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Explicit polynomial expansions of regular real functions by means of even order Bernoulli polynomials and boundary values

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Abstract

For a function $f \in C^{2n+1}([a, b])$ an explicit polynomial interpolant in a and in the even derivatives up to the order $2n - 1$ at the end-points of the interval is derived. Explicit Cauchy and Peano representations and bounds for the error are given and the analysis of the remainder term allows to find sufficient conditions on f so that the polynomial approximant converges to f. These results are applied to derive a new summation formula with application to rectangular quadrature rule. The polynomial interpolant is related to a fairly interesting boundary value problem for ODEs. We will exhibit solutions for this problem in some special situations.

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1. Status of the problem

In recent years polynomial expansions for sufficiently smooth real functions in polygonal domains by means of boundary values have been investigated in several papers. In particular in [6] there is an expansion in Bernoulli polynomials, i.e., the polynomials of the sequence defined recursively by means

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of relations (see, for example, [12,13])

$$
\begin{aligned}
&B_0(x)=1,\\
&B_n'(x)=nB_{n-1}(x), \quad n\geqslant 1,\\
&\int_0^1 B_n(x)\,\mathrm{d}x=0, \quad n\geqslant 1,
\end{aligned} \tag{1}
$$

for functions in the class $C^n([a,a+h]),\ a\in\mathbb{R}, h>0$:

$$
\begin{aligned}
f(x)=f(a)&+\sum_{k=1}^{n}\frac{B_k\left(\frac{x-a}{h}\right)-B_k}{k!}h^{k-1}(f^{(k-1)}(a+h)-f^{(k-1)}(a))\\
&-\frac{h^{n-1}}{n!}\int_0^1 f^{(n)}(t)(B_n^*(x-t)+(-1)^n B_n(t))\,\mathrm{d}t,
\end{aligned} \tag{2}
$$

where

$$
B_n^*(x)=\begin{cases} B_n(x), & 0\leqslant x<1,\\ B_n^*(x+1) & \text{otherwise}\end{cases}
$$

is the periodic Bernoulli function. Formula (2) is a sort of generalized Taylor formula: the polynomial approximant tends to the Taylor polynomial of f with initial point in a when $h\to 0$. In [7,8], the previous formula has been generalized to the bivariate case, when the approximating function is defined and sufficiently smooth on rectangular or triangular domains. Such kind of expansions join the well-known two-points univariate expansions, like Taylor or Lidstone one, and its generalizations to multivariate polygonal domains [5,9,14]. The interest in this topic lies in the fact that these expansions find applications to several problems of numerical analysis (approximation of solutions of some boundary value problems; polynomial approximation; construction of splines with application to finite elements; etc.). For this reason, in this note we give a further contribution to the problem with new expansions for univariate functions that are, in some sense, symmetric to the Lidstone one.

Let us recall that for functions in the class $C^{2n}([0,1])$ the Lidstone approximation formula [2] can be written as follows:

$$
f(x)=\sum_{k=0}^{n-1} f^{(2k)}(0)\Lambda_k(1-x)+f^{(2k)}(1)\Lambda_k(x)+\int_0^1 G_n(x,t)f^{(2n)}(t)\,\mathrm{d}t, \tag{3}
$$

where [2]

$$
G_n(x,t)=-\begin{cases}\displaystyle\sum_{k=0}^{n-1}\frac{(1-t)^{2(n-k)-1}}{(2(n-k)-1)!}\Lambda_k(x), & x\leqslant t,\\ \displaystyle\sum_{k=0}^{n-1}\frac{t^{2(n-k)-1}}{(2(n-k)-1)!}\Lambda_k(1-x), & x\geqslant t,\end{cases}
$$

and $\{\Lambda_n(x)\}_{n\in\mathbb{N}}$ is the sequence of Lidstone polynomials in the interval [0, 1], which can be defined by means of the recursive relations

$$\begin{aligned}&\Lambda_0(x)=x,\\&\Lambda_n''(x)=\Lambda_{n-1}(x), \quad n\geqslant 1,\\&\Lambda_n(0)=\Lambda_n(1)=0, \quad n\geqslant 1.\end{aligned} \tag{4}$$

The polynomials $\Lambda_n(x)$ are related to the odd degree Bernoulli polynomials $B_{2n+1}(x)$ by the relations [15]

$$\Lambda_n(x)=\frac{2^{2n+1}}{(2n+1)!}\,B_{2n+1}\left(\frac{x+1}{2}\right), \tag{5}$$

for each $n\in\mathbb{N}$. Note that both expansions in (2) and (3) give polynomial approximations to f by means of Bernoulli polynomials using only values of this function and its derivatives at the boundary of the interval. Moreover, denoting by $B_n[f](x)$ and $L_n[f](x)$ the polynomial approximants, respectively, in (2) and (3), the following interpolation conditions hold:

$$B_n[f](0)=f(0), \quad B_n[f]^{(k)}(1)-B_n[f]^{(k)}(0)=f^{(k)}(1)-f^{(k)}(0) \tag{6}$$

and

$$L_n[f]^{(2k)}(0)=f^{(2k)}(0), \quad L_n[f]^{(2k)}(1)=f^{(2k)}(1), \tag{7}$$

for each $k=0,1,\ldots,n-1$. Note that conditions (6) include the interpolation of the function f at the point $x=1$.

In Section 2, we introduce the new polynomial expansion, determine the related property of interpolation and study the remainder term. The analysis of the remainder allows us to find sufficient conditions on f so that the polynomial approximant converges to f. In Section 3 we finally apply previous results to derive a new summation formula with application to rectangular quadrature rule. The polynomial interpolant is related to a fairly interesting boundary value problem for ODEs. We exhibit solutions for this problem in some special situations.

2. Construction of the expansion

2.1. Preliminary results

Let us consider the polynomial sequence defined recursively by means of relations

$$\begin{aligned}&v_0(x)=1,\\&v_k'(x)=\int_0^x v_{k-1}(t)\,\mathrm{d}t, \quad k\geqslant 1,\\&\int_0^1 v_k(x)\,\mathrm{d}x=0, \quad k\geqslant 1.\end{aligned} \tag{8}$$

By definition, $v_k(x)$ is a polynomial of degree not greater than $2k$ and by (8) it is easy to calculate the first polynomials of the sequence:

$$v_0(x)=1,\quad v_1(x)=\frac{x^2}{2}-\frac{1}{6},\quad v_2(x)=\frac{x^4}{24}-\frac{x^2}{12}+\frac{7}{360},\ldots\ .$$

The polynomial sequence $\{v_k(x)\}$ is related to the Lidstone polynomials by the following

Proposition 1. *For each* $k\geqslant 1$ *we have*

$$v_k'(x)=\Lambda_{k-1}(x), \tag{9}$$

where $\Lambda_{k-1}(x)$ *is the Lidstone polynomial of degree* $2k-1$.

Proof. In fact, by (8) we have

$$\begin{aligned}&v_1'(x)=x,\\ &\frac{d^2}{dx^2}v_k'(x)=v_{k-1}'(x),\quad k>1,\\ &v_k'(0)=v_k'(1)=0,\quad k>1,\end{aligned}$$

and Eq. (9) follows for each $k=1,2,\ldots$ by the uniqueness of Lidstone polynomial sequence (4) [2]. □

The polynomial sequence (8) is related to Bernoulli polynomials of even degree by the following:

Proposition 2. *For each* $k\geqslant 1$

$$v_k(x)=\frac{2^{2k}}{(2k)!}B_{2k}\left(\frac{1+x}{2}\right). \tag{10}$$

Proof. By (9) and relations (5) by integration of the second equation in (8) it follows

$$\begin{aligned}v_k(x)&=v_k(0)+\int_0^x v_k'(t)\,dt\\ &=v_k(0)+\frac{2^{2k-1}}{(2k-1)!}\int_0^x B_{2k-1}\left(\frac{1+t}{2}\right)dt\\ &=v_k(0)+\frac{2^{2k}}{(2k-1)!}\int_{1/2}^{(1+x)/2}B_{2k-1}(t)\,dt\\ &=v_k(0)+\frac{2^{2k}}{(2k)!}\left(B_{2k}\left(\frac{1+x}{2}\right)-B_{2k}\left(\frac{1}{2}\right)\right)\end{aligned}$$

and by the third equation in (8)

$$\begin{aligned}0 &= \int_0^1 v_k(x)\,\mathrm{d}x \\ &= \int_0^1 \left(v_k(0) + \frac{2^{2k}}{(2k)!}\left(B_{2k}\left(\frac{1+x}{2}\right) - B_{2k}\left(\frac{1}{2}\right)\right)\right)\mathrm{d}x \\ &= v_k(0) + \frac{2^{2k}}{(2k)!}\left(\int_0^1 B_{2k}\left(\frac{1+x}{2}\right)\mathrm{d}x - B_{2k}\left(\frac{1}{2}\right)\right) \\ &= v_k(0) - \frac{2^{2k}}{(2k)!}B_{2k}\left(\frac{1}{2}\right)\end{aligned}$$

since

$$\int_0^1 B_{2k}\left(\frac{1+x}{2}\right)\mathrm{d}x = \frac{2}{(2k+1)}\left(B_{2k+1}(1) - B_{2k+1}\left(\frac{1}{2}\right)\right) = 0$$

for each $k \geqslant 1$, in force of the following well-known properties of Bernoulli polynomials [1]:

$$B_n\left(\tfrac{1}{2}\right) = -(1-2^{1-n})B_n(0), \quad n \geqslant 1, \tag{11}$$

$$B_n = B_n(0) = B_n(1), \quad n \geqslant 1, \tag{12}$$

$$B_{2n+1}(0) = 0, \quad n \geqslant 1. \quad \square \tag{13}$$

2.2. The main theorem

Now we can prove the following

Theorem 3 (*Main theorem*). *Let us denote by* $\{v_k(x)\}_{k=0,1,2,\ldots}$ *the polynomial sequence defined recursively by means of relations* (8); *then for each* $f \in C^{2n+1}([0,1])$ *the following identity holds*:

$$f(x) = P_{0,n}[f](x) + R_{0,n}(f,x), \tag{14}$$

where $P_{0,n}[f](x)$ *is the polynomial defined by*

$$\begin{aligned}P_{0,n}[f](x) = f(0) + \sum_{j=1}^{n}[&f^{(2j-1)}(1)(v_j(x) - v_j(0)) \\ &- f^{(2j-1)}(0)(v_j(1-x) - v_j(1))]\end{aligned} \tag{15}$$

and the remainder $R_{0,n}(f,x)$ *in its Peano's and Cauchy's representation is given respectively by*

$$R_{0,n}(f,x) = \int_0^1 f^{(2n+1)}(t)K_{0,n}(x,t)\,\mathrm{d}t \tag{16}$$

with

$$K_{0,n}(x,t)=\begin{cases}-\sum_{j=1}^{n}(v_j(x)-v_j(0))\dfrac{(1-t)^{2n-2j+1}}{(2n-2j+1)!}, & x\leqslant t,\\ \dfrac{t^{2n}}{(2n)!}+\sum_{j=1}^{n}(v_j(1-x)-v_j(1))\dfrac{t^{2n-2j+1}}{(2n-2j+1)!}, & t\leqslant x\end{cases} \tag{17}$$

and

$$R_{0,n}(f,x)=\frac{Q_{0,2n+1}(x)}{(2n)!}\,f^{(2n+1)}(\xi),\quad \xi\in(0,1), \tag{18}$$

where we have set

$$Q_{0,2n+1}(x)=\frac{2}{(2n+1)(2n+2)}\left(B_{2n+2}(x)-B_{2n+2}-2^{2n+2}\left(B_{2n+2}\left(\frac{x}{2}\right)-B_{2n+2}\right)\right). \tag{19}$$

In addition, the polynomial $P_{0,n}[f](x)$ *satisfies the following interpolation conditions*:

$$\begin{aligned}&P_{0,n}[f](0)=f(0),\\ &\frac{d^{2k-1}}{dx^{2k-1}}P_{0,n}[f](0)=\frac{d^{2k-1}}{dx^{2k-1}}f(0),\quad k=1,\ldots,n,\\ &\frac{d^{2k-1}}{dx^{2k-1}}P_{0,n}[f](1)=\frac{d^{2k-1}}{dx^{2k-1}}f(1),\quad k=1,\ldots,n.\end{aligned} \tag{20}$$

Proof. Let $f\in C^{2n+1}([0,1])$ be fixed. We start by proving that the polynomial (15) satisfies the interpolation conditions (20): the first of conditions (20) holds trivially; on the other hand for each $j=0,1,\ldots$ and $k=1,2,\ldots$, by an iteration of the second equation in (8) we obtain

$$\frac{d^{2(k-1)}}{dx^{2(k-1)}}v_j(x)=\begin{cases}v_{j-k+1}(x), & k\leqslant j+1,\\ 0, & j\leqslant k-2,\end{cases}$$

so that

$$\frac{d^{2k-1}}{dx^{2k-1}}v_j(x)=v'_{j-k+1}(x)=\int_0^x v_{j-k}(t)\,dt,$$

$$\frac{d^{2k-1}}{dx^{2k-1}}v_j(1-x)=\frac{d}{dx}v_{j-k+1}(1-x)=-v'_{j-k+1}(1-x)=-\int_0^{1-x}v_{j-k}(t)\,dt,$$

for each $k\leqslant j$ and

$$\left.\frac{d^{2k-1}}{dx^{2k-1}}P_{0,n}[f](x)\right|_{x=0}=\sum_{j=k}^{n}\left[f^{(2j-1)}(1)\int_0^0 v_{j-k}(t)\,dt+f^{(2j-1)}(0)\int_0^1 v_{j-k}(t)\,dt\right]=f^{(2k-1)}(0),$$

the third of conditions (20) can be proved by analogy. Let us now prove that the Cauchy's representation (18) holds. In fact, according to the interpolation conditions (20), the function $x \mapsto R_{0,n}(f,x) = f(x) - P_{0,n}[f](x)$ satisfies

$$R_{0,n}(f,0) = 0,$$
$$\left.\frac{\mathrm{d}^{2k-1}}{\mathrm{d}x^{2k-1}} R_{0,n}(f,x)\right|_{x=0} = \left.\frac{\mathrm{d}^{2k-1}}{\mathrm{d}x^{2k-1}} R_{0,n}(f,x)\right|_{x=1} = 0, \quad k = 1,\ldots,n, \tag{21}$$

and analogue conditions are satisfied, for each $n \geqslant 1$, by the $(2n+1)$-degree polynomial $Q_{0,2n+1}$ as defined in (19):

$$Q_{0,2n+1}(0) = 0,$$
$$\left.\frac{\mathrm{d}^{2k-1}}{\mathrm{d}t^{2k-1}} Q_{0,2n+1}(t)\right|_{t=0} = \left.\frac{\mathrm{d}^{2k-1}}{\mathrm{d}t^{2k-1}} Q_{0,2n+1}(t)\right|_{t=1} = 0, \quad k = 1,\ldots,n, \tag{22}$$

in virtue of (11)–(13). Let now $x \in (0,1)$ be fixed; by (21), (22) the function

$$\Phi(\tau) = R_{0,n}(f,\tau) - \frac{R_{0,n}(f,x)}{Q_{0,2n+1}(x)} Q_{0,2n+1}(\tau) \tag{23}$$

satisfies the conditions

$$\Phi(0) = 0,$$
$$\left.\frac{\mathrm{d}^{2k-1}}{\mathrm{d}\tau^{2k-1}} \Phi(\tau)\right|_{\tau=0} = \left.\frac{\mathrm{d}^{2k-1}}{\mathrm{d}\tau^{2k-1}} \Phi(\tau)\right|_{\tau=1} = 0, \quad k = 1,\ldots,n,$$

and, in addition, Φ vanishes also at $\tau = x$, so that by the Rolle's theorem the first derivative $\Phi'(\tau)$ must vanish at a point $\xi_1 : 0 < \xi_1 < x$. The three zeros $0, \xi_1, 1$ of $\Phi'(\tau)$ imply, by a repeated application of Rolle's theorem, the existence of a point $\xi_2 \in (0,1)$ s.t. $\Phi^{(3)}(\xi_2) = 0$ and the three zeros $0, \xi_2, 1$ of $\Phi^{(3)}(\tau)$ imply the existence of a zero $\xi_3 \in (0,1)$ of $\Phi^{(5)}(\tau)$ and so on. Now, by induction, $\Phi^{(2n+1)}(\tau)$ must vanish at a point $\xi \in (0,1)$, so that differentiating $(2n+1)$ times the function in (23) we obtain

$$0 = f^{(2n+1)}(\xi) - \frac{R_{0,n}(f,x)}{Q_{0,2n+1}(x)} (2n)!,$$

that is (18). It remains to prove the Peano's representation (16). For this, we have to note that, by virtue of previous results, for each fixed $x \in [0,1]$ the linear functional

$$f \mapsto R_{0,n}(f,x)$$

is a Peano's type functional on $C^{2n+1}([a,b])$, i.e.,

$$R_{0,n}(p,x) = 0$$

for each p in the space $\mathscr{P}^{2n}$ of the polynomials in x of degree not greater than $2n$. By applying the well known Peano's kernel theorem [10, p. 70] Eq. (16) is certainly true if

$$K_{0,n}(x,t) = \frac{1}{(2n)!} R_{0,n}((x-t)_+^{2n}, x),$$

where $R_{0,n}$ is applied to $(x-t)_+^{2n}$ considered as a function of x:

$$(x-t)_+^{2n} = \begin{cases} 0, & x < t, \\ (x-t)^{2n}, & x \geqslant t; \end{cases}$$

on the other hand,

$$\begin{aligned} &\frac{R_{0,n}((x-t)_+^{2n}, x)}{(2n)!} \\ &= \begin{cases} -\sum_{j=1}^{n} (v_j(x) - v_j(0)) \dfrac{(1-t)^{2n-2j+1}}{(2n-2j+1)!}, & x < t, \\ \dfrac{(x-t)^{2n}}{(2n)!} - \sum_{j=1}^{n} (v_j(x) - v_j(0)) \dfrac{(1-t)^{2n-2j+1}}{(2n-2j+1)!}, & x \geqslant t, \end{cases} \\ &= \begin{cases} -\sum_{j=1}^{n} (v_j(x) - v_j(0)) \dfrac{(1-t)^{2n-2j+1}}{(2n-2j+1)!}, & x < t, \\ \dfrac{t^{2n}}{(2n)!} + \sum_{j=1}^{n} (v_j(1-x) - v_j(1)) \dfrac{t^{2n-2j+1}}{(2n-2j+1)!}, & x \geqslant t, \end{cases} \end{aligned} \tag{24}$$

that is Eq. (16) holds true. The last equation in (24) is obtained using the expansion

$$\begin{aligned} \frac{(x-t)^{2n}}{(2n)!} = \frac{(-t)^{2n}}{(2n)!} + \sum_{j=1}^{n} \Bigg[&\frac{(1-t)^{2n-2j+1}}{(2n-2j+1)!} (v_j(x) - v_j(0)) \\ &- \frac{(-t)^{2n-2j+1}}{(2n-2j+1)!} (v_j(1-x) - v_j(1)) \Bigg], \end{aligned}$$

which holds since $(x-t)^{2n}/(2n)!$, considered as a function of x, is a polynomial of degree $2n$, and hence it must coincide with its polynomial expansion (15). □

Remark 4. For each $f \in C^{2n+1}([0, 1])$ a symmetric expansion (with respect to axis $x = \frac{1}{2}$) to that in Theorem 3 is also possible; in particular, for each $x \in [0, 1]$ the following identity holds:

$$f(x) = P_{1,n}[f](x) + R_{1,n}(f, x), \tag{25}$$

where $P_{1,n}[f](x)$ is the polynomial defined by

$$\begin{aligned} P_{1,n}[f](x) = f(1) + \sum_{j=1}^{n} [&f^{(2j-1)}(1)(v_j(x) - v_j(1)) \\ &- f^{(2j-1)}(0)(v_j(1-x) - v_j(0))]; \end{aligned}$$

for the remainder $R_{1,n}(f, x)$ there are the following representations:

(1) Peano's representation of the error:

$$R_{1,n}(f, x) = \int_0^1 f^{(2n+1)}(t) K_{1,n}(x, t)\, \mathrm{d}t,$$

where

$$K_{1,n}(x,t)=\begin{cases}-\dfrac{(1-t)^{2n}}{(2n)!}-\sum\limits_{j=1}^{n}(v_j(x)-v_j(1))\dfrac{(1-t)^{2n-2j+1}}{(2n-2j+1)!}, & x\leqslant t,\\ \sum\limits_{j=1}^{n}(v_j(1-x)-v_j(0))\dfrac{t^{2n-2j+1}}{(2n-2j+1)!}, & t\leqslant x.\end{cases}$$

(2) Cauchy representation of the error: for each $x\in(0,1)$ there exists $\xi\in(0,1)$ s.t.

$$R_{1,n}(f,x)=\frac{Q_{1,2n+1}(x)}{(2n)!}f^{(2n+1)}(\xi),$$

where we have set

$$Q_{1,2n+1}(x)=\frac{2}{(2n+1)(2n+2)}\left(B_{2n+2}(x)-B_{2n+2}\right.$$
$$\left.-2^{2n+2}\left(B_{2n+2}\left(\frac{x}{2}\right)-B_{2n+2}\left(\frac{1}{2}\right)\right)\right).$$

The polynomial $P_{1,n}[f](x)$ satisfies the following interpolation conditions:

$$P_{1,n}[f](1)=f(1),$$
$$\frac{d^{2k-1}}{dx^{2k-1}}P_{1,n}[f](0)=\frac{d^{2k-1}}{dx^{2k-1}}f(0),\quad k=1,\ldots,n,$$
$$\frac{d^{2k-1}}{dx^{2k-1}}P_{1,n}[f](1)=\frac{d^{2k-1}}{dx^{2k-1}}f(1),\quad k=1,\ldots,n.$$

These results follow by analogy from (14)–(20) after the change of variable $x\mapsto 1-x$.

Remark 5. If $f\in C^{2n+1}([a,b])$ similar expansions to (14), (25) on $[a,b]$ can be obtained by means of a linear transformation of the variable; in particular, if we set $h=(b-a)$ we get from (14), (15), (18)

$$f(x)=f(a)+\sum_{j=1}^{n}\left[h^{2j-1}f^{(2j-1)}(b)\left(v_j\left(\frac{x-a}{h}\right)-v_j(0)\right)\right.$$
$$\left.-h^{2j-1}f^{(2j-1)}(a)\left(v_j\left(\frac{b-x}{h}\right)-v_j(1)\right)\right]$$
$$+h^{2n+1}\frac{Q_{0,2n+1}\left(\frac{x-a}{h}\right)}{(2n)!}f^{(2n+1)}(\xi),\tag{26}$$

where the point $\xi\in(a,b)$ in the reminder term depends on x. The remainder, in its Peano's representation, is given by

$$R_{0,n}(f,x)=h^{2n+1}\int_0^1 f^{(2n+1)}(a+th)K_{0,n}\left(\frac{x-a}{h},t\right)dt,\tag{27}$$

with $K_{0,n}(x,t)$ as defined in (17).

The Cauchy's representation of the error stated in Theorem 3 allows us to derive bounds for the remainder; consequently, we can give sufficient conditions on f so that the polynomial approximant (15) converges to f.

Proposition 6 (*Bound for remainder*). *If* $f \in C^{2n+1}([0,1])$, *then for the error function* $R_{0,n}(f,x)$ *it holds*

$$|R_{0,n}(f,x)| \leqslant 4\,\frac{(2^{2n+2}-1)}{(2n+2)!}|B_{2n+2}| \max_{x\in[0,1]} |f^{(2n+1)}(x)|. \tag{28}$$

Proof. We have

$$\begin{aligned}|Q_{0,2n+1}(x)| &= \frac{2}{(2n+1)(2n+2)}\left|B_{2n+2}(x)-B_{2n+2}\right. \\ &\quad \left. -2^{2n+2}\left(B_{2n+2}\left(\frac{x}{2}\right)-B_{2n+2}\right)\right| \\ &\leqslant \frac{2}{(2n+1)(2n+2)}\left((2^{2n+2}-1)|B_{2n+2}|\right. \\ &\quad \left. +\left|B_{2n+2}(x)-2^{2n+2}B_{2n+2}\left(\frac{x}{2}\right)\right|\right) \\ &\leqslant \frac{4}{(2n+1)(2n+2)}(2^{2n+2}-1)|B_{2n+2}|\end{aligned}$$

since by (11)–(13) the function $B_{2n+2}(x)-2^{2n+2}B_{2n+2}\left(\frac{x}{2}\right)$ assumes its maximum modulus at the boundary of [0, 1]. The thesis follows from (18). □

Corollary 7. *Let* $f \in C^{\infty}([0,1])$ *and suppose that there exist a positive constant* $p < \pi$ *and an integer* $v > 0$ *s.t.*

$$f^{(2n+1)}(x) = \mathrm{O}(p^{2n+1}) \quad \textit{for all } n \geqslant v,\ x \in [0,1],$$

then the polynomial sequence $\{P_{0,n}[f](x)\}_n$ *absolutely and uniformly converges to* $f(x)$ in [0, 1].

Proof. In fact by the Euler's formula [11, p. 5]

$$B_{2n} = (-1)^{n+1}\frac{2(2n)!}{(2\pi)^{2n}}\sum_{k=1}^{\infty}k^{-2n}, \quad n \geqslant 1,$$

it follows that

$$|B_{2n+2}| \leqslant \frac{2(2n+2)!}{(2\pi)^{2n+2}}, \quad n \geqslant 0,$$

so that the right member of (28) is bounded from above by $8(1/\pi)^{2n+2}\max_{x\in[0,1]}|f^{(2n+1)}(x)|$. The thesis follows. □

Example 8. For each $|p| < \pi$ the expansion

$$\cos px = 1 + \sum_{j=1}^{\infty} (-1)^j p^{2j-1} \sin p(v_j(x) - v_j(0)) \tag{29}$$

holds and the right member of (29) absolutely and uniformly converges to $\cos px$ in $[0, 1]$. If $p = \pm\pi$ the convergence of the related series to $\cos \pi x$ fails. For each $p : |p| > \pi$ it can be proved that series in the second member of (29) does not converge absolutely in [0, 1].

3. Applications

3.1. Summation formula and its application to numerical quadrature

For each $l \in \mathbb{N}$ set in (26) with remainder (27) $a = l$, $b = l + 1$, $\xi = \xi_l$; by integrating from l to $l + 1$ and using relations (10), (1), (11)–(13) to compute integrals at the right member we obtain

$$\begin{aligned}\int_l^{l+1} f(x)\,\mathrm{d}x = f(l) + \sum_{j=1}^{n} \Bigg[& f^{(2j-1)}(l+1) \left(-\frac{2^{2j}}{(2j)!} B_{2j}\left(\frac{1}{2}\right)\right) \\ & - f^{(2j-1)}(l) \left(-\frac{2^{2j}}{(2j)!} B_{2j}(1) \right) \Bigg] \\ & + \int_l^{l+1} \int_0^1 f^{(2n+1)}(l+t) K_{0,n}(x-l, t)\,\mathrm{d}t\,\mathrm{d}x,\end{aligned}$$

hence by summing from $l = 0$ to $m - 1$ and inverting the order of summation at the right member it follows that

$$\begin{aligned}\sum_{l=0}^{m-1} f(l) = \int_0^m f(x)\,\mathrm{d}x & - \sum_{j=1}^{n} f^{(2j-1)}(0) \frac{2^{2j}}{(2j)!} B_{2j} \\ & - \sum_{j=1}^{n} \sum_{l=1}^{m-2} f^{(2j-1)}(l) 2 \frac{2^{2j}-1}{(2j)!} B_{2j} - \sum_{j=1}^{n} f^{(2j-1)}(m) \frac{2^{2j}-2}{(2j)!} B_{2j} \\ & - \sum_{l=0}^{m-1} \int_l^{l+1} \int_0^1 f^{(2n+1)}(l+t) K_{0,n}(x-l, t)\,\mathrm{d}t\,\mathrm{d}x.\end{aligned}$$

If we are working under the hypothesis that $f^{(2n+1)}(x)$ is continuous in $[0, \infty)$ we can change the order of integration in the last term of previous equation, and after some calculation we thus obtain

Theorem 9. *Let f be a function defined for $x \geqslant 0$ having $2n+1$ continuous derivatives. Then the following identity holds for $m = 1, 2, \dots$*

$$\begin{aligned}\sum_{l=0}^{m-1} f(l) = &\int_0^m f(x)\,\mathrm{d}x - \sum_{j=1}^{n} f^{(2j-1)}(0)\,\frac{2^{2j}}{(2j)!}\,B_{2j} \\ &- \sum_{j=1}^{n}\sum_{l=1}^{m-2} f^{(2j-1)}(l)2\,\frac{2^{2j}-1}{(2j)!}\,B_{2j} - \sum_{j=1}^{n} f^{(2j-1)}(m)\,\frac{2^{2j}-2}{(2j)!}\,B_{2j} \\ &- \int_0^1 \left(\int_0^1 \sum_{l=0}^{m-1} f^{(2n+1)}(l+t)\right) K_{0,n}(s,t)\,\mathrm{d}s\,\mathrm{d}t,\end{aligned} \tag{30}$$

where $K_{0,n}(s,t)$ is defined as in (17).

Example 10. Let g be a $(2n+1)$ times continuously differentiable function on $[0, 1]$. We apply the summation formula (30) to

$$f(x) = g(hx),$$

where $h = \frac{1}{m}$. Writing $x_l = lh$, $l = 0, 1, \dots, m$, there results

$$\begin{aligned}h\sum_{l=0}^{m-1} g(x_l) = &\int_0^1 g(x)\,\mathrm{d}x - \sum_{j=1}^{n} h^{2j} g^{(2j-1)}(0)\,\frac{2^{2j}}{(2j)!}\,B_{2j} \\ &- \sum_{j=1}^{n}\sum_{l=1}^{m-2} h^{2j} g^{(2j-1)}(x_l)2\,\frac{2^{2j}-1}{(2j)!}\,B_{2j} - \sum_{j=1}^{n} h^{2j} g^{(2j-1)}(1)\,\frac{2^{2j}-2}{(2j)!}\,B_{2j} \\ &- h^{2n+2}\int_0^1 \left(\int_0^1 \sum_{l=0}^{m-1} g^{(2n+1)}(h(l+t))\right) K_{0,n}(s,t)\,\mathrm{d}s\,\mathrm{d}t\end{aligned}$$

and inverting the order of summation at the right member

$$h\sum_{l=0}^{m-1} g(x_l) = \int_0^1 g(x)\,\mathrm{d}x - \sum_{j=1}^{n} h^{2j}\left(g^{(2j-1)}(0)\frac{2^{2j}}{(2j)!}B_{2j}\right. \tag{31}$$

$$\left. - \sum_{l=1}^{m-2} g^{(2j-1)}(x_l)2\,\frac{2^{2j}-1}{(2j)!}\,B_{2j} - g^{(2j-1)}(1)\,\frac{2^{2j}-2}{(2j)!}\,B_{2j}\right) \tag{32}$$

$$- h^{2n+2}\int_0^1\int_0^1 \left(\sum_{l=0}^{m-1} g^{(2n+1)}(h(l+t))\right) K_{0,n}(s,t)\,\mathrm{d}s\,\mathrm{d}t. \tag{33}$$

The left side of the above equation is the result of the numerical integration $\int_0^1 g(x)\,\mathrm{d}x$ by applying the rectangular rule to m equal subintervals. The previous formula exhibits the error of this approximation. The remainder term in the second member clearly is $\mathrm{O}(h^{2n+2})$. If g has derivatives of all orders, formula

(33) holds true for $n = 1, 2, \ldots$ and thus it expresses the fact that the rectangular procedure admits the asymptotic expansion

$$h \sum_{l=0}^{m-1} g(x_l) \approx \int_0^1 g(x)\,\mathrm{d}x - \sum_{j=1}^{n} h^{2j} \left(g^{(2j-1)}(0) \frac{2^{2j}}{(2j)!} B_{2j} - \sum_{l=1}^{m-2} g^{(2j-1)}(x_l) 2 \frac{2^{2j}-1}{(2j)!} B_{2j} - g^{(2j-1)}(1) \frac{2^{2j}-2}{(2j)!} B_{2j} \right)$$

as $h \to 0, h > 0$. Then it is possible, using the derivatives of g, to increase the accuracy of the rectangular procedure by evaluating some terms of this expansion.

Remark 11. In the papers [3,4] there are investigated generalizations of the Euler–MacLaurin formula that involve even-order instead of odd-order derivatives of the function to be integrated; in these formulas the coefficients multiplying the derivatives are connected with Appell polynomials instead of Bernoulli numbers.

3.2. A related boundary value problem

For each fixed $n \geqslant 1$ let us consider the $(2n+1)$th order differential equation

$$y^{(2n+1)}(x) = \Phi(x, y(x), y'(x), \ldots, y^{(k)}(x)), \quad 0 \leqslant k \leqslant 2n-1, \tag{34}$$

with the boundary conditions

$$\begin{aligned} &y(0) = \alpha, \\ &y^{(2i-1)}(0) = \beta_i, \quad y^{(2i-1)}(1) = \gamma_i, \quad i = 1, \ldots, n, \end{aligned} \tag{35}$$

where Φ is continuous at least in the domain of interest and $\alpha, \ \beta_i, \ \gamma_i$ are real numbers. If $\Phi = 0$ it follows by Theorem 3 that problem (34), (35) has a unique solution $y(x) = P_{0,n}[f](x)$; another result related to problem (34), (35) in the particular case when $\Phi = \Phi(x)$ is also a direct consequence of the main theorem:

If $\Phi = \Phi(x)$ *is a real continuous function on* $[0, 1]$ *then the problem*

$$\begin{aligned} &y^{(2n+1)}(x) = \Phi(x), \\ &y(0) = 0, \\ &y^{(2i-1)}(0) = 0, \quad y^{(2i-1)}(1) = 0, \quad i = 1, \ldots, n \end{aligned}$$

has a unique solution given by

$$y(x) = \int_0^1 f^{(2n+1)}(t) K_{0,n}(x,t)\,\mathrm{d}t,$$

where $K_{0,n}(x,t)$ *is defined as* (17).

In our opinion the general boundary value problem (34), (35) is fairly interesting and for this reason the problem of the existence and finding approximated solutions, when they exist, can be studied separately in a successive paper.

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