Available online at www.sciencedirect.com
ScienceDirect
International Journal of Solids and Structures 45 (2008) 2322–2354
INTERNATIONAL JOURNAL OF SOLIDS AND STRUCTURES
www.elsevier.com/locate/ijsolstr

Variational formulations, convergence and stability properties in nonlocal elastoplasticity

Francesco Marotti de Sciarra *

Dipartimento di Ingegneria Strutturale, Università di Napoli Federico II, via Claudio, 21, 80125 Napoli, Italy

Received 3 August 2007; received in revised form 26 November 2007
Available online 8 December 2007

Abstract

A thermodynamically consistent formulation of nonlocal plasticity in the framework of the internal variable theories of inelastic behaviors of associative type is presented. A family of mixed variational formulations, with different combinations of state variables, is provided starting from the finite-step nonlocal elastoplastic structural problem. It is shown that a suitable minimum principles provides a rational basis to exploit the iterative elastic predictor-plastic corrector algorithm in terms of the dissipation functional. A sufficient condition is proved for the convergence of the iterative elastic predictor-plastic corrector algorithm based on a suitable choice of the elastic operator in the prediction phase and a necessary and sufficient condition for the existence of a unique solution (if any) of the nonlocal problem at hand is then provided. The nonlinear stability analysis of the nonlocal problem is carried out following the concept of nonexpansivity proposed in local plasticity.

Keywords: Nonlocal plasticity; Elastoplastic structural model; Variational formulations; Convergence; Stability

1. Introduction

Many materials exhibit a softening behavior which consists in a loss of positive definiteness of the tangent stiffness operator, i.e. a decrease of stress at increasing strain. Such a behavior is coupled with the strain localization which determines the growing of narrow regions where plastic strains tend to concentrate whereas the remainder part of the body unloads elastically.

In classical plasticity theories, the deformation can localize in a zone which is infinitely small so that a displacement discontinuity can develop. In fact, classical plasticity does not contain information about the size of the localization zone which tends to become infinitely thin in the continuum approach or takes on the size of the smallest finite element in the FE-approach.

One possibility to overcome these shortcomings with local plasticity consists in introducing an internal length scale parameter into the continuum model of plasticity. Nonlocal effects can then be modeled by defin-

* Tel.: +39 081 7683337; fax: +39 081 7683332.
E-mail address: marotti@unina.it

0020-7683/$ - see front matter
doi:10.1016/j.ijsolstr.2007.11.022

ing suitably weighted averages, such as in nonlocal formulations, or by defining suitably gradients, such as in gradient approaches, of a collection of static and/or kinematic fields linked to the inelastic processes.

Contributions to the development of the nonlocal theory can be found in Pijaudier-Cabot and Bažant (1987), Bažant and Lin (1988) and, more recently, in Planas et al. (1993), Vermeer and Brinkgreve (1994), Nilsson (1997, 1999), Svedberg and Runesson (1998), Borino et al. (1999), Borino and Failla (2000), Jirásek and Rolshoven (2003). Gradient plasticity has been discussed in Aifantis (1987), de Borst and Pamin (1996), Acharya and Bassani (2000), Bassani et al. (2001), de Borst (2001), Fleck and Hutchinson (2001), Liebe and Steinmann (2001).

In the existing literature on nonlocal plasticity, only few papers (see e.g. Borino et al. (1999) for rate nonlocal plasticity) have been devoted to provide a solid variational basis to the nonlocal theory of plasticity so that most of the FE-algorithms for the resolution of finite-step nonlocal formulations of elastoplasticity are based on extensions of the relations pertaining to local elastoplasticity.

The purpose of the present paper is to address a general structural model based on the nonlinear constitutive model of nonlocal elastoplasticity recently contributed in Marotti de Sciarra (in press) in order to unify several models of nonlocal elastoplasticity existing in the literature. The nonlocal elastoplastic model is formulated in a geometrically linear range and is based on the internal variable theories of inelastic behaviors of associative type (Halphen and Nguyen, 1975).

Performing the time integration of the plastic flow rule according to a fully implicit integration strategy, the relevant finite-step nonlocal elastoplastic structural problem is provided and the related mixed nonlocal variational formulation in the complete set of state variables is directly derived from the structural model following a general procedure. A family of mixed variational formulations, with different combinations of state variables, can then be obtained by enforcing the fulfilment of field equations and constraint conditions.

Two mixed nonlocal variational principles are then specialized in order to recover the finite-step counterpart of the corresponding two principles contributed in Borino et al. (1999) in nonlocal rate plasticity.

It is worth noting that, in the field of local plasticity, formulations and algorithms presented in Bird and Martin (1990) and Reddy and Martin (1991) provide a dual viewpoint of the ones pursued in Ortiz and Popov (1985) and Simo and Govindjee (1991) since, in the former papers, variational and algorithmic aspects are based on the evolution law expressed in terms of the dissipation functional and in the latter papers the elastoplastic problem is expressed in terms of the convex yield function, normality rule and plastic multiplier.

Hence, a further contribution of this paper is devoted to provide a generalized and unified account of the nonlocal elastoplastic structural problem since it is shown that a suitable minimum principles provides a rational basis to exploit the iterative nonlocal elastic predictor-plastic corrector algorithm in terms of the dissipation functional.

Moreover, a sufficient condition is proved for the convergence of the iterative elastic predictor-plastic corrector algorithm based on a suitable choice of the elastic operator in the prediction phase and a necessary and sufficient condition for the uniqueness of the solution (if any) of the nonlocal problem at hand is then provided.

A stability analysis of the nonlocal problem is then carried out following the concept of nonexpansivity proposed in local plasticity by Simo and Govindjee (1991) and Reddy and Martin (1991). The analysis shows that the nonlocal evolution problem meets the property of nonexpansivity for a given displacement history. In particular, the stability is proved for each of the state variables of the model: stresses, kinematic internal variables and dual static internal variables.

Finally a model with a nonlocal variable governing the degradation of the yield limit is considered. A new expression of the space weight function, recently proposed in the literature, is adopted in order to avoid the use of nonstandard and nonsymmetric weight functions for plastic zones close to the boundary. It is shown that the considered nonlocal model reduces to a cohesive model and an example is provided with reference to a one-dimensional bar in tension.

The paper is organized as follows. Section 2 summarizes the basic nonlocal relations. Sections 3 provides the nonlocal constitutive relations in a fully nonlinear form which are specialized to the case of linear elasticity with linear hardening and softening. In Section 4 the nonlocal plastic yielding laws are introduced. The structural model of nonlocal elastoplasticity is presented in Section 5 and its finite-step counterpart based on a fully implicit integration scheme is then addressed. The mixed variational formulation in the complete set of state variables is proved. In Section 6 a family of variational formulations are presented. In Section 7, two general mixed variational formulations are specialized in order to recover the finite-step counterpart of two principles contributed in Borino et al. (1999). A one-dimensional bar in tension is considered to show the localization properties of the model and of the adopted spatial weight function. In Section 8 the convergence criterion is proved and in Section 9 the stability analysis is carried out. The paper is closed by an Appendix A devoted to some basic results of convex analysis adopted in the paper.

2. Nonlocal averaging

A quasi-static evolution process under isothermal conditions is considered for an elastoplastic body subject to a given load history. The nonlocal elastoplastic model is defined on a regular bounded domain $\mathcal{B}$ of an Euclidean space. The time is conceived as a monotonically increasing parameter which orders successive events. Accordingly, a time-independent mechanical behavior of the body is assumed.

Let $\mathcal{D}$ denote the linear space of strains ε and $\mathcal{S}$ denote the dual space of stresses σ. As usual, the total strain ε is assumed to be the sum of an elastic strain e and of a plastic strain p.

In an internal variable approach of associated type (Halphen and Nguyen, 1975), the plastic behavior is described in terms of dual kinematic and static internal variables which account for the evolution of the hardening/softening phenomena. The kinematic (strain-like) internal variables are denoted by $\kappa \in \mathcal{Y}$, $\alpha_1 \in \mathcal{Y}_1$, $\alpha_2 \in \mathcal{Y}_2$, $\alpha_3 \in \mathcal{Y}_3$ and the dual (stress-like) static internal variables are $X \in \mathcal{Y}'$, $\chi_1 \in \mathcal{Y}'_1$, $\chi_2 \in \mathcal{Y}'_2$, $\chi_3 \in \mathcal{Y}'_3$, respectively. The generalized space of kinematic and static internal variables are denoted by $\widehat{\mathcal{D}} = \mathcal{Y}_1 \times \mathcal{Y}_2 \times \mathcal{Y}_3 \times \mathcal{Y}$ and $\widehat{\mathcal{S}} = \mathcal{Y}'_1 \times \mathcal{Y}'_2 \times \mathcal{Y}'_3 \times \mathcal{Y}'$, respectively.

The symbol $((\cdot,\cdot))$ denotes the inner product in the dual spaces and has the mechanical meaning of the internal virtual work. For the Cauchy model it results:

$$((\bullet,\bullet)) = \int_{\mathcal{B}} \bullet \cdot \bullet \, \mathrm{d}\mathcal{B}$$

where $\cdot$ denotes the simple (double) index saturation operation between vectors (tensors).

Nonlocal constitutive theories are based on the assumption that the local value of the state variables at a given point cannot be sufficient to evaluate the state of the material at that point. In fact long-range interactions do exist in real media and, in some circumstances, these interactions may be relevant in the quality of the response of the material.

If the strain field is sufficiently smooth, as often happens in the elastic range, the standard local theory provides a good approximation and it is not necessary the recourse to nonlocal theories. After strain localization, nonlocal effects become meaningful so that nonlocal averaging is applied only to internal variables linked to dissipative processes and nonlocal elastic effects are neglected (see e.g. Bažant and Lin, 1988; Strömber and Ristinmaa, 1996; Svedberg and Runesson, 1998).

A nonlocal field $\bar{\alpha} \in \mathcal{Z}$ will be denoted by a superimposed bar and it can be obtained as a spatial weighted average of another local variable, say $\alpha \in \mathcal{Y}$, by the following parametric relation:

$$\bar{\alpha}(\boldsymbol{x}) = (\mathbf{R}\alpha)(\boldsymbol{x}) \tag{1}$$

where $\mathbf{R} : \mathcal{Y} \to \mathcal{Z}$ denotes a suitable linear regularization operator (see Pijaudier-Cabot and Bažant, 1987; Strömber and Ristinmaa, 1996; Borino et al., 1999; Jirásek and Rolshoven, 2003 for an overview). The kinematic internal variable $\bar{\alpha}$ turns out to be nonlocal since its value at the point $\boldsymbol{x}$ of the body $\mathcal{B}$ depends on the entire field α.

The expression (1) encompasses both nonlocal and gradient plasticity as shown in Marotti de Sciarra (2004). In the present paper attention is focused on integral nonlocal averages of the form:

$$\bar{\alpha}(\boldsymbol{x}) = (\mathbf{R}\alpha)(\boldsymbol{x}) = \int_{\mathcal{B}} \beta_{\boldsymbol{x}}(\boldsymbol{y})\alpha(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}$$

where $\beta_{\boldsymbol{x}}(\boldsymbol{y})$ is a spatial weighting function depending on a material parameter called the internal length scale. Different expressions can be given to the spatial weight function β, such as Gauss-like function. Anyway, for the subsequent developments it is not necessary to assume any explicit expression for β.

Since a nonlocal behavior must be present for high space variation of the local variable α, it is assumed that $\mathbf{R} = I$ for uniform fields α being I the identity operator.

The dual regularization operator $\mathbf{R}' : \mathcal{Z}' \to \mathcal{Y}'$ is defined by the relation:

$$((\xi, \mathbf{R}\alpha)) = ((\mathbf{R}'\xi, \alpha)).$$

The static internal variable $\mathbf{R}'\xi$ is nonlocal since its pointwise value depends upon the entire field ξ over the body $\mathcal{B}$, i.e. $\bar{\xi}(\boldsymbol{x}) = (\mathbf{R}'\xi)(\boldsymbol{x})$ for any $\boldsymbol{x} \in \mathcal{B}$.

For the Cauchy model, an explicit expression of the dual regularization operator $\mathbf{R}'$ can be deduced from the following equalities:

$$\begin{aligned}((\xi, \mathbf{R}\alpha)) &= \int_{\mathcal{B}} \int_{\mathcal{B}} \beta_{\boldsymbol{x}}(\boldsymbol{y})\alpha(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\,\xi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \int_{\mathcal{B}} \alpha(\boldsymbol{y}) \int_{\mathcal{B}} \beta_{\boldsymbol{x}}(\boldsymbol{y})\xi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y} = \int_{\mathcal{B}} \alpha(\boldsymbol{x}) \int_{\mathcal{B}} \beta_{\boldsymbol{y}}(\boldsymbol{x})\xi(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x} \\ &= ((\mathbf{R}'\xi, \alpha))\end{aligned}$$

where

$$(\mathbf{R}'\xi)(\boldsymbol{x}) = \int_{\mathcal{B}} \beta_{\boldsymbol{y}}(\boldsymbol{x})\xi(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}.$$

If the spatial weighting function $\beta_{\boldsymbol{x}}(\boldsymbol{y})$ depends only on the difference $\|\boldsymbol{x} - \boldsymbol{y}\|$, the regularization operator and its dual coincide, that is the operator $\mathbf{R}$ is self-adjoint, i.e. $\mathbf{R} = \mathbf{R}'$.

3. The nonlocal constitutive model

A linear relation is assumed to relate the plastic strain p to the internal variable α_1 by means of a linear operator $A : \mathcal{Y}_1 \to \mathcal{D}$ such that:

$$p = A\alpha_1.$$

The operator A is assumed to have a null kernel in order to ensure that the plastic strains can be univocally deduced from internal variables.

In order to define the free energy, let us introduce the linear operator $L : \mathcal{D} \times \mathcal{Y}_1 \to \mathcal{D}$ and its dual $L' : \mathcal{S} \to \mathcal{S} \times \mathcal{Y}_1'$ defined by:

$$L = [I_D \quad -A], \qquad L' = \begin{bmatrix} I_S \\ -A' \end{bmatrix} \tag{2}$$

where $A' : \mathcal{S} \to \mathcal{Y}_1'$ is the dual operator of A. Accordingly, the additive decomposition of the total strain ε can be written in the form:

$$e = L(\varepsilon, \alpha_1) = \varepsilon - A\alpha_1.$$

The nonlocal free energy is assumed to be the sum of two functionals: the convex elastic energy Φ_{el} and the saddle functional Φ_{in} which accounts for inelastic phenomena:

$$\Phi(\varepsilon, \alpha_1, \alpha_2, \alpha_3, \kappa) = (\Phi_{\mathrm{el}} \circ L)(\varepsilon, \alpha_1) + \Phi_{\mathrm{in}}(\alpha_1, \mathbf{R}\alpha_2, \alpha_3, \mathbf{R}\kappa). \tag{3}$$

The free energy component $(\Phi_{\mathrm{el}} \circ L)$ is convex in (ε, α_1) and Φ_{in} is convex in $(\alpha_1, \alpha_2, \alpha_3)$ and concave in κ.

3.1. Thermodynamic considerations

The first principle of thermodynamics (see e.g. Lemaitre and Chaboche, 1994) for a nonlocal elastoplastic model and isothermal processes can be written pointwise in $\mathcal{B}$ in the following form:

$$\dot{\mathbf{e}} = \sigma \cdot \dot{\varepsilon} + P$$

where the explicit dependence on the point has been dropped for simplicity. The internal energy density $\mathbf{e}$ depends on strain ε, entropy s and internal variables $(\alpha_1, \alpha_2, \alpha_3, \kappa)$ related to isotropic hardening/softening behavior. The nonlocality residual function P takes into account the energy exchanges between neighbor particles (see e.g. Edelen and Laws, 1971; Polizzotto et al., 2006). Being the body a thermodynamically isolated system with reference to energy exchanges due to nonlocality, the following insulation condition holds:

$$\int_{\mathcal{B}} P\,\mathrm{d}\boldsymbol{x} = 0. \tag{4}$$

The second principle of thermodynamics for isothermal processes and for a nonlocal behavior, must be written in global form so that the body energy dissipation W is given by:

$$W = \int_{\mathcal{B}} \dot{s}T\,\mathrm{d}\boldsymbol{x} = ((\sigma, \dot{\varepsilon})) - \int_{\mathcal{B}} \dot{\Phi}\,\mathrm{d}\boldsymbol{x} \geqslant 0 \tag{5}$$

where $\dot{s}$ is the internal entropy production rate per unit volume, T is the absolute temperature and $\Phi = \mathbf{e} - sT$ is the Helmholtz free energy. Following the approach developed in Edelen and Laws (1971) and Borino et al. (1999), the body energy dissipation (5) can be rewritten pointwise by taking into account the nonlocality residual function. Accordingly the dissipation at a given point of the body is:

$$D = \dot{s}T = \sigma \cdot \dot{\varepsilon} - \dot{\Phi} + P \geqslant 0 \tag{6}$$

which represents the Clausius–Duhem inequality for isothermal processes differing from its classical format by the presence of the nonlocality residual function P to guarantee the nonnegativeness of the dissipation and to account for material nonlocality. Expanding the inequality (6) and recalling the expression (3) of the free energy, it results:

$$D = \sigma \cdot \dot{\varepsilon} - d_{\varepsilon}\Phi \cdot \dot{\varepsilon} - d_{\alpha_1}\Phi \cdot \dot{\alpha}_1 - d_{\alpha_2}\Phi \cdot \dot{\alpha}_2 - d_{\alpha_3}\Phi \cdot \dot{\alpha}_3 - d_{\kappa}\Phi \cdot \dot{\kappa} + P \geqslant 0. \tag{7}$$

Using standard arguments (see e.g. Lemaitre and Chaboche, 1994) the following state laws hold:

$$\begin{aligned} &\sigma = d_{\varepsilon}\Phi, \quad -\chi_1 = d_{\alpha_1}\Phi, \quad \bar{\chi}_2 = d_{\alpha_2}\Phi \\ &\chi_3 = d_{\alpha_3}\Phi, \quad -\overline{X} = d_{\kappa}\Phi. \end{aligned} \tag{8}$$

By substituting the relations (8) into the expression (7), the dissipation becomes:

$$D = \chi_1 \cdot \dot{\alpha}_1 - \bar{\chi}_2 \cdot \dot{\alpha}_2 - \chi_3 \cdot \dot{\alpha}_3 + \overline{X} \cdot \dot{\kappa} + P \geqslant 0. \tag{9}$$

At every point where an irreversible mechanism develops, the dissipation can be assumed in the following bilinear form:

$$D = \chi_1 \cdot \dot{\alpha}_1 - \chi_2 \cdot \dot{\eta}_2 - \chi_3 \cdot \dot{\alpha}_3 + X \cdot \dot{\eta} \geqslant 0 \tag{10}$$

where χ_2 and X are (local) variables thermodynamically conjugated to the variables η_2 and η whose expressions are hereafter identified. In such a way the nonlocality residual function has disappeared from the pointwise expression of the dissipation. By comparing (9) and (10), the nonlocality residual function is:

$$P = \bar{\chi}_2 \cdot \dot{\alpha}_2 - \overline{X} \cdot \dot{\kappa} - \chi_2 \cdot \dot{\eta}_2 + X \cdot \dot{\eta} \tag{11}$$

and, employing the insulation condition (4), it results:

$$((\bar{\chi}_2, \dot{\alpha}_2)) - ((\overline{X}, \dot{\kappa})) - ((\chi_2, \dot{\eta}_2)) + ((X, \dot{\eta})) = 0. \tag{12}$$

Noting that the following duality conditions hold:

$$((\bar{\chi}_2, \dot{\alpha}_2)) = ((\chi_2, \dot{\bar{\alpha}}_2)), \quad ((\overline{X}, \dot{\kappa})) = ((X, \dot{\bar{\kappa}})) \tag{13}$$

and substituting the equalities (13) in (12), it results:

$$((\chi_2, \dot{\bar{\alpha}}_2)) - ((X, \dot{\bar{\kappa}})) - ((\chi_2, \dot{\eta}_2)) + ((X, \dot{\eta})) = 0$$

for any possible static variable χ_2 and X. Hence the identifications $\dot{\eta}_2 = \dot{\bar{\alpha}}_2$ and $\dot{\eta} = \dot{\bar{\kappa}}$ hold true. The nonlocality residual function (11) and the dissipation (10) can be given the following explicit expressions at a given point of the body $\mathcal{B}$:

$$
\begin{aligned}
P &= \bar{\chi}_2 \cdot \dot{\alpha}_2 - \overline{X} \cdot \dot{\kappa} - \chi_2 \cdot \dot{\bar{\alpha}}_2 + X \cdot \dot{\bar{\kappa}} \\
D &= \chi_1 \cdot \dot{\alpha}_1 - \chi_2 \cdot \dot{\bar{\alpha}}_2 - \chi_3 \cdot \dot{\alpha}_3 + X \cdot \dot{\bar{\kappa}} \geqslant 0.
\end{aligned} \tag{14}
$$

If the nonlocal variables are constant in space, the regularization operator becomes the identity operator and the nonlocal variables turn out to be coincident to their local counterparts, i.e. $\dot{\bar{\alpha}}_2 = \dot{\alpha}_2$, $\dot{\bar{\kappa}} = \dot{\kappa}$, $\bar{\chi}_2 = \chi_2$ and $\overline{X} = X$. As a consequence the relation $(14)_1$ shows that the nonlocality residual function P vanishes, the inequality $(14)_2$ provides the expression of the dissipation in terms of local variables and the relations (8) yield the constitutive laws in terms of local fields.

Hence the above analysis shows that the constant field requirement on the operator $\mathbf{R}$ and the insulation condition guarantee that the nonlocal model behaves in all aspects as a local one under uniform fields.

3.2. Nonlocal constitutive relations

Stresses and static internal variables are related to strains and kinematic internal variables by means of the multi-valued relation:

$$
\begin{aligned}
&(\sigma, -\chi_1, \bar{\chi}_2, \chi_3, -\overline{X}) \in \partial \Phi(\varepsilon, \alpha_1, \alpha_2, \alpha_3, \kappa) \Longleftrightarrow \\
&(\sigma, -\chi_1, \bar{\chi}_2, \chi_3) \times (-\overline{X}) \in \partial_1 \Phi(\varepsilon, \alpha_1, \alpha_2, \alpha_3, \kappa) \times \partial_2 \Phi(\varepsilon, \alpha_1, \alpha_2, \alpha_3, \kappa)
\end{aligned} \tag{15}
$$

where $\partial_1 \Phi$ denotes the subdifferential (Hiriart-Urruty and Lemarechal, 1993) of Φ with respect to $(\varepsilon, \alpha_1, \alpha_2, \alpha_3)$ and $\partial_2 \Phi$ denotes the superdifferential of Φ with respect to κ. For simplicity, the superdifferential of a concave functional is denoted by the same symbol ∂ of the subdifferential of a convex functional. Moreover the subdifferential and the superdifferential are both referred to as subdifferential if no confusion can arise. In the sequel, the free energy is assumed to be a differentiable functional so that the subdifferentials ∂ appearing in (15) become the usual derivative d.

Recalling the expressions (1) and (3), the nonlocal constitutive relations are given by:

$$
\begin{aligned}
&(\sigma, -\chi_1, \bar{\chi}_2, \chi_3, -\overline{X}) = d\Phi(\varepsilon, \alpha_1, \alpha_2, \alpha_3, \kappa) \\
&\Longleftrightarrow \begin{cases}
\begin{bmatrix} \sigma \\ -\chi_1 \end{bmatrix} = d_{(\varepsilon,\alpha_1)} \Phi(\varepsilon, \alpha_1, \alpha_2, \alpha_3, \kappa) \\
\bar{\chi}_2 = d_{\alpha_2} \Phi(\varepsilon, \alpha_1, \alpha_2, \alpha_3, \kappa) = d_{\alpha_2} \Phi_{\text{in}}(\alpha_1, \mathbf{R}\alpha_2, \alpha_3, \mathbf{R}\kappa) \\
\chi_3 = d_{\alpha_3} \Phi(\varepsilon, \alpha_1, \alpha_2, \alpha_3, \kappa) = d_{\alpha_3} \Phi_{\text{in}}(\alpha_1, \mathbf{R}\alpha_2, \alpha_3, \mathbf{R}\kappa) \\
\overline{X} = -d_{\kappa} \Phi(\varepsilon, \alpha_1, \alpha_2, \alpha_3, \kappa) = -d_{\kappa} \Phi_{\text{in}}(\alpha_1, \mathbf{R}\alpha_2, \alpha_3, \mathbf{R}\kappa)
\end{cases}
\end{aligned} \tag{16}
$$

The derivative of $\Phi(\varepsilon, \alpha_1, \alpha_2, \alpha_3, \kappa)$ with respect to the pair (ε, α_1) can be evaluated by means of a chain rule of differential calculus (see Appendix A) to get:

$$
\begin{aligned}
d_{(\varepsilon,\alpha_1)} \Phi(\varepsilon, \alpha_1, \alpha_2, \alpha_3, \kappa) &= d_{(\varepsilon,\alpha_1)} (\Phi_{\text{el}} \circ L)(\varepsilon, \alpha_1) + d_{\alpha_1} \Phi_{\text{in}}(\alpha_1, \mathbf{R}\alpha_2, \alpha_3, \mathbf{R}\kappa) \\
&= L' d\Phi_{\text{el}}(L(\varepsilon, \alpha_1)) + d_{\alpha_1} \Phi_{\text{in}}(\alpha_1, \mathbf{R}\alpha_2, \alpha_3, \mathbf{R}\kappa) \\
&= \begin{bmatrix} d\Phi_{\text{el}}(\varepsilon - A\alpha_1) \\ -A' d\Phi_{\text{el}}(\varepsilon - A\alpha_1) \end{bmatrix} + \begin{bmatrix} 0 \\ d_{\alpha_1} \Phi_{\text{in}}(\alpha_1, \mathbf{R}\alpha_2, \alpha_3, \mathbf{R}\kappa) \end{bmatrix}.
\end{aligned}
$$

Then the constitutive relations assume the form:

$$
\begin{cases}
\sigma = d\Phi_{\text{el}}(\varepsilon - A\alpha_1) \\
-\chi_1 = -A'\sigma + d_{\alpha_1} \Phi_{\text{in}}(\alpha_1, \mathbf{R}\alpha_2, \alpha_3, \mathbf{R}\kappa) \\
\bar{\chi}_2 = \mathbf{R}' d_{\bar{\alpha}_2} \Phi_{\text{in}}(\alpha_1, \bar{\alpha}_2, \alpha_3, \mathbf{R}\kappa) = \mathbf{R}'\chi \\
\chi_3 = d_{\alpha_3} \Phi_{\text{in}}(\alpha_1, \mathbf{R}\alpha_2, \alpha_3, \mathbf{R}\kappa) \\
\overline{X} = -\mathbf{R}' d_{\bar{\kappa}} \Phi_{\text{in}}(\alpha_1, \mathbf{R}\alpha_2, \alpha_3, \bar{\kappa}) = -\mathbf{R}'\xi
\end{cases} \tag{17}
$$

where we have set:

$$
\chi = d_{\bar{\alpha}2} \Phi_{\text{in}}(\alpha_1, \bar{\alpha}_2, \alpha_3, \mathbf{R}\kappa), \quad \xi = d_{\bar{\kappa}} \Phi_{\text{in}}(\alpha_1, \mathbf{R}\alpha_2, \alpha_3, \bar{\kappa}).
$$

Note that the static internal variable $\bar{\chi}_2$ is a nonlocal variable since its pointwise value depends upon the entire field χ over the body $\mathcal{B}$, i.e. $\bar{\chi}_2(\boldsymbol{x}) = (\mathbf{R}'\chi)(\boldsymbol{x})$ for any $\boldsymbol{x} \in \mathcal{B}$. The nonlocal field $\bar{\chi}_2$ is dual of the (local) kinematic internal variable α_2 and $\bar{\chi}_2$ is the $\mathbf{R}'$-transformed of the static internal variable χ which is conjugate to $\bar{\alpha}_2$. Analogously the nonlocal internal variable $\overline{X} = -\mathbf{R}'\xi$ is the dual of the kinematic internal variable κ and turns out to be the opposite of the $\mathbf{R}'$-transformed of the static internal variable ξ which is conjugate to $\bar{\kappa}$.

In the case of a linear elastic behavior with linear hardening and softening, the free energy is expressed by:

$$\Phi(\varepsilon, \alpha_1, \alpha_2, \alpha_3, \kappa) = \frac{1}{2}((\mathbf{E}(\varepsilon - A\alpha_1), \varepsilon - A\alpha_1)) + \frac{1}{2}((\mathbf{H}_1\alpha_1, \alpha_1)) + \frac{1}{2}((h_2\mathbf{R}\alpha_2, \mathbf{R}\alpha_2)) + \frac{1}{2}((h_3\alpha_3, \alpha_3)) + \frac{1}{2}((h\mathbf{R}\kappa, \mathbf{R}\kappa)) \quad (18)$$

where the operator $\mathbf{H}_1$ is positive definite, the moduli h_2 and h_3 are positive and the softening modulus h is negative. Then the nonlocal constitutive relations (17) turn out to be:

$$\begin{cases} \sigma = \mathbf{E}(\varepsilon - A\alpha_1) \\ -\chi_1 = -A'\sigma + \mathbf{H}_1\alpha_1 \\ \bar{\chi}_2 = \mathbf{R}'h_2\mathbf{R}\alpha_2 = \mathbf{R}'\chi \\ \chi_3 = h_3\alpha_3 \\ \overline{X} = -\mathbf{R}'h\mathbf{R}\kappa = -\mathbf{R}'\xi \end{cases} \quad (19)$$

where $\chi = h_2\mathbf{R}\alpha_2$ and $\xi = h\mathbf{R}\kappa$.

The expression $(19)_1$ is the elastic relation being $e = \varepsilon - A\alpha_1$ where $p = A\alpha_1$. The expression $(19)_2$ yields the static internal variable χ_1 in terms of σ and α_1. The expression $(19)_3$ provides the nonlocal static internal force $\bar{\chi}_2$, conjugate to α_2, which is the $\mathbf{R}'$-transformed of the dissipative force $\chi = h_2\bar{\alpha}_2$. The expression $(19)_4$ provides the static internal force χ_3 conjugate to α_3 and the expression $(19)_5$ yields the nonlocal static internal force $\overline{X}$, conjugate to κ, which is the opposite of the $\mathbf{R}'$-transformed of the dissipative force $\xi = h\bar{\kappa}$.

In order to develop a variational formulation for this class of nonlocal problems, alternative expressions of the constitutive relations (16) have to be provided.

To this end, the conjugate of the nonlocal free energy provides the *complementary* nonlocal elastic energy which is the saddle functional $\Phi^* : \mathcal{S} \times \widehat{\mathcal{S}} \to \overline{\Re}(\overline{\Re} = \{-\infty\} \cup \Re \cup \{+\infty\})$, convex in the state variables $(\sigma, \chi_1, \bar{\chi}_2, \chi_3)$ and concave in $\overline{X}$, given by:

$$\Phi^*(\sigma, \chi_1, \bar{\chi}_2, \chi_3, \overline{X}) = \inf_{\delta} \sup_{(\eta, \delta_1, \delta_2, \delta_3)} \{((\sigma, \eta)) + ((\chi_1, \delta_1)) + ((\bar{\chi}_2, \delta_2)) + ((\chi_3, \delta_3)) + ((\overline{X}, \delta)) - \Phi(\eta, \delta_1, \delta_2, \delta_3, \delta)\}.$$

The partial conjugates of the nonlocal free energy with respect to the state variables $(\varepsilon, \alpha_1, \alpha_2, \alpha_3)$ and with respect to the kinematic variable κ provide the *semicomplementary* nonlocal free energies, respectively, expressed by the convex functionals $\Psi : \mathcal{S} \times \mathcal{Y}'_1 \times \mathcal{Y}'_2 \times \mathcal{Y}'_3 \times \mathcal{Y} \to \Re \cup \{+\infty\}$ and $\Psi^* : \mathcal{D} \times \mathcal{Y}_1 \times \mathcal{Y}_2 \times \mathcal{Y}_3 \times \mathcal{Y}' \to \Re \cup \{+\infty\}$ which are defined by:

$$\begin{aligned} \Psi(\sigma, \chi_1, \bar{\chi}_2, \chi_3, \kappa) &= -\inf_{\overline{Y}} \{((\overline{Y}, \kappa)) - \Phi^*(\sigma, \chi_1, \bar{\chi}_2, \chi_3, \overline{Y})\} \\ &= \sup_{(\eta, \delta_1, \delta_2, \delta_3)} \{((\sigma, \eta)) + ((\chi_1, \delta_1)) + ((\bar{\chi}_2, \delta_2)) + ((\chi_3, \delta_3)) - \Phi(\eta, \delta_1, \delta_2, \delta_3, \kappa)\} \\ \Psi^*(\varepsilon, \alpha_1, \alpha_2, \alpha_3, -\overline{X}) &= -\inf_{\delta} \{((\overline{X}, \delta)) - \Phi(\varepsilon, \alpha_1, \alpha_2, \alpha_3, \delta)\} \\ &= \sup_{(\tau, \xi_1, \bar{\xi}_2, \xi_3)} \{((\tau, \varepsilon)) + ((\xi_1, \alpha_1)) + ((\bar{\xi}_2, \alpha_2)) + ((\xi_3, \alpha_3)) - \Phi^*(\tau, \xi_1, \bar{\xi}_2, \xi_3, \overline{X})\}. \end{aligned} \quad (20)$$

The nonlocal elastoplastic constitutive relations (16) can then be formulated according to the following equivalent expressions:

$$\begin{aligned}
&(\sigma,-\chi_1,\bar{\chi}_2,\chi_3,-\overline{X}) = d\Phi(\varepsilon,\alpha_1,\alpha_2,\alpha_3,\kappa),\\
&(\varepsilon,\alpha_1,\alpha_2,\alpha_3,\kappa) = d\Phi^*(\sigma,-\chi_1,\bar{\chi}_2,\chi_3,-\overline{X}),\\
&(\varepsilon,\alpha_1,\alpha_2,\alpha_3,\overline{X}) = d\Psi(\sigma,-\chi_1,\bar{\chi}_2,\chi_3,\kappa),\\
&(\sigma,-\chi_1,\bar{\chi}_2,\chi_3,\kappa) = d\Psi^*(\varepsilon,\alpha_1,\alpha_2,\alpha_3,\overline{X})
\end{aligned} \tag{21}$$

which can be equivalently rewritten in terms of Fenchel's equalities:

$$\begin{aligned}
&-\Psi(\sigma,-\chi_1,\bar{\chi}_2,\chi_3,\kappa)+\Phi^*(\sigma,-\chi_1,\bar{\chi}_2,\chi_3,-\overline{X}) = -((\overline{X},\kappa)),\\
&\Psi(\sigma,-\chi_1,\bar{\chi}_2,\chi_3,\kappa)+\Phi(\varepsilon,\alpha_1,\alpha_2,\alpha_3,\kappa) = ((\sigma,\varepsilon))-((\chi_1,\alpha_1))+((\bar{\chi}_2,\alpha_2))+((\chi_3,\alpha_3)),\\
&-\Psi^*(\varepsilon,\alpha_1,\alpha_2,\alpha_3,\overline{X})+\Phi(\varepsilon,\alpha_1,\alpha_2,\alpha_3,\kappa) = -((\overline{X},\kappa)),\\
&\Psi^*(\varepsilon,\alpha_1,\alpha_2,\alpha_3,\overline{X})+\Phi^*(\sigma,-\chi_1,\bar{\chi}_2,\chi_3,-\overline{X}) = ((\sigma,\varepsilon))-((\chi_1,\alpha_1))+((\bar{\chi}_2,\alpha_2))+((\chi_3,\alpha_3)),\\
&\Phi(\varepsilon,\alpha_1,\alpha_2,\alpha_3,\kappa)+\Phi^*(\sigma,-\chi_1,\bar{\chi}_2,\chi_3,-\overline{X}) = ((\sigma,\varepsilon))-((\chi_1,\alpha_1))+((\bar{\chi}_2,\alpha_2))+((\chi_3,\alpha_3))-((\overline{X},\kappa)),\\
&\Psi(\sigma,-\chi_1,\bar{\chi}_2,\chi_3,\kappa)+\Psi^*(\varepsilon,\alpha_1,\alpha_2,\alpha_3,\overline{X}) = ((\sigma,\varepsilon))-((\chi_1,\alpha_1))+((\bar{\chi}_2,\alpha_2))+((\chi_3,\alpha_3))+((\overline{X},\kappa)).
\end{aligned} \tag{22}$$

4. The plastic yielding laws

The elastic domain C is defined in the space of static internal variables $(\chi_1,\bar{\chi}_2,\chi_3,\overline{X})$ as the level set of a convex yield mode $G:\widehat{\mathcal{S}}\to\Re\cup\{+\infty\}$ in the form:

$$C=\{(\chi_1,\bar{\chi}_2,\chi_3,\overline{X})\in\widehat{\mathcal{S}}:G(\chi_1,\bar{\chi}_2,\chi_3,\overline{X})\leqslant 0\} \tag{23}$$

provided that the minimum of G is negative.

In the applications the nonlocal yield mode G is usually written in terms of the convex yield function g in the form:

$$G(\chi_1,\bar{\chi}_2,\chi_3,\overline{X})=g(\chi_1)-\bar{\chi}_2-\chi_3-\overline{X}-\chi_0\leqslant 0 \tag{24}$$

where χ_0 represents a constant scalar value which characterize the initial yield limit. The choice of the function g depends on the particular yield criterion adopted for the material, see e.g. Salençon (1983). Note that the yield function g is expressed in terms of the static internal variable χ_1 which is linked to the stress σ by the relation $(17)_2$ and the static internal variables $\bar{\chi}_2$, χ_3 and $\overline{X}$ control the size of the elastic domain.

In the proposed model some of the static internal variables are treated as local and some as nonlocal in order to get a nonlocal elastoplastic problem which can be specialized to several existing model of nonlocal plasticity as provided in Marotti de Sciarra (in press). The variable χ_1 is linked to the stress as reported in $(17)_2$ so that a kinematic hardening is introduced in the model since the stress σ is admissible if the associated thermodynamic force $A'\sigma$ belongs to the transformation of the elastic domain C by the amount $d_{\alpha_1}\Phi_{\text{in}}$. Depending on the expression of the free energy, the static internal variable χ_1 can coincide to the stress σ so that the yield function g is expressed in terms of stresses.

If, besides the static internal variable χ_1, one local internal variable is introduced in the model, say χ_3, the free energy is given by $\Phi(\varepsilon,\alpha_3)$ and the elastic domain is given in the form $G(\chi_1,\chi_3)=g(\chi_1)-\chi_3-\chi_0\leqslant 0$ so that the model is, trivially, local. If one nonlocal internal variables is considered, say $\overline{X}$, the free energy is given by $\Phi(\varepsilon,\kappa)$ and the elastic domain is given as $G(\chi_1,\overline{X})=g(\chi_1)-\overline{X}-\chi_0\leqslant 0$. The model does not act as a genuine localization limiter since it does not prevent localization of plastic strains into a set of zero measure. A theoretical and computational discussion on this issue is provided in Section 7.

As a consequence two internal variables χ_3 and $\overline{X}$ must be related to plastic softening. An additional variable $\bar{\chi}_2$ is introduced in the nonlocal model since the internal variables $\bar{\chi}_2$, χ_3 and $\overline{X}$ can be combined in various ways in the expression of the free energy so that several existing softening models can be recovered as special cases of the present model (Marotti de Sciarra, in press).

A mechanical interpretation of the dual kinematic internal variables is given in the sequel after the specification of the flow rule.

The generalized flow rule is given in the following three equivalent forms:

$$\begin{aligned}&(\dot{\alpha}_1, -\dot{\alpha}_2, -\dot{\alpha}_3, \dot{\kappa}) \in N_C(\chi_1, \bar{\chi}_2, \chi_3, \overline{X})\\ &(\chi_1, \bar{\chi}_2, \chi_3, \overline{X}) \in \partial D(\dot{\alpha}_1, -\dot{\alpha}_2, -\dot{\alpha}_3, \dot{\kappa})\\ &\sqcup_C(\chi_1, \bar{\chi}_2, \chi_3, \overline{X}) + D(\dot{\alpha}_1, -\dot{\alpha}_2, -\dot{\alpha}_3, \dot{\kappa}) = ((\chi_1, \dot{\alpha}_1)) - ((\bar{\chi}_2, \dot{\alpha}_2)) - ((\chi_3, \dot{\alpha}_3)) + ((\overline{X}, \dot{\kappa}))\end{aligned} \tag{25}$$

where $\sqcup_C(\chi_1, \bar{\chi}_2, \chi_3, \overline{X})$ is the indicator of the elastic domain and $D : \widehat{\mathcal{D}} \to \Re \cup \{+\infty\}$ has the physical meaning of dissipation associated with a given rate $(\dot{\alpha}_1, -\dot{\alpha}_2, -\dot{\alpha}_3, \dot{\kappa})$ of internal kinematic variables. The functional D is the support functional of the elastic domain C:

$$D(\dot{\alpha}_1, -\dot{\alpha}_2, -\dot{\alpha}_3, \dot{\kappa}) = \sup\{((\tau_1, \dot{\alpha}_1)) - ((\bar{\tau}_2, \dot{\alpha}_2)) - ((\tau_3, \dot{\alpha}_3)) + ((\overline{Y}, \dot{\kappa})) \quad \text{s.t. } (\tau_1, \bar{\tau}_2, \tau_3, \overline{Y}) \in C\}, \tag{26}$$

where "s.t." stands for "subject to". The sup operation is then performed with respect to the state variables $(\tau_1, \bar{\tau}_2, \tau_3, \overline{Y})$ belonging to the elastic domain C.

The dissipation D turns out to be nonnegative if and only if the null static internal variables belong to the elastic domain C. Moreover the functional D is strictly positive if and only if the null static internal variables lie in the interior of the elastic domain C (see Romano et al., 1993b, in the case of local plasticity).

The dissipation attains its maximum at the point $(\chi_1, \bar{\chi}_2, \chi_3, \overline{X})$ which fulfils the normality rule with the rate of the kinematic internal variables $(\dot{\alpha}_1, -\dot{\alpha}_2, -\dot{\alpha}_3, \dot{\kappa})$ and is given by (Marotti de Sciarra, in press):

$$D(\dot{\alpha}_1, -\dot{\alpha}_2, -\dot{\alpha}_3, \dot{\kappa}) = ((\chi_1, \dot{\alpha}_1)) - ((\bar{\chi}_2, \dot{\alpha}_2)) - ((\chi_3, \dot{\alpha}_3)) + ((\overline{X}, \dot{\kappa})). \tag{27}$$

Recalling the expression (24) of the yield mode, the flow rule $(25)_1$ can be equivalently rewritten in terms of the plastic multiplier λ. In fact, being $\sqcup_C(\chi_1, \bar{\chi}_2, \chi_3, \overline{X}) = (\sqcup_{\Re^-} \circ G)(\chi_1, \bar{\chi}_2, \chi_3, \overline{X})$, where $\Re^-$ denotes the set of nonpositive scalars, it results (Marotti de Sciarra, 2004):

$$\partial\sqcup_C(\chi_1, \bar{\chi}_2, \chi_3, \overline{X}) = \partial(\sqcup_{\Re^-} \circ G)(\chi_1, \bar{\chi}_2, \chi_3, \overline{X}) = \partial\sqcup_{\Re^-}[(G\chi_1, \bar{\chi}_2, \chi_3, \overline{X})](dG\chi_1, \bar{\chi}_2, \chi_3, \overline{X}).$$

Noting that $\partial\sqcup_{\Re^-}[(G\chi_1, \bar{\chi}_2, \chi_3, \overline{X})] = N_{\Re^-}[(G\chi_1, \bar{\chi}_2, \chi_3, \overline{X})]$, where $N_{\Re^-}$ denotes the normal cone to the set $\Re^-$, the flow rule $(25)_1$ can be rewritten in the equivalent forms:

$$\begin{aligned}&(\dot{\alpha}_1, -\dot{\alpha}_2, -\dot{\alpha}_3, \dot{\kappa}) \in \partial\sqcup_C(\chi_1, \bar{\chi}_2, \chi_3, \overline{X})\\ &(\dot{\alpha}_1, -\dot{\alpha}_2, -\dot{\alpha}_3, \dot{\kappa}) = \lambda dG(\chi_1, \bar{\chi}_2, \chi_3, \overline{X}) \quad \text{s.t. } \lambda \in N_{\Re^-}[G(\chi_1, \bar{\chi}_2, \chi_3, \overline{X})]\\ &(\dot{\alpha}_1, -\dot{\alpha}_2, -\dot{\alpha}_3, \dot{\kappa}) = \lambda dG(\chi_1, \bar{\chi}_2, \chi_3, \overline{X})\\ &\quad \text{s.t. } \lambda \geqslant 0, \ g(\chi_1) - \bar{\chi}_2 - \chi_3 - \overline{X} - \chi_0 \leqslant 0\\ &\qquad \lambda[g(\chi_1) - \bar{\chi}_2 - \chi_3 - \overline{X} - \chi_0] = 0.\end{aligned} \tag{28}$$

Recalling the expression (24), the flow rule $(28)_3$ can be explicitly rewritten as:

$$\begin{cases} \dot{\alpha}_1 = \lambda d_{\chi_1} G(\chi_1, \bar{\chi}_2, \chi_3, \overline{X}) = \lambda dg(\chi_1)\\ -\dot{\alpha}_2 = \lambda d_{\bar{\chi}_2} G(\chi_1, \bar{\chi}_2, \chi_3, \overline{X}) = -\lambda\\ -\dot{\alpha}_3 = \lambda d_{\chi_3} G(\chi_1, \bar{\chi}_2, \chi_3, \overline{X}) = -\lambda\\ \dot{\kappa} = \lambda d_{\overline{X}} G(\chi_1, \bar{\chi}_2, \chi_3, \overline{X}) = -\lambda \end{cases} \tag{29}$$

under the complementarity conditions:

$$\begin{aligned}&\lambda \geqslant 0, \quad g(\chi_1) - \bar{\chi}_2 - \chi_3 - \overline{X} - \chi_0 \leqslant 0\\ &\lambda[g(\chi_1) - \bar{\chi}_2 - \chi_3 - \overline{X} - \chi_0] = 0.\end{aligned} \tag{30}$$

As a result, the kinematic internal variable α_1 coincides to the plastic strain if $A = I$ and the flow rule $(29)_1$ is expressed in terms of the plastic strain rate. Moreover the rate of the kinematic internal variables $\dot{\alpha}_2$, $\dot{\alpha}_3$ and $-\dot{\kappa}$ coincide to the plastic multiplier λ. Since in the elastic range $\alpha_2 = \alpha_3 = \kappa = 0$, the relations $(29)_{2\text{–}4}$ yield the equalities $\alpha_2 = \alpha_3 = -\kappa$ and they coincide to the actual value of the plastic multiplier.

If the yield mode fulfils the condition $\|dg(\chi_1)\| = 1$, it results $\dot{\alpha}_2 = \dot{\alpha}_3 = -\dot{\kappa} = \|\dot{\alpha}_1\|$ being $\|\dot{\alpha}_1\| = \lambda\|dg(\chi_1)\| = \lambda$. Hence the kinematic internal variables α_2, α_3 and κ can be related to the accumulated generalized plastic strain in the following form:

$$\alpha_2 = \alpha_3 = -\kappa = \int_0^t \|\dot{\alpha}_1(\tau)\| \mathrm{d}\tau = \alpha^p.$$

In most of nonlocal elastoplastic models, the flow rule is expressed in terms of plastic strains. This requirement can be fulfilled by assuming that the operator A coincides to the identity one so that $\alpha_1 = p$ and the accumulated generalized plastic strain coincides to the effective plastic strain ε^p, i.e. $\alpha^p = \varepsilon^p = \int_0^t \|\dot{p}(\tau)\| \mathrm{d}\tau$. Accordingly the kinematic internal variables $\alpha_2 = \alpha_3 = -\kappa$ have the physical meaning of the effective plastic strain. In this case the size of the elastic domain is driven by the sum of the static internal variables $\bar{\chi}_2 + \chi_3 + \overline{X}$ which depend on the effective plastic strain.

Assuming a linear behavior with linear hardening/softening and a macroscopically homogeneous material for which the moduli h, h_2 and h_3 are constant in space, the dissipative force is given by:

$$q = \bar{\chi}_2 + \chi_3 + \overline{X} = \mathbf{R}' h_2 \mathbf{R} \alpha_2 + h_3 \alpha_3 - \mathbf{R}' h \mathbf{R} \kappa = h_3 \alpha_2 + \mathbf{R}'(h_2 + h) \mathbf{R} \alpha_2.$$

The size of the elastic domain is then controlled by the sum of a local variable and of a nonlocal one so that a localized plastic zone of nonzero measure can be obtained.

If the material parameters are such that $h_2 + h = \beta m$ and $h_3 = (1 - \beta) m$, where β is a suitable material parameter and $m < 0$ is a plastic modulus, softening is driven by the linear combination

$$q = (1 - \beta) m \alpha_2 + \beta m \tilde{\alpha}_2$$

where $\tilde{\alpha}_2 = \mathbf{R}' \mathbf{R} \alpha_2$. A model similar to the one proposed by Vermeer and Brinkgreve (1994) is thus recovered. For $\beta = 0$ the local model of plasticity is recovered and for $\beta = 1$ the cohesive model is obtained. For $0 \leqslant \beta \leqslant 1$, Planas et al. (1993) proved in the one-dimensional setting that the plastic strain localizes into a set of zero measure that is a single cross section of the bar. The plastic zone has a nonzero measure if $\beta > 1$.

The size of the elastic domain is controlled by the variable q so that the kinematic internal variables α_2 can be assumed as the driving variable of the softening law.

5. The structural problem for nonlocal elastoplasticity

Let one assume that displacements u belong to the Sobolev space $\mathcal{U} = \mathcal{H}^m(\mathcal{B})$ of fields which are square integrable in $\mathcal{B}$ together with their distributional derivatives up to the order m (Brezis, 1983). Conforming displacement fields fulfil linear constraint conditions and belong to a closed linear subspace $\mathcal{L} \subset \mathcal{U}$.

The kinematic operator $\mathbf{B} \in \mathrm{Lin}\{\mathcal{U}, \mathcal{D}\}$ is a bounded linear operator from $\mathcal{U}$ to the Hilbert space of square integrable strain fields $\varepsilon \in \mathcal{D}$.

Denoting by $\mathcal{F}$ the subspace of external forces, which is dual of $\mathcal{U}$, the continuous operator $\mathbf{B}' \in \mathrm{Lin}\{\mathcal{S}, \mathcal{F}\}$, dual of $\mathbf{B}$, is the equilibrium operator. The symbol $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $\mathcal{U}$ and its dual $\mathcal{F}$.

Let $\ell = \{\mathbf{t}, \mathbf{b}\} \in \mathcal{F}$ be the load functional where $\mathbf{t}$ and $\mathbf{b}$ denote the tractions and the body forces. For simplicity, imposed strains and displacements are not considered.

The equilibrium equation and the compatibility condition are:

$$f = \mathbf{B}' \sigma \text{ where } f \in \mathcal{F}, \sigma \in \mathcal{S}, \quad \varepsilon = \mathbf{B} u \text{ where } \varepsilon \in \mathcal{D}, u \in \mathcal{U}.$$

The external relation between reactions and displacements is assumed to be given by:

$$r \in \partial \Upsilon(u)$$

being $\Upsilon : \mathcal{U} \to \Re \cup \{-\infty\}$ a concave functional. Accordingly, the inverse relation is expressed as:

$$u \in \partial \Upsilon^*(r)$$

where the concave functional $\Upsilon^* : \mathcal{F} \to \Re \cup \{-\infty\}$ represents the conjugate of Υ and Fenchel's relation holds:

$$\Upsilon(u) + \Upsilon^*(r) = \langle r, u \rangle. \tag{31}$$

Different expressions can be given to the functional Υ depending on the type of external constraints such as bilateral, unilateral, elastic or convex. A survey of the particular expression assumed by the functional in each of these cases can be found in Romano (2002). For future reference we report the expressions of Υ and Υ^* in the case of external frictionless bilateral constraints with homogeneous boundary conditions. Noting that the subspace of the external constraint reactions R is the orthogonal complement of the subspace of conforming displacements $\mathcal{L}$, that is $R = \mathcal{L}^{\perp}$, the functional Υ turns out to be:

$$\Upsilon(u) = \sqcap_{\mathcal{L}}(u) = \begin{cases} 0 & \text{if } u \in \mathcal{L} \\ -\infty & \text{otherwise} \end{cases}$$

and

$$\Upsilon^*(r) = \sqcap_{\mathcal{L}^{\perp}}(r) = \begin{cases} 0 & \text{if } r \in \mathcal{L}^{\perp} = R \\ -\infty & \text{otherwise.} \end{cases}$$

Accordingly the relation $r \in \partial\Upsilon(u)$ is equivalent to state $u \in \mathcal{L}$ and $r \in R = \mathcal{L}^{\perp}$, i.e. $\langle r, u\rangle = 0$ for any conforming displacement $u \in \mathcal{L}$.

The relations governing the nonlocal elastoplastic structural problem for a given load history $\ell(t)$ are:

$$\begin{cases} \mathbf{B}'\sigma = \ell + r & \text{equilibrium} \\ \mathbf{B}u = \varepsilon & \text{compatibility} \\ (\dot{\alpha}_1, -\dot{\alpha}_2, -\dot{\alpha}_3, \dot{\kappa}) \in N_C(\chi_1, \bar{\chi}_2, \chi_3, \overline{X}) & \text{flow rule} \\ (\sigma, -\chi_1, \bar{\chi}_2, \chi_3, -\overline{X}) = d\Phi(\varepsilon, \alpha_1, \alpha_2, \alpha_3, \kappa) & \text{constitutive relation} \\ u \in \partial\Upsilon^*(r) & \text{external relation.} \end{cases} \tag{32}$$

The evolution analysis of a nonlocal elastoplastic problem can be performed by solving a sequence of problems in which the load increment is applied and the state variables are updated at the end of each increment (see e.g. Reddy and Martin, 1991; Simo et al., 1988).

Attention is focused on a single step of the procedure for which the load increment is given. Accordingly one needs to evaluate the finite increments of the unknown variables corresponding to the increment of strain when their values are assigned at the beginning of the step. Let $(\cdot)_o$ denote the known quantities $(\cdot)$ at the beginning of each step. In order to formulate the finite-step counterpart of the flow rule $(32)_3$, the time derivative is replaced by the finite increment ratio $(\Delta\alpha_1, -\Delta\alpha_2, -\Delta\alpha_3, \Delta\kappa)/\Delta t$ where $\Delta\bullet = (\bullet) - (\bullet)_o$. Adopting a fully implicit time integration scheme, the flow rule of the nonlocal model is enforced at the end of the step according to the relation:

$$\frac{1}{\Delta t}(\Delta\alpha_1, -\Delta\alpha_2, -\Delta\alpha_3, \Delta\kappa) \in N_C(\chi_1, \bar{\chi}_2, \chi_3, \overline{X})$$

which, being $N_C(\chi_1, \bar{\chi}_2, \chi_3, \overline{X})$ a convex cone, can be rewritten in the equivalent forms:

$$\begin{aligned} &(\Delta\alpha_1, -\Delta\alpha_2, -\Delta\alpha_3, \Delta\kappa) \in N_C(\chi_1, \bar{\chi}_2, \chi_3, \overline{X}) \\ &(\chi_1, \bar{\chi}_2, \chi_3, \overline{X}) \in \partial D(\Delta\alpha_1, -\Delta\alpha_2, -\Delta\alpha_3, \Delta\kappa) \\ &\sqcup_C(\chi_1, \bar{\chi}_2, \chi_3, \overline{X}) + D(\Delta\alpha_1, -\Delta\alpha_2, -\Delta\alpha_3, \Delta\kappa) = ((\chi_1, \Delta\alpha_1)) - ((\bar{\chi}_2, \Delta\alpha_2)) - ((\chi_3, \Delta\alpha_3)) + ((\overline{X}, \Delta\kappa)). \end{aligned} \tag{33}$$

The finite-step nonlocal elastoplastic model is then given by

$$\begin{cases} \mathbf{B}'\sigma = \ell + r & \text{equilibrium} \\ \mathbf{B}u = \varepsilon & \text{compatibility} \\ (\Delta\alpha_1, -\Delta\alpha_2, -\Delta\alpha_3, \Delta\kappa) \in N_C(\chi_1, \bar{\chi}_2, \chi_3, \overline{X}) & \text{finite-step flow rule} \\ (\sigma, -\chi_1, \bar{\chi}_2, \chi_3, -\overline{X}) = d\Phi(\varepsilon, \alpha_1, \alpha_2, \alpha_3, \kappa) & \text{constitutive relation} \\ u \in \partial\Upsilon^*(r) & \text{external relation.} \end{cases} \tag{34}$$

Introducing the product space $\mathcal{W} = \mathcal{U} \times \mathcal{S} \times \widehat{\mathcal{S}} \times \mathcal{D} \times \widehat{\mathcal{D}} \times \mathcal{F}$ and its dual space $\mathcal{W}'$, the finite-step structural problem (34) can be arranged to build up a global multi-valued structural operator $\boldsymbol{S} : \mathcal{W} \to \mathcal{W}'$ governing the whole problem:

$$\mathbf{0} \in \boldsymbol{S}(\boldsymbol{w}) = \widehat{\boldsymbol{S}}(\boldsymbol{w}) + \boldsymbol{w}_{\mathrm{o}}, \qquad \boldsymbol{w} \in \mathcal{W}, \quad \boldsymbol{w}_{\mathrm{o}} \in \mathcal{W}'.$$

The explicit expression of the structural operator $\widehat{\boldsymbol{S}}$ is:

$$\widehat{\boldsymbol{S}} = \left[\begin{array}{cccccccccccc}
0 & \mathbf{B}' & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I_{\mathcal{F}} \\
\mathbf{B} & 0 & 0 & 0 & 0 & 0 & -I_{\mathcal{D}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & & & & & 0 & I_{\mathcal{Y}_1} & 0 & 0 & 0 & 0 \\
0 & 0 & & & & & 0 & 0 & -I_{\mathcal{Y}_2} & 0 & 0 & 0 \\
 & & & \multicolumn{2}{c}{-\partial \sqcup_C} & & & & & & & \\
0 & 0 & & & & & 0 & 0 & 0 & -I_{\mathcal{Y}_3} & 0 & 0 \\
0 & 0 & & & & & 0 & 0 & 0 & 0 & I_{\mathcal{Y}} & 0 \\
0 & -I_{\mathcal{S}} & 0 & 0 & 0 & 0 & 0 & & & & & 0 \\
0 & 0 & I_{\mathcal{Y}'_1} & 0 & 0 & 0 & 0 & & & & & 0 \\
0 & 0 & 0 & -I_{\mathcal{Y}'_2} & 0 & 0 & 0 & & & & & 0 \\
 & & & & & & & & \multicolumn{2}{c}{d\Phi} & & \\
0 & 0 & 0 & 0 & -I_{\mathcal{Y}'_3} & 0 & 0 & & & & & 0 \\
0 & 0 & & & & I_{\mathcal{Y}'} & 0 & & & & & 0 \\
-I_{\mathcal{U}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial \Upsilon^*
\end{array}\right],$$

and the vectors $\boldsymbol{w}$ and $\boldsymbol{w}_{\mathrm{o}}$ are given by

$$\begin{aligned}
\boldsymbol{w}^{\mathrm{T}} &= \begin{bmatrix} u & \sigma & \chi_1 & \bar{\chi}_2 & \chi_3 & \overline{X} & \varepsilon & \alpha_1 & \alpha_2 & \alpha_3 & \kappa & r \end{bmatrix}, \\
\boldsymbol{w}_{\mathrm{o}} &= [-\ell \quad 0 \quad -\alpha_{1\mathrm{o}} \quad \alpha_{2\mathrm{o}} \quad \alpha_{3\mathrm{o}} \quad -\kappa_{\mathrm{o}} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0].
\end{aligned}$$

It is apparent that the present nonlocal model is governed by single-valued and multi-valued operators so that it is necessary the recourse to the potential theory for multi-valued operators (Romano et al., 1993a) in order to derive, in a direct manner, the general mixed functional whose generalized gradient yields back the field equations and the constitutive relations (34).

Accordingly the conservativity of the structural operator follows from the duality existing between the pairs $(\mathbf{B}, \mathbf{B}')$, $(I_{\mathcal{D}}, I_{\mathcal{S}})$, $(I_{\mathcal{Y}_1}, I_{\mathcal{Y}'_1})$, $(I_{\mathcal{Y}_2}, I_{\mathcal{Y}'_2})$, $(I_{\mathcal{Y}_3}, I_{\mathcal{Y}'_3})$, $(I_{\mathcal{Y}}, I_{\mathcal{Y}'})$, $(I_{\mathcal{U}}, I_{\mathcal{F}})$, the conservativity of $d\Phi$ and the conservativity of the subdifferentials $\partial \sqcup_C$ and $\partial \Upsilon^*$.

The related potential can be evaluated by summing up the potentials of each component operator so that it turns out to be:

$$\boldsymbol{\Omega}(\boldsymbol{w}) = \int_0^1 ((\boldsymbol{S}(t\boldsymbol{w}), \boldsymbol{w}))\,\mathrm{d}t = \int_0^1 ((\widehat{\boldsymbol{S}}(t\boldsymbol{w}), \boldsymbol{w}))\,\mathrm{d}t - \langle \ell, u \rangle - ((\chi_1, \alpha_{1\mathrm{o}})) + ((\bar{\chi}_2, \alpha_{2\mathrm{o}})) + ((\chi_3, \alpha_{3\mathrm{o}})) - ((\overline{X}, \kappa_{\mathrm{o}})),$$

to get:

$$\begin{aligned}
\boldsymbol{\Omega}(u, \sigma, \chi_1, \bar{\chi}_2, \chi_3, \overline{X}, \varepsilon, \alpha_1, \alpha_2, \alpha_3, \kappa, r) = {} & \Phi(\varepsilon, \alpha_1, \alpha_2, \alpha_3, \kappa) - \sqcup_C(\chi_1, \bar{\chi}_2, \chi_3, \overline{X}) - ((\sigma, \varepsilon)) \\
& + ((\sigma, \mathbf{B}u)) + ((\chi_1, \alpha_1 - \alpha_{1\mathrm{o}})) - ((\bar{\chi}_2, \alpha_2 - \alpha_{2\mathrm{o}})) - ((\chi_3, \alpha_3 - \alpha_{3\mathrm{o}})) \\
& + ((\overline{X}, \kappa - \kappa_{\mathrm{o}})) + \Upsilon^*(r) - \langle \ell + r, u \rangle.
\end{aligned}$$

The potential $\mathbf{\Omega}$ turns out to be linear in (u,σ), jointly convex with respect to the state variables $(\varepsilon,\alpha_1,\alpha_2,\alpha_3)$ and jointly concave with respect to $(\chi_1,\bar{\chi}_2,\chi_3,\overline{X},\kappa,r)$. The stationary conditions of $\mathbf{\Omega}$ enforced at the point $(u,\sigma,\chi_1,\bar{\chi}_2,\chi_3,\overline{X},\varepsilon,\alpha_1,\alpha_2,\alpha_3,\kappa,r)$ provides the operator form of the structural problem (34). In fact the stationary condition:

$$\mathbf{0}\in\partial\mathbf{\Omega}(\boldsymbol{w}) \iff \begin{cases} 0\in\partial_u\mathbf{\Omega}(\boldsymbol{w}) \\ 0\in\partial_\sigma\mathbf{\Omega}(\boldsymbol{w}) \\ (0,0,0,0)\in\partial_{(\chi_1,\bar{\chi}_2,\chi_3,\overline{X})}\mathbf{\Omega}(\boldsymbol{w}) \\ (0,0,0,0,0)\in\partial_{(\varepsilon,\alpha_1,\alpha_2,\alpha_3,\kappa)}\mathbf{\Omega}(\boldsymbol{w}) \\ 0\in\partial_r\mathbf{\Omega}(\boldsymbol{w}) \end{cases}$$

yields the relations:

$$\begin{cases} \mathbf{B}'\sigma=\ell+r \\ \mathbf{B}u=\varepsilon \\ (\Delta\alpha_1,-\Delta\alpha_2,-\Delta\alpha_3,\Delta\kappa)\in\partial\sqcup_C(\chi_1,\bar{\chi}_2,\chi_3,\overline{X}) \\ (\sigma,-\chi_1,\bar{\chi}_2,\chi_3,-\overline{X})=d\Phi(\varepsilon,\alpha_1,\alpha_2,\alpha_3,\kappa) \\ u\in\partial\Upsilon^*(r). \end{cases}$$

By reverting the steps above, it can be shown that a solution of the finite-step nonlocal elastoplastic problem makes the functional $\mathbf{\Omega}$ stationarity. The following statement then holds.

Proposition 1. *The set of state variables* $(u,\sigma,\chi_1,\bar{\chi}_2,\chi_3,\overline{X},\varepsilon,\alpha_1,\alpha_2,\alpha_3,\kappa,r)$ *is a solution of the saddle problem*

$$\min_{(\varepsilon,\alpha_1,\alpha_2,\alpha_3)}\ \max_{(\chi_1,\bar{\chi}_2,\chi_3,\overline{X},\kappa,r)}\ \operatorname*{stat}_{(u,\sigma)}\mathbf{\Omega}(u,\sigma,\chi_1,\bar{\chi}_2,\chi_3,\overline{X},\varepsilon,\alpha_1,\alpha_2,\alpha_3,\kappa,r)$$

if and only if it is a solution of the finite-step nonlocal elastoplastic problem (34).

A family of potentials can be recovered from the potential $\mathbf{\Omega}$ by enforcing the field equations, the constitutive relations and the external relation. All these functionals assume the same value when they are evaluated at a solution point of the nonlocal elastoplastic structural problem.

6. Variational principles

Quite a few variational formulations have been proposed for the nonlocal elastoplastic problem (see e.g. Borino et al., 1999 for rate nonlocal elastoplasticity, Mühlhaus and Aifantis, 1991 for gradient plasticity). In the sequel different functionals in a reduced number of state variables and the related variational principles are derived starting from the potential $\mathbf{\Omega}$. Such formulations are compared with the corresponding ones existing in the literature (if any) in the case of nonlocal or local plasticity.

Imposing in the expression of the potential $\mathbf{\Omega}$ the finite-step flow rule $(34)_3$, in terms of Fenchel's equality $(33)_3$, and the constitutive relation $(34)_4$, in terms of Fenchel's equation $(22)_3$, we get the following variational formulation.

Proposition 2. *The set* $(u,\sigma,\overline{X},\varepsilon,\alpha_1,\alpha_2,\alpha_3,\kappa,r)$ *is a solution of the saddle problem*

$$\min_{(\varepsilon,\alpha_1,\alpha_2,\alpha_3,\kappa,\overline{X})}\ \max_{r}\ \operatorname*{stat}_{(u,\sigma)}\mathbf{\Omega}_1(u,\sigma,\overline{X},\varepsilon,\alpha_1,\alpha_2,\alpha_3,\kappa,r)$$

where

$$\begin{aligned}\mathbf{\Omega}_1(u,\sigma,\overline{X},\varepsilon,\alpha_1,\alpha_2,\alpha_3,\kappa,r)=\ &\Psi^*(\varepsilon,\alpha_1,\alpha_2,\alpha_3,\overline{X})-((\overline{X},\kappa))+D(\Delta\alpha_1,-\Delta\alpha_2,-\Delta\alpha_3,\Delta\kappa)-((\sigma,\varepsilon))\\ &+((\sigma,\mathbf{B}u))+\Upsilon^*(r)-\langle\ell+r,u\rangle\end{aligned}$$

if and only if it is a solution of the finite-step nonlocal elastoplastic problem (34).

The stationary conditions of $\mathbf{\Omega}_1$ enforced at the point $(u,\sigma,\overline{X},\varepsilon,\alpha_1,\alpha_2,\alpha_3,\kappa,r)$ are given by:

$$
(0,0,0,0,0,0,0,0,0) \in \partial \mathbf{\Omega}_1(u,\sigma,\overline{X},\varepsilon,\alpha_1,\alpha_2,\alpha_3,\kappa,r)
$$
$$
\Longleftrightarrow \begin{cases} 0 \in \partial_u \mathbf{\Omega}_1(u,\sigma,\overline{X},\varepsilon,\alpha_1,\alpha_2,\alpha_3,\kappa,r) \\ 0 \in \partial_\sigma \mathbf{\Omega}_1(u,\sigma,\overline{X},\varepsilon,\alpha_1,\alpha_2,\alpha_3,\kappa,r) \\ 0 \in \partial_{\overline{X}} \mathbf{\Omega}_1(u,\sigma,\overline{X},\varepsilon,\alpha_1,\alpha_2,\alpha_3,\kappa,r) \\ 0 \in \partial_\varepsilon \mathbf{\Omega}_1(u,\sigma,\overline{X},\varepsilon,\alpha_1,\alpha_2,\alpha_3,\kappa,r) \\ (0,0,0,0) \in \partial_{(\alpha_1,\alpha_2,\alpha_3,\kappa)} \mathbf{\Omega}_1(u,\sigma,\overline{X},\varepsilon,\alpha_1,\alpha_2,\alpha_3,\kappa,r) \\ 0 \in \partial_r \mathbf{\Omega}_1(u,\sigma,\overline{X},\varepsilon,\alpha_1,\alpha_2,\alpha_3,\kappa,r) \end{cases}
$$

which yield the relations:

$$
\begin{cases} \mathbf{B}'\sigma = \ell + r \\ \mathbf{B}u = \varepsilon \\ \kappa = d_{\overline{X}} \Psi^*(\varepsilon,\alpha_1,\alpha_2,\alpha_3,\overline{X}) \\ \sigma = d_\varepsilon \Psi^*(\varepsilon,\alpha_1,\alpha_2,\alpha_3,-\overline{X}) \\ (-d_{(\alpha_1,\alpha_2,\alpha_3)} \Psi^*(\varepsilon,\alpha_1,\alpha_2,\alpha_3,-\overline{X}),\overline{X}) \in \partial_{(\alpha_1,\alpha_2,\alpha_3,\kappa)} D(\Delta\alpha_1,-\Delta\alpha_2,-\Delta\alpha_3,\Delta\kappa) \\ u \in \partial \Upsilon^*(r). \end{cases} \tag{35}
$$

The relation $(35)_1$ provides the equilibrium equation. The relation $(35)_2$ represents the compatibility condition. The relations $(35)_{3\text{–}4}$ yield the constitutive relations in terms of stresses σ and of nonlocal kinematic internal variables κ, according to $(21)_4$. The relation $(35)_5$ shows, according to $(21)_4$, that the triplet $(-\chi_1,\bar{\chi}_2,\chi_3) = d_{(\alpha_1,\alpha_2,\alpha_3)}\Psi^*(\varepsilon,\alpha_1,\alpha_2,\alpha_3,\overline{X})$ is such that $(\chi_1,-\bar{\chi}_2,-\chi_3,\overline{X}) \in \partial_{(\alpha_1,\alpha_2,\alpha_3,\kappa)} D(\Delta\alpha_1,-\Delta\alpha_2,-\Delta\alpha_3,\Delta\kappa)$ which is equivalent to the flow rule $(\chi_1,\bar{\chi}_2,\chi_3,\overline{X}) \in \partial D(\Delta\alpha_1,-\Delta\alpha_2,-\Delta\alpha_3,\Delta\kappa)$ where the subdifferential is performed with respect to the arguments of D. The relation $(35)_6$ provides the external constraint. The structural model is thus obtained. By reverting the steps above, a solution of the finite-step nonlocal elastoplastic problem makes the functional $\mathbf{\Omega}_1$ stationarity.

The direct proof of the subsequent variational principles is omitted for conciseness and it can be performed following the reasoning shown above for Proposition 2. It is worth noting that a new variational formulation is obtained from an existing one by enforcing the fulfilment of the field equations and of the constitutive relations pertaining to the nonlocal model (34). Hence the variational formulations turn out to be equivalent each other.

A variational principle in terms of the displacements, kinematic internal variables, static internal variable $\overline{X}$ and external reactions can be obtained from the expression of the potential $\mathbf{\Omega}_1$ by imposing the compatibility condition $(34)_2$ to get the next variational formulation.

Proposition 3. *The set* $(u,\alpha_1,\alpha_2,\alpha_3,\kappa,\overline{X},r)$ *is a solution of the saddle problem*

$$
\min_{(u,\alpha_1,\alpha_2,\alpha_3,\kappa,\overline{X})} \max_r \mathbf{\Omega}_2(u,\alpha_1,\alpha_2,\alpha_3,\kappa,\overline{X},r)
$$

where

$$
\mathbf{\Omega}_2(u,\alpha_1,\alpha_2,\alpha_3,\kappa,\overline{X},r) = \Psi^*(\mathbf{B}u,\alpha_1,\alpha_2,\alpha_3,\overline{X}) - ((\overline{X},\kappa)) + D(\Delta\alpha_1,-\Delta\alpha_2,-\Delta\alpha_3,\Delta\kappa) + \Upsilon^*(r) - \langle \ell + r, u \rangle
$$

if and only if it is a solution of the finite-step nonlocal elastoplastic problem (34).

Imposing in the expression of the potential $\mathbf{\Omega}_2$ the external relation $(34)_5$ in terms of Fenchel's equality (31), it turns out to be the following minimum principle holds.

Proposition 4. *The set* $(u,\alpha_1,\alpha_2,\alpha_3,\kappa,\overline{X})$ *is a solution of the convex optimization problem*

$$
\min_{(u,\alpha_1,\alpha_2,\alpha_3,\kappa,\overline{X})} \mathbf{\Omega}_3(u,\alpha_1,\alpha_2,\alpha_3,\kappa,\overline{X})
$$

where

$$
\mathbf{\Omega}_3(u,\alpha_1,\alpha_2,\alpha_3,\kappa,\overline{X}) = \Psi^*(\mathbf{B}u,\alpha_1,\alpha_2,\alpha_3,\overline{X}) - ((\overline{X},\kappa)) + D(\Delta\alpha_1,-\Delta\alpha_2,-\Delta\alpha_3,\Delta\kappa) - \Upsilon(u) - \langle \ell, u \rangle
$$

if and only if it is a solution of the finite-step nonlocal elastoplastic problem (34).

Remark 5. It is of interest to investigate the condition for the finite-step nonlocal elastoplastic structural problem (34) to admit a unique solution. In fact, minimum principles in structural mechanics are relevant since solution techniques can be exploited and existence and uniqueness results can be provided by recourse to functional analysis. In particular, uniqueness of the solution is ensured if the functional to be minimized is strictly convex. If the semicomplementary free energy Ψ^* pertaining to the nonlocal constitutive model is strictly convex, the functional $\mathbf{\Omega}_3$ turns out to be strictly convex and the finite-step nonlocal elastoplastic structural model (34) admits a unique solution (if any). Since the constitutive model is usually formulated in terms of the free energy Φ, it is useful to provide the uniqueness condition in terms of the free energy. Noting that a convex functional is strictly convex if and only if its conjugate is differentiable, the functional Ψ^* turns out to be strictly convex if and only if the free energy Φ is differentiable with respect to the kinematic internal variable κ and strictly convex with respect to the state variables $(\varepsilon, \alpha_1, \alpha_2, \alpha_3)$. Existence of the solution is still an open problem.

The extension to the present nonlocal context of the classical one-field variational formulation in stress rates, known as Prager–Hodge principle (Prager and Hodge, 1951) can be obtained from the potential $\mathbf{\Omega}$ by imposing the equilibrium equation $(34)_1$ and the constitutive relation $(34)_4$ in terms of Fenchel's equality $(22)_2$:

$$\begin{aligned}\mathbf{\Omega}_4(\sigma, \chi_1, \bar{\chi}_2, \chi_3, \overline{X}) &= -\Phi^*(\sigma, -\chi_1, \bar{\chi}_2, \chi_3, -\overline{X}) - \sqcup_C(\chi_1, \bar{\chi}_2, \chi_3, \overline{X}) - ((\chi_1, \alpha_{1\mathrm{o}})) + ((\bar{\chi}_2, \alpha_{2\mathrm{o}})) + ((\chi_3, \alpha_{3\mathrm{o}})) \\ &\quad - ((\overline{X}, \kappa_{\mathrm{o}})) + \Upsilon^*(\mathbf{B}'\sigma - \ell) \\ &= -W^*(\sigma, \chi_1, \bar{\chi}_2, \chi_3, \overline{X}) + \Upsilon^*(\mathbf{B}'\sigma - \ell)\end{aligned}$$

where

$$\begin{aligned}W^*(\sigma, \chi_1, \bar{\chi}_2, \chi_3, \overline{X}) &= \Phi^*(\sigma, -\chi_1, \bar{\chi}_2, \chi_3, -\overline{X}) + \sqcup_C(\chi_1, \bar{\chi}_2, \chi_3, \overline{X}) + ((\chi_1, \alpha_{1\mathrm{o}})) - ((\bar{\chi}_2, \alpha_{2\mathrm{o}})) - ((\chi_3, \alpha_{3\mathrm{o}})) \\ &\quad + ((\overline{X}, \kappa_{\mathrm{o}})).\end{aligned}$$

It is worth noting that the potential $\mathbf{\Omega}_4$ is concave in $(\sigma, \chi_1, \bar{\chi}_2, \chi_3)$ and convex in $\overline{X}$ if the static internal variables belong to the elastic domain since the indicator of the elastic domain vanishes. The following result thus holds.

Proposition 6 (Finite-step nonlocal Prager–Hodge principle). *The set* $(\sigma, \chi_1, \bar{\chi}_2, \chi_3, \overline{X})$ *is a solution of the optimization problem*:

$$\max_{(\sigma, \chi_1, \bar{\chi}_2, \chi_3)} \operatorname*{stat}_{\overline{X}} \mathbf{\Omega}_4(\sigma, \chi_1, \bar{\chi}_2, \chi_3, \overline{X})$$

if and only if it is a solution of the finite-step nonlocal elastoplastic problem (34).

The extension to the present nonlocal context of the classical one-field variational formulation in displacement rates, due to Greenberg (Greenberg, 1949), can be obtained from the potential $\mathbf{\Omega}_3$ by performing the following minimization:

$$W(\mathbf{B}u) = \inf_{(\alpha_1, \alpha_2, \alpha_3, \kappa, \overline{X})} \left\{ \Psi^*(\mathbf{B}u, \alpha_1, \alpha_2, \alpha_3, \overline{X}) - ((\overline{X}, \kappa)) + D(\Delta\alpha_1, -\Delta\alpha_2, -\Delta\alpha_3, \Delta\kappa) \right\}.$$

Substituting the functional W into the expression of the potential $\mathbf{\Omega}_3$, the following convex functional is obtained

$$\mathbf{\Omega}_5(u) = W(\mathbf{B}u) - \Upsilon(u) - \langle \ell, u \rangle$$

and the next statement then holds.

Proposition 7 (Finite-step nonlocal Greenberg principle). *The displacement u is a solution of the convex optimization problem*:

$$\min_u \mathbf{\Omega}_5(u)$$

if and only if it is a solution of the finite-step nonlocal elastoplastic problem (34).

The corresponding of the finite-step Greenberg principle in which the plastic multiplier λ explicitly appears as an independent variable is derived hereafter.

To this end the complementarity conditions (30) can be rewritten in the form:

$$\lambda \in \partial \sqcup_{\mathfrak{R}^-}[G(\chi_1, \bar{\chi}_2, \chi_3, \overline{X})] \iff G(\chi_1, \bar{\chi}_2, \chi_3, \overline{X}) \in \partial \sqcup_{\mathfrak{R}}^+(\lambda) \tag{36}$$

being $\sqcup_{\mathfrak{R}^-}$ and $\sqcup_{\mathfrak{R}^+}$ conjugate functionals. By virtue of Fenchel's equality (Hiriart-Urruty and Lemarechal, 1993), the relation (36) between λ and G can be equivalently formulated in the form:

$$\sqcup_{\mathfrak{R}^-}[G(\chi_1, \bar{\chi}_2, \chi_3, \overline{X})] + \sqcup_{\mathfrak{R}^+}(\lambda) = ((\lambda, G(\chi_1, \bar{\chi}_2, \chi_3, \overline{X}))). \tag{37}$$

Recalling that $\sqcup_C(\chi_1, \bar{\chi}_2, \chi_3, \overline{X}) = \sqcup_{\mathfrak{R}^-}[G(\chi_1, \bar{\chi}_2, \chi_3, \overline{X})]$ and inserting (37) in the expression of the functional W, it results:

$$\begin{aligned} W(\mathbf{B}u, \lambda) &= \inf_{(\alpha_1, \alpha_2, \alpha_3 \kappa)} \operatorname*{stat}_{\kappa} \{\Phi(\mathbf{B}u, \alpha_1, \alpha_2, \alpha_3, \kappa) + D(\Delta\alpha_1, -\Delta\alpha_2, -\Delta\alpha_3, \Delta\kappa)\} \\ &= \inf_{(\alpha_1, \alpha_2, \alpha_3, \kappa)} \sup_{(\chi_1, \bar{\chi}_2, \chi_3, \overline{X})} \{\Phi(\mathbf{B}u, \alpha_1, \alpha_2, \alpha_3, \kappa) - \sqcup_C(\chi_1, \bar{\chi}_2, \chi_3, \overline{X}) + ((\chi_1, \Delta\alpha_1)) - ((\bar{\chi}_2, \Delta\alpha_2)) \\ &\quad - ((\chi_3, \Delta\alpha_3)) + ((\overline{X}, \Delta\kappa))\} \\ &= \inf_{(\alpha_1, \alpha_2, \alpha_3, \kappa)} \sup_{(\chi_1, \bar{\chi}_2, \chi_3, \overline{X})} \{\Phi(\mathbf{B}u, \alpha_1, \alpha_2, \alpha_3, \kappa) - ((\lambda, G(\chi_1, \bar{\chi}_2, \chi_3, \overline{X}))) + ((\chi_1, \Delta\alpha_1)) - ((\bar{\chi}_2, \Delta\alpha_2)) \\ &\quad - ((\chi_3, \Delta\alpha_3)) + ((\overline{X}, \Delta\kappa))\} + \sqcup_{\mathfrak{R}^+}(\lambda). \end{aligned}$$

The variational principle in terms of (u, λ) is the nonlocal counterpart of the variational formulation introduced by Capurso (1969), Capurso and Maier (1970) for local rate plasticity. It follows from $\mathbf{\Omega}_5$ by considering the above expression of the functional W to get the next variational principle.

Proposition 8. *The set* (u, λ) *is a solution of the convex optimization problem*:

$$\min_{(u,\lambda)} \mathbf{\Omega}_6(u, \lambda)$$

where

$$\mathbf{\Omega}_6(u, \lambda) = W(\mathbf{B}u, \lambda) - \Upsilon(u) - \langle \ell, u \rangle$$

if and only if it is a solution of the finite-step nonlocal elastoplastic problem (34).

Minimum principles are relevant in structural mechanics since, as previously shown, uniqueness can be inferred by the recourse to properties of convex functionals, and, moreover, existence results can be provided under suitable conditions and solution techniques based on minimization procedure can be exploited.

From a computational point of view, algorithms of convex optimization can be adopted for the numerical solution of the optimization finite-step problem which arises from a space discretization by the finite element method.

In particular, the variational formulation stated in Proposition 4 has been accounted for in the discussion of uniqueness and will be resorted to for the consistent formulation of the predictor-corrector algorithm and for the convergence analysis of the method in Section 8. The results of Propositions 6–8 are the generalization to the present nonlocal context of the classical principles of Prager–Hodge, Greenberg and Capurso and Maier for local rate plasticity. From the Proposition 8 a convex optimization problem will be derived (see Section 7) in order to show that the present general theory can be specialized to recover existing variational formulations for nonlocal models contributed in the literature.

7. A model with a nonlocal softening variable

The general constitutive model adopted in this paper can be specialized to several existing softening models of plasticity based upon an integral approach (Marotti de Sciarra, in press). The proposed variational formulations can then be used to recover in a straightforward manner the structural model associated with the considered constitutive model. The procedure consists in the following steps: (i) define the linear operator L introduced in (2) in order to provide the correspondence between the plastic strain p and the kinematic internal variable α_1; (ii) specialize the expression of the free energy (3); (iii) specialize the expression of the elastic domain C given by (24); (iv) specialize the general variational formulation expressed in terms of the potential Ω. Such a procedure can be repeated with reference to the other functionals provided in this paper.

Hereafter the model contributed in Borino et al. (1999) is considered in which the softening behavior is governed by one nonlocal internal variable.

The linear operator L is defined in Table 2(i) so that the kinematic internal variable α_1 has the mechanical meaning of the plastic strain p. The additive expression (3) of the free energy is reported in Table 2(ii) where the elastic component $(\Phi_{\text{el}} \circ L)$ is convex in (ε, p) and the inelastic component $(\Phi_{\text{in}} \circ \mathbf{R})$ is concave in κ.

The convex elastic domain C is defined in the space of static internal variables $(\chi_1, \overline{X})$ as the level set of a convex yield mode G in the form:

$$C = \{(\chi_1, \overline{X}) \in \mathcal{Y}_1' \times \mathcal{Y}' : G(\chi_1, \overline{X}) \leqslant 0\}$$

where G is reported in Table 2(iii).

Imposing in the expression of the functional $\mathbf{\Omega}$ the compatibility condition $(34)_2$ and the external relation $(34)_5$ in terms of Fenchel's equality (31), the potential $\mathbf{\Omega}_{1B}$ reported in Table 2(iv) is recovered.

The potential $\mathbf{\Omega}_{1B}$ turns out to be jointly convex with respect to the state variables (u, p) and jointly concave with respect to $(\chi_1, \overline{X}, \kappa)$. The stationary conditions of $\mathbf{\Omega}_{1B}$ enforced at the point $(u, \chi_1, \overline{X}, p, \kappa)$ in the form:

$$\begin{cases} 0 \in \partial_u \mathbf{\Omega}_{1B}(u, \chi_1, \overline{X}, p, \kappa) \\ 0 \in \partial_p \mathbf{\Omega}_{1B}(u, \chi_1, \overline{X}, p, \kappa) \\ 0 \in \partial_\kappa \mathbf{\Omega}_{1B}(u, \chi_1, \overline{X}, p, \kappa) \\ (0,0) \in \partial_{(\chi_1, \overline{X})} \mathbf{\Omega}_{1B}(u, \chi_1, \overline{X}, p, \kappa) \end{cases}$$

provide the structural problem associated with the model of Borino et al. (1999):

$$\begin{cases} \mathbf{B}' d\Phi_{\text{el}}(\mathbf{B}u - p) - \ell \in \partial \Upsilon(u) \\ \chi_1 = d\Phi_{\text{el}}(\mathbf{B}u - p) \\ \overline{X} = -\mathbf{R}' d\Phi_{\text{in}}(\bar{\kappa}) \\ \Delta p = \lambda dg(\chi_1), \quad \Delta\kappa = -\lambda, \end{cases} \tag{38}$$

Table 1
Independent fields appearing in the variational principles

	Independent fields												
	u	σ	χ_1	$\bar{\chi}_2$	χ_3	$\overline{X}$	ε	α_1	α_2	α_3	κ	r	λ
$\mathbf{\Omega}$	$\nearrow$	$\nearrow$	$\frown$	$\frown$	$\frown$	$\frown$	$\smile$	$\smile$	$\smile$	$\smile$	$\frown$	$\frown$	
$\mathbf{\Omega}_1$	$\nearrow$	$\nearrow$				$\smile$	$\smile$	$\smile$	$\smile$	$\smile$	$\smile$	$\frown$	
$\mathbf{\Omega}_2$	$\smile$					$\smile$		$\smile$	$\smile$	$\smile$	$\smile$	$\frown$	
$\mathbf{\Omega}_3$	$\smile$					$\smile$		$\smile$	$\smile$	$\smile$	$\smile$		
$\mathbf{\Omega}_4$		$\frown$	$\frown$	$\frown$	$\frown$	$\smile$							
$\mathbf{\Omega}_5$	$\smile$												
$\mathbf{\Omega}_6$	$\smile$												$\smile$

Table 2
Model with a local plastic strain

(i)	$L = [I_D \quad -I_D], \quad e = L(\varepsilon, \alpha_1) = \varepsilon - \alpha_1 = \varepsilon - p$
(ii)	$\Phi(\varepsilon, p, \kappa) = \Phi_{\text{el}}(\varepsilon - p) + \Phi_{\text{in}}(\mathbf{R}\kappa)$
(iii)	$G(\chi_1, \overline{X}) = g(\chi_1) - \overline{X} - \chi_0$
(iv)	$\mathbf{\Omega}_{1B}(u, \chi_1, \overline{X}, p, \kappa) = \Phi_{\text{el}}(\mathbf{B}u - p) + \Phi_{\text{in}}(\mathbf{R}\kappa) - \sqcup_C(\chi_1, \overline{X}) + ((\chi_1, p - p_{\text{o}})) + ((\overline{X}, \kappa - \kappa_{\text{o}})) - \Upsilon(u) - \langle \ell, u \rangle$

s.t. $\lambda \geqslant 0$, $G(\chi_1, \overline{X}) \leqslant 0$ and $\lambda G(\chi_1, \overline{X}) = 0$.

The relation $(38)_1$ shows that there exist a strain $\varepsilon = \mathbf{B}u$, a stress $\sigma = d\Phi_{\rm el}(\mathbf{B}u - p)$ and an external reaction $r \in \partial\Upsilon(u)$ which fulfil the equilibrium equation $\mathbf{B}'\sigma = \ell + r$. The equality $(38)_2$ shows that the static internal variable χ_1 coincides to the stress σ. The relation $(38)_3$ shows that the nonlocal static internal force $\overline{X}$ is the opposite of the $\mathbf{R}'$-transformed of the dissipative force $d_{\bar\kappa}\Phi_{\rm in}(\bar\kappa)$. The equality $(38)_4$ provides the finite-step flow rule. The model proposed in Borino et al. (1999) is thus recovered.

From a mechanical point of view, the kinematic internal variable α_1 coincides to the plastic strain and its conjugate static variable χ_1 turns out to be equal to the stress σ. The kinematic internal variable κ is the softening variable and has the physical meaning of the opposite of the cumulative plastic strain for suitable choices of the yield mode. The related dual force $\overline{X}$ is the variable that controls the size of the elastic domain.

Being $\sigma = \chi_1$, the elastic domain and the finite-step flow rule $(38)_4$ can be expressed in terms of stresses and the following variational principle holds.

Proposition 9. *The set* $(u, \sigma, \overline{X}, p, \kappa)$ *is a solution of the saddle problem*:

$$\min_{(u,p)} \max_{(\chi_1, \overline{X}, \kappa)} \mathbf{\Omega}_{1B}(u, \sigma, \overline{X}, p, \kappa)$$

if and only if it is a solution of the finite-step nonlocal elastoplastic problem (38)*.*

The mixed min–max problem above is the finite-step counterpart of the mixed-type principle reported in Borino et al. (1999) for the rate problem.

Form the mechanical point of view the kinematic internal variable κ is related to the effective plastic strain ε^p if the yield mode is such that $\|dg(\sigma)\| = 1$. In fact, being $\|\dot{p}\| = \lambda\|dg(\sigma)\| = \lambda$ and noting the equalities $-\dot{\kappa} = \lambda = \|\dot{p}\|$, the opposite of the kinematic internal variable κ has the meaning of the accumulated plastic strain $-\kappa = \int_0^t \|\dot{p}(\tau)\|\,\mathrm{d}\tau = \varepsilon^p$. As a consequence, the degradation of the yield stress is driven by the nonlocal variable $\overline{X}$ which depends on the effective plastic strain ε^p.

The finite-step counterpart of the kinematic-type variational principle reported in Borino et al. (1999) for the rate problem can be obtained from the potential $\mathbf{\Omega}_6$. In fact it turns out to be:

$$W(\mathbf{B}u, \lambda) = \Phi\big(\mathbf{B}u, p_{\rm o} + \lambda d_\sigma G(\sigma, \overline{X}), \kappa_{\rm o} + \lambda d_{\overline{X}} G(\sigma, \overline{X})\big) \tag{39}$$

subject to $\lambda \geqslant 0$, $G(\sigma, \overline{X}) \leqslant 0$ and $\lambda G(\sigma, \overline{X}) = 0$. In the case of linear elasticity and softening, assuming the expression of Table 2(iii) for the yield mode G with $\sigma = \chi_1$ and external frictionless bilateral constraints, the potential $\mathbf{\Omega}_6$, recalling the relation (39), becomes:

Proposition 10. *The set* (u, λ) *is a solution of the convex optimization problem*:

$$\min_{(u,\lambda)} \mathbf{\Omega}_{2B}(u, \lambda)$$

where

$$\mathbf{\Omega}_{2B}(u, \lambda) = \frac{1}{2}((\mathbf{E}(\mathbf{B}u - p(\lambda)), \mathbf{B}u - p(\lambda))) + \frac{1}{2}((h_{\rm NL}\mathbf{R}\kappa(\lambda), \mathbf{R}\kappa(\lambda))) - \langle \ell, u \rangle$$

under the conditions $u \in \mathcal{L}$ *and*

$$\begin{cases} p - p_{\rm o} = \lambda dg(\sigma) & \kappa - \kappa_{\rm o} = -\lambda \\ \lambda \geqslant 0, G(\sigma, \overline{X}) \leqslant 0, & \lambda G(\sigma, \overline{X}) = 0, \end{cases}$$

with $\sigma = \mathbf{E}(\mathbf{B}u - p(\lambda))$, $\overline{X} = -\mathbf{R}' h_{\rm NL}\mathbf{R}\kappa(\lambda)$ *being* $h_{\rm NL}$ *a negative modulus, if and only if it is a solution of the finite-step nonlocal elastoplastic problem* (38)*.*

The above minimum principle is also the nonlocal finite-step counterpart of the principle given by Mühlhaus and Aifantis (1991) for gradient plasticity.

7.1. Bar under uniaxial tension

To fix the ideas, let the nonlocal counterpart $\bar\kappa$ of the kinematic variable κ, linked to $\overline{X}$ which governs the size of the elastic domain, be defined in the following form:

$$\bar{\kappa}(\boldsymbol{x}) = (\mathbf{R}\kappa)(\boldsymbol{x}) = \int_{\mathcal{B}} \beta_{\boldsymbol{x}}(\xi)\kappa(\xi)\,\mathrm{d}\xi \quad \forall\, \boldsymbol{x} \in \mathcal{B} \tag{40}$$

where the volume $\mathcal{B}$ of the structure is referred to a Cartesian orthogonal co-ordinate system $\boldsymbol{x}$. The linear regularization operator $\mathbf{R} : \mathcal{Y} \to \mathcal{Z}$ transforms the local kinematic internal variable κ into the related nonlocal variable $\bar{\kappa}$ since its value at the point $\boldsymbol{x}$ of the body $\mathcal{B}$ depends on the entire field κ.

In the equality (40), β is the space weight function which describes the mutual long-range elastic interaction. From a mechanical standpoint, the space weight function β is positive, have its maximum for $\boldsymbol{x} = \xi$ and decreases monotonically and rapidly to zero approaching the boundary of the interaction zone. For $\|\boldsymbol{x} - \xi\| \geqslant R$, where R is the chosen influence distance, the space weight function β vanishes.

It is worth emphasizing that a nonlocal behavior must be present for high space variation of a local field so that it should be $\mathbf{R} = I$ for uniform fields being I the identity operator. Accordingly the weight function β must fulfil the normalizing condition:

$$\int_{\mathcal{B}} \beta_{\boldsymbol{x}}(\xi)\,\mathrm{d}\xi = 1 \tag{41}$$

for any $\boldsymbol{x}$ in $\mathcal{B}$. In order to impose such a condition, also for points close to the boundary of the body in which the interaction zone is deprived of a contribution, the following nonstandard weight function is commonly assumed (see e.g. Pijaudier-Cabot and Bažant, 1987; Jirásek and Rolshoven, 2003):

$$\beta_{\boldsymbol{x}}(\xi) = \frac{1}{V(\boldsymbol{x})} g(\boldsymbol{x}, \xi) \tag{42}$$

where the scalar function $g(\boldsymbol{x}, \xi)$ is the attenuation (or influence) function and

$$V(\boldsymbol{x}) = \int_{\mathcal{B}} g(\boldsymbol{x}, \xi)\,\mathrm{d}\xi \tag{43}$$

is called the representative volume.

The attenuation function $g(\boldsymbol{x}, \xi)$ usually depends upon the Euclidean distance $\|\boldsymbol{x} - \xi\|$ and is governed by the material internal length scale l since the regularization takes place if the distance between the source point ξ, at which a local variable is considered, and the point $\boldsymbol{x}$, where the nonlocal effect is considered, is less than the influence distance R which is a multiple of the internal length. A more refined dependence of g on the pair $(\boldsymbol{x}, \xi)$ in the case of nonhomogeneous materials and for a body having cracks or gaps in the convexity of the domain $\mathcal{B}$ occupied by the structure can be found in Polizzotto et al. (2004, 2006).

Note that the nonstandard weight function (42) is not symmetric, that is $\beta_{\boldsymbol{x}}(\xi) \neq \beta_{\xi}(\boldsymbol{x})$ due to the requirement to accommodate the uniform field condition near to the boundary of the body.

Typical choices for the attenuation functions are the Gauss-like function:

$$g(\boldsymbol{x}, \xi) = \frac{1}{l\sqrt{2\pi}} \exp\left(-\frac{\|\boldsymbol{x} - \xi\|^2}{2l^2}\right), \tag{44}$$

the bi-exponential function:

$$g(\boldsymbol{x}, \xi) = \frac{1}{2l} \exp\left(-\frac{\|\boldsymbol{x} - \xi\|}{l}\right) \tag{45}$$

where it is assumed $l = R/6$ in the examples, or the bell-shaped polynomial function:

$$g(\boldsymbol{x}, \xi) = \frac{15}{16R}\left(1 - \frac{\|\boldsymbol{x} - \xi\|^2}{R^2}\right)^2, \tag{46}$$

if $\|\boldsymbol{x} - \xi\| \leqslant R$ and $g(\boldsymbol{x}, \xi) = 0$ if $\|\boldsymbol{x} - \xi\| > R$.

The Gauss-like and the bi-exponential functions have an unbounded support so that the nonlocal interactions have effects at arbitrary distances. Since the decay of the exponential function for increasing $\|\boldsymbol{x} - \xi\|/l$ is very fast, from a computational point of view, it is possible to assume that the attenuation function $g(\boldsymbol{x}, \xi)$ is vanishing if $\|\boldsymbol{x}-\boldsymbol{\xi}\|>R$, being R the interaction distance. On the contrary the bell-shaped polynomial function (46) has a bounded support and $g(\boldsymbol{x},\boldsymbol{\xi})$ vanishes for $\|\boldsymbol{x}-\boldsymbol{\xi}\|>R$.

In order to get a symmetric weight function β, the following expression is considered in the sequel:

$$\beta_{\boldsymbol{x}}(\boldsymbol{\xi})=\left[1-\alpha\frac{V(\boldsymbol{x})}{V_\infty}\right]\delta(\boldsymbol{x},\boldsymbol{\xi})+\frac{\alpha}{V_\infty}g(\boldsymbol{x},\boldsymbol{\xi}) \tag{47}$$

where the symbol $\delta(\boldsymbol{x},\boldsymbol{\xi})$ denotes the Dirac delta function, V_∞ is the value assumed by the representative volume V for an unbounded body and α is an dimensional scalar parameter which can be calibrated by suitable identification tests. Accordingly the regularization operator is self-adjoint, i.e. $\mathbf{R}=\mathbf{R}'$. The expression (47) follows from the one proposed in Borino et al. (2003) within the context of nonlocal damage.

Let us consider a one-dimensional bar of length $L=100$ cm which is fixed at one end and loaded by an applied displacement at the opposite end. The material length scale is $l=2$ cm, the interaction distance $R=12$ cm and $\alpha=1$. The uniaxial strain distribution is uniform along the bar before the peak of the stress–strain curve. After the peak the stress remains uniform and decreases but the plastic zone is localized into a narrow band and, sometimes, into a single cross section. Typically it is necessary to distinguish between the case in which the plastic zone is far from the boundaries so that the analysis can be carried out as if the bar is infinite and the case in which the plastic region is close to the boundary and the scaling effects of the nonlocal weight must be considered. On the contrary, in the proposed approach, the two cases depicted above can be analyzed in a unitary framework since the symmetric weight function (47) is adopted.

For a uniform kinematic internal variable, Fig. 1a–c report the plots of the functions $W(\boldsymbol{x})=\int_{\mathcal{B}}\beta_{\boldsymbol{x}}(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi}$, $(1-\alpha V(\boldsymbol{x})/V_\infty)$, $\alpha V(\boldsymbol{x})/V_\infty$ and of the representative volume $V(\boldsymbol{x})$ in which the attenuation function g is, respectively, chosen as the Gauss-like function (44), the bi-exponential function (45) and the bell-shaped polynomial function (46). It is worth emphasizing that the considered weight function preserve constant fields since the normalizing condition (41) is fulfilled as shown in Fig. 1 from a computational point of view independent of the choice of the attenuation functions g. Moreover, the dual averaging preserves constant field since the regularization operator is self-adjoint.

To analyze the possible localization of the plastic strain into a part of the bar, the structural problem given by the relations (38) is considered assuming a linear elastic and softening behavior.

In a uniaxial tensile test the yield condition can be written in the form $\sigma-\overline{X}-\sigma_{\mathrm{o}}=0$. From $(38)_{3-4}$ it results $\overline{X}=-\mathbf{R}'h\mathbf{R}\kappa$ and $p=-\kappa$ so that the kinematic internal variable κ fulfils the following integral condition at a localization point x_p of the bar determined by random imperfections:

$$\frac{\sigma-\sigma_{\mathrm{o}}}{h}=(\mathbf{R}'\mathbf{R}\kappa)(x_p)=\int_{\mathcal{B}}Z(x_p,\xi)\kappa(\xi)\,\mathrm{d}\xi \tag{48}$$

being

$$(\mathbf{R}'\mathbf{R}\kappa)(x)=\int_{\mathcal{B}}\beta_x(z)\left[\int_{\mathcal{B}}\beta_z(\xi)\kappa(\xi)\,\mathrm{d}\xi\right]\mathrm{d}z=\int_{\mathcal{B}}\left[\int_{\mathcal{B}}\beta_x(z)\beta_z(\xi)\,\mathrm{d}z\right]\kappa(\xi)\,\mathrm{d}\xi=\int_{\mathcal{B}}Z(x,\xi)\kappa(\xi)\,\mathrm{d}\xi$$

where

$$Z(x,\xi)=\int_{\mathcal{B}}\beta_x(z)\beta_z(\xi)\,\mathrm{d}z \tag{49}$$

is the symmetric double weight function. The double weight function derived from the bell-shaped attenuation function g is plotted in Fig. 2, together with the attenuation function g itself, at the middle section of the previously introduced bar.

It is worth noting that the nonlocal averaging with the proposed double weight function (49) transforms a uniform local variable defined in a finite domain into a nonlocal variable which is uniform, also near the boundary. In fact it results:

$$\int_{\mathcal{B}}Z(x,\xi)\,\mathrm{d}\xi=\int_{\mathcal{B}}\int_{\mathcal{B}}\beta_x(z)\beta_z(\xi)\,\mathrm{d}z\,\mathrm{d}\xi=\int_{\mathcal{B}}\beta_x(z)\left(\int_{\mathcal{B}}\beta_z(\xi)\,\mathrm{d}\xi\right)\mathrm{d}z=\int_{\mathcal{B}}\beta_x(z)\,\mathrm{d}z=1.$$

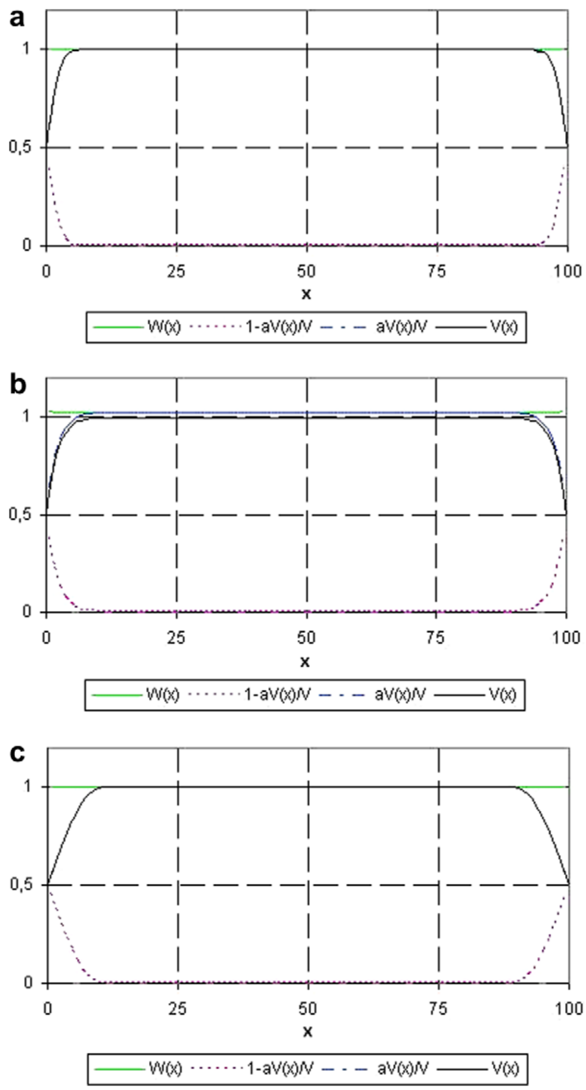

Fig. 1. Plots of the symmetric weight function $W(x)$, of the two contributions $(1 - \alpha V(x)/V_\infty)$ and $\alpha V(x)/V_\infty$ and of the representative volume $V(x)$ for a one-dimensional bar with $l = 2$ cm, $R = 12$ cm and $\alpha = 1$ for different attenuation functions: (a) Gauss-like function; (b) bi-exponential function; (c) bell-shaped polynomial function.

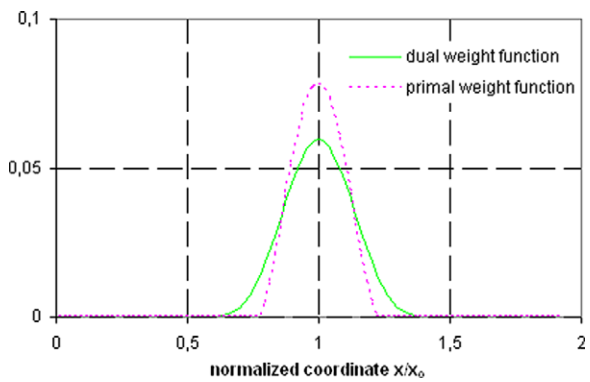

Fig. 2. Double weight function derived from the bell-shaped attenuation function g and the attenuation function g at the middle section of the one-dimensional bar with $L = 100$ cm and $R = 12$ cm.

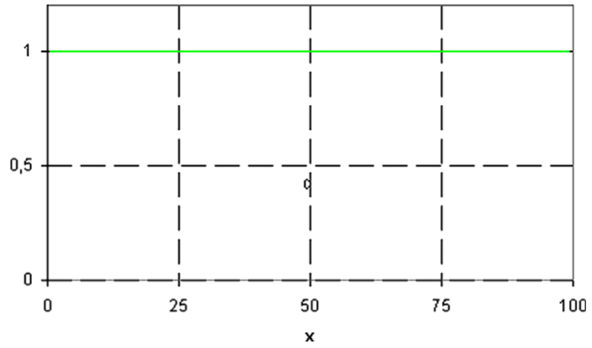

Fig. 3. Numerical evaluation of the function $\int_B Z(x,\xi)\,d\xi$ which turns out to be equal to *1* along the one-dimensional bar even in the boundary layer.

Fig. 3 shows the numerically evaluated integral above as a function of the variable x which turns out to be equal to *1* along the one-dimensional bar. On the contrary usual approaches provide symmetric double functions that do not meet the previous condition in the sense that a uniformly distributed local variable is transformed into a nonlocal variable which is not uniform in the vicinity of the boundary (see e.g. Jirásek and Rolshoven, 2003) since it is adopted a nonstandard weight function affected by scaling only in the boundary layer.

The internal variable κ fulfilling the relation (48) is then given by:

$$\kappa(x) = -p(x) = \frac{\sigma - \sigma_o}{hZ(x_p, x_p)}\delta(x - x_p) \tag{50}$$

where x_p is the localization point and δ is the Dirac distribution centered at x_p.

The nonlocal static internal variable $\overline{X}$ governing the degradation of the elastic domain turns then out to be:

$$\overline{X}(x) = (\mathbf{R}'h\mathbf{R}p)(x) = -\frac{\sigma - \sigma_o}{Z(x_p, x_p)}\int_B Z(x,\xi)\delta(\xi - x_p)\,d\xi = -\frac{\sigma - \sigma_o}{Z(x_p, x_p)}Z(x, x_p) \tag{51}$$

and it is a multiple of the double weight function centered at the localization point x_p. The plastic zone is concentrated at the single cross section of the bar placed at $x = x_p$. Since the stress is constant, the current softening law $\sigma_o + \overline{X}$ must reach its minimum at the localization point x_p and changes not only in the plastic region but also in its neighborhood since $\overline{X}$ is different form zero at points at which the local counterpart X is vanishing. Accordingly the double weight function Z must attain its maximum at $x = x_p$. This is the characteristic feature of the function Z so that the solutions (50) and (51) are valid.

The plots of the double weight function $Z(x_p, \xi)$, as a function of ξ, for the considered one-dimensional bar and for the localization point x_p placed far from the boundary, right on the boundary and set in the vicinity of the boundary are reported in Figs. 4–6 for different attenuation functions. In particular, the localization point x_p is assumed far from the end cross sections at $x_p = 51.56$ cm $(x_p/R = 4.29)$ in Fig. 4, coincident to the end cross section at $x_p = 0$ in Fig. 5 and in the boundary layer at $x_p = 6.77$ cm $(x_p/R = 1.77)$ in Fig. 6. It is apparent that the maximum of the double weight function $Z(x_p, \cdot)$ is always in correspondence of the considered localization point x_p even if the plastic zone localizes into a cross section of the bar which is affected by the presence of the boundary. On the contrary in the usual models, see e.g. Jirásek and Rolshoven (2003) for a comprehensive overview, the double weight function has a maximum at a point which is different from the localization point if it is placed at the end cross section or in the boundary layer due to the nonstandard form of the weight function.

The double weight functions derived from the bi-exponential and Gauss-like functions provide a similar distribution of the nonlocal static internal variable $\overline{X}$. Such a distribution is sharper than the one corresponding to the double weight functions derived from the bell-shaped function for each of the considerate localization points of the plastic strain $\kappa = -p$. The peak appearing in the distribution of the nonlocal static internal variable $\overline{X}$ derived from the Gauss-like function is due to the presence of a term containing the Dirac delta

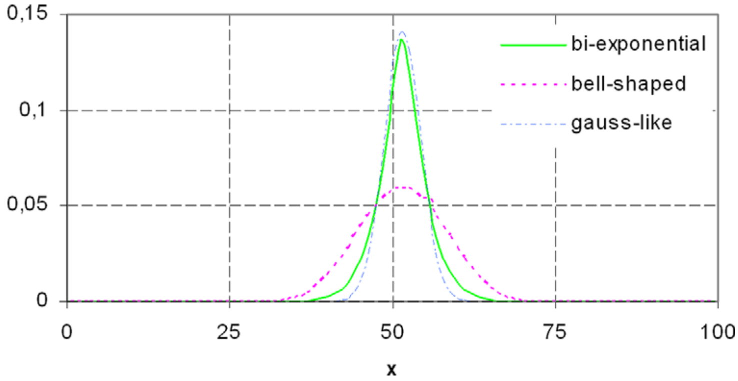

Fig. 4. Plot of the double weight function $Z(x_p, \xi)$ for the considered one-dimensional bar at the localization point $x_p = 51.56$ (far from the boundary) for different attenuation functions: Gauss-like function; bi-exponential function; bell-shaped polynomial function.

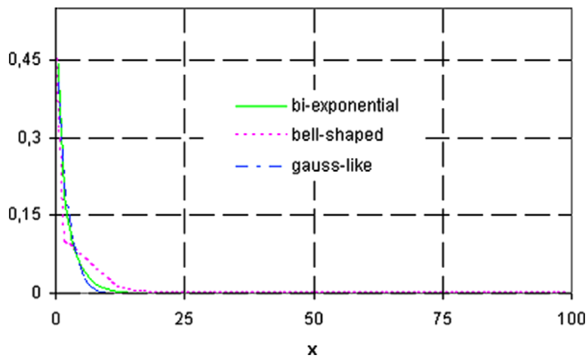

Fig. 5. Plot of the double weight function $Z(x_p, \xi)$ for the considered one-dimensional bar at the localization point placed at the boundary $x_p = 0$ for different attenuation functions: Gauss-like function; bi-exponential function; bell-shaped polynomial function. The maximum of the double weight function $Z(x_p, \xi)$ is in correspondence of the localized cross section x_p.

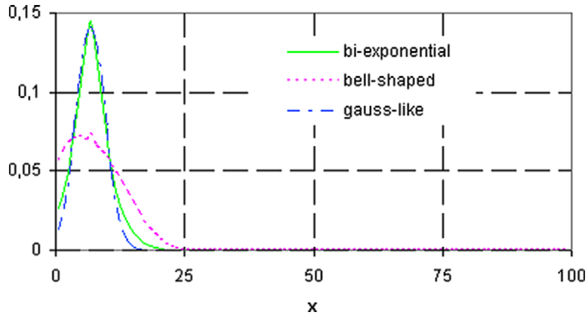

Fig. 6. Plot of the double weight function $Z(x_p, \xi)$ for the considered one-dimensional bar at the localization point $x_p = 6.77$, placed in the boundary layer, for different attenuation functions: Gauss-like function; bi-exponential function; bell-shaped polynomial function. The maximum of the double weight function $Z(x_p, \xi)$ is in correspondence of the localized cross section x_p.

$\delta(x - x_p)$ in the expression (49). Further analyses on multi-dimensional examples have to be carried out and they are the subject of ongoing researches.

The solution of the problem (48) leads to a displacement field of the form:

$$u(x) = E^{-1}\sigma(x) + H_{x_p}(x)$$

where H_{x_p} is the Heaviside function centered at x_p. The displacement jump at x_p is then given by:

$$[[u]](x_p) = \frac{\sigma - \sigma_o}{hZ(x_p, x_p)}$$

which leads to a cohesive traction-separation law:

$$\sigma = \sigma_o + hZ(x_p, x_p)[[u]](x_p).$$

The stress transmitted by the cohesive zone vanishes if the displacement jump reaches the critical value $[[u]]_0(x_p) = -\sigma/(hZ(x_p, x_p))$.

Hence the presented nonlocal formulation can be interpreted as a cohesive model where the plastic strain is localized into a set of zero measure.

8. A convergence criterion

The elastic predictor followed by the plastic corrector is the classical scheme of computational plasticity for the numerical solution of local and nonlocal elastoplastic finite-step structural problems, see e.g. Simo et al. (1988) and Mühlhaus and Aifantis (1991).

In this section, the predictor-corrector scheme in term of dissipation is consistently derived by minimizing the functional $\mathbf{\Omega}_3$ with respect to displacements and to internal variables alternatively. It is further shown that the minimum principle involving the functional $\mathbf{\Omega}_3$ leads to a sufficient condition for the convergence of the iterative procedure and to a criterion for the choice of the material elastic stiffness in the prediction phase.

In order to show the equivalence of the minimization of the functional $\mathbf{\Omega}_3$ and the predictor-corrector algorithm, the load path is divided in finite increments and the rth step (from $r-1$ to r) of the load history is considered. The corresponding increment of the applied load is denoted by $\Delta\ell^r$. The increments of the displacements and of the internal variables in the rth step are denoted by $(\Delta u^r, \Delta\alpha_1^r, \Delta\alpha_2^r, \Delta\alpha_3^r, \Delta\kappa^r, \Delta\overline{X}^r)$.

To fix the ideas, let us focus the attention in correspondence of the ith iteration (from $i-1$ to i) of the rth load step. Denoting by $\Delta\bullet^{i-1}$ the known increment of the state variable $\bullet$ at the end of the iteration $i-1$ (or equivalently at the beginning of the iteration i) and by $\delta\bullet^i$ the relevant unknown increment in the iteration ith, the increments of the state variables can be written as:

$$\Delta u^i = \Delta u^{i-1} + \delta u^i, \quad \Delta\alpha_1^i = \Delta\alpha_1^{i-1} + \delta\alpha_1^i, \quad \Delta\alpha_2^i = \Delta\alpha_2^{i-1} + \delta\alpha_2^i,$$
$$\Delta\alpha_3^i = \Delta\alpha_3^{i-1} + \delta\alpha_3^i, \quad \Delta\kappa^i = \Delta\kappa^{i-1} + \delta\kappa^i, \quad \Delta\overline{X}^i = \Delta\overline{X}^{i-1} + \delta\overline{X}^i.$$

In the case of nonlinear elastic and hardening/softening behavior and external frictionless bilateral constrains, the variational formulation associated with the potential $\mathbf{\Omega}_3$ can then be modified to get:

Proposition 11. *The set* $(\delta u^i, \delta\alpha_1^i, \delta\alpha_2^i, \delta\alpha_3^i, \delta\kappa^i, \delta\overline{X}^i)$ *is a solution of the convex optimization problem*

$$\min_{(\delta u^i, \delta\alpha_1^i, \delta\alpha_2^i, \delta\alpha_3^i, \delta\kappa^i, \delta\overline{X}^i)} \widehat{\mathbf{\Omega}}_3(\delta u^i, \delta\alpha_1^i, \delta\alpha_2^i, \delta\alpha_3^i, \delta\kappa^i, \delta\overline{X}^i)$$

where

$$\begin{aligned}\widehat{\mathbf{\Omega}}_3(\delta u^i, \delta\alpha_1^i, \delta\alpha_2^i, \delta\alpha_3^i, \delta\kappa^i, \delta\overline{X}^i) = {} & \Psi^*(\mathbf{B}(\Delta u^{i-1} + \delta u^i), \Delta\alpha_1^{i-1} + \delta\alpha_1^i, \Delta\alpha_2^{i-1} + \delta\alpha_2^i, \Delta\alpha_3^{i-1} + \delta\alpha_3^i, \Delta\overline{X}^{i-1} + \delta\overline{X}^i) \\ & - ((\Delta\overline{X}^{i-1} + \delta\overline{X}^i, \Delta\kappa^{i-1} + \delta\kappa^i)) + D(\delta\alpha_1^i, -\delta\alpha_2^i, -\delta\alpha_3^i, \delta\kappa^i) \\ & - \langle \Delta\ell^r, \Delta u^{i-1} + \delta u^i \rangle,\end{aligned}$$

with $(\Delta u^{i-1} + \delta u^i) \in \mathcal{L}$, *if and only if it is a solution of the finite-step nonlocal elastoplastic problem in terms of increments of the state variables in the ith iteration.*

The relations governing the finite-step nonlocal structural problem in terms of increments of the state variables within the ith iteration can then be recovered by performing the stationarity of the potential $\widehat{\mathbf{\Omega}}_3$. Hence it turns out to be:

$$(0,0,0,0,0,0) \in \partial\widehat{\boldsymbol{\Omega}}_3\left(\delta u^i, \delta\alpha_1^i, \delta\alpha_2^i, \delta\alpha_3^i, \delta\kappa^i, \delta\overline{X}^i\right)$$
$$\Longleftrightarrow \begin{cases} \mathbf{B}' d_{\mathbf{B}\delta u^i}\Psi^* = \Delta\ell^r \\ \left(-d_{\delta\alpha_1^i}\Psi^*, -d_{\delta\alpha_2^i}\Psi^*, -d_{\delta\alpha_3^i}\Psi^*, \Delta\overline{X}^{i-1} + \delta\overline{X}^i\right) \in \partial_{\left(\delta\alpha_1^i,\delta\alpha_2^i,\delta\alpha_3^i,\delta\kappa^i\right)} D\left(\delta\alpha_1^i, -\delta\alpha_2^i, -\delta\alpha_3^i, \delta\kappa^i\right) \\ \Delta\kappa^{i-1} + \delta\kappa^i = d_{\delta\overline{X}^i}\Psi^* \end{cases} \tag{52}$$

where the arguments of the functional Ψ^* have been dropped for simplicity. Recalling $(21)_4$, the nonlocal structural problem in the ith iteration (52) is given in the following explicit form:

$$\begin{cases} \mathbf{B}\delta u^i = \delta\varepsilon^i \\ \Delta\sigma^{i-1} + \delta\sigma^i = d_{\mathbf{B}\delta u^i}\Psi^* \\ \mathbf{B}'(\Delta\sigma^{i-1} + \delta\sigma^i) = \Delta\ell^r \\ -\Delta\chi_1^{i-1} - \delta\chi_1^i = d_{\delta\alpha_1^i}\Psi^* \\ \Delta\bar{\chi}_2^{i-1} + \delta\bar{\chi}_2^i = d_{\delta\alpha_2^i}\Psi^* \\ \Delta\chi_3^{i-1} + \delta\chi_3^i = d_{\delta\alpha_3^i}\Psi^* \\ \left(\Delta\chi_1^{i-1} + \delta\chi_1^i, \Delta\bar{\chi}_2^{i-1} + \delta\bar{\chi}_2^i, \Delta\chi_3^{i-1} + \delta\chi_3^i, \Delta\overline{X}^{i-1} + \delta\overline{X}^i\right) \in \partial D\left(\delta\alpha_1^i, -\delta\alpha_2^i, -\delta\alpha_3^i, \delta\kappa^i\right) \\ \Delta\kappa^{i-1} + \delta\kappa^i = d_{\delta\overline{X}^i}\Psi^* \end{cases} \tag{53}$$

where $(\Delta u^{i-1} + \delta u^i) \in \mathcal{L}$.

The increments of the displacements and of the kinematic and static internal variables in the ith iteration can be determined by a sequence of alternated minimization of the functional $\widehat{\boldsymbol{\Omega}}_3$ with respect to displacements and to kinematic and static internal variables.

– The *prediction phase* consists in the minimization of $\widehat{\boldsymbol{\Omega}}_3$ with respect to δu^i in which the increments of the kinematic and static internal variables are held constant to the initial values $(\Delta\alpha_1^{i-1}, \Delta\alpha_2^{i-1}, \Delta\alpha_3^{i-1}, \Delta\kappa^{i-1}, \Delta\overline{X}^{i-1})$:

$$\min_{\delta u^i} \widehat{\boldsymbol{\Omega}}_3(\delta u^i, 0, 0, 0, 0, 0).$$

The relevant stationary condition is provided by the relation $(52)_1$ in the form:

$$\mathbf{B}' d_{\mathbf{B}\delta u^i}\Psi^*\left(\mathbf{B}(\Delta u^{i-1} + \delta u^i), \Delta\alpha_1^{i-1}, \Delta\alpha_2^{i-1}, \Delta\alpha_3^{i-1}, \Delta\overline{X}^{i-1}\right) = \Delta\ell^r \tag{54}$$

with $(\Delta u^{i-1} + \delta u^i) \in \mathcal{L}$, which amounts to solving an elastic problem for the given increment of the external load in the ith iteration. In fact, being

$$\Delta\sigma^{i-1} + \delta\sigma^i = d_{\mathbf{B}\delta u^i}\Psi^*\left(\mathbf{B}(\Delta u^{i-1} + \delta u^i), \Delta\alpha_1^{i-1}, \Delta\alpha_2^{i-1}, \Delta\alpha_3^{i-1}, \Delta\overline{X}^{i-1}\right)$$

the condition (54) can be rewritten in the form:

$$\mathbf{B}'\delta\sigma^i = \Delta\ell^r - \mathbf{B}'\Delta\sigma^{i-1}.$$

The difference $\Delta\ell^r - \mathbf{B}'\Delta\sigma^{i-1}$ represents the residual load in the ith iteration, that is the difference between the applied load and the fictitious load associated with the increment of the stress at the end of the iteration $i-1$.

– The *correction phase* can be obtained by minimizing the potential $\widehat{\boldsymbol{\Omega}}_3$ with respect to the internal variables:

$$\min_{\left(\delta\alpha_1^i,\delta\alpha_2^i,\delta\alpha_3^i,\delta\kappa^i,\delta\overline{X}^i\right)} \widehat{\boldsymbol{\Omega}}_3\left(\delta u^i, \delta\alpha_1^i, \delta\alpha_2^i, \delta\alpha_3^i, \delta\kappa^i, \delta\overline{X}^i\right)$$

where δu^i is the solution of the prediction phase.

The relevant stationary conditions:

$$(0,0,0,0,0) \in \partial_{\left(\delta\alpha_1^i,\delta\alpha_2^i,\delta\alpha_3^i,\delta\kappa^i,\delta\overline{X}^i\right)} \widehat{\boldsymbol{\Omega}}_3\left(\delta u^i, \delta\alpha_1^i, \delta\alpha_2^i, \delta\alpha_3^i, \delta\kappa^i, \delta\overline{X}^i\right)$$

yield the relations $(52)_{2-3}$ which are equivalent to the relations $(53)_{4-8}$.

A convergence criterion can now be presented in order to extend to the present nonlocal variational framework an analogous criterion derived by Comi and Maier (1990) in the context of finite elements for local elastoplasticity.

Let us assume a linear elastic and hardening/softening behavior. The expression of the potential $\widehat{\mathbf{\Omega}}_3$ becomes:

$$\begin{aligned}\widehat{\mathbf{\Omega}}_3\left(\delta u^i,\delta\alpha_1^i,\delta\alpha_2^i,\delta\alpha_3^i,\delta\kappa^i,\delta\overline{X}^i\right) = &\frac{1}{2}\left(\left(\mathbf{E}\left(\mathbf{B}(\Delta u^{i-1}+\delta u^i)-A\left(\Delta\alpha_1^{i-1}+\delta\alpha_1^i\right)\right),\mathbf{B}(\Delta u^{i-1}+\delta u^i)-A\left(\Delta\alpha_1^{i-1}+\delta\alpha_1^i\right)\right)\right)\\ &+\frac{1}{2}\left(\left(\mathbf{H}_1\left(\Delta\alpha_1^{i-1}+\delta\alpha_1^i\right),\Delta\alpha_1^{i-1}+\delta\alpha_1^i\right)\right)\\ &+\frac{1}{2}\left(\left(h_2\mathbf{R}(\Delta\alpha_2^{i-1}+\delta\alpha_2^i),\mathbf{R}\left(\Delta\alpha_2^{i-1}+\delta\alpha_2^i\right)\right)\right)\\ &+\frac{1}{2}\left(\left(h_3\left(\Delta\alpha_3^{i-1}+\delta\alpha_3^i\right),\Delta\alpha_3^{i-1}+\delta\alpha_3^i\right)\right)\\ &-\frac{1}{2}\left(\left((\mathbf{R}'h\mathbf{R})^{-1}\left(\Delta\overline{X}^{i-1}+\delta\overline{X}^i\right),\left(\Delta\overline{X}^i+\delta\overline{X}^{i+1}\right)\right)\right)\\ &-\left(\left(\Delta\overline{X}^{i-1}+\delta\overline{X}^i,\Delta\kappa^{i-1}+\delta\kappa^i\right)\right)+D\left(\delta\alpha_1^i,-\delta\alpha_2^i,-\delta\alpha_3^i,\delta\kappa^i\right)\\ &-\langle\Delta\ell^r,\Delta u^{i-1}+\delta u^i\rangle.\end{aligned}$$

The above minimum principle associated with the potential $\widehat{\mathbf{\Omega}}_3$ is used hereafter to prove a theorem that establishes a sufficient condition for the convergence of the iterative procedure elastic predictor-plastic corrector and a criterion for the choice of the material elastic stiffness in the elastic prediction phase.

The theorem is based on a global convergence proposition reported in Luenberger (1973) for iterative descent algorithms. By iterative one means that the algorithm generates a series of points, each point being calculated on the basis of the points preceding it. By descent one means that as each new point is generated by the algorithm, the corresponding value of some function, evaluated at the most recent point, decreases in value. The global convergence theorem ensures that the sequence of points generated by the algorithm in this way converges to a solution of the original problem.

In order to ensure the convergence of the iterative elastic predictor-plastic corrector algorithm, it is necessary to show that the elastic prediction followed by the plastic correction is a descent algorithm. The next statement is devoted to this issue.

Theorem 12. *The potential* $\widehat{\mathbf{\Omega}}_3$ *is monotonously decreasing in the ith iteration*:

$$\widehat{\mathbf{\Omega}}_3(0,0,0,0,0,0)-\widehat{\mathbf{\Omega}}_3\left(\delta u^i,\delta\alpha_1^i,\delta\alpha_2^i,\delta\alpha_3^i,\delta\kappa^i,\delta\overline{X}^i\right)\geqslant 0 \tag{55}$$

if the material elastic stiffness is chosen as follows:

$$\begin{cases}E^i=E & \text{if } i=0\\ E^i=\gamma E & \text{if } i>0 \text{ such that } E^i \text{ is symmetric and positive definite}\end{cases}$$

with $\gamma>1/2$.

Proof. Let us show that both the prediction and the correction phases are monotonously decreasing.

- Prediction phase: The difference (55) in the prediction phase can be rewritten, after some algebra, in the form:

$$\begin{aligned}\widehat{\mathbf{\Omega}}_3(0,0,0,0,0,0)-\widehat{\mathbf{\Omega}}_3(\delta u^i,0,0,0,0,0)=&\frac{1}{2}((\mathbf{E}(\mathbf{B}\Delta u^{i-1}-A\Delta\alpha_1^{i-1}),\mathbf{B}\Delta u^{i-1}-A\Delta\alpha_1^{i-1}))\\ &-\langle\Delta\ell^r,\Delta u^{i-1}\rangle-\frac{1}{2}((\mathbf{E}(\mathbf{B}(\Delta u^{i-1}+\delta u^i)-A\Delta\alpha_1^{i-1}),\mathbf{B}(\Delta u^{i-1}\\ &+\delta u^i)-A\Delta\alpha_1^{i-1}))+\langle\Delta\ell^r,\Delta u^{i-1}+\delta u^i\rangle=\\ &-((\mathbf{B}'E\mathbf{B}\Delta u^{i-1},\delta u^i))-\frac{1}{2}((\mathbf{B}'E\mathbf{B}\delta u^i,\delta u^i))+((E\mathbf{B}\delta u^i,A\Delta\alpha_1^{i-1}))\\ &+\langle\Delta\ell^r,\delta u^i\rangle.\end{aligned} \tag{56}$$

Denoting by $\mathbf{K}^i = \mathbf{B}'E^i\mathbf{B}$ the elastic stiffness operator adopted in the prediction phase of the ith iteration, the stationarity of (56) with respect to δu^i yields:

$$\Delta\ell^r = \mathbf{K}^i\delta u^i + \mathbf{B}'E\mathbf{B}\Delta u^{i-1} - \mathbf{B}'EA\Delta\alpha_1^{i-1} = \mathbf{K}^i\delta u^i + \mathbf{B}'E(\mathbf{B}\Delta u^{i-1} - A\Delta\alpha_1^{i-1}). \tag{57}$$

Hence, substituting (57) in the relation (56), it turns out to be after some algebra:

$$\widehat{\mathbf{\Omega}}_3(0,0,0,0,0,0) - \widehat{\mathbf{\Omega}}_3(\delta u^i,0,0,0,0,0) = -\frac{1}{2}((\mathbf{K}\delta u^i,\delta u^i)) + ((\mathbf{K}^i\delta u^i,\delta u^i)) = (((\mathbf{K}^i - \frac{1}{2}\mathbf{K})\delta u^i,\delta u^i)).$$

being $\mathbf{K} = \mathbf{B}'E\mathbf{B}$ the elastic stiffness operator.

Noting that the operator

$$\mathbf{K}^i - \frac{1}{2}\mathbf{K} = \mathbf{B}'\left(E^i - \frac{1}{2}E\right)\mathbf{B}$$

is positive by assumption, it results:

$$\widehat{\mathbf{\Omega}}_3(0,0,0,0,0,0) - \widehat{\mathbf{\Omega}}_3(\delta u^i,0,0,0,0,0) \geqslant 0$$

for any conforming displacement, i.e. $(\Delta u^{i-1} + \delta u^i) \in \mathcal{L}$, and the equality holds if and only if $\delta u^i = 0$.

- Correction phase: The difference $\widehat{\mathbf{\Omega}}_3(\delta u^i,0,0,0,0,0) - \widehat{\mathbf{\Omega}}_3(\delta u^i,\ \delta\alpha_1^i,\ \delta\alpha_2^i,\ \delta\alpha_3^i,\ \delta\kappa^i,\ \delta\overline{X}^i)$ turns out to be nonnegative since the potential $\widehat{\mathbf{\Omega}}_3$ attains its minimum at the solution point $(\delta u^i,\delta\alpha_1^i,\delta\alpha_2^i,\delta\alpha_3^i,\delta\kappa^i,\delta\overline{X}^i)$, so that it results:

$$\widehat{\mathbf{\Omega}}_3(\delta u^i,0,0,0,0,0) - \widehat{\mathbf{\Omega}}_3\left(\delta u^i,\delta\alpha_1^i,\delta\alpha_2^i,\delta\alpha_3^i,\delta\kappa^i,\delta\overline{X}^i\right) \geqslant 0.$$

The proof is thus complete since the difference reported in (55) turns out to be nonnegative.□

Hence the elastic prediction-plastic correction generates a sequence of points $(\delta u^i,\delta\alpha_1^i,\delta\alpha_2^i,\delta\alpha_3^i,\delta\kappa^i,\delta\overline{X}^i)$ which converges to a solution of the structural problem.

9. Stability

In local plasticity, the property of nonexpansivity as a suitable measure of nonlinear stability of the evolution equations was introduced in Nguyen (1977). Then it is shown in Simo and Govindjee (1991) that the evolution equations in both local hardening plasticity and viscoplasticity exhibit the property of nonexpansivity and, further, it is proved that the return mapping algorithm obeys the so-called property of B-stability which is the discrete counterpart of nonexpansivity. The same issue is addressed in Reddy and Martin (1991) for local elastoplasticity in the framework of internal variables coupled with the use of the evolution law in terms of dissipation.

The property of nonexpansivity is now examined with reference to the finite-step nonlocal elastoplastic problem addressed in this paper following the approach proposed in Reddy and Martin (1991) for local plasticity.

Given a function $f : t \to f(t) = f_t \in \mathcal{X}$, the flow f_t is said to be nonexpansive with respect to the scalar product generated by a positive-definite symmetric operator $\boldsymbol{M}$ if the following inequality holds:

$$\|f(t) - \underline{f}(t)\|_{\boldsymbol{M}} \leqslant \|f(0) - \underline{f}(0)\|_{\boldsymbol{M}} \quad \text{for all } t \geqslant 0, \tag{58}$$

where $f(t)$ and $\underline{f}(t)$ are flows corresponding to distinct initial conditions $f(0)$ and $\underline{f}(0)$, respectively. The nonexpansivity condition (58) ensures that two flows generated by two nearby sets of initial conditions will be, at any time t, at least as near to each other as they were at the initial time. It is worth noting that the condition (58) is a nonlinear stability condition and no linearization will be carried out in the sequel in order to assess its validity.

A sufficient condition for (58) to hold is given by the inequality:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|f(t) - \underline{f}(t)\|^2_{\boldsymbol{M}} \leqslant 0 \quad \text{for all } t \geqslant 0$$

or, equivalently, by the condition:

$$((f(t)-\underline{f}(t),\dot{f}(t)-\dot{\underline{f}}(t)))_{\boldsymbol{M}}\leqslant 0\quad \text{for all } t\geqslant 0. \tag{59}$$

For a given strain history $\varepsilon(t)$, the nonexpansivity of the nonlocal finite-step elastoplastic model is analyzed for linear elasticity and hardening/softening behavior. For simplicity, the relations (19) and $(33)_2$ pertaining to the nonlocal finite-step model are rewritten hereafter:

$$\begin{cases} \sigma = E(\varepsilon - \Delta\alpha_1) \\ \chi_1 = A'\sigma - \mathbf{H}_1\alpha_1 = A'E\varepsilon - (A'EA + \mathbf{H}_1)\alpha_1 \\ \bar{\chi}_2 = \mathbf{R}'h_2\mathbf{R}\alpha_2 \\ \chi_3 = h_3\alpha_3 \\ \overline{X} = -\mathbf{R}'h\mathbf{R}\kappa \\ (\chi_1,\bar{\chi}_2,\chi_3,\overline{X}) \in \partial D(\Delta\alpha_1,-\Delta\alpha_2,-\Delta\alpha_3,\Delta\kappa) \end{cases} \tag{60}$$

in which all the state variables are functions of time.

By virtue of the relations between convex and saddle functional, the dissipation D can be expressed in terms of a saddle functional $B:\mathcal{Y}_1\times\mathcal{Y}_2\times\mathcal{Y}_3\times\mathcal{Y}'\to\overline{\Re}$, convex with respect to $(\Delta\alpha_1,-\Delta\alpha_2,-\Delta\alpha_3)$ and concave with respect to $\overline{X}$, in the form:

$$D(\Delta\alpha_1,-\Delta\alpha_2,-\Delta\alpha_3,\Delta\kappa) = -\inf_{\overline{Y}}\{((\overline{Y},-\Delta\kappa)) - B(\Delta\alpha_1,-\Delta\alpha_2,-\Delta\alpha_3,\overline{Y})\}$$

so that the finite-step flow rule $(33)_2$ can be equivalently rewritten as follows:

$$(\chi_1,\bar{\chi}_2,\chi_3,\overline{X}) \in \partial D(\Delta\alpha_1,-\Delta\alpha_2,-\Delta\alpha_3,\Delta\kappa) \Longleftrightarrow (\chi_1,\bar{\chi}_2,\chi_3,-\Delta\kappa) \in \partial B(\Delta\alpha_1,-\Delta\alpha_2,-\Delta\alpha_3,\overline{X}). \tag{61}$$

The subdifferential relation $(61)_2$ is then equivalent to state:

$$(\chi_1,\bar{\chi}_2,\chi_3) \in \partial_1 B(\Delta\alpha_1,-\Delta\alpha_2,-\Delta\alpha_3,\overline{X}) - \Delta\kappa \in \partial_2 B(\Delta\alpha_1,-\Delta\alpha_2,-\Delta\alpha_3,\overline{X}) \tag{62}$$

where $\partial_1 B$ denotes the subdifferential of B with respect to the variables $(\Delta\alpha_1,-\Delta\alpha_2,-\Delta\alpha_3)$ and $\partial_2 B$ denotes the superdifferential of B with respect to the variable $\overline{X}$. As a consequence, the multi-valued maps $\partial_1 B$ and $-\partial_2 B$ are cyclically monotone and hence monotone.

Let us now consider two nonlocal finite-step elastoplastic problems arising from two distinct initial conditions. The relevant state variables, following from the two distinct initial conditions, are denoted by unbarred and underbarred symbols, respectively. Then the relations (62) and the monotonicity (see Appendix A) of $\partial_1 B$ and $-\partial_2 B$ yield:

$$\begin{cases} ((\chi_1-\underline{\chi}_1,\Delta\alpha_1-\Delta\underline{\alpha}_1)) + ((\bar{\chi}_2-\underline{\bar{\chi}}_2,-\Delta\alpha_2+\Delta\underline{\alpha}_2)) + ((\chi_3-\underline{\chi}_3,-\Delta\alpha_3+\Delta\underline{\alpha}_3)) \geqslant 0 \\ ((\overline{X}-\underline{\overline{X}},\Delta\kappa-\Delta\underline{\kappa})) \leqslant 0. \end{cases} \tag{63}$$

Substitution of the static internal variables $(60)_{2\text{–}4}$ in $(63)_1$ gives, after some algebra, the inequality:

$$((\mathbf{D}(\alpha_1-\underline{\alpha}_1),\Delta\alpha_1-\Delta\underline{\alpha}_1)) + ((\mathbf{R}'h_2\mathbf{R}(\alpha_2-\underline{\alpha}_2),\Delta\alpha_2-\Delta\underline{\alpha}_2)) + ((h_3(\alpha_3-\underline{\alpha}_3),\Delta\alpha_3-\Delta\underline{\alpha}_3)) \leqslant 0 \tag{64}$$

where $\mathbf{D} = A'EA + \mathbf{H}_1$. If the operators A, $\mathbf{H}_1$ and $\mathbf{R}'h_2\mathbf{R}$ are definite positive, the relations (64) can be rewritten in the form:

$$\left(\left(\begin{bmatrix}\alpha_1-\underline{\alpha}_1\\ \alpha_2-\underline{\alpha}_2\\ \alpha_3-\underline{\alpha}_3\end{bmatrix},\begin{bmatrix}\Delta\alpha_1-\Delta\underline{\alpha}_1\\ \Delta\alpha_2-\Delta\underline{\alpha}_2\\ \Delta\alpha_3-\Delta\underline{\alpha}_3\end{bmatrix}\right)\right)_{\mathbb{M}} \leqslant 0$$

where $\mathbb{M} = \mathrm{diag}[\mathbf{D},\mathbf{R}'h_2\mathbf{R},h_3]$. Hence the evolution of the kinematic internal variables $(\alpha_1,\alpha_2,\alpha_3)$ is nonexpansive with respect to the scalar product induced by $\mathbb{M}$.

The nonexpansivity in terms of static internal variables follows from the inequality (64) which can be rewritten in terms of static internal variables by means of the constitutive relations $(60)_{2\text{–}4}$ to get:

$$((\chi_1-\underline{\chi}_1,\mathbf{D}^{-1}(\Delta\chi_1-\Delta\underline{\chi}_1))) + ((\bar{\chi}_2-\underline{\bar{\chi}}_2,(\mathbf{R}'h_2\mathbf{R})^{-1}(\Delta\bar{\chi}_2-\Delta\underline{\bar{\chi}}_2))) + ((\chi_3-\underline{\chi}_3,h_3^{-1}(\Delta\chi_3-\Delta\underline{\chi}_3))) \leqslant 0 \tag{65}$$

or equivalently:

$$\left(\left(\begin{bmatrix} \chi_1-\underline{\chi}_1 \\ \bar{\chi}_2-\underline{\bar{\chi}}_2 \\ \chi_3-\underline{\chi}_3 \end{bmatrix}, \begin{bmatrix} \Delta\chi_1-\Delta\underline{\chi}_1 \\ \Delta\bar{\chi}_2-\Delta\underline{\bar{\chi}}_2 \\ \Delta\chi_3-\Delta\underline{\chi}_3 \end{bmatrix}\right)\right)_{\mathbb{M}^{-1}} \leqslant 0$$

which ensures the nonexpansivity of the static internal variables $(\chi_1, \bar{\chi}_2, \chi_3)$ with respect to the scalar product induced by $\mathbb{M}^{-1}$.

In order to investigate the nonexpansivity of the evolution of the stress, the constitutive relations $(60)_{1-2}$ yield:

$$\sigma-\underline{\sigma} = EA\mathbf{D}^{-1}(\chi_1-\underline{\chi}_1). \tag{66}$$

Substituting the difference $\chi_1-\underline{\chi}_1$ recovered from the equality (66) into the inequality (65), it results:

$$((\sigma-\underline{\sigma}, E^{-1}A'^{-1}\mathbf{D}A^{-1}E^{-1}(\Delta\sigma-\Delta\underline{\sigma}))) + ((\bar{\chi}_2-\underline{\bar{\chi}}_2, (\mathbf{R}'h_2\mathbf{R})^{-1}(\Delta\bar{\chi}_2-\Delta\underline{\bar{\chi}}_2)))$$
$$+ ((\chi_3-\underline{\chi}_3, h_3^{-1}(\Delta\chi_3-\Delta\underline{\chi}_3))) \leqslant 0$$

or equivalently:

$$\left(\left(\begin{bmatrix} \sigma-\underline{\sigma} \\ \bar{\chi}_2-\underline{\bar{\chi}}_2 \\ \chi_3-\underline{\chi}_3 \end{bmatrix}, \begin{bmatrix} \Delta\sigma-\Delta\underline{\sigma} \\ \Delta\bar{\chi}_2-\Delta\underline{\bar{\chi}}_2 \\ \Delta\chi_3-\Delta\underline{\chi}_3 \end{bmatrix}\right)\right)_{\mathbb{N}^{-1}} \leqslant 0$$

being $\mathbb{N} = \text{diag}\,[EA\mathbf{D}^{-1}A'E, \mathbf{R}'h_2\mathbf{R}, h_3]$, which ensures the nonexpansivity of the stress σ and of the static internal variables $(\bar{\chi}_2, \chi_3)$ with respect to the scalar product induced by $\mathbb{N}^{-1}$. Note that the operators appearing in $\mathbb{N}^{-1}$ which act on the variables $\bar{\chi}_2$ and χ_3 coincide to the relevant ones of $\mathbb{M}^{-1}$.

Finally there arises the question of whether the evolution of the kinematic internal variable κ is nonexpansive and if so with respect to which scalar product. By virtue of the relations between convex and saddle functional, the dissipation D can be expressed in terms of a saddle functional $C : \mathcal{Y}_1 \times \mathcal{Y}_2' \times \mathcal{Y}_3' \times \mathcal{Y} \to \overline{\Re}$, convex with respect to $(\Delta\alpha_1, \Delta\kappa)$ and concave with respect to $(\bar{\chi}_2, \chi_3)$, in the form:

$$D(\Delta\alpha_1, -\Delta\alpha_2, -\Delta\alpha_3, \Delta\kappa) = -\inf_{(\overline{Y}_2, Y_3)}\{((\overline{Y}_2, \Delta\alpha_2)) + ((Y_3, \Delta\alpha_3)) - C(\Delta\alpha_1, \overline{Y}_2, Y_3, \Delta\kappa)\}$$

so that the finite-step flow rule $(33)_2$ can be equivalently rewritten as follows:

$$(\chi_1, \bar{\chi}_2, \chi_3, \overline{X}) \in \partial D(\Delta\alpha_1, -\Delta\alpha_2, -\Delta\alpha_3, \Delta\kappa) \iff (\chi_1, \Delta\alpha_2, \Delta\alpha_3, \overline{X}) \in \partial C(\Delta\alpha_1, \bar{\chi}_2, \chi_3, \Delta\kappa). \tag{67}$$

Hence the subdifferential relation $(67)_2$ is equivalent to state:

$$(\chi_1, \overline{X}) \in \partial_1 C(\Delta\alpha_1, \bar{\chi}_2, \chi_3, \Delta\kappa) \;\; (\Delta\alpha_2, \Delta\alpha_3) \in \partial_2 C(\Delta\alpha_1, \bar{\chi}_2, \chi_3, \Delta\kappa) \tag{68}$$

where $\partial_1 C$ denotes the subdifferential of C with respect to the variables $(\Delta\alpha_1, \Delta\kappa)$ and $\partial_2 C$ denotes the superdifferential of C with respect to $(\bar{\chi}_2, \chi_3)$. The monotonicity of the multi-valued map $-\partial_2 C$ implies:

$$((\chi_2-\underline{\chi}_2, \Delta\alpha_2-\Delta\underline{\alpha}_2)) + ((\chi_3-\underline{\chi}_3, \Delta\alpha_3-\Delta\underline{\alpha}_3)) \leqslant 0. \tag{69}$$

Such an inequality can, also, be assessed starting from $(63)_1$ by considering two nonlocal finite-step elastoplastic problems such that $\chi_1 = \underline{\chi}_1$.

Recalling the relations $(60)_{3-4}$ and the equalities $\alpha_2 = \alpha_3 = -\kappa$, the inequality (69) yields:

$$(((\mathbf{R}'h_2\mathbf{R} + h_3)(\kappa-\underline{\kappa}), \Delta\kappa-\Delta\underline{\kappa})) \leqslant 0 \tag{70}$$

so that, for any symmetric operator $Q : \mathcal{Y} \to \mathcal{Y}'$, it turns out to be:

$$((Q(\kappa-\underline{\kappa}), \Delta\kappa-\Delta\underline{\kappa})) \leqslant 0, \tag{71}$$

provided that $Q = \mathbf{R}'h_2\mathbf{R} + h_3$ using (70). Accordingly a sufficient condition for the nonexpansivity of the kinematic internal variable κ is that there exists an operator Q such that:

$$Q = \mathbf{R}'h_2\mathbf{R} + h_3.$$

Then the kinematic internal variable κ is nonexpansive with respect to the scalar product induced by Q.

The nonexpansivity in terms of the static internal variable $\overline{X}$ follows from the inequality (71) and the relation $(60)_5$:

$$(((\mathbf{R}'h\mathbf{R})^{-1}Q(\mathbf{R}'h\mathbf{R})^{-1}(\overline{X}-\underline{X}),\Delta\overline{X}-\Delta\underline{X}))\leqslant 0$$

which ensures that the nonexpansivity of the static internal variable $\bar{X}$ with respect to the scalar product induced by $(\mathbf{R}'h\mathbf{R})^{-1}Q(\mathbf{R}'h\mathbf{R})^{-1}$.

10. Closure

The response of a structural nonlocal elastoplastic problem under assigned loads is provided. A family of mixed variational principles with different combinations of state variables is addressed and a comparison between the mixed variational formulations presented in this paper, which differs for the type of independent fields, is summarized in Table 1 where the potentials are reported on the left side and, for each of them, the variables appearing in the related variational formulations are listed. The symbols $\nearrow$, $\smile$ or $\frown$ mean that the potential is linear, convex or concave in the corresponding variable. It is worth noting that many other variational formulations can be obtained following the procedure outlined in this paper.

The nonlocal elastic predictor-plastic corrector procedure is developed with reference to the evolution law expressed in terms of dissipation. A convergence criterion for the elastic prediction-plastic correction is proposed and a discussion on the uniqueness of the solution is provided. Finally the stability analysis of the nonlocal problem is analyzed.

A one-dimensional example is carried out to show the effectiveness of a recently proposed spatial weight function which allows one to treat the plastic zone close or far from the boundary in a unitary framework without the recourse to nonstandard weight functions. Multi-dimensional examples will be the subject of subsequent research works.

The proposed treatment of nonlocal plasticity can provide a basis for further developments, to be achieved elsewhere, as numerical analyses to validate the theory and computational comparisons with existing models.

The nonlocal model turns out to be rather versatile due to its thermodynamic basis and can be used to model different material behaviors such as nonlocal elasticity, nonlocal elasticity with damage, nonlocal elastoplasticity with damage and, in general, any material behavior which cast in the framework of the internal variable theories.

Acknowledgments

The study presented in this paper was developed within the activities of Rete dei Laboratori Universitari di Ingegneria Sismica – ReLUIS for the research program founded by the Dipartimento di Protezione Civile – Progetto Esecutivo 2005–2008.

This paper is dedicated to Angela on the occasion of her 12th birthday.

Appendix A

Some basic definitions and properties of convex analysis which are referred to in the paper are briefly recalled here. A comprehensive treatment of the subject can be found in Hiriart-Urruty and Lemarechal (1993).

Let (X, X') be a pair of locally convex topological vector spaces placed in separating duality by a bilinear form $((\cdot,\cdot))$. The subdifferential of the convex functional $f : X \to \Re \cup \{+\infty\}$ is the set $\partial f \subseteq X'$ given by

$$x^* \in \partial f(x_o) \Longleftrightarrow f(y) - f(x_o) \geqslant ((x^*, y - x_o)) \quad \forall\, y \in X.$$

In particular, if the functional f is differentiable at $x_o \in X$, the subdifferential is a singleton and reduces to the usual differential.

A graph G is a nonempty subset of the product space $X \times X'$. A graph $G \subseteq X \times X'$ is said to be monotone if:

$$((x_2^* - x_1^*, x_2 - x_1)) \geqslant 0 \quad \forall\ (x_i, x_i^*) \in G; \qquad i = 1, 2.$$

Moreover, a graph $G \subseteq X \times X'$ is cyclically monotone if it results:

$$\sum_{i=0}^{n} ((x_i^*, x_{i+1} - x_i)) \leqslant 0 \Longleftrightarrow \sum_{i=0}^{n} ((x_{i+1}^* - x_i^*, x_{i+1})) \geqslant 0$$

for every $(x_i, x_i^*) \in G$ with $i = 0, \ldots, n, n+1 \equiv 0$. It is apparent that a cyclically monotone graph is also monotone. It can be proved that the subdifferential of a proper convex function is cyclically monotone and, hence, monotone.

The following rules holds for subdifferentiability.

Chain rule: Given a differentiable operator $A : X \to Y$ and a convex functional $f : Y \to \Re \cup \{+\infty\}$ which turns out to be subdifferentiable at $y = A(x)$, it results:

$$\partial(f \circ A)(x) = [dA(x)]' \partial f(A(x)),$$

where $dA(x)$ is the derivative of the operator A at the point x and $[dA(x)]'$ is the dual operator;

Additivity: Given two convex functional $f_1 : X \to \Re \cup \{+\infty\}$ and $f_2 : X \to \Re \cup \{+\infty\}$ which are subdifferentiable at $x \in X$, it results:

$$\partial(f_1 + f_2)(x) = \partial f_1(x) + \partial f_2(x).$$

The conjugate of a convex functional f is the convex functional $f^* : X' \to \Re \cup \{+\infty\}$ defined by:

$$f^*(x^*) = \sup\{((x^*, y)) - f(y) \quad \text{with } y \in X\},$$

so that Fenchel's inequality holds:

$$f(y) + f^*(x^*) \geqslant ((x^*, y)) \quad \forall\ y \in X,\ \forall\ x^* \in X'.$$

The elements x, x^* for which Fenchel's inequality holds as an equality are said to be conjugate and the following relations are equivalent if f is closed:

$$f(x) + f^*(x^*) = ((x^*, x)), \quad x^* \in \partial f(x), \quad x \in \partial f(x^*).$$

Analogous results holds for concave functional by interchanging the role of $+\infty$, $\geqslant$ and sup with those of $-\infty$, $\leqslant$ and inf. The prefix *sub* used in the convex case has to be replaced by *super*. The same symbol ∂ is used to denote subdifferential (superdifferential) of a convex (concave) functional when no ambiguity can arise.

A relevant case of conjugate functionals associated with a convex set C is provided by the indicator functional:

$$\sqcup_C(x) = \begin{cases} 0 & \text{if } x \in C \\ +\infty & \text{otherwise} \end{cases}$$

and the support functional:

$$D(x^*) = \sup\{((x^*, x)) \quad \text{with } x \in C\}.$$

It is worth noting that the subdifferential of the indicator of a convex set C at a point $x \in C$ coincides to the normal cone to C at x:

$$\partial \sqcup_C(x) = N_C(x) = \begin{cases} \{x^* \in X' : ((x^*, y - x)) \leqslant 0 \quad \forall\ y \in X\} & \text{if } x \in C \\ \varnothing & \text{otherwise.} \end{cases}$$

A functional $k : X \times Y \to \overline{\Re}$ is said to be saddle (convex–concave) if $k(x, y)$ is a convex functional of $x \in X$ for each $y \in Y$ and a concave functional of y for each x. The subdifferential of the convex functional $k(\cdot, y)$ at x is defined as $\partial_1 k(x, y)$ or $\partial_x k(x, y)$ and the superdifferential of the concave functional $k(x, \cdot)$ at y is defined as $\partial_2 k(x, y)$ or $\partial_y k(x, y)$. The subdifferential of the saddle functional k at the point (x, y) is defined as follows:

$$\partial k(x, y) = \partial_x k(x, y) \times \partial_y k(x, y).$$

References

Acharya, A., Bassani, J.L., 2000. Incompatibility and crystal plasticity. Mech. Phys. Solids 48, 1565–1595.

Aifantis, E.C., 1987. The physics of plastic deformation. Int. J. Eng. Plast. 3, 211–247.

Bassani, J.L., Needleman, A., Van der Giessen, E., 2001. Plastic flow in a composite: a comparison of nonlocal continuum and discrete dislocation predictions. Int. J. Solids Struct. 38, 833–853.

Bažant, Z.P., Lin, F.-B., 1988. Nonlocal yield-limit degradation. Int. J. Numer. Methods Eng. 26, 1805–1823.

Bird, W.W., Martin, J.B., 1990. Consistent predictors and the solution of piecewise holonomic incremental problem in elastoplasticity. Eng. Struct. 12, 9–14.

Borino, G., Fuschi, P., Polizzotto, C., 1999. A thermodynamic approach to nonlocal plasticity and related variational principles. J. Appl. Mech. (ASME) 66, 952–963.

Borino, G., Failla, B., 2000. Thermodynamic consistent plasticity models with local and nonlocal internal variables. In: European Congress an Computational Methods in Applied Sciences and Engineering, CD-Rom Proceedings.

Borino, G., Failla, B., Parrinello, F., 2003. A symmetric nonlocal damage theory. Int. J. Solids Struct. 40, 3621–3645.

Brezis, H., 1983. Analyse Fonctionnelle. In: Théorie et Applications. Masson Editeur, Paris.

Capurso, M., 1969. Principi di minimo per la soluzione incrementale dei problemi elastoplasticiti. Parte I e II, Rend. Acc. Lincei, XLVI, Aprile-Maggio.

Capurso, M., Maier, G., 1970. Incremental elastoplastic analysis and quadratic optimization. Meccanica V (1), 107–117.

Comi, C., Maier, G., 1990. Extremum theorem and convergence criterion for an iterative solution to finite-step problem in elastoplasticity with mixed nonlinear hardening. Eur. J. Mech. A/Solids 9 (6), 563–585.

de Borst, R., 2001. Some recent issues in computational failure mechanics. Int. J. Numer. Methods Eng. 52, 63–95.

de Borst, R., Pamin, J., 1996. Some novel developments in finite element procedures for gradient-dependent plasticity. Int. J. Numer. Methods Eng. 39, 2477–2505.

Edelen, D.G.B., Laws, N., 1971. On the thermodynamics of systems with nonlocality. Arch. Ration Mech. Anal. 43, 24–35.

Fleck, N.A., Hutchinson, J.W., 2001. A reformulation of strain gradient plasticity. J. Mech. Phys. Solids 49, 2245–2271.

Greenberg, H.J., 1949. Complementary minimum principles for an elastic-plastic material. Q. Appl. Math. 7, 85.

Halphen, B., Nguyen, Q.S., 1975. Sur les matériaux standards géné ralisés. J. Mech. Theor. 14, 39–63.

Hiriart-Urruty, J.B., Lemarechal, C., 1993. Convex Analysis and Minimization Algorithms I–II. Springer-Verlag, New York.

Jirásek, M., Rolshoven, S., 2003. Comparison of integral-type nonlocal plasticity models for strain-softening materials. Int. J. Eng. Sci. 41, 1553–1602.

Lemaitre, J., Chaboche, J.L., 1994. Mechanics of Solids Materials. Cambridge University Press, Cambridge, UK.

Liebe, T., Steinmann, P., 2001. Theory and numerics of a thermodynamically consistent framework for geometrically linear gradient plasticity. Int. J. Numer. Methods Eng. 51, 1437–1467.

Luenberger, D.G., 1973. Linear and Nonlinear Programming. Addison-Wesley Publishing Company, London.

Marotti de Sciarra, F., 2004. Nonlocal and gradient plasticity. Int. J. Solids Struct. 41, 7329–7349.

Marotti de Sciarra, F., in press. A general theory for nonlocal softening plasticity of integral-type. Int. J. Plasticity. doi:10.1016/j.ijplas.2007.

Mühlhaus, H.B., Aifantis, E.C., 1991. A variational principle for gradient plasticity. Int. J. Solids Struct. 28, 845–857.

Nguyen, Q.S., 1977. On elastic-plastic initial boundary value problem and its numerical integration. Int. J. Numer. Methods Eng. 11, 817–832.

Nilsson, C., 1997. Nonlocal strain softening bar revisited. Int. J. Solids Struct. 34, 4399–4419.

Nilsson, C., 1999. Author's closure. Int. J. Solids Struct. 36, 3093–3100.

Ortiz, M., Popov, E.P., 1985. Accuracy and stability of integration algorithms for elastoplastic constitutive relations. Int. J. Numer. Methods Eng. 21, 1561–1576.

Pijaudier-Cabot, G., Bažant, Z.P., 1987. Nonlocal damage theory. J. Eng. Mech. (ASCE) 113, 127–144.

Polizzotto, C., Fuschi, P., Pisano, A.A., 2004. A strain-difference-based nonlocal elasticity model. Int. J. Solids Struct. 41, 2383–2401.

Polizzotto, C., Fuschi, P., Pisano, A.A., 2006. A nonhomogeneous nonlocal elasticity model. Eur. J. Mech. A/Solids 25, 308–333.

Planas, J., Elices, M., Guinea, G.V., 1993. Cohesive cracks versus nonlocal models: closing the gap. Int. J. Fracture 63, 173–187.

Prager, W., Hodge, P.G., 1951. Theory of Perfect Plastic Solids. John Wiley, New York.

Reddy, B.D., Martin, J.B., 1991. Algorithms for the solution of internal variable problems in plasticity. Comp. Methods Appl. Mech. Eng. 93, 253–273.

Romano, G., Rosati, L., Marotti de Sciarra, F., Bisegna, P., 1993a. A potential theory for monotone multi-valued operators. Q. Appl. Math. 4, 613–631.

Romano, G., Rosati, L., Marotti de Sciarra, F., 1993b. An internal variable theory of inelastic behaviour derived from the uniaxial rigid-perfectly plastic law. Int. J. Eng. Sci. 31, 1105–1120.

Romano, G., 2002. In: Structural Mechanics, vol. 2. Hevelius, Naples (in Italian).

Salençon, J., 1983. Calculà la Rupture et Analyse Limite. Presses Ponts et Chaussées, Paris.

Simo, J.C., Kennedy, J., Govindjee, S., 1988. Non-smooth multisurface plasticity and viscoplasticity. Loading/unloading conditions and numerical algorithms. Int. J. Numer. Methods Eng. 26, 2161–2185.

Simo, J.C., Govindjee, S., 1991. Non-linear B-stability and symmetry preserving return mapping algorithms for plasticity and viscoplasticity. Int. J. Numer. Methods Eng. 31, 151–176.

Strömber, L., Ristinmaa, M., 1996. FE-formulation of a nonlocal plasticity theory. Comp. Methods Appl. Mech. Eng. 136, 127–144.

Svedberg, T., Runesson, K., 1998. Thermodynamically Consistent Nonlocal and Gradient Formulations of Plasticity, EUROMECH Colloqium 378, Mulhouse, France.

Vermeer, P.A., Brinkgreve, R.B.J., 1994. A new effective non-local strain measure for softening plasticity. In: Chambon, R., Desrues, J., Vardoulakis, I. (Eds.), Localisation and Bifurcation Theory for Soils and Rocks. Balkema, Rotterdam, pp. 9–100.