which can be arbitrarily placed in the complex plane (subject to conjugate symmetry). ☐

4. An example

In this section, we illustrate the reduced order exponential observer design for nonlinear systems with a physical example, namely the nonlinear pendulum without friction.

Consider the nonlinear system described by

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\sin(\omega_0^2 x_1) \tag{13}$$
$$y = x_1$$

where x_1 is the measured state, and x_2 is the unmeasured state. In other words, we assume that only the angular velocity of the nonlinear pendulum $x_1 = \theta$ is available for measurement.

In this example, we will illustrate the procedure outlined in Theorem 1 to build an exponential observer that estimates only the unmeasured state x_2 of the nonlinear pendulum.

First, we shall verify that the nonlinear pendulum satisfies the assumptions **(H1)** and **(H2)** of Theorem 1. It is easy to see that the nonlinear pendulum satisfies the assumption **(H1)** as the equilibrium state $(x_1, x_2) = (0, 0)$ is clearly Lyapunov stable, as we know that the nonlinear pendulum without friction will undergo oscillations. In fact, this can also be seen by considering the total energy function

$$V(x_1, x_2) = \frac{1}{\omega_0^2}[1 - \cos(\omega_0^2 x_1)] + \frac{1}{2}x_2^2$$

and noting that $\dot{V} \equiv 0$ along the trajectories of the pendulum dynamics.

Linearizing the pendulum equations given in (13) at $x = 0$, we obtain

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \omega_0^2 & 0 \end{bmatrix}.$$

It is trivial to see that the pair (C, A) is observable. By Lemma 1, it is equivalent to say that the pair (A_{12}, A_{22}) is observable. In fact, note that the pair

$$A_{12} = 1 \quad \text{and} \quad A_{22} = 0$$

is trivially observable. Thus, the assumption **(H2)** is also satisfied.

Hence, by Theorem 1, we can build a reduced order exponential state estimator of the form (12) for the given pendulum with steady-state error dictated by any pre-assigned error pole, say, $\lambda = -5\omega_0$.

Note that the system matrix governing the error dynamics is given by

$$A_{22} - LA_{12} = 0 - L = -L$$

which has the characteristic equation $s + L = 0$.

Hence, it is obvious that we should take the observer gain as $L = 5\omega_0$ to meet the design specification.

By Theorem 1, the reduced order estimator equation is given by

$$\dot{\zeta}_2 = -\sin(\omega_0^2 y) - 5\omega_0(\zeta_2 + 5\omega_0 y)$$

and the state estimate is given by

$$z_2 = \zeta_2 + 5\omega_0 y. \quad \square$$

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