# Strip Planarity Testing

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**Abstract.** In this paper we introduce and study the *strip planarity testing* problem, which takes as an input a planar graph $G(V, E)$ and a function $\gamma : V \to \{1, 2, \ldots, k\}$ and asks whether a planar drawing of $G$ exists such that each edge is monotone in the $y$-direction and, for any $u, v \in V$ with $\gamma(u) < \gamma(v)$, it holds $y(u) < y(v)$. The problem has strong relationships with some of the most deeply studied variants of the planarity testing problem, such as *clustered planarity*, *upward planarity*, and *level planarity*. We show that the problem is polynomial-time solvable if $G$ has a fixed planar embedding.

## 1 Introduction

Testing the planarity of a given graph is one of the oldest and most deeply investigated problems in algorithmic graph theory. A celebrated result of Hopcroft and Tarjan [20] states that the planarity testing problem is solvable in linear time.

A number of interesting variants of the planarity testing problem have been considered in the literature [25]. Such variants mainly focus on testing, for a given planar graph $G$, the existence of a planar drawing of $G$ satisfying certain *constraints*. For example the *partial embedding planarity* problem [1,22] asks whether a plane drawing $\mathcal{G}$ of a given planar graph $G$ exists in which the drawing of a subgraph $H$ of $G$ in $\mathcal{G}$ coincides with a given drawing $\mathcal{H}$ of $H$. *Clustered planarity testing* [10,23], *upward planarity testing* [5,16,21], *level planarity testing* [24], *embedding constraints planarity testing* [17], *radial level planarity testing* [4], and *clustered level planarity testing* [14] are further examples of problems falling in this category.

In this paper we introduce and study the *strip planarity testing* problem, which is defined as follows. The input of the problem consists of a planar graph $G(V, E)$ and of a function $\gamma : V \to \{1, 2, \ldots, k\}$. The problem asks whether a *strip planar* drawing of $(G, \gamma)$ exists, i.e. a planar drawing of $G$ such that each edge is monotone in the $y$-direction and, for any $u, v \in V$ with $\gamma(u) < \gamma(v)$, it holds $y(u) < y(v)$. The name "strip" planarity comes from the fact that, if a strip planar drawing $\Gamma$ of $(G, \gamma)$ exists, then $k$ disjoint horizontal strips $\gamma_1, \gamma_2, \ldots, \gamma_k$ can be drawn in $\Gamma$ so that $\gamma_i$ lies below $\gamma_{i+1}$, for $1 \leq i \leq k-1$, and so that $\gamma_i$ contains a vertex $x$ of $G$ if and only if $\gamma(x) = i$, for $1 \leq i \leq k$. It is not difficult to argue that strips $\gamma_1, \gamma_2, \ldots, \gamma_k$ can be given as part of the input, and the problem is to decide whether $G$ can be planarly drawn so that each edge is monotone in the $y$-direction and each vertex $x$ of $G$ with $\gamma(x) = i$ lies in the strip $\gamma_i$. That is, arbitrarily predetermining the placement of the strips does not alter the possibility of constructing a strip planar drawing of $(G, \gamma)$.

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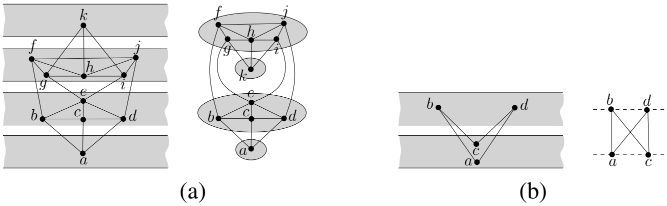


**Fig. 1.** (a) A negative instance $(G, \gamma)$ of the strip planarity testing problem whose associated clustered graph $C(G, T)$ is $c$-planar. (b) A positive instance $(G, \gamma)$ of the strip planarity testing problem that is not level planar.

Before presenting our result, we discuss the strong relationships of the strip planarity testing problem with three famous graph drawing problems.

**Strip Planarity and Clustered Planarity.** The *c-planarity testing* problem takes as an input a *clustered graph* $C(G, T)$, that is a planar graph $G$ together with a rooted tree $T$, whose leaves are the vertices of $G$. Each internal node $\mu$ of $T$ is called *cluster* and is associated with the set $V_\mu$ of vertices of $G$ in the subtree of $T$ rooted at $\mu$. The problem asks whether a *c-planar drawing* exists, that is a planar drawing of $G$ together with a drawing of each cluster $\mu \in T$ as a simple closed region $R_\mu$ so that: (i) if $v \in V_\mu$, then $v \in R_\mu$; (ii) if $V_\nu \subset V_\mu$, then $R_\nu \subset R_\mu$; (iii) if $V_\nu \cap V_\mu = \emptyset$, then $R_\nu \cap R_\mu = \emptyset$; and (iv) each edge of $G$ intersects the border of $R_\mu$ at most once. Determining the time complexity of testing the $c$-planarity of a given clustered graph is a long-standing open problem. See [10,23] for two recent papers on the topic. An instance $(G, \gamma)$ of the strip planarity testing problem naturally defines a clustered graph $C(G, T)$, where $T$ consists of a root having $k$ children $\mu_1, \dots, \mu_k$ and, for every $1 \leq j \leq k$, cluster $\mu_j$ contains every vertex $x$ of $G$ such that $\gamma(x) = j$. The $c$-planarity of $C(G, T)$ is a necessary condition for the strip planarity of $(G, \gamma)$, since suitably bounding the strips in a strip planar drawing of $(G, \gamma)$ provides a $c$-planar drawing of $C(G, T)$. However, the $c$-planarity of $C(G, T)$ is not sufficient for the strip planarity of $(G, \gamma)$ (see Fig. 1(a)). It turns out that strip planarity testing *coincides* with a special case of a problem opened by Cortese et al. [8,9] and related to $c$-planarity testing. The problem asks whether a graph $G$ can be planarly embedded "inside" an host graph $H$, which can be thought as having "fat" vertices and edges, with each vertex and edge of $G$ drawn inside a prescribed vertex and a prescribed edge of $H$, respectively. It is easy to see that the strip planarity testing problem coincides with this problem in the case in which $H$ is a path.

**Strip Planarity and Level Planarity.** The *level planarity testing* problem takes as an input a planar graph $G(V, E)$ and a function $\gamma : V \rightarrow \{1, 2, \dots, k\}$ and asks whether a planar drawing of $G$ exists such that each edge is monotone in the $y$-direction and each vertex $u \in V$ is drawn on the horizontal line $y = \gamma(u)$. The level planarity testing (and embedding) problem is known to be solvable in linear time [24], although a sequence of incomplete characterizations by forbidden subgraphs [15,18] (see also [13]) has revealed that the problem is not yet fully understood. The similarity of the level planarity testing problem with the strip planarity testing problem is evident: They have the same input, they both require planar drawings with $y$-monotone edges, and they both

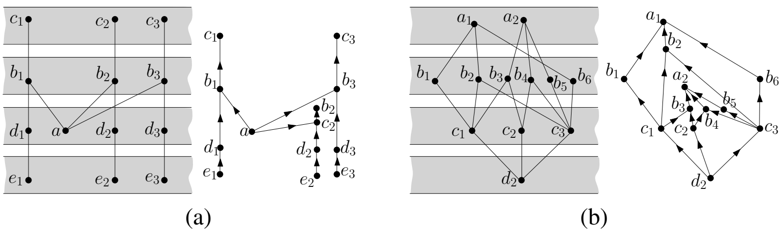


**Fig. 2.** Two negative instances $(G_1, \gamma_1)$ (a) and $(G_2, \gamma_2)$ (b) whose associated directed graphs are upward planar, where $G_1$ is a tree and $G_2$ is a subdivision of a triconnected plane graph

constrain the vertices to lie in specific regions of the plane; they only differ for the fact that such regions are horizontal lines in one case, and horizontal strips in the other one. Clearly the level planarity of an instance $(G, \gamma)$ is a sufficient condition for the strip planarity of $(G, \gamma)$, as a level planar drawing is also a strip planar drawing. However, it is easy to construct instances $(G, \gamma)$ that are strip planar and yet not level planar, even if we require that the instances are *strict*, i.e., no edge $(u, v)$ is such that $\gamma(u) = \gamma(v)$. See Fig. 1(b). Also, the approach of [24] seems to be not applicable to test the strip planarity of a graph. Namely, Jünger et al. [24] visit the instance $(G, \gamma)$ one level at a time, representing with a PQ-tree [7] the possible orderings of the vertices in level $i$ that are consistent with a level planar embedding of the subgraph of $G$ induced by levels $\{1, 2, \ldots, i\}$. However, when visiting an instance $(G, \gamma)$ of the strip planarity testing problem one strip at a time, PQ-trees seem to be not powerful enough to represent the possible orderings of the vertices in strip $i$ that are consistent with a strip planar embedding of the subgraph of $G$ induced by strips $\{1, 2, \ldots, i\}$.

**Strip Planarity and Upward Planarity.** The *upward planarity testing* problem asks whether a given directed graph $\overrightarrow{G}$ admits an *upward planar drawing*, i.e., a drawing which is planar and such that each edge is represented by a curve monotonically increasing in the $y$-direction, according to its orientation. Testing the upward planarity of a directed graph $\overrightarrow{G}$ is an $\mathcal{NP}$-hard problem [16], however it is polynomial-time solvable, e.g., if $\overrightarrow{G}$ has a fixed embedding [5], or if it has a single-source [21]. A strict instance $(G, \gamma)$ of the strip planarity testing problem naturally defines a directed graph $\overrightarrow{G}$, by directing an edge $(u, v)$ of $G$ from $u$ to $v$ if $\gamma(u) < \gamma(v)$. It is easy to argue that the upward planarity of $\overrightarrow{G}$ is a necessary and not sufficient condition for the strip planarity of $(G, \gamma)$ (see Fig.s 2(a) and 2(b)). Roughly speaking, in an upward planar drawing different parts of the graph are free to "nest" one into the other, while in a strip planar drawing, such a nesting is only allowed if coherent with the strip assignment.

In this paper, we show that the strip planarity testing problem is polynomial-time solvable for planar graphs with a fixed planar embedding. Our approach consists of performing a sequence of modifications to the input instance $(G, \gamma)$ (such modifications consist mainly of insertions of graphs inside the faces of $G$) that ensure that the instance satisfies progressively stronger constraints while not altering its strip planarity. Eventually, the strip planarity of $(G, \gamma)$ becomes equivalent to the upward planarity of its associated directed graph, which can be tested in polynomial time.

The paper is organized as follows. In Section 2 we give some preliminaries; in Section 3 we prove our result; finally, in Section 4 we conclude and present open problems. For space limitations, proofs are sketched or omitted; refer to [3] for complete proofs.

## 2 Preliminaries

A planar drawing of a graph determines a circular ordering of the edges incident to each vertex. Two drawings of the same graph are *equivalent* if they determine the same circular orderings around each vertex. A *planar embedding* (or *combinatorial embedding*) is an equivalence class of planar drawings. A planar drawing partitions the plane into topologically connected regions, called *faces*. The unbounded face is the *outer face*. Two planar drawings with the same combinatorial embedding have the same faces. However, such drawings could still differ for their outer faces. A *plane embedding* of a graph $G$ is a planar embedding of $G$ together with a choice for its outer face. In this paper, we will assume all the considered graphs to have a prescribed plane embedding.

For the sake of simplicity of description, in the following we assume that the considered plane graphs are 2-*connected*, unless otherwise specified. We will sketch in the conclusions how to extend our results to simply-connected and even non-connected plane graphs. We now define some concepts related to strip planarity.

An instance $(G, \gamma)$ of the strip planarity testing problem is *strict* if it contains no intra-strip edge, where an edge $(u, v)$ is *intra-strip* f $\gamma(u) = \gamma(v)$. An instance $(G, \gamma)$ of strip planarity is *proper* if, for every edge $(u, v)$ of $G$, it holds $\gamma(v) - 1 \leq \gamma(u) \leq \gamma(v) + 1$. Given any non-proper instance of strip planarity, one can replace every edge $(u, v)$ such that $\gamma(u) = \gamma(v) + j$, for some $j \geq 2$, with a path $(v = u_1, u_2, \dots, u_{j+1} = u)$ such that $\gamma(u_{i+1}) = \gamma(u_i) + 1$, for every $1 \leq i \leq j$, thus obtaining a proper instance $(G', \gamma')$ of the strip planarity testing problem. It is easy to argue that $(G, \gamma)$ is strip planar if and only if $(G', \gamma')$ is strip planar. In the following, we will assume all the considered instances of the strip planarity testing problem to be proper.

Let $(G, \gamma)$ be an instance of the strip planarity testing problem. A path $(u_1, \dots, u_j)$ in $G$ is *monotone* if $\gamma(u_i) = \gamma(u_{i-1}) + 1$, for every $2 \leq i \leq j$. For any face $f$ in $G$, we denote by $C_f$ the simple cycle delimiting the border of $f$. Let $f$ be a face of $G$, let $u$ be a vertex incident to $f$, and let $v$ and $z$ be the two neighbors of $u$ on $C_f$. We say that $u$ is a *local minimum* for $f$ if $\gamma(v) = \gamma(z) = \gamma(u) + 1$, and it is a *local maximum* for $f$ if $\gamma(v) = \gamma(z) = \gamma(u) - 1$. Also, we say that $u$ is a *global minimum* for $f$ (a *global maximum* for $f$) if $\gamma(w) \geq \gamma(u)$ (resp. $\gamma(w) \leq \gamma(u)$), for every vertex $w$ incident to $f$. A global minimum $u_m$ and a global maximum $u_M$ for a face $f$ are *consecutive* in $f$ if no global minimum and no global maximum exists in one of the two paths connecting $u_m$ and $u_M$ in $C_f$. A local minimum $u_m$ and a local maximum $u_M$ for a face $f$ are *visible* if one of the paths $P$ connecting $u_m$ and $u_M$ in $C_f$ is such that, for every vertex $u$ of $P$, it holds $\gamma(u_m) < \gamma(u) < \gamma(u_M)$.

**Definition 1.** *An instance $(G, \gamma)$ of the strip planarity problem is* quasi-jagged *if it is strict and if, for every face $f$ of $G$ and for any two visible local minimum $u_m$ and local maximum $u_M$ for $f$, one of the two paths connecting $u_m$ and $u_M$ in $C_f$ is monotone.*

**Definition 2.** *An instance* $(G, \gamma)$ *of the strip planarity problem is* jagged *if it is strict and if, for every face* $f$ *of* $G$*, any local minimum for* $f$ *is a global minimum for* $f$*, and every local maximum for* $f$ *is a global maximum for* $f$*.*

## 3 How to Test Strip Planarity

In this section we show an algorithm to test strip planarity.

### 3.1 From a General Instance to a Strict Instance

In this section we show how to reduce a general instance of the strip planarity testing problem to an equivalent strict instance.

**Lemma 1.** *Let* $(G, \gamma)$ *be an instance of the strip planarity testing problem. Then, there exists a polynomial-time algorithm that either constructs an equivalent strict instance* $(G^*, \gamma^*)$ *or decides that* $(G, \gamma)$ *is not strip planar.*

Consider any intra-strip edge $(u, v)$ in $G$, if it exists. We distinguish two cases.

In *Case 1*, $(u, v)$ is an edge of a 3-cycle $(u, v, z)$ that contains vertices in its interior in $G$. Observe that, $\gamma(u) - 1 \leq \gamma(z) \leq \gamma(u) + 1$. Denote by $G'$ the plane subgraph of $G$ induced by the vertices lying outside cycle $(u, v, z)$ together with $u$, $v$, and $z$ (this graph might coincide with cycle $(u, v, z)$ if such a cycle delimits the outer face of $G$); also, denote by $G''$ the plane subgraph of $G$ induced by the vertices lying inside cycle $(u, v, z)$ together with $u$, $v$, and $z$. Also, let $\gamma'(x) = \gamma(x)$, for every vertex $x$ in $G'$, and let $\gamma''(x) = \gamma(x)$, for every vertex $x$ in $G''$. We have the following:

**Claim 1.** $(G, \gamma)$ *is strip planar if and only if* $(G', \gamma')$ *and* $(G'', \gamma'')$ *are both strip planar.*

The strip planarity of $(G'', \gamma'')$ can be tested in linear time as follows.

If $\gamma''(z) = \gamma''(u)$, then $(G'', \gamma'')$ is strip planar if and only if $\gamma''(x) = \gamma''(u)$ for every vertex $x$ of $G''$ (such a condition can clearly be tested in linear time). For the necessity, 3-cycle $(u, v, z)$ is entirely drawn in $\gamma''(u)$, hence all the internal vertices of $G''$ have to be drawn inside $\gamma''(u)$ as well. For the sufficiency, $G''$ has a plane embedding by assumption, hence any planar $y$-monotone drawing (e.g. a straight-line drawing where no two vertices have the same $y$-coordinate) respecting such an embedding and contained in $\gamma''(u)$ is a strip planar drawing of $(G'', \gamma'')$.

If $\gamma''(z) = \gamma''(u) - 1$ (the case in which $\gamma''(z) = \gamma''(u) + 1$ is analogous), then we argue as follows: First, a clustered graph $C(G'', T)$ can be defined such that $T$ consists of two clusters $\mu$ and $\nu$, respectively containing every vertex $x$ of $G''$ such that $\gamma''(x) = \gamma''(u) - 1$, and every vertex $x$ of $G''$ such that $\gamma''(x) = \gamma''(u)$. We show that $(G'', \gamma'')$ is strip planar if and only if $C(G'', T)$ is $c$-planar. For the necessity, it suffices to observe that a strip planar drawing of $(G'', \gamma'')$ is also a $c$-planar drawing of $C(G'', T)$. For the sufficiency, if $C(G'', T)$ admits a $c$-planar drawing, then it also admits a $c$-planar *straight-line* drawing $\Gamma(C)$ in which the regions $R(\mu)$ and $R(\nu)$ representing $\mu$ and $\nu$, respectively, are *convex* [2,12]. Assuming w.l.o.g. up to a rotation of $\Gamma(C)$ that $R(\mu)$ and $R(\nu)$ can be separated by a horizontal line, we have that disjoint horizontal strips can be drawn containing $R(\mu)$ and $R(\nu)$. Slightly perturbing the positions of the vertices so that no two of them have the same $y$-coordinate ensures that the the edges are $y$-monotone, thus resulting in a strip planar drawing of $(G'', \gamma'')$. Finally, the $c$-planarity of a clustered graph containing two clusters can be decided in linear time, as independently proved by Biedl et al. [6] and by Hong and Nagamochi [19].

In *Case 2*, a 3-cycle $(u, v, z)$ exists that contains no vertices in its interior in $G$. Then, *contract* $(u, v)$, that is, identify $u$ and $v$ to be the same vertex $w$, whose incident edges are all the edges incident to $u$ and $v$, except for $(u, v)$; the clockwise order of the edges incident to $w$ is: All the edges that used to be incident to $u$ in the same clockwise order starting at $(u, v)$, and then all the edges that used to be incident to $v$ in the same clockwise order starting at $(v, u)$. Denote by $G'$ the resulting graph. Since $G$ is plane, $G'$ is plane; since $G$ contains no 3-cycle $(u, v, z)$ that contains vertices in its interior, $G'$ is simple. Let $\gamma'(x) = \gamma(x)$, for every vertex $x \neq u, v$ in $G$, and let $\gamma'(w) = \gamma(u)$. We have the following.

**Claim 2.** *$(G', \gamma')$ is strip planar if and only if $(G, \gamma)$ is strip planar.*

Claims 1 and 2 imply Lemma 1. Namely, if $(G, \gamma)$ has no intra-strip edge, there is nothing to prove. Otherwise, $(G, \gamma)$ has an intra-strip edge $(u, v)$, hence either Case 1 or Case 2 applies. If Case 2 applies to $(G, \gamma)$, then an instance $(G', \gamma')$ is obtained in linear time containing one less vertex than $(G, \gamma)$. By Claim 2, $(G', \gamma')$ is equivalent to $(G, \gamma)$. Otherwise, Case 1 applies to $(G, \gamma)$. Then, either the non-strip planarity of $(G, \gamma)$ is deduced (if $(G'', \gamma'')$ is not strip planar), or an instance $(G', \gamma')$ is obtained containing at least one less vertex than $(G, \gamma)$ (if $(G'', \gamma'')$ is strip planar). By Claim 1, $(G', \gamma')$ is equivalent to $(G, \gamma)$. The repetition of such an argument either leads to conclude in polynomial time that $(G, \gamma)$ is not strip planar, or leads to construct in polynomial time a strict instance $(G^*, \gamma^*)$ of strip planarity equivalent to $(G, \gamma)$.

### 3.2 From a Strict Instance to a Quasi-Jagged Instance

In this section we show how to reduce a strict instance of the strip planarity testing problem to an equivalent quasi-jagged instance. Again, for the sake of simplicity of description, we assume that every considered instance $(G, \gamma)$ is 2-connected.

**Lemma 2.** *Let $(G, \gamma)$ be a strict instance of the strip planarity testing problem. Then, there exists a polynomial-time algorithm that constructs an equivalent quasi-jagged instance $(G^*, \gamma^*)$ of the strip planarity testing problem.*

Consider any face $f$ of $G$ containing two visible local minimum and maximum $u_m$ and $u_M$, respectively, such that no path connecting $u_m$ and $u_M$ in $C_f$ is monotone. Insert a monotone path connecting $u_m$ and $u_M$ inside $f$. Denote by $(G^+, \gamma^+)$ the resulting instance of the strip planarity testing problem. We have the following claim:

**Claim 3.** *$(G^+, \gamma^+)$ is strip planar if and only if $(G, \gamma)$ is strip planar.*

**Proof Sketch:** The necessity is trivial. For the sufficiency, consider any strip planar drawing $\Gamma$ of $(G, \gamma)$. Denote by $P$ the path connecting $u_m$ and $u_M$ along $C_f$ and such that $\gamma(u_m) < \gamma(v) < \gamma(u_M)$ holds for every internal vertex $v$ of $P$. Because of the existence of some parts of the graph that "intermingle" with $P$, it might not be possible to draw a $y$-monotone curve inside $f$ connecting $u_m$ and $u_M$ in $\Gamma$. Thus, a part of $\Gamma$ has to be horizontally shrunk, so that it moves "far away" from $P$, thus allowing for the monotone path connecting $u_m$ and $u_M$ to be drawn as a $y$-monotone curve inside $f$. This results in a strip planar drawing of $(G^+, \gamma^+)$. □

Claim 3 implies Lemma 2, as proved in the following.

First, the repetition of the above described augmentation leads to a quasi-jagged instance $(G^*, \gamma^*)$. In fact, whenever the augmentation is performed, the number of triples $(v_m, v_M, g)$ such that vertices $v_m$ and $v_M$ are visible local minimum and maximum for face $g$, respectively, and such that both paths connecting $v_m$ and $v_M$ along $C_f$ are not monotone decreases by 1, thus eventually the number of such triples is zero, and the instance is quasi-jagged.

Second, $(G^*, \gamma^*)$ can be constructed from $(G, \gamma)$ in polynomial time. Namely, the number of pairs of visible local minima and maxima for a face $g$ of $G$ is polynomial in the number of vertices of $g$. Hence, the number of triples $(v_m, v_M, g)$ such that vertices $v_m$ and $v_M$ are visible local minimum and maximum for face $g$, over all faces of $G$, is polynomial in $n$. Since a linear number of vertices are introduced in $G$ whenever the augmentation described above is performed, it follows that the the construction of $(G^*, \gamma^*)$ from $(G, \gamma)$ can be accomplished in polynomial time.

Third, $(G^*, \gamma^*)$ is an instance of the strip planarity testing problem that is equivalent to $(G, \gamma)$. This directly comes from repeated applications of Claim 3.

### 3.3 From a Quasi-Jagged Instance to a Jagged Instance

In this section we show how to reduce a quasi-jagged instance of the strip planarity testing problem to an equivalent jagged instance. Again, for the sake of simplicity of description, we assume that every considered instance $(G, \gamma)$ is 2-connected.

**Lemma 3.** *Let $(G, \gamma)$ be a quasi-jagged instance of the strip planarity testing problem. Then, there exists a polynomial-time algorithm that constructs an equivalent jagged instance $(G^*, \gamma^*)$ of the strip planarity testing problem.*

Consider any face $f$ of $G$ that contains some local minimum or maximum which is not a global minimum or maximum for $f$, respectively. Assume that $f$ contains a local minimum $v$ which is not a global minimum for $f$. The case in which $f$ contains a local maximum which is not a global maximum for $f$ can be discussed analogously. Denote by $u$ (denote by $z$) the first global minimum or maximum for $f$ that is encountered when walking along $C_f$ starting at $v$ while keeping $f$ to the left (resp. to the right).

We distinguish two cases, namely the case in which $u$ is a global minimum for $f$ and $z$ is a global maximum for $f$ (Case 1), and the case in which $u$ and $z$ are both global maxima for $f$ (Case 2). The case in which $u$ is a global maximum for $f$ and $z$ is a global minimum for $f$, and the case in which $u$ and $z$ are both global minima for $f$ can be discussed symmetrically.

In *Case 1*, denote by $Q$ the path connecting $u$ and $z$ in $C_f$ and containing $v$. Consider the internal vertex $v'$ of $Q$ that is a local minimum for $f$ and that is such that

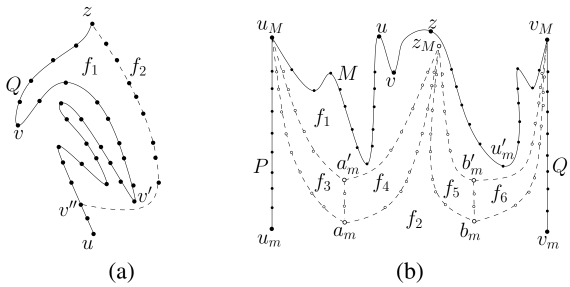


**Fig. 3.** Augmentation of $(G, \gamma)$ inside a face $f$ in: (a) Case 1 and (b) Case 2

$\gamma(v') = \min_{u'} \gamma(u')$ among all the internal vertices $u'$ of $Q$ that are local minima for $f$. Traverse $Q$ starting from $u$, until a vertex $v''$ is found with $\gamma(v'') = \gamma(v')$. Notice that, the subpath of $Q$ between $u$ and $v''$ is monotone. Insert a monotone path connecting $v''$ and $z$ inside $f$. See Fig. 3(a). Denote by $(G^+, \gamma^+)$ the resulting instance of the strip planarity testing problem. We have the following claim:

**Claim 4.** *Suppose that Case 1 is applied to a quasi-jagged instance $(G, \gamma)$ to construct an instance $(G^+, \gamma^+)$. Then, $(G^+, \gamma^+)$ is strip planar if and only if $(G, \gamma)$ is strip planar. Also, $(G^+, \gamma^+)$ is quasi-jagged.*

**Proof Sketch:** The necessity is trivial. For the sufficiency, consider any strip planar drawing $\Gamma$ of $(G, \gamma)$. First, $\Gamma$ is modified so that $v''$ has $y$-coordinate smaller than every local minimum of $Q$ different from $u$. Then, a $y$-monotone curve can be drawn inside $f$ connecting $v''$ and $z$, thus resulting in a strip planar drawing of $(G^+, \gamma^+)$. □

In *Case 2*, denote by $M$ a *maximal* path that is part of $C_f$, whose end-vertices are two global maxima $u_M$ and $v_M$ for $f$, that contains $v$ in its interior, and that does not contain any global minimum in its interior. By the assumptions of Case 2, such a path exists. Assume, w.l.o.g., that face $f$ is to the right of $M$ when walking along $M$ starting at $u_M$ towards $v_M$. Possibly $u_M = u$ and/or $v_M = z$. Let $u_m$ ($v_m$) be the global minimum for $f$ such that $u_m$ and $u_M$ (resp. $v_m$ and $v_M$) are consecutive global minimum and maximum for $f$. Possibly, $u_m = v_m$. Denote by $P$ the path connecting $u_m$ and $u_M$ along $C_f$ and not containing $v$. Also, denote by $Q$ the path connecting $v_m$ and $v_M$ along $C_f$ and not containing $v$. Since $M$ contains a local minimum among its internal vertices, and since $(G, \gamma)$ is quasi-jagged, it follows that $P$ and $Q$ are monotone.

Insert the plane graph $A(u_M, v_M, f)$ depicted by white circles and dashed lines in Fig. 3(b) inside $f$. Consider a local minimum $u'_m \in M$ for $f$ such that $\gamma(u'_m) = \min_{v'_m} \gamma(v'_m)$ among the local minima $v'_m$ for $f$ in $M$. Set $\gamma(z_M) = \gamma(u_M)$, set $\gamma(a_m) = \gamma(b_m) = \gamma(u_m)$, and set $\gamma(a'_m) = \gamma(b'_m) = \gamma(u'_m)$. The dashed lines connecting $a_m$ and $u_M$, connecting $a'_m$ and $u_M$, connecting $a_m$ and $z_M$, connecting $a'_m$ and $z_M$, connecting $b_m$ and $z_M$, connecting $b'_m$ and $z_M$, connecting $b_m$ and $v_M$, connecting $b'_m$ and $v_M$, connecting $a_m$ and $a'_m$, and connecting $b_m$ and $b'_m$ represent monotone paths. Denote by $(G^+, \gamma^+)$ the resulting instance of the strip planarity testing problem. We have the following claim:

**Claim 5.** *Suppose that Case 2 is applied to a quasi-jagged instance $(G, \gamma)$ to construct an instance $(G^+, \gamma^+)$. Then, $(G^+, \gamma^+)$ is strip planar if and only if $(G, \gamma)$ is strip planar. Also, $(G^+, \gamma^+)$ is quasi-jagged.*

**Proof Sketch:** The necessity is trivial. For the sufficiency, consider any strip planar drawing $\Gamma$ of $(G, \gamma)$. If $P$ is to the left of $Q$, then a region $R$ is defined as the region delimited by $P$, by $M$, by $Q$, and by the horizontal line delimiting $\gamma(u_m)$ from above. Then, the part of $\Gamma$ that lies inside $R$ is redrawn so that it lies inside a region $R_Q \subset R$ arbitrarily close to $Q$. Such a redrawing "frees" space for the drawing of $A(u_M, v_M, f)$ inside $f$, which results in a strip planar drawing of $(G^+, \gamma^+)$. If $P$ is to the right of $Q$, then $M$ might "wiggle" to the right of $P$ and to the left of $Q$. Thus, we first horizontally shrink a part of $\Gamma$ that "intermingles" with $P$ and $Q$, and we then draw $A(u_M, v_M, f)$ using its four monotone paths connecting global minima with global maxima in order to "circumvent" $M$. This results in a strip planar drawing of $(G^+, \gamma^+)$. □

Claims 4–5 imply Lemma 3, as proved in the following.

First, we prove that the repetition of the above described augmentation leads to a jagged instance $(G^*, \gamma^*)$ of the strip planarity testing problem. For an instance $(G, \gamma)$ and for a face $g$ of $G$, denote by $n(g)$ the number of vertices that are local minima for $g$ but not global minima for $g$, plus the number of vertices that are local maxima for $g$ but not global maxima for $g$. Also, let $n(G) = \sum_g n(g)$, where the sum is over all faces $g$ of $G$. We claim that, when one of the augmentations of Cases 1 and 2 is performed and instance $(G, \gamma)$ is transformed into an instance $(G^+, \gamma^+)$, we have $n(G^+) \leq n(G) - 1$. The claim implies that eventually $n(G^*) = 0$, hence $(G^*, \gamma^*)$ is jagged.

We prove the claim. When a face $f$ of $G$ is augmented as in Case 1 or in Case 2, for each face $g \neq f$ and for each vertex $u$ incident to $g$, vertex $u$ is a local minimum, a local maximum, a global minimum, or a global maximum for $g$ in $(G^+, \gamma^+)$ if and only if it is a local minimum, a local maximum, a global minimum, or a global maximum for $g$ in $(G, \gamma)$, respectively. Hence, it suffices to prove that $\sum n(f_i) \leq n(f) - 1$, where the sum is over all the faces $f_i$ that are created from the augmentation inside $f$.

Suppose that Case 1 is applied to insert a monotone path between vertices $v''$ and $z$ inside $f$. Such an insertion splits $f$ into two faces, which we denote by $f_1$ and $f_2$, as in Fig. 3(a). Face $f_2$ is delimited by two monotone paths, hence $n(f_2) = 0$. Every vertex inserted into $f$ is neither a local maximum nor a local minimum for $f_1$. As a consequence, no vertex $x$ exists such that $x$ contributes to $n(f_1)$ and $x$ does not contribute to $n(f)$. Further, vertex $v'$ is a global minimum for $f_1$, by construction, and it is a local minimum but not a global minimum for $f$. Hence, $v'$ contributes to $n(f)$ and does not contribute to $n(f_1)$. It follows that $n(f_1) + n(f_2) \leq n(f) - 1$.

Suppose that Case 2 is applied to insert plane graph $A(u_M, v_M, f)$ inside face $f$. Such an insertion splits $f$ into six faces, which are denoted by $f_1, \ldots, f_6$, as in Fig. 3(b). Every vertex of $A(u_M, v_M, f)$ incident to a face $f_i$, for some $1 \leq i \leq 6$, is either a global maximum for $f_i$, or a global minimum for $f_i$, or it is neither a local maximum nor a local minimum for $f_i$. As a consequence, no vertex $x$ exists such that $x$ contributes to some $n(f_i)$ and $x$ does not contribute to $n(f)$. Further, for each vertex $x$ that contributes to $n(f)$, there exists at most one face $f_i$ such that $x$ contributes to $n(f_i)$. Finally, vertex $u'_m$ of $M$ is a global minimum for $f_1$, by construction, and it is a local minimum but not a global minimum for $f$. Hence, $u'_m$ contributes to $n(f)$ and does not contribute to $n(f_i)$, for any $1 \leq i \leq 6$. It follows that $\sum_{i=1}^{6} n(f_i) \leq n(f) - 1$.

Second, $(G^*, \gamma^*)$ can be constructed from $(G, \gamma)$ in polynomial time. Namely, the number of local minima (maxima) for a face $f$ that are not global minima (maxima) for $f$ is at most the number of vertices of $f$. Hence, the number of such minima and maxima over all the faces of $G$, which is equal to $n(G)$, is linear in $n$. Since a linear number of vertices are introduced in $G$ whenever the augmentation described above is performed, and since the augmentation is performed at most $n(G)$ times, it follows that the construction of $(G^*, \gamma^*)$ can be accomplished in polynomial time.

Third, $(G^*, \gamma^*)$ is an instance of the strip planarity testing problem that is equivalent to $(G, \gamma)$. This directly comes from repeated applications of Claims 4 and 5.

### 3.4 Testing Strip Planarity for Jagged Instances

In this section we show how to test in polynomial time whether a jagged instance $(G, \gamma)$ of the strip planarity testing problem is strip planar. Recall that the associated directed graph of $(G, \gamma)$ is the directed plane graph $\overrightarrow{G}$ obtained from $(G, \gamma)$ by orienting each edge $(u, v)$ in $G$ from $u$ to $v$ if and only if $\gamma(v) = \gamma(u) + 1$. We have the following:

**Lemma 4.** *A jagged instance $(G, \gamma)$ of the strip planarity testing problem is strip planar if and only if the associated directed graph $\overrightarrow{G}$ of $(G, \gamma)$ is upward planar.*

**Proof Sketch:** The necessity is trivial. For the sufficiency, we first insert dummy edges in $\overrightarrow{G}$ to augment it to a *plane st-digraph* $\overrightarrow{G}_{st}$, which is an upward planar directed graph with exactly one source $s$ and one sink $t$ incident to its outer face [11]. Each face $f$ of $\overrightarrow{G}_{st}$ consists of two monotone paths, called *left path* and *right path*, where the left path has $f$ to the right when traversing it from its source to its sink. The inserted dummy edges only connect two sources or two sinks of each face of $\overrightarrow{G}$. Since $(G, \gamma)$ is jagged, the end-vertices of each dummy edge are in the same strip.

We divide the plane into $k$ horizontal strips. We compute an upward planar drawing of $\overrightarrow{G}_{st}$ starting from a $y$-monotone drawing of the leftmost path of $\overrightarrow{G}_{st}$ and adding to the drawing one face at a time, in an order corresponding to any linear extension of the partial order of the faces induced by the directed dual graph of $\overrightarrow{G}_{st}$ [11]. When a face is added to the drawing, its left path is already drawn as a $y$-monotone curve. We draw the right path of $f$ as a $y$-monotone curve in which each vertex $u$ lies inside strip $\gamma(u)$, hence the rightmost path of the graph in the current drawing is always represented by a $y$-monotone curve. A strip planar drawing of $(G, \gamma)$ can be obtained from the drawing of $\overrightarrow{G}_{st}$ by removing the dummy edges. □

We thus obtain the following:

**Theorem 1.** *The strip planarity testing problem can be solved in polynomial time for instances $(G, \gamma)$ such that $G$ is a plane graph.*

**Proof:** By Lemmata 1–3, it is possible to reduce in polynomial time any instance of the strip planarity testing problem to an equivalent jagged instance $(G, \gamma)$. By Lemma 4, $(G, \gamma)$ is strip planar if and only if the associated directed plane graph $\overrightarrow{G}$ of $(G, \gamma)$ is upward planar. Finally, by the results of Bertolazzi et al. [5], the upward planarity of $\overrightarrow{G}$ can be tested in polynomial time. □

## 4 Conclusions

In this paper, we introduced the strip planarity testing problem and showed how to solve it in polynomial time if the input graph is 2-connected and has a prescribed plane embedding. We now sketch how to remove the 2-connectivity requirement.

Suppose that the input graph $(G, \gamma)$ is simply-connected (possibly not 2-connected). The algorithmic steps are the same. The transformation of a general instance into a strict instance is exactly the same. The transformation of a strict instance into a quasi-jagged instance has some differences with respect to the 2-connected case. In fact, the visibility between local minima and maxima for a face $f$ of $G$ is redefined with respect to *occurrences* of such minima and maxima along $f$. Thus, the goal of such a transformation is to create an instance in which, for every face $f$ and for every pair of visible occurrences $\sigma_i(u_m)$ and $\sigma_j(u_M)$ of a local minimum $u_m$ and a local maximum $u_M$ for $f$, respectively, there is a monotone path between $\sigma_i(u_m)$ and $\sigma_j(u_M)$ in $C_f$. This is done with the same techniques as in Claim 3. The transformation of a quasi-jagged instance into a jagged instance is almost the same as in the 2-connected case, except that the 2-connected components of $G$ inside a face $f$ have to be suitably squeezed along the monotone paths of $f$ to allow for a drawing of a monotone path between $v''$ and $z$ or for a drawing of plane graph $A(u_M, v_M, f)$ This is done with the same techniques as in Claims 4 and 5. Finally, the proof of the equivalence between the strip planarity of a jagged instance and the upward planarity of its associated directed graph holds as it is.

Suppose now that the input graph $(G, \gamma)$ is not connected. Test individually the strip planarity of each connected component of $(G, \gamma)$. If one of the tests fails, then $(G, \gamma)$ is not strip planar. Otherwise, construct a strip planar drawing of each connected component of $(G, \gamma)$. Place the drawings of the connected components containing edges incident to the outer face of $G$ side by side. Repeatedly insert connected components in the internal faces of the currently drawn graph $(G', \gamma)$ as follows. If a connected component $(G_i, \gamma)$ of $(G, \gamma)$ has to be placed inside an internal face $f$ of $(G', \gamma)$, check whether $\gamma(u_M) \leq \gamma(u^f_M)$ and whether $\gamma(u_m) \geq \gamma(u^f_m)$, where $u_M$ ($u_m$) is a vertex of $(G_i, \gamma)$ such that $\gamma(u_M)$ is maximum (resp. $\gamma(u_m)$ is minimum) among the vertices of $G_i$, and where $u^f_M$ ($u^f_m$) is a vertex of $C_f$ such that $\gamma(u^f_M)$ is maximum (resp. $\gamma(u^f_m)$ is minimum) among the vertices of $C_f$. If the test fails, then $(G, \gamma)$ is not strip planar. Otherwise, using a technique analogous to the one of Claim 3, a strip planar drawing of $(G', \gamma)$ can be modified so that two consecutive global minimum and maximum for $f$ can be connected by a $y$-monotone curve $\mathcal{C}$ inside $f$. Suitably squeezing a strip planar drawing of $(G_i, \gamma)$ and placing it arbitrarily close to $\mathcal{C}$ provides a strip planar drawing of $(G' \cup G_i, \gamma)$. Repeating such an argument leads either to conclude that $(G, \gamma)$ is not strip planar, or to construct a strip planar drawing of $(G, \gamma)$.

The main question raised by this paper is whether the strip planarity testing problem can be solved in polynomial time or is rather $\mathcal{NP}$-hard for graphs without a prescribed plane embedding. The problem is intriguing even if the input graph is a tree.

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